Available online at www.sciencedirect.com
ScienceDirect
Journal of Algebra 318 (2007) 680–691
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Engel graph associated with a group ☆

Alireza Abdollahi [a,b]

[a] *Department of Mathematics, University of Isfahan, Isfahan 81746-73441, Iran*
[b] *Institute for Studies in Theoretical Physics and Mathematics (IPM), Tehran, Iran*

Received 26 January 2007

Available online 27 September 2007

Communicated by Gernot Stroth

Abstract

Let G be a non-Engel group and let $L(G)$ be the set of all left Engel elements of G. Associate with G a graph $\mathcal{E}_G$ as follows: Take $G\backslash L(G)$ as vertices of $\mathcal{E}_G$ and join two distinct vertices x and y whenever $[x,_k y] \neq 1$ and $[y,_k x] \neq 1$ for all positive integers k. We call $\mathcal{E}_G$, the Engel graph of G. In this paper we study the graph theoretical properties of $\mathcal{E}_G$.

Keywords: Left Engel elements; Engel groups; Nilpotent groups

1. Introduction

Let G be a group and $x_1, \dots, x_n \in G$. For all $n > 0$ we define inductively $[x_1, \dots, x_n]$ as follows: $[x_1] = x_1$ and

$$[x_1, \dots, x_n] = [x_1, \dots, x_{n-1}]^{-1} x_n^{-1} [x_1, \dots, x_{n-1}] x_n \quad \text{for all } n > 1.$$

If $x_2 = \cdots = x_n$, then we denote $[x_1, \dots, x_n]$ by $[x_1,_{n-1} x_2]$. Note that $[x_1] = [x_1,_0 x_2] = x_1$ and $[x_1, x_2] = x_1^{-1} x_2^{-1} x_1 x_2$.

An element x of G is called *left Engel* if for every element $a \in G$, there exists a positive integer k such that $[a,_k x] = 1$. If the integer k is fixed for any element a, then the element x is called *left k-Engel*. An element x is called *bounded left Engel* if it is *left k-Engel* for some

☆ This research was in part supported by a grant from IPM (No. 86200021).
E-mail address: a.abdollahi@math.ui.ac.ir.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2007.09.007

positive integer k. The sets of all left Engel and bounded left Engel elements of G are denoted by $L(G)$ and $\overline{L}(G)$, respectively. A group G is called an *Engel* group, if $L(G) = G$.

Associate with a non-Engel group G a (simple) graph $\mathcal{E}_G$ as follows: Take $G \backslash L(G)$ as vertices of $\mathcal{E}_G$ and join two distinct vertices x and y whenever $[x,_k y] \neq 1$ and $[y,_k x] \neq 1$ for all positive integers k. We call $\mathcal{E}_G$, the Engel graph of G.

In this paper we study the graph theoretical properties of $\mathcal{E}_G$. One of our motivations for associating with a group such kind of graph is a problem posed by Erdös: For a group G, consider a graph Γ whose vertex set is G and join two distinct elements if they do not commute. Then he asked: Is there a finite bound for the cardinalities of cliques in Γ, if Γ has no infinite clique? (By a clique of a graph Δ we mean a set of vertices of Δ which are pairwise adjacent. The largest size (if it exists) of cliques of Δ is called the clique number of Δ; it will be denoted by $\omega(\Delta)$).

Neumann [12] answered positively Erdös' problem by proving that such groups are exactly the center-by-finite groups and the index of the center can be considered as the requested bound in the problem.

In fact, groups with some conditions on cliques of their Engel graphs have been already studied, without explicitly specifying that such a graph had been under consideration. For example, Longobardi and Maj [10] (also Endimioni [5]) proved that if G is a finitely generated soluble group in which every infinite subset contains two distinct elements x and y such that $[x,_k y] = 1$ for some integer $k = k(x, y)$, then G is finite-by-nilpotent. The following result is an easy consequence of the latter which may be considered as an answer to a Erdös like question on Engel graphs.

Theorem 1.1. *If G is a non-Engel finitely generated soluble group, then $\mathcal{E}_G$ has no infinite clique if and only if the clique number of $\mathcal{E}_G$ is finite.*

In [1], the condition $\mathcal{E}(n)$ ($n \in \mathbb{N}$) on groups was considered by the author: Every subset of elements consisting of more than n elements possesses a pair x, y such that $[x,_k y] = 1$ for some $k = k(x, y) \in \mathbb{N}$. This simply means that we were studying groups G such that every clique of $\mathcal{E}_G$ has size at most n. It is shown in [1] that all finite groups and all finitely generated soluble groups satisfying $\mathcal{E}(n)$ are nilpotent for $n \leqslant 2$ and that all finite groups in $\mathcal{E}(n)$ are soluble for $n \leqslant 15$. Therefore we can summarize the latter results in terms of Engel graph as following.

Theorem 1.2. *Let G be a finite or a finitely generated soluble non-Engel group. Then $\omega(\mathcal{E}_G) \geqslant 3$. If G is finite and $\omega(\mathcal{E}_G) \leqslant 15$, then G is soluble.*

In [2], it is proved that if G is a finitely generated soluble group satisfying $\mathcal{E}(n)$, then the index of the hypercenter of G is bounded by a function of n. Also it is proved that if G is a finite group satisfying $\mathcal{E}(n)$, then the index of the Fitting subgroup of G is bounded by a function of n. Note that if G is either a finite group or a soluble group, then $L(G)$ is a subgroup and coincides with the Fitting subgroup of G (see [3] and [7, Proposition 3]).

