NORTH-HOLLAND

On the Continuity of Generalized Inverses of Linear Operators in Hilbert Spaces

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This paper is dedicated to the memory of Professor Y. Y. Tseng.

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ABSTRACT

Let X and Y be Hilbert spaces, and let $T : X \to Y$ be a bounded linear operator with closed range. We study the continuity problem of the generalized inverse of T and related least squares solutions to the operator equation $Tx = y$. © Elsevier Science Inc., 1997

1. INTRODUCTION

Let X and Y be two Hilbert spaces, let $L(X, Y)$ be the vector space of all bounded linear operators $T : X \to Y$, and let $LC(X, Y)$ be the set of all $T \in L(X, Y)$ such that the range of T, $R(T)$, is closed. In this paper, we shall investigate the continuity of the generalized inverse of $T \in LC(X, Y)$ and the related least squares problem

$$\|Tx - y\| = \min_{z \in X} \|Tz - y\|. \tag{1}$$

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The concept of generalized inverses of matrices was first proposed by Moore in the 1920s, and a generalization of his original idea to the bounded linear operators between Hilbert spaces with closed range was mainly due to his student Tseng in the 1930s and 1940s in a series of papers (see [1] for more details). It is Nashed [10] who gave a systematical study of the perturbation and approximations of generalized inverses of linear operators between more general Banach spaces. The theory and computation of generalized inverses of matrices (finite dimensional linear operators) is complete, and several excellent monographs (e.g., [1] and [8]) have summarized the modern results in this subfield.

The perturbation analysis of the generalized inverse is important from the viewpoint of both pure and computational mathematics. The book by Stewart and Sun [11] presented a complete matrix perturbation theory. With the appearance of Wei's first paper [13] on the perturbation analysis of the Moore-Penrose generalized inverse of matrices of deficient rank, a series of papers [3, 4, 14, 15] have appeared on this subject. Recently the perturbation of the generalized inverse of infinite dimensional bounded linear operators in Hilbert spaces has been studied in [5, 6, 2]. In [6], error estimates were given for small perturbations which preserve the dimension of the null space or the range of the original bounded linear operator, and in [2] equivalent conditions, namely type I and type II perturbations, were proposed for the perturbation results. Basically the same results were obtained from different points of view, and it was implied in the papers that the generalized inverse is not continuous with respect to the operator norm.

Based on the previous results, we further explore the continuity of the map from T to $T^{\dagger}$ in this paper. We shall show that with a new topology on $\mathrm{LC}(X, Y)$, the generalized inverse is continuous. We also investigate the upper semicontinuity of the least squares solutions with respect to the operator norm.

After introducing some concepts in the next section, we present the continuity result in Section 3. Section 4 will be devoted to the upper semicontinuity of the solution set to the least squares problem (1).

2. $\gamma(T)$ AND GENERALIZED INVERSES

Let $T \in L(X, Y)$ be given with the operator norm $\|T\| = \sup\{\|Tx\| : \|x\| = 1\}$, where $\|$ q is the norm of X or Y induced by its respective inner product $(\cdot, \cdot)$. Let $N(T)$ be the null space of T, and $N(T)^{\perp}$ the orthogonal complement of $N(T)$ in X.

The number $\gamma(T)$ defined below is needed in the study of the generalized inverse of T. Let $S_M = \{x \in M : \|x\| = 1\}$ for $M \subseteq X$.

DEFINITION 2.1. Let $T \in L(X, Y)$. Define

$$\gamma(T) = \inf\{\|Tx\| : x \in S_{N(T)^\perp}\}.$$

REMARK 2.1. An equivalent definition is

$$\gamma(T) = \inf\{\|Tx\| : \operatorname{dist}(x, N(T)) = 1\},$$

where $\operatorname{dist}(x, N(T)) = \inf\{\|x - y\| : y \in N(T)\}$ is the distance of x to $N(T)$. This definition, however, is more general in the setting of Banach spaces.

Some important properties of γ are listed in the following. For more details, see the monograph of Kato [9].

