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Sofic groups and direct finiteness

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Abstract

We construct an analogue of von Neumann's affiliated algebras for sofic group algebras over arbitrary fields. Consequently, we settle Kaplansky's direct finiteness conjecture for sofic groups.

Keywords: Sofic groups; Direct finiteness; Continuous von Neumann regular rings; Group algebras

1. Introduction

The following conjecture is due to Kaplansky.

Conjecture 1. *For any group G and commutative field K, the group algebra $K(G)$ is directly finite. That is $ab = 1$ in $K(G)$ implies $ba = 1$.*

Recently Ara, O'Meara and Perera [1] settled the conjecture for residually amenable groups even in the case of group algebras $D(G)$, where D is a division ring. They also proved that $\mathrm{Mat}_{n\times n}(D(G))$ is directly finite as well. It is important to note that Conjecture 1 holds for any group G if $D = \mathbb{C}$, the complex field [6]. Indeed, $\mathbb{C}(G)$ is a subalgebra of the von Neumann algebra $N(G)$. The algebra $N(G)$ always satisfies the Ore-condition

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with respect to its non-zero divisors. Hence one can consider its classical ring of fractions $U(G)$. The algebra $U(G)$ is the so-called *affiliated algebra* of G and it is a continuous von Neumann regular ring [3,7], hence it is known to be directly finite. Let us turn to another conjecture due to Gottschalk [4].

Conjecture 2. *Let G be a countable group and X a finite set. Consider the compact metrizable space X^G of X-valued functions on G equipped with the product topology. Let $f : X^G \to X^G$ be a continuous map that commutes with the natural right G-action. Then if f is injective, it is surjective as well.*

In [5] Gromov proved Gottschalk's conjecture in the case of *sofic groups* (the name "sofic groups" was coined by Weiss [9]). We shall review the definitions and basic properties of sofic groups in Section 4, nevertheless, let us note that residually amenable groups are sofic groups as well, and in [2] we constructed sofic groups that are not residually amenable. On the other hand, there seems to be no example yet of a group which is not sofic. Let us observe that Conjecture 2 implies Conjecture 1 for finite fields F. Indeed, it is enough to prove Conjecture 1 for countable groups. Then any element a of the group algebra $F(G)$ induces a continuous linear map on F^G commuting with the right G-action. Simply, a acts as convolution on the left. Then $F(G)$ can be identified with the dense set of elements in F^G having only finitely many non-zero values. If $ab = 1$ on this dense subset then ab must be equal to the identity on the whole F^G. Therefore b is injective and thus it is a bijective continuous map by our assumption. Consequently, a is the inverse of b, thus $ba = 1$. The goal of this paper is to replace the notion of the affiliated algebras of complex group algebras with something similar for group algebras of sofic groups over arbitrary division rings.

Theorem 1. *Let G be a sofic group and let D be a division ring. Then $D(G)$ can be embedded into a simple continuous von Neumann regular ring $R(G)$.*

Therefore we extend the result of Ara, O'Meara and Perera to the class of sofic groups.

2. Continuous von Neumann regular rings

In this section we give a brief summary of the theory of continuous von Neumann regular rings, based upon the monograph of Goodearl [3]. Recall that a unital ring R is *von Neumann regular* if for any $x \in R$ there exists $y \in R$ such that $xyx = x$. It is equivalent to say that any finitely generated right ideal of R can be generated by one single idempotent. A ring R is called *unit-regular* if for any $x \in R$ there exists a unit y such that $xyx = x$. A ring is called *directly finite* if $xy = 1$ implies $yx = 1$ and it is called *stably finite* if $\mathrm{Mat}_{n\times n}(R)$ is directly finite for all $n \geqslant 1$. Any unit-regular ring is necessarily stably finite. A lattice L is called a *continuous geometry* if it is modular, complete, complemented and

$$a \wedge \left(\bigvee_{\alpha\in I} b_\alpha\right) = \bigvee_{\alpha\in I} (a \wedge b_\alpha),$$

$$a \vee \left(\bigwedge_{\alpha \in I} b_\alpha \right) = \bigwedge_{\alpha \in I} (a \vee b_\alpha)$$

for any linearly ordered subset $\{b_\alpha\}_{\alpha \in I} \subset L$. A von Neumann regular ring is called *continuous* if the lattices of both its finitely generated right ideals and left ideals are continuous geometries. The continuous von Neumann regular rings are unit-regular, hence they are stably finite as well. Division rings and matrix rings over division rings are the simplest examples of continuous von Neumann regular rings. The first simple continuous von Neumann rings which do not satisfy either the ascending or the descending chain condition had already been constructed by John von Neumann [8]. The following proposition summarizes what we need to know about such rings.

Proposition 2.1. *If R is a simple continuous, von Neumann regular ring that does not satisfy either the ascending or the descending chain condition, then*

- $K_0(R) = \mathbb{R}$, *in fact there exists a unique non-negative real-valued dimension function* $\dim_R$ *on the set of finitely generated projective right modules over R taking all non-negative values such that*
  (1) $\dim_R(R) = 1$.
  (2) $\dim_R(0) = 0$.
  (3) $\dim_R(A \oplus B) = \dim_R(A) + \dim_R(B)$.
  (4) $\dim_R(A) = \dim_R(B)$ *if and only if* $A \simeq B$.
- *If A, B are finitely generated submodules of a projective module then so is $A \cap B$, and*

$$\dim_R(A \cap B) + \dim_R(A + B) = \dim_R(A) + \dim_R(B).$$

- *If $a \in R$ then* $\mathrm{Ann}(a) = \{x \in R\colon ax = 0\}$ *is a principal right ideal and*

$$\dim_R\bigl(\mathrm{Ann}(a)\bigr) + \dim_R(aR) = 1.$$

- *If $A \leqslant B$ are finitely generated projective modules and* $\dim_R(A) = \dim_R(B)$ *then* $A = B$.