Theorem 1.3. *Let G be a finite or a finitely generated soluble non-Engel group such that $\omega(\mathcal{E}_G)$ is finite. Then the index of the Fitting subgroup of G is finite and bounded by a function of $\omega(\mathcal{E}_G)$.*

Also in [2] finite centerless groups G satisfying the condition $\mathcal{E}(n)$ for $n \leqslant 15$ are characterized. This of course implies that we have a characterization of all finite centerless groups G with $\omega(\mathcal{E}_G) \leqslant 15$.

In Section 2, we study the connectedness of $\mathcal{E}_G$ for a non-Engel group G. We do not know whether there is a non-Engel group G such that $\mathcal{E}_G$ is not connected. We give some classes of groups whose Engel graphs are connected. Throughout Section 2 we generalize some known results on the set of left Engel elements by defining new types of Engel elements. Most of the results are about the connectedness of certain subgraphs of a non-Engel group. In Section 3, we characterize finite groups whose Engel graphs are planar. In Section 4 we shortly study groups with isomorphic Engel graphs and we show that the Engel graph of a finite group cannot be isomorphic to the Engel graph of an infinite one.

2. Connectedness of Engel graph

In this section, we study the connectedness of the Engel graph $\mathcal{E}_G$ for a non-Engel group G. Before starting to show the results, we recall some concepts for a simple graph Δ. A *path* P in Δ is a sequence $v_0 - v_1 - \cdots - v_k$ whose terms are vertices of Δ such that for any $i \in \{1, \ldots, k\}$, v_{i-1} and v_i are adjacent. In this case P is called a path between v_0 and v_k. The number k is called the *length* of P. If v and w are vertices in Δ, then by definition $d(v, v) = 0$ and whenever $v \neq w$, $d(v, w)$ denotes the length of the shortest path between v and w if a path exists, otherwise $d(v, w) = \infty$ and we call $d(v, w)$ the distance between v and w in Δ. We say that Δ is *connected* if there is a path between each pair of vertices of Δ. If Δ is connected, then the largest distance between all pairs of the vertices of Δ is called the *diameter* of Δ, and it is denoted by $\mathrm{diam}(\Delta)$.

Some results and examples suggest that the Engel graph of any finite non-Engel group must be connected. We have checked the Engel graphs of all non-Engel finite groups of orders at most 570 (except ones of order 384!) by GAP [6] using the package GRAPE and the following program written in GAP. All these groups have connected Engel graphs with diameter 1 or 2.

```
c:=function(a,b,G)
local n,L,s,i,A,M;
n:=Size(G); M:=1; s:=0;
if Comm(a,b)=Identity(G) then s:=1; fi;
A:=a;
while s=0 do
  A:=Comm(A,b); M:=M+1;
  if A=Identity(G) then s:=1; fi;
  if M>n and A<>Identity(G) then s:=-1; fi;
od;
if s=0 or s=-1 then L:=true; fi;
if s=1 then L:=false; fi;
return L;
end;

c2:=function(a,b,G)
return c(a,b,G) and c(b,a,G);
end;

LoadPackage("GRAPE");
```

```
EngelGraph:=function(G)
return Graph(G,Difference(G,FittingSubgroup(G)),OnPoints,
  function(x,y) return c2(x,y,G); end);
end;

DiametersOfEngelGraphs:=function(n,m)
return Union(List([n..m],j->List(AllSmallGroups(j,IsNilpotent,
  false),i->Diameter(EngelGraph(i)))));
end;
```

The difficulty to settle the question of whether Engel graphs of finite non-Engel groups are connected, is that the relation $\sim$ on G defined as

$$x \sim y \quad \Leftrightarrow \quad [x,_k y] = 1 \quad \text{for some } k \in \mathbb{N}$$

is not symmetric, for example in the symmetric group of degree 3, we have $(1,2) \sim (1,2,3)$ but $(1,2,3) \nsim (1,2)$. Indeed, if it is true that $\mathcal{E}_G$ is connected, it should be true that $\mathcal{E}_G$ has no isolated vertex, i.e. a vertex v such that $d(v,w) = \infty$ for every vertex $w \neq v$. We were unable to prove the latter for all groups, but we shall prove it for some wide classes of groups. Our results generalize some well-known results on left Engel elements and Engel sets, as we consider a new type of Engel elements in groups. In the definition of edges of the complement of an Engel graph, since we want to have a simple graph, we use the symmetrized of the relation $\sim$, i.e.

$$x \sim' y \quad \Leftrightarrow \quad \text{either } [x,_k y] = 1 \quad \text{or} \quad [y,_k x] = 1 \quad \text{for some } k \in \mathbb{N}.$$

This motivates to consider the following types of Engel conditions on elements of a group.