PROPOSITION 2.1. *Let $T \in L(X, Y)$. Then*

(i) $\gamma(T) > 0$ *if and only if* $T \in \mathrm{LC}(X, Y)$;
(ii) $\gamma(T^*) = \gamma(T)$, *where T^* is the adjoint of T.*

DEFINITION 2.2. Let $T \in \mathrm{LC}(X, Y)$. The bounded linear operator $T^\dagger : Y \to X$ defined by

$$T^\dagger T x = x \quad \text{for} \quad x \in N(T)^\perp$$

and

$$T^\dagger y = 0 \quad \text{for} \quad y \in R(T)^\perp$$

is called the Moore-Penrose generalized inverse of T.

It is well know that $x = T^\dagger y$ is the minimal norm solution to the least squares problem (1), and all solutions to (1) constitute the affine space $T^\dagger y + N(T)$. A characterization of $T^\dagger$ is given by

PROPOSITION 2.2. *Let $T \in \mathrm{LC}(X, Y)$. Then $T^\dagger$ is the unique operator in $L(Y, X)$ such that*

$$T^\dagger T = P_{R(T^\dagger)} \quad \text{and} \quad T T^\dagger = P_{R(T)},$$

where P_M is the orthogonal projector on M.

The importance of studying $\gamma(T)$ arises from the following simple relation between $\gamma(T)$ and $\|T^\dagger\|$. For a proof, see [5].

PROPOSITION 2.3. *Let* $T \in \mathrm{LC}(X, Y)$. *Then*

$$\|T^\dagger\| = \gamma(T)^{-1}. \tag{2}$$

3. THE CONTINUITY OF $T^\dagger$

It was proved in [5] that the map $T \to T^\dagger$ is continuous at T under the operator norm if T is one-to-one or onto. But it is discontinuous elsewhere even in the finite dimensional case (see [11]). We shall show that, under a new topology for $\mathrm{LC}(X, Y)$ defined by a distance function, the above map is continuous everywhere.

The concept of the *distance* between two closed subspaces in a Banach space was introduced in [9]. Here we state it without the closedness assumption on the subspaces, in the spirit of Lemma 3.2 of [6]. Suppose X is a Banach space. for any two subspaces A and B of X, let $\delta(A, B) = \sup_{x \in S_A} \mathrm{dist}(x, B)$.

DEFINITION 3.1. The quantity

$$\hat{\delta}(A, B) = \max\{\delta(A, B), \delta(B, A)\}$$

is called the gap between A and B.

REMARK 3.1. From Lemma 3.2 of [6], $\delta(A, B) = \delta(\bar{A}, \bar{B})$. Hence,

$$\hat{\delta}(A, B) = \hat{\delta}(\bar{A}, \bar{B}).$$

PROPOSITION 3.1. *The distance function has the following properties*:

(i) $\delta(A, B) = \delta(B^\perp, A^\perp)$. *Hence,*

$$\hat{\delta}(A, B) = \hat{\delta}(A^\perp, B^\perp).$$

(ii) $\delta(A, B) < 1$ *implies* $\dim A \leqslant \dim B$. *Thus,*

$$\hat{\delta}(A, B) < 1 \quad \Rightarrow \quad \dim A = \dim B.$$

(iii) $\delta(A, B) < 1$ *implies* $A \cap B^{\perp} = \{0\}$. *Therefore,*

$$\hat{\delta}(A, B) < 1 \quad \Rightarrow \quad (A \cap B^{\perp}) \cup (B \cap A^{\perp}) = \{0\}.$$

Proof. (i) and (ii) are shown in [9]. To prove (iii), note that if $x \in A \cap B^{\perp}$ and $x \neq 0$, then for $y = x/\|x\|$,

$$\delta(A, B) \geqslant \operatorname{dist}(y, B) = \|y\| = 1. \qquad ■$$

DEFINITION 3.2. Let X and Y be Hilbert spaces and $T, S \in L(X, Y)$. Define

$$d_1(T, S) = \hat{\delta}(N(T), N(S)) + \|T - S\| \tag{3}$$

and

$$d_2(T, S) = \hat{\delta}(R(T), R(S)) + \|T - S\|, \tag{4}$$

each of which is called a distance between T and S.