3. A pseudo-rank function on the direct product of matrix rings

Let I be a set and let $\{A_\alpha\}_{\alpha \in I}$ be finitely generated right D-modules, where D is a division ring. Consider the direct product $E = \prod_{\alpha \in I} \mathrm{End}_D(A_\alpha)$, where $\mathrm{End}_D(A_\alpha)$ is the endomorphism ring of A_α. This ring is directly finite, von Neumann regular, right and left self-injective. Now we recall the notion of a pseudo-rank function [3].

Definition 3.1. A pseudo-rank function on a von Neumann regular ring R is a map $N : R \to [0, 1]$ such that

(a) $N(1) = 1$, $N(0) = 0$.

(b) $N(xy) \leqslant N(x)$, $N(xy) \leqslant N(y)$, for all $x, y \in R$.
(c) $N(e+f) = N(e) + N(f)$ for all orthogonal idempotents $e, f \in R$.

Before stating our proposition let us recall the notion of ultralimit as well. Let ω be an ultrafilter on the set I. Then $\lim_\omega$ is the unique real valued functional on the space of bounded real sequences $\{a_\alpha\}_{\alpha\in I}$ such that: if $\lim_\omega(\{a_\alpha\}_{\alpha\in I}) = t$, then for any $\varepsilon > 0$,

$$\bigl\{\alpha \in I \mid a_\alpha \in [t-\varepsilon, t+\varepsilon]\bigr\} \in \omega.$$

Note that $\lim_\omega(\{a_\alpha + b_\alpha\}_{\alpha\in I}) = \lim_\omega(\{a_\alpha\}_{\alpha\in I}) + \lim_\omega(\{b_\alpha\}_{\alpha\in I})$.

Proposition 3.2. *Define $N : E \to [0,1]$ the following way. If $r_\alpha \in \operatorname{End}_D(A_\alpha)$,*

$$N\bigl(\{r_\alpha\}_{\alpha\in I}\bigr) = \lim_\omega \frac{\dim_D(\operatorname{Ran}(r_\alpha))}{\dim_D(A_\alpha)},$$

where $\dim_D$ denotes the dimension function on finite dimensional right D-modules. Then N is a pseudo-rank function.

Proof. Clearly, $N(1) = 1$ and $N(0) = 0$. Let $\underline{r} = \{r_\alpha\}_{\alpha\in I}$, $\underline{s} = \{s_\alpha\}_{\alpha\in I}$. Then

$$\dim_D\bigl(\operatorname{Ran}(r_\alpha s_\alpha)\bigr) \leqslant \dim_D\bigl(\operatorname{Ran}(r_\alpha)\bigr) \quad \text{and} \quad \dim_D\bigl(\operatorname{Ran}(r_\alpha s_\alpha)\bigr) \leqslant \dim_D\bigl(\operatorname{Ran}(s_\alpha)\bigr).$$

Hence $N(\underline{r}\,\underline{s}) \leqslant N(\underline{r})$, $N(\underline{r}\,\underline{s}) \leqslant N(\underline{s})$. Now let $\underline{e} = \{e_\alpha\}_{\alpha\in I}$, $\underline{f} = \{f_\alpha\}_{\alpha\in I} \in E$ be orthogonal idempotents. Then for any $\alpha \in I$, e_α and f_α are orthogonal idempotents in $\operatorname{End}_D(A_\alpha)$. Thus

$$\dim_D\bigl(\operatorname{Ran}(e_\alpha + f_\alpha)\bigr) = \dim_D\bigl(\operatorname{Ran}(e_\alpha)\bigr) + \dim_D\bigl(\operatorname{Ran}(f_\alpha)\bigr).$$

Consequently, $N(\underline{e} + \underline{f}) = N(\underline{e}) + N(\underline{f})$. □

Note that $\operatorname{Ker}(N) = \{\underline{x} \in E \mid N(\underline{x}) = 0\}$ is a two-sided ideal of E.

Proposition 3.3. *The ring $R_N = E/\operatorname{Ker}(N)$ is a simple continuous von Neumann regular ring.*

Proof. The direct product ring E is a right and left self-injective von Neumann regular ring, therefore by Corollary 13.5 [3], E is continuous. If M is a maximal two-sided ideal of E, then by Corollary 13.27 [3], E/M is a simple continuous von Neumann regular ring. Thus it is enough to show that $\operatorname{Ker}(N)$ is a maximal ideal. Let $Q \subset E$ be the following two-sided ideal: $\{a_\alpha\}_{\alpha\in I} \in Q$ if $\{\alpha \in I \mid q_\alpha = 0\} \in \omega$. Then of course E/Q is just the ultraproduct of the rings $\operatorname{End}_D(A_\alpha)$. Note that Q is a prime ideal