Let G be a group. An element a of G is called *randomly power Engel* if for every $g \in G$, there exists a sequence $a_1, \dots, a_k$ of elements of $\langle a \rangle$ with $\langle a \rangle = \langle a_i \rangle$ for all $i \in \{1, \dots, k\}$ such that either $[a_1^g, a_2, \dots, a_k] = 1$ or $[a_1, a_2^g, \dots, a_k^g] = 1$. The word "randomly" has been selected for the name of such an element, because of the "either … or" sentence involved in the definition and the word "power" is for the fact that, the elements $a_1, \dots, a_n$ are indeed powers of a. If the integer k in the above is the same for all $g \in G$, then we say that a is a *randomly power k-Engel* element. If all the elements $a_1, \dots, a_k$ can be always chosen equal to a, then a is called a *randomly Engel element* of G, *randomly k-Engel* if k is the same for all $g \in G$. An element a of G is called *bounded randomly (power) Engel* if it is randomly (power) k-Engel for some positive integer k. The set of randomly power Engel elements in a group G is denoted by $\mathcal{L}(G)$ and we denote by $\overline{\mathcal{L}}(G)$ the set of bounded randomly power Engel elements of G. Clearly we have $L(G) \subseteq \mathcal{L}(G)$, $\overline{L}(G) \subseteq \overline{\mathcal{L}}(G)$ and these sets are invariant under conjugation of elements of G.

We need the following lemma in the sequel.

Lemma 2.1. *Let a and g be elements of a group such that the normal closure of a in $\langle a, g \rangle$ is abelian. If $[a, g^{t_1}, \dots, g^{t_k}] = 1$ for some $t_1, \dots, t_k \in \mathbb{N}$, then $[a,_k g^m] = 1$, where m is any positive integer divisible by the least common multiple of $t_1, \dots, t_k$.*

Proof. Since $\langle a \rangle^{\langle a,g \rangle}$ is abelian, we may write

$$\left[a, g^m, g^{t_2}, \dots, g^{t_k}\right] = \left[\left[a, g^{t_1}\right]^{(g^{t_1})^{\frac{m}{t_1}-1} + (g^{t_1})^{\frac{m}{t_1}-2} + \dots + g^{t_1} + 1}, g^{t_2}, \dots, g^{t_k}\right]$$

$$\left[a, g^{t_1}, g^{t_2}, \ldots, g^{t_k}\right]^{(g^{t_1})^{\frac{m}{t_1}-1}+(g^{t_1})^{\frac{m}{t_1}-2}+\cdots+g^{t_1}+1} = 1.$$

Now since the normal closure of $[a, g^m]$ in $\langle a, g\rangle$ is also abelian, by induction on k, we have that $[a,_k g^m] = 1$. This completes the proof. □

Lemma 2.2. *Let A be a normal abelian subgroup of a group G and let $g \in G$.*

(1) *If $g \in \mathcal{L}(A\langle g\rangle)$, then $A\langle g\rangle$ is locally nilpotent.*
(2) *If g is a randomly power k-Engel element of $A\langle g\rangle$, then $A\langle g\rangle$ is nilpotent of class at most $k+1$.*

Proof. Let $K = A\langle g\rangle$. To prove (1) and (2) it is enough to show that $g \in L(K)$ and g is a left k-Engel element, respectively.

Let $a \in A$, $k \in \mathbb{N}$ and $t_1, t_2, \ldots, t_k \in \mathbb{Z}$. Then, since A is a normal abelian subgroup of G, we can write

$$\begin{aligned}\left[(g^{t_1})^a, g^{t_2}, \ldots, g^{t_k}\right] &= \left[\left[g^{t_1}, a\right], g^{t_2}, \ldots, g^{t_k}\right] \\ &= \left[\left[a, g^{t_1}\right]^{-1}, g^{t_2}, \ldots, g^{t_k}\right] = \left[a, g^{t_1}, g^{t_2}, \ldots, g^{t_k}\right]^{-1},\end{aligned}$$

and

$$\begin{aligned}\left[g^{t_1}, (g^{t_2})^a, \ldots, (g^{t_k})^a\right] &= \left[(g^{t_1})^{a^{-1}}, g^{t_2}, \ldots, g^{t_k}\right]^a \\ &= \left[\left[g^{t_1}, a^{-1}\right], g^{t_2}, \ldots, g^{t_k}\right]^a = \left[a, g^{t_1}, g^{t_2}, \ldots, g^{t_k}\right].\end{aligned}$$

Since $g \in \mathcal{L}(K)$, it follows from these equalities that for any $a \in A$, there exists a sequence $t_1, \ldots, t_k$ of integers with $\langle g^{t_i}\rangle = \langle g\rangle$ for every $i \in \{1, \ldots, k\}$ such that

$$\left[a, g^{t_1}, g^{t_2}, \ldots, g^{t_k}\right] = 1. \tag{$*$}$$

Now if g is of infinite order, then $t_1, \ldots, t_k \in \{1, -1\}$ and by $(*)$ it is easy to see that for any $a \in A$, there exists a positive integer k such that $[a,_k g] = 1$. Since A is abelian and normal in K, it follows that $g \in L(K)$. Now assume that g is of finite order and let m be the product of all positive integers $t \leqslant |g|$ such that $\gcd(t, |g|) = 1$. It follows from $(*)$ and Lemma 2.1 that $[a,_k g^m] = 1$, which implies that $g^m \in L(K)$. Now by a result of Gruenberg [7, Proposition 3], $L(K)$ is a subgroup of G and so $g \in L(K)$. This shows that, in any case $g \in L(K)$. The above argument also shows that if g is a randomly power k-Engel element of K, then g is a left k-Engel element of K. This completes the proofs of (1) and (2). □

Theorem 2.3. *Let G be a nilpotent-by-abelian non-Engel group. If $\mathcal{E}_{\frac{G}{H'}}$ is connected for some nilpotent subgroup H containing G', then $\mathcal{E}_G$ is connected. Moreover, in this case, the diameter of $\mathcal{E}_G$ is at most* $\max\{\operatorname{diam}(\mathcal{E}_{\frac{G}{H'}}), 2\}$.