REMARK 3.2. Since $d_i(T, S) \geqslant \|T - S\|$ for $i = 1, 2$, the topology defined by d_i is stronger than that defined by the operator norm.

PROPOSITION 3.2. *The map* $d_i : L(X, Y) \to R^{+}$ *defines a metric on* $L(X, Y)$ *so that* $(L(X, Y), d_i)$ *is a metric space.*

Proof. It is clear that $d_i(T, S) = d_i(S, T)$, and $d(T, S) = 0$ if and only if $T = S$. To prove the triangle inequality, it is enough to note the fact that for any two closed subspaces A, B of a Hilbert space,

$$\hat{\delta}(A, B) = \|P_A - P_B\|,$$

which follows from Theorem I.6.34 of [9]. ■

Now let T, $\bar{T} = T + \delta T \in L(X, Y)$ be given.

LEMMA 3.1.

(i) $\gamma(\bar{T}) \geqslant \gamma(T)\{1 - [\delta(N(T), N(\bar{T}))]^2\}^{1/2} - \|\delta T\|$.
(ii) $\gamma(\bar{T}) \geqslant \gamma(T)\{1 - [\delta(R(\bar{T}), R(T)]^2\}^{1/2} - \|\delta T\|$.

Proof. See [6]. ■

PROPOSITION 3.3.

(i) $\delta(N(\bar{T}), N(T)) < 1$ *implies*

$$-\gamma(T)\delta\big(N(T), N(\bar{T})\big) - \|\delta T\| \leqslant \gamma(\bar{T}) - \gamma(T)$$

$$\leqslant \frac{\gamma(T)\delta\big(N(\bar{T}), N(T)\big) + \|\delta T\|}{1 - \delta\big(N(\bar{T}), N(T)\big)}.$$

(ii) $\delta(R(T), R(\bar{T})) < 1$ *implies*

$$-\gamma(T)\delta\big(R(\bar{T}), R(T)\big) - \|\delta T\| \leqslant \gamma(\bar{T}) - \gamma(T)$$

$$\leqslant \frac{\gamma(T)\delta\big(R(T), R(\bar{T})\big) + \|\delta T\|}{1 - \delta\big(R(T), R(\bar{T})\big)}.$$

Proof. The left inequality in (i) is from Lemma 3.1(i) and the inequality $(1 - a^2)^{1/2} \geqslant 1 - a$ for $0 \leqslant a \leqslant 1$. To prove the right inequality in (i), we interchange T and $\bar{T}$ in Lemma 3.1(i) to get

$$\gamma(T) \geqslant \gamma(\bar{T})\Big\{1 - \big[\delta\big(N(\bar{T}), N(T)\big)\big]^2\Big\}^{1/2} - \|\delta T\|.$$

Since $\delta(N(\bar{T}), N(T)) < 1$,

$$\gamma(\bar{T}) \leqslant \frac{\gamma(T) + \|\delta T\|}{\Big\{1 - \big[\delta\big(N(\bar{T}), N(T)\big)\big]^2\Big\}^{1/2}} \leqslant \frac{\gamma(T) + \|\delta T\|}{1 - \delta\big(N(\bar{T}), N(T)\big)},$$

from which it follows that

$$\gamma(\bar{T}) - \gamma(T) \leqslant \frac{\gamma(T)\delta\big(N(\bar{T}), N(T)\big) + \|\delta T\|}{1 - \delta\big(N(\bar{T}), N(T)\big)}.$$

The proof of (ii) is similar. ■

Denote $\hat{\delta}_N = \hat{\delta}(N(T), N(\bar{T}))$ and $\hat{\delta}_R = \hat{\delta}(R(T), R(\bar{T}))$.

COROLLARY 3.1.