Proof. For every $a \in G$, $a \in L(G)$ if and only if $aH' \in L(\langle a\rangle H/H')$; for if $aH' \in L(\langle a\rangle H/H')$, then $\langle a\rangle H/H'$ is locally nilpotent by Lemma 2.2. Since H is nilpotent, a Hall-type result of Plotkin [14] (see also [17]) implies that $\langle a\rangle H$ is locally nilpotent. Since $G' \leqslant H$, $\langle a\rangle^G \leqslant \langle a\rangle H$

and so $a \in L(G)$. It follows that xH' is a vertex of $\mathcal{E}_{\frac{G}{H'}}$ if and only if x is a vertex of $\mathcal{E}_G$. Now let x and y be two distinct vertices in $\mathcal{E}_G$. If $xH' = yH'$, then there exists $z \in G$ such that zH' is an adjacent vertex to xH'. It follows that $x - z - y$ is a path of length 2 between x and y in $\mathcal{E}_G$. Now if $xH' \neq yH'$, then there exists a path P of length $d \leqslant \operatorname{diam}(\mathcal{E}_{\frac{G}{H'}})$ between xH' and yH' in $\mathcal{E}_{\frac{G}{H'}}$. Now any set of preimages of the vertices of P in G (under the natural epimorphism $G \to \frac{G}{H'}$) forms a path of length d between x and y in G. This completes the proof. □

Theorem 2.4. *If G is a nilpotent-by-cyclic non-Engel group, then $\mathcal{E}_G$ is connected and its diameter is at most 6. If G is nilpotent-by-abelian and $x \notin L(G)$, then $\langle x\rangle^G$ is non-Engel, $\mathcal{E}_{\langle x\rangle^G}$ is connected with diameter at most 6, and the induced subgraph of $\mathcal{E}_G$ on the conjugacy class of x in G is connected with diameter at most 2.*

Proof. Let $G = A\langle g\rangle$, where A is a normal nilpotent subgroup of G and $g \in G$. By Theorem 2.3 we may assume that A is abelian. Let g_1 and g_2 be two distinct vertices of $\mathcal{E}_G$ such that $g_1 = a_1 g^n$ and $g_2 = a_2 g^n$ for some $a_1, a_2 \in A$ and $n \in \mathbb{Z}$. Since A is a normal abelian subgroup of G, g_1 is adjacent to g_2 if and only if

$$\left[a_1 a_2^{-1},_k g^n\right] \neq 1 \quad \text{for all } k \in \mathbb{N}.$$

Since g_1 is a vertex of $\mathcal{E}_G$, $g_1 \notin \mathcal{L}(G)$ by Lemma 2.2. Thus g_1 is adjacent to g_1^y for some $y \in G$. Note that $g_1^y = a_3 g^n$ for some $a_3 \in A$. Therefore $[a_1 a_3^{-1},_k g^n] \neq 1$ for all $k \in \mathbb{N}$. Now if g_1 and g_2 are not adjacent, then $[a_1 a_2^{-1},_m g^n] = 1$ for some $m \in \mathbb{N}$. Thus $[a_2 a_3^{-1},_k g^n] \neq 1$ for all $k \geqslant m$ and so clearly we have that $[a_2 a_3^{-1},_k g^n] \neq 1$ for all $k \in \mathbb{N}$. Hence g_2 is adjacent to g_1^y and $g_1 - g_1^y - g_2$ is a path of length 2 between g_1 and g_2. This implies that any two distinct vertices of $\mathcal{E}_G$ of forms $a_1 g^n$ and $a_2 g^n$ are either adjacent or there is a path of length 2 between them so that the middle vertex in this path can be a suitable conjugate of either g_1 or g_2 $(*)$.

Now we prove that if $a_1 g^n$ is a vertex of $\mathcal{E}_G$ (with $a_1 \in A$ and $n \in \mathbb{Z}$), then there is a path of length at most 3 between $a_1 g^n$ and g. As $g^n \notin \mathcal{L}(G)$ by Lemma 2.2, there exists $a \in A$ such that $(g^n)^a$ is adjacent to g^n. This is equivalent to $[a,_k g^n] \neq 1$ for all $k \in \mathbb{N}$. Now Lemma 2.1 implies that $[a, g^n,_k g] \neq 1$ and $[a, g,_k g^n] \neq 1$ for all $k \in \mathbb{N}$. Therefore $[(g^n)^a,_k g] \neq 1$ and $[g,_k (g^n)^a] \neq 1$ for all $k \in \mathbb{N}$ and so $(g^n)^a$ is adjacent to g. By the previous part, there is a path of length at most 2 between $a_1 g^n$ and $(g^n)^a$ and so there is a path of length 3 between $a_1 g^n$ and g.

Now let $a_2 g^m$ be another vertex in $\mathcal{E}_G$, where $a_2 \in A$. Therefore by the latter paragraph, there is a path of length 3 between $a_2 g^m$ and g. Thus $d(a_1 g^n, a_2 g^m) \leqslant 6$. This completes the proof of the first statement of the theorem.

Suppose now that G is nilpotent-by-abelian and $x \notin L(G)$. Then $x \notin L(\langle x\rangle^G)$, otherwise, by a result of Gruenberg [7, Proposition 3], x lies in the Hirsch–Plotkin radical of $\langle x\rangle^G$, and since the latter subgroup is normal in G, x lies in the Hirsch–Plotkin radical of G, which contradicts $x \notin L(G)$. This implies that $\langle x\rangle^G$ is non-Engel. As $\langle x\rangle^G \leqslant \langle x\rangle G'$, we have that $\langle x\rangle^G$ is nilpotent-by-cyclic, and so by the previous part $\mathcal{E}_{\langle x\rangle^G}$ is connected and $\operatorname{diam}(\mathcal{E}_{\langle x\rangle^G}) \leqslant 6$.

The last part of the theorem easily follows from $(*)$. □

As far as we know, it is an open problem whether the set of left Engel elements of an arbitrary group G forms a subgroup. However there are classes of groups G in which not only $L(G)$ and $\mathcal{L}(G)$ are subgroups but also they are very well-behaved. We shall see some of these classes in the following.

Theorem 2.5. *Let G be a soluble group.*

(1) *$\mathcal{L}(G)$ coincides with the Hirsch–Plotkin radical of G and is a Gruenberg group. In particular, $\mathcal{L}(G) = L(G)$.*
(2) *$\overline{\mathcal{L}}(G)$ coincides with the Baer radical of G. In particular, $\overline{\mathcal{L}}(G) = \overline{L}(G)$.*

Proof. Let $g_1 \in \mathcal{L}(G)$ and $g_2 \in \overline{\mathcal{L}}(G)$. By a result of Gruenberg [7, Proposition 3] (see also [15, 12.3.3]), it is enough to show that $g_1 \in L(G)$ and $g_2 \in \overline{L}(G)$. We argue by induction on the derived length d of G. If $d \leqslant 1$, then G is abelian and $g_1, g_2 \in \overline{L}(G)$; thus we can assume $d > 1$ and write $A = G^{(d-1)}$. Now obviously $g_1 A \in L(G/A)$ and $g_2 A \in \overline{L}(G/A)$. It remains to prove that $g_1 \in L(\langle A, g_1\rangle)$ and $g_2 \in \overline{L}(\langle A, g_2\rangle)$: this immediately follows from Lemma 2.2. □

In [3] Baer proved that in every group G satisfying the maximal condition on all subgroups, $L(G)$ coincides with the Fitting subgroup of G, and in [13], Peng by using an argument of Baer, generalized the latter result as following.

Theorem 2.6. *If G is a group satisfying the maximal condition on abelian subgroups, then $L(G)$ coincides with the Fitting subgroup of G.*

Recently a weaker version of Theorem 2.6 has been proved in [11, Theorem 1]. In the following we generalize Theorem 2.6 to randomly power Engel elements.

Theorem 2.7. *If G is a group satisfying the maximal condition on abelian subgroups, then $L(G) = \mathcal{L}(G)$. In particular, $\mathcal{L}(G)$ coincides with the Fitting subgroup of G.*

Proof. Let $a \in \mathcal{L}(G)$. It is enough to show that $\langle a\rangle^G$ is nilpotent.

The proof is similar (mostly *mutatis mutandis*) to that given for [16, Part 2, Lemma 7.22], but it needs some little change. With the notation of that proof, one has to consider the subgroups

$$W_U = \langle N_U(I) \cap a^G\rangle \quad \text{and} \quad W_V = \langle N_V(I) \cap a^G\rangle,$$

where a^G denotes the set of all conjugates of a in G, and fix elements $v \in (N_V(I) \cap a^G)\backslash I$ and $u \in (N_U(I) \cap a^G)\backslash I$. Then make use of Theorem 2.5 and of the fact that if $\langle a^t\rangle = \langle a\rangle$ for some integer t, then $\langle u^t\rangle = \langle u\rangle$ and $\langle v^t\rangle = \langle v\rangle$. □

In [16, Part 2, p. 55], a subset S of a group G is called an *Engel set* if given x and y in S there is an integer $n = n(x, y)$ such that $[x,_n y] = 1$. As a corollary to [16, Part 2, Lemma 7.22] (Theorem 2.6, here), normal Engel sets in a group G satisfying the maximal condition on abelian subgroups are characterized in [16, Part 2, Theorem 7.23] as the subsets of the Fitting subgroup of G. Recall that a normal set in a group is a set closed under conjugation. We call a subset R of a group G a *randomly Engel set* if given x and y in R there is an integer $n = n(x, y)$ such that either $[x,_n y] = 1$ or $[y,_n x] = 1$. Generally a randomly Engel set is not an Engel set, consider for example $\{(1, 2), (1, 2, 3)\}$ in the symmetric group of degree 3, but we shall see in the following result that the normal ones are the same in certain groups.