(i) *If* $\hat{\delta}_N < 1$, *then*

$$|\gamma(\bar{T}) - \gamma(T)| \leqslant \frac{\gamma(T)\hat{\delta}_N + \|\delta T\|}{1 - \hat{\delta}_N}. \tag{5}$$

(ii) *If* $\hat{\delta}_R < 1$, *then*

$$|\gamma(\bar{T}) - \gamma(T)| \leqslant \frac{\gamma(T)\hat{\delta}_R + \|\delta T\|}{1 - \hat{\delta}_R}. \tag{6}$$

Thus, $\gamma : (L(X, Y), d_i) \to R^+$ *is continuous, and* LC(X, Y) *is an open subset of* $L(X, Y)$ *under the metric* d_i.

COROLLARY 3.2. *If* $N(\bar{T}) = N(T)$ *or* $R(\bar{T}) = R(T)$, *then*

$$|\gamma(\bar{T}) - \gamma(T)| \leqslant \|\delta T\|.$$

Since $\|T^\dagger\| = \gamma(T)^{-1}$, it follows that the map $T \to \|T^\dagger\|$ is continuous on $(\mathrm{LC}(X, Y), d_i)$. Using the following decomposition, we can show that $T \to T^\dagger$ is a continuous map from $(\mathrm{LC}(X, Y), d_i)$ to $(\mathrm{LC}(X, Y), \|\ \|)$.

LEMMA 3.2. *Let* $T, \bar{T} = T + \delta T \in \mathrm{LC}(X, Y)$. *Then*

$$\bar{T}^\dagger - T^\dagger = -\bar{T}^\dagger \delta T T^\dagger + \bar{T}^\dagger(\bar{T}^\dagger)^*(\delta T)^*(I - TT^\dagger)$$

$$+(I - \bar{T}^\dagger \bar{T})(\delta T)^*(T^\dagger)^* T^\dagger.$$

THEOREM 3.1. *Let X and Y be Hilbert spaces, let* $T \in \mathrm{LC}(X, Y)$, *and let* $\bar{T} = T + \delta T \in L(X, Y)$. *Then*:

(i) $\hat{\delta}_N + \|\delta T\| \, \|T^{\dagger}\| < 1$ *implies that* $\bar{T} \in \mathrm{LC}(X, Y)$ *and*

$$\|\bar{T}^{\dagger} - T^{\dagger}\| \leqslant \left(\frac{1}{1 - \hat{\delta}_N - \|\delta T\| \, \|T^{\dagger}\|} + \frac{1}{\left[1 - \hat{\delta}_N - \|\delta T\| \, \|T^{\dagger}\|\right]^2} + 1 \right) \|T^{\dagger}\|^2 \|\delta T\|.$$

(ii) $\hat{\delta}_R + \|\delta T\| \, \|T^{\dagger}\| < 1$ *implies that* $\bar{T} \in \mathrm{LC}(X, Y)$ *and*

$$\|\bar{T}^{\dagger}\| T^{\dagger} - \leqslant \left(\frac{1}{1 - \hat{\delta}_R - \|\delta T\| \, \|T^{\dagger}\|} + \frac{1}{\left[1 - \hat{\delta}_R - \|\delta T\| \, \|T^{\dagger}\|\right]^2} + 1 \right) \|T^{\dagger}\|^2 \|\delta T\|.$$

Proof. It is enough to prove (i). From Lemma 3.1(i),

$$\|\bar{T}^{\dagger}\| \leqslant \frac{\|T^{\dagger}\|}{1 - \hat{\delta}_N - \|\delta T\| \, \|T^{\dagger}\|}.$$

Thus, Lemma 3.2 gives

$$\begin{aligned} \|\bar{T}^{\dagger} - T^{\dagger}\| &\leqslant \|\bar{T}^{\dagger}\| \, \|T^{\dagger}\| \, \|\delta T\| + \|\bar{T}^{\dagger}\|^2 \|\delta T\| + \|\delta T\| \, \|T^{\dagger}\|^2 \\ &\leqslant \left(\frac{1}{1 - \hat{\delta}_N - \|\delta T\| \, \|T^{\dagger}\|} + \frac{1}{\left(1 - \hat{\delta}_N - \|\delta T\| \, \|T^{\dagger}\|\right)^2} + 1 \right) \|T^{\dagger}\|^2 \|\delta T\|. \end{aligned}$$

This completes the proof. ■

COROLLARY 3.3. *The map $T \to T^\dagger$ is continuous from the metric space* $(\mathrm{LC}(X,Y), d_i)$ *to the normed space* $(\mathrm{LC}(Y,X), \|\ \|)$.