Theorem 2.8. *Let G be a group satisfying the maximal condition on abelian subgroups* (*so G may be a finite group*). *Then a normal subset of G is a randomly Engel set if and only if it* *is contained in the Fitting subgroup of G. In particular, an element x of G lies in the Fitting subgroup of G if and only if for every $g \in G$, there exists a positive integer k such that either $[x^g,_k x] = 1$ or $[x,_k x^g] = 1$.*

Proof. Suppose that the normal subset S of G is a randomly Engel set and let $a \in S$. If $g \in G$, then $a^g \in S$ and so either $[a^g,_n a] = 1$ or $[a,_n a^g] = 1$ for some integer n. It follows that a is a randomly Engel element of G and so Theorem 2.7 implies that $\langle a\rangle^G$ is nilpotent. Therefore S is contained in the Fitting subgroup of G. The converse is clear. □

Now using Theorem 2.8, it is easy to generalize [8, Satz 1] (see also [16, Part 2, Theorem 7.24]). The proof is mostly *mutatis mutandis* the proof of [16, Part 2, Theorem 7.24] so we will omit it.

Theorem 2.9. *Let G be a group satisfying the minimal condition on subgroups and suppose that the elements whose orders are powers of p form a randomly Engel set for each prime p. Then G is a hypercentral Černikov group.*

Corollary 2.10. *Let G be a group in which every two-generated subgroup of G is either soluble or satisfies the maximal condition on its abelian subgroups. Then $L(G) = \mathcal{L}(G)$. In particular, if G is non-Engel, then $\mathcal{E}_G$ has no isolated vertex.*

Proof. The first part follows from Theorems 2.5 and 2.7.

If G is non-Engel and a is a vertex of $\mathcal{E}_G$, then a is not a randomly power element of G. Thus there is a conjugate of a which is adjacent to a. This completes the proof. □

The following result, which is of independent interest, generalizes [9, Theorem 1(1)].

Theorem 2.11. *Let G be a finite group and p a prime number. Let x be a p-element and P a Sylow p-subgroup of G such that for every $y \in G$ with $x^y \in P$, the set $\{x, x^y\}$ is a randomly Engel set. If G is p-soluble, then $x \in P$.*

Proof. Suppose that G is a counterexample of minimum order. Hence $O_p(G) = 1$ and $G = \langle P, x\rangle$. Let S be any minimal normal subgroup of G. By minimality of G, we have that $G = PS$. Since G is p-soluble and $O_p(G) = 1$, S is a normal subgroup of order relatively prime to p. There exists $s \in S$ such that $x^s \in P$. Thus $[x^s,_k x] = 1$ or $[x,_k x^s] = 1$ for some $k \in \mathbb{N}$. Suppose that $[x^s,_k x] = 1$ and k is the least non-negative integer with this property; obviously $k > 0$. Then

$$\big[[x, s],_k x\big] = 1 \tag{1}$$

and $[[x, s],_{k-1} x] \neq 1$. Now we prove that if $k \geqslant 1$, then we lead to a contradiction. Put $|x| = q$, then using (1) we can write

$$1 = \big[[x, s],_{k-1} x^q\big] = \big[[x, s],_{k-1} x\big]^{q(x^{q-1}+\cdots+x+1)^{k-2}} = \big[[x, s],_{k-1} x\big]^{q^{k-1}}.$$

Since $[[x, s],_{k-1} x] \in S$ and S is a p'-group, it follows that $[[x, s],_{k-1} x] = 1$ which contradicts the minimality of k. This completes the proof, in this case. If $[x,_k x^s] = 1$, then $[x^{s^{-1}},_k x] = 1$ and a similar argument completes the proof. □

Theorem 2.12. *Let G be a non-Engel group.*

(1) *If $L(G)$ is a subgroup of G and x is a vertex of $\mathcal{E}_G$ such that x is adjacent to every vertex of $\mathcal{E}_G$, then $x^2=1$ and $C_G(x)=\langle x\rangle$.*
(2) *If $L(G)$ is a subgroup of G, then* $\operatorname{diam}(\mathcal{E}_G)=1$ *if and only if $L(G)$ is a normal abelian subgroup of G without elements of order 2 and for any vertex x of $\mathcal{E}_G$ we have $G=L(G)\langle x\rangle$, $L(G)\cap\langle x\rangle=1$, $x^2=1$ and $g^x=g^{-1}$ for all $g\in L(G)$.*
(3) *If $L(G)$ is a subgroup of G and G is periodic, then $\mathcal{E}_G$ contains a vertex which is adjacent to all other vertices in $\mathcal{E}_G$ if and only if* $\operatorname{diam}(\mathcal{E}_G)=1$.

Proof. (1) Since $L(G)\leqslant G$, x^{-1} is also a vertex and since x is adjacent to all vertices, then $x=x^{-1}$. Now let $y\in C_G(x)$ and $y\neq x$. Since $[x,y]=1$ and x is adjacent to every vertex, y is not a vertex and so $y\in L(G)$. If y is non-trivial, then $yx\neq x$ and by a similar argument, $yx\in L(G)$. As $L(G)$ is a subgroup of G, it follows that $x\in L(G)$, a contradiction. Hence $C_G(x)=\langle x\rangle$.

(2) Let $\operatorname{diam}(\mathcal{E}_G)=1$. By part (1), $a^2=1$ for every vertex a of $\mathcal{E}_G$. Now suppose a and b are two distinct vertices of $\mathcal{E}_G$. Then $\langle ab\rangle$ is a normal subgroup in the dihedral group $\langle a,b\rangle$ and so if ab is a vertex of $\mathcal{E}_G$, then it is not adjacent to a. It follows that $G/L(G)$ is of order 2 which implies that $G=L(G)\langle x\rangle$ and $L(G)\cap\langle x\rangle=1$, for every vertex x of $\mathcal{E}_G$. If $g\in L(G)$, then gx is a vertex of $\mathcal{E}_G$. It follows that $(gx)^2=1$, or $g^x=g^{-1}$.

The converse is straightforward.

(3) Let x be a vertex of G adjacent to all other vertices of $\mathcal{E}_G$. By part (1), we have $C_G(x)=\langle x\rangle$ and $x^2=1$. It follows that $\langle x\rangle\cap\langle x\rangle^g=1$ for all $g\in G\backslash\langle x\rangle$. Now by Theorem 5 of [18], $A=G\backslash\{x^g\mid g\in G\}$ is a normal abelian subgroup of G, such that $G=A\langle x\rangle$ and obviously $A\cap\langle x\rangle=1$. Thus G is a soluble periodic group which implies that G is locally finite. Let $a\in A$ be a non-trivial element of A, then $B=\langle a,a^x\rangle$ is a finite abelian normal subgroup of G. Hence x induces a fixed-point-free automorphism of order 2 on B, which implies that B is an abelian group of odd order and $b^x=b^{-1}$ for all $b\in B$. Therefore $a^x=a^{-1}$ for all $a\in A$. Obviously we have that $L(G)=A$. Now part (2) completes the proof. □

Question 2.13. Is the hypothesis "$L(G)\leqslant G$" necessary in Theorem 2.12?

Question 2.14. Let G be a finite non-nilpotent group. Is it true that $\mathcal{E}_G$ is connected? If so, is it true that $\operatorname{diam}(\mathcal{E}_G)\leqslant 2$?

Question 2.15. In which classes of groups, the Engel graph of every non-Engel group has no isolated vertex?

3. Groups whose Engel graphs are planar

A *planar graph* is a graph which can be drawn in the plane so that its edges intersect only at end vertices.

In this section we prove

Theorem 3.1. *Let G be a finite non-Engel group. Then $\mathcal{E}_G$ is planar if and only if $G\cong S_3$, D_{12} or $T=\langle x,y\mid x^6=x^3y^{-2}=x^yx=1\rangle$.*

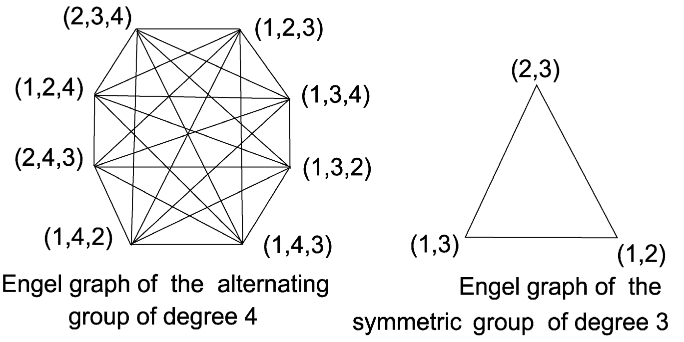

Fig. 1.

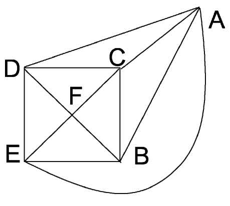

Fig. 2.

Proof. Suppose that $\mathcal{E}_G$ is planar. Then $\mathcal{E}_G$ has no subgraph isomorphic to K_5 (the complete graph with 5 vertices) or $K_{3,3}$ (the complete bipartite graph whose parts have the same size 3) (see [4, Corollary 4.2.11]). This implies that $\omega(\mathcal{E}_G) \leqslant 4$. Now it follows from Proposition 1.4 of [2] and Theorem 1.2, that $\overline{G} = \frac{G}{Z^*(G)} \cong S_3$ or A_4, where $Z^*(G)$ is the hypercenter of G. As we see in Fig. 1, $\mathcal{E}_{A_4}$ has a subgraph isomorphic to $K_{3,3}$. Thus $\overline{G} \cong S_3$. Now put $\bar{x} = xZ^*(G)$ for every $x \in G$ and let $a, b \in G$ such that $\overline{G} = \langle \bar{a}, \bar{b} \rangle \cong S_3$ where $\bar{a}^3 = \bar{b}^2 = \bar{1}$. Then since every element of $Z^*(G)$ is a right Engel element of G, we have that $L(G) = \langle a \rangle Z^*(G)$ and

$$G \backslash L(G) = bZ^*(G) \cup abZ^*(G) \cup a^2bZ^*(G).$$

Now because $\{\bar{b}, \bar{a}\bar{b}, \bar{a}^2\bar{b}\}$ is a clique in $\mathcal{E}_{\overline{G}}$ (see Fig. 1) and every element of $Z^*(G)$ is right Engel, we have that every element in $a^ibZ^*(G)$ is adjacent to every element in $a^jbZ^*(G)$ for distinct $i, j \in \{0, 1, 2\}$. It follows that $|Z^*(G)| \leqslant 2$, otherwise $\mathcal{E}_G$ contains a subgraph isomorphic to $K_{3,3}$. Thus G is a non-nilpotent group of order 6 or a non-nilpotent group of order 12 with $|Z^*(G)| = 2$. Since $Z(A_4) = 1$, the proof of "only if" part is complete.