THEOREM 3.2. *The map $T \to T^\dagger$ is continuous from* $(\mathrm{LC}(X,Y), d_i)$ *to* $(\mathrm{LC}(Y,X), d_j)$, *where $i \neq j$.*

Proof. Since $N(\overline{T}^\dagger) = R(\overline{T})^\perp$ and $N(T^\dagger) = R(T)^\perp$,

$$\begin{aligned} d_1(\overline{T}^\dagger, T^\dagger) &= \hat{\delta}\big(N(\overline{T}^\dagger), N(T^\dagger)\big) + \|\overline{T}^\dagger - T^\dagger\| \\ &= \hat{\delta}\big(R(\overline{T})^\perp, R(T)^\perp\big) + \|\overline{T}^\dagger - T^\dagger\| \\ &= \hat{\delta}\big(R(\overline{T}), R(T)\big) + \|\overline{T}^\dagger - T^\dagger\|. \end{aligned}$$

Thus, from Theorem 3.1(ii), there is a constant K such that, for $d_2(\overline{T}, T)$ small enough,

$$d_1(\overline{T}^\dagger, T^\dagger) \leqslant K d_2(\overline{T}, T),$$

which implies that the map $T \to T^\dagger$ is continuous from $(\mathrm{LC}(X,Y), d_2)$ to $(\mathrm{LC}(Y,X), d_1)$. Similarly, from $R(\overline{T}^\dagger) = N(\overline{T})^\perp$ and $R(T^\dagger) = N(T)^{\Pi}$, we have

$$d_2(\overline{T}^\dagger, T^\dagger) \leqslant L d_1(\overline{T}, T)$$

for some constant L. ■

Before ending this section, we show that the two metrics d_1 and d_2 are actually equivalent. For this purpose, we need the following result. Its proof is found in [2].

LEMMA 3.3. *Let $T, \overline{T} = T + \delta T \in L(X,Y)$. Then*

$$\gamma(T)\,\delta\big(N(\overline{T}), N(T)\big) \leqslant \|\delta T\|, \tag{7}$$

$$\gamma(T)\,\delta\big(R(T), R(\overline{T})\big) \leqslant \|\delta T\|. \tag{8}$$

PROPOSITION 3.4. *Let* $T \in \mathrm{LC}(X,Y)$ *and* $T_n \in L(X,Y)$. *Then* $\lim_{n\to\infty} d_1(T_n, T) = 0$ *if and only if* $\lim_{n\to\infty} d_2(T_n, T) = 0$.

Proof. Suppose $\lim_{n\to\infty} d_1(T_n, T) = 0$. Then $\|T_n - T\| \to 0$ and $\hat{\delta}(N(T_n), N(T)) \to 0$. By Theorem 3.1(i),

$$\|T_n^\dagger - T^\dagger\| \to 0,$$

so $\|T_n^\dagger\|$ is uniformly bounded. From (8),

$$\lim_{n\to\infty} \delta(R(T), R(T_n)) = 0.$$

And (8) also gives

$$\gamma(T_n)\,\delta(R(T_n), R(T)) \leqslant \|T_n - T\|,$$

which together with (2) implies that

$$\lim_{n\to\infty} \delta(R(T_n), R(T)) = 0.$$

Therefore, $\lim_{n\to\infty} d_2(T_n, T) = 0$. Similarly, using (7), we can show that $d_2(T_n, T) \to 0$ implies $d_1(T_n, T) \to 0$. ∎

4. THE UPPER SEMICONTINUITY OF LEAST SQUARES SOLUTIONS

Although the map $T \to T^\dagger$ is continuous with respect to the metric topology d_i, that is not the case under the operator norm alone. In this section, we first establish the upper semicontinuity of $\gamma(T)$ with respect to the usual operator norm. Then we investigate the upper semicontinuity of the least squares solutions to the problem (1).