Conversely, in Fig. 1 we have the Engel graph of S_3; the Engel graphs of $D_{12} = \langle s, r \mid s^6 = r^2 = s^r s = 1 \rangle$ and T are isomorphic to the graph shown in Fig. 2, where (A, B, C, D, E, F) is equal to $(r, rs, rs^2, rs^4, rs^5, rs^3)$ in $\mathcal{E}_{D_{12}}$ and $(y, yx, yx^2, yx^4, yx^5, y^3)$ in $\mathcal{E}_T$, respectively. These graphs are visibly planar. This completes the proof. □

We end this section with the following question.

Question 3.2. Is there an infinite non-Engel group whose Engel graph is planar?

4. Groups with the same Engel graph

In this section we study groups with isomorphic Engel graphs. In fact we consider the following question.

Question 4.1. Let G and H be two non-Engel groups such that $\mathcal{E}_G \cong \mathcal{E}_H$. For which group property $\mathcal{P}$, if G has $\mathcal{P}$, then H also has $\mathcal{P}$?

At the moment we give the positive answer to Question 4.1, when $\mathcal{P}$ is the property of being finite.

Theorem 4.2. *Let G and H be two non-Engel groups such that $\mathcal{E}_G \cong \mathcal{E}_H$. If G is finite, then H is also a finite group. Moreover $|L(H)|$ divides $|G| - |L(G)|$.*

Proof. Since $\mathcal{E}_G \cong \mathcal{E}_H$, $|H \backslash L(H)| = |G \backslash L(G)|$. Then $H \backslash L(H)$ is finite. If $h \in H \backslash L(H)$, then $\{h^x \mid x \in H\} \subseteq H \backslash L(H)$, since $L(H)$ is closed under conjugation. Thus every element in $H \backslash L(H)$ has finitely many conjugates in H. It follows that $K = C_H(H \backslash L(H))$ has finite index in H. By Corollary 2.10, $\mathcal{E}_H$ has no isolated vertex. Thus there exist two adjacent vertices h_1 and h_2 in $\mathcal{E}_H$. Now if $s \in K$, then $s \in C_H(h_1, h_2)$. It follows that

$$[sh_{1,k} h_2] = [h_{1,k} h_2] \neq 1 \quad \text{and} \quad [h_{2,k} sh_1] = [h_{2,k} h_1] \neq 1 \quad \text{for all } k \in \mathbb{N}.$$

Therefore $Kh_1 \subseteq H \backslash L(H)$ and so K is finite. Hence H is finite and we have that $|H| - |L(H)| = |G| - |L(G)|$. Now since H is finite, it follows from [3] that $L(H)$ is a subgroup of H and so $|L(H)|$ divides $|H|$. This completes the proof. □

Acknowledgments

The author is indebted to the referee for his valuable comments. The author thanks the Centre of Excellence for Mathematics, University of Isfahan.

References

[1] A. Abdollahi, Some Engel conditions on finite subsets of certain groups, Houston J. Math. 27 (3) (2001) 511–522.
[2] A. Abdollahi, Groupes satisfaisant une condition d'Engel, J. Algebra 283 (2) (2005) 431–446.
[3] R. Baer, Engelsche Elemente Noetherscher Gruppen, Math. Ann. 133 (1957) 256–270.
[4] R. Diestel, Graph Theory, second ed., Grad. Texts in Math., vol. 173, Springer-Verlag, New York, 2000.
[5] G. Endimioni, Groups covered by finitely many nilpotent subgroups, Bull. Austral. Math. Soc. 50 (3) (1994) 459–464.
[6] The GAP Group, GAP—Groups, Algorithms, and Programming, version 4.4, http://www.gap-system.org, 2005.
[7] K.W. Gruenberg, The Engel elements of a soluble group, Illinois J. Math. 3 (1959) 151–168.
[8] D. Held, Nilpotenz- und Verstreutheitskriterien für artinsche Gruppen, Math. Z. 87 (1965) 49–61.
[9] C.Y. Ho, On the p-elements of a finite group, Proc. Amer. Math. Soc. 48 (1) (1975) 61–66.
[10] P. Longobardi, M. Maj, Finitely generated soluble groups with an Engel condition on infinite subsets, Rend. Sem. Mat. Univ. Padova 89 (1993) 97–102.
[11] A.S. Mamontov, An analog of the Baer–Suzuki theorem for infinite groups, Siberian Math. J. 45 (2) (2004) 327–330.
[12] B.H. Neumann, A problem of Paul Erdös on groups, J. Aust. Math. Soc. Ser. A 21 (4) (1976) 467–472.
[13] T.A. Peng, Engel elements of groups with maximal condition on abelian subgroups, Nanta Math. 1 (1966) 23–28.

[14] B.I. Plotkin, Some properties of automorphisms of nilpotent groups, Dokl. Akad. Nauk SSSR 137 (1961) 1303–1306.
[15] D.J.S. Robinson, A Course in the Theory of Groups, Springer-Verlag, New York, 1982.
[16] D.J.S. Robinson, Finiteness Conditions and Generalized Soluble Groups, Parts 1 and 2, Springer-Verlag, Berlin, 1972.
[17] D.J.S. Robinson, A property of lower central series of a group, Math. Z. 107 (1968) 225–231.
[18] V.P. Šunkov, On a periodic group with an almost regular involution, Algebra Logika 7 (1) (1968) 113–121.