DEFINITION 4.1. Let B be a Banach space. A real function $f: B \to R$ is said to be upper semicontinuous at $a \in B$ if

$$\limsup_{x\to a} f(x) \leqslant f(a).$$

A point-to-set map $F: B \to 2^B$ is said to be upper semicontinuous at $a \in B$ if for any $\epsilon > 0$ and $r > 0$, there is $\delta > 0$ such that

$$F(a + B_\delta) \cap B_r \subset F(a) + B_\epsilon,$$

where 2^B denotes the collection of all subsets of B, and B_r is the open ball centered at 0 of radius r in B.

The next result indicates that if the condition of the following proposition is kept satisfied in the perturbation, then $\lim_{\delta T \to 0} \gamma(T + \delta T) = 0$, which gives the discontinuity of the generalized inverse.

PROPOSITION 4.1. *If* $N(T) \cap N(\bar{T})^{\perp} \neq \{0\}$, *then*

$$\gamma(\bar{T}) \leqslant \|\delta T\|. \tag{9}$$

That is, $\|\bar{T}^{\dagger}\| \geqslant 1/\|\delta T\|$ *if* $\bar{T}^{\dagger}$ *is well defined. More generally, if* $\delta(N(T), N(\bar{T})) = 1$, *then* (9) *is still true.*

Proof. First suppose $N(T) \cap N(\bar{T})^{\perp} \neq \{0\}$. Then there is $x \in S_{N(\bar{T})^{\perp}}$ such that $Tx = 0$. Hence,

$$\gamma(\bar{T}) \leqslant \inf\left\{\|Tx\| : x \in S_{N(\bar{T})^{\perp}}\right\} + \|\delta T\| = \|\delta T\|.$$

Now if $\delta(N(T), N(\bar{T})) = 1$, then $\delta(N(\bar{T})^{\perp}, N(T)^{\perp}) = 1$ by Proposition 3.1(i). Therefore, there is a sequence $\{x_n\} \subset S_{N(\bar{T})^{\perp}}$ with $x_n = y_n + z_n$, $y_n \in N(T)^{\perp}$, $z_n \in N(T)$ such that $\|x_n - y_n\| \to 1$. From

$$|1 - \|y_n\|| = |\|x_n\| - \|y_n\|| \leqslant \|x_n - y_n\| \to 1$$

we see that $\|y_n\| \to 0$. Thus,

$$\|Tx_n\| \leqslant \|Tx_n - Ty_n\| + \|Ty_n\| = \|Ty_n\| \to 0.$$

This completes the proof. ∎

THEOREM 4.1. $\gamma : (\mathrm{LC}(X, Y), \|\ \|) \to R^{+}$ *is upper semicontinuous.*

Proof. From the proof of Proposition 4.1, if $\|\delta T\|\,\|T^{\dagger}\| < 1$, then $\delta(N(\bar{T}), N(T)) < 1$. By Proposition 3.3(i),

$$\gamma(\bar{T}) - \gamma(T) \geqslant \frac{\gamma(T)\,\delta(N(\bar{T}), N(T)) + \|\delta T\|}{1 - \delta(N(\bar{T}), N(T))} \leqslant \frac{2\|\delta T\|}{1 - \|\delta T\|\,\|T^{\dagger}\|}.$$

Thus γ is upper semicontinuous under the operator norm.

Now we study the upper semicontinuity of the least squares solutions to (1). Let $T \in \mathrm{LC}(X, Y)$ and $y \in Y$. Define

$$M(T, y) = \left\{x \in X : \|Tx - y\| = \min_{z \in X}\|Tz - y\|\right\}.$$

Then M is a point-to-set map from $\mathrm{LC}(X, Y) \times Y$ to 2^X,

$$M(T, y) = T^{\dagger}y + N(T),$$

and $M(T, y)$ is the solution set of the so-called *normal equation*

$$T^*Tx = T^*y$$

of the problem (1). Let $\overline{T} = T + \delta T \in \mathrm{LC}(X, Y)$ and $\bar{y} = y + \delta y \in Y$.

THEOREM 4.2. *For any nonzero $\bar{x} \in M(\overline{T}, \bar{y})$, there is $x \in M(T, y)$ such that*

$$\frac{\|\bar{x} - x\|}{\|\bar{x}\|} \leqslant \|T^{\dagger}\|^2\left[\|\delta T\|\left(\frac{\|y\| + \|\delta y\|}{\|\bar{x}\|} + \|T\| + \|\delta T\|\right) + \|T\|\left(\frac{\|\delta y\|}{\|\bar{x}\|} + \|\delta T\|\right)\right]. \tag{10}$$

Proof. Let x_N be the orthogonal projection of $\bar{x} \in M(\overline{T}, \bar{y})$ onto $N(T)$, and let $x = T^{\dagger}y + x_N$. Then

$$\bar{x} - x \in N(T)^{\perp} = N(T^*T)^{\perp}. \tag{11}$$

Subtract $T^*Tx = T^*y$ from $\overline{T}^*\overline{T}\bar{x} = \overline{T}^*\bar{y}$, we have

$$T^*T(\bar{x} - x) = (\delta T)^*[y + \delta y - (T + \delta T)\bar{x}] + T^*(\delta y - \delta T\bar{x}).$$

Since $(T^*T)^{\dagger}T^*T$ is the projector onto $N(T^*T)^{\perp}$, by (11),

$$\bar{x} - x = (T^*T)^{\dagger}\{(\delta T)^*[y + \delta y - (T + \delta T)\bar{x}] + T^*(\delta y - \delta T\bar{x})\}.$$

Hence, noting that $\|(T^*T)^\dagger\| = \|T^\dagger\|^2$,

$$\begin{aligned}\|\bar{x} - x\| &\leqslant \|(T^*T)^\dagger\| \{\|(\delta T)^*[y + \delta y - (T + \delta T)\bar{x}]\| \\ &\qquad + \|T^*(\delta y - \delta T\bar{x}\|)\} \\ &\leqslant \|T^\dagger\|^2 \{\|\delta T\|[\|y\| + \|\delta y\| + (\|T\| + \|\delta T\|)\|\bar{x}\|] \\ &\qquad + \|T\|(\|\delta y\| + \|\delta T\|\,\|\bar{x}\|)\}.\end{aligned}$$

This proves (10). ∎

COROLLARY 4.1. *The point-to-set map $M : \mathrm{LC}(X, Y) \times Y \to 2^X$ is upper semicontinuous in the sense of Definition* 4.1.

Using the same technique, we can prove the following general perturbation result for the consistent linear operator equation $Ax = b$.

PROPOSITION 4.2. *Let* A, $\bar{A} = A + \delta A \in \mathrm{LC}(X, Y)$. *Suppose* $b \in R(A)$ *and* $\bar{b} \in R(\bar{A})$. *Then for any nonzero solution* $\bar{x}$ *to the equation* $\bar{A}\bar{x} = \bar{b}$, *there is a solution* x *to the equation* $Ax = b$ *such that*

$$\frac{\|\bar{x} - x\|}{\|\bar{x}\|} \leqslant \|A^\dagger\| \left(\frac{\|\delta b\|}{\|\bar{x}\|} + \|\delta A\| \right). \tag{12}$$

Proof. Let x satisfy $Ax = b$ and $\bar{x} - x \in N(A)^\perp$. Then from

$$A(\bar{x} - x) + \delta A \bar{x} = \delta b,$$

we have

$$\bar{x} - x = A^\dagger(\delta b - \delta A \bar{x}),$$

which gives (12). ∎

5. CONCLUSIONS

In this paper we have studied the continuity and the upper semicontinuity problem of the generalized inverse of a bounded linear operator on a Hilbert space with a closed range and the related least squares problem. We have shown that the generalized inverse is continuous if the perturbed operator is close to the original operator not only in the operator norm, but also with respect to the null space, and that if only the operator norm is to measure the perturbation, we have only the upper semicontinuity for the least squares problem. Error estimates under different topologies have been presented.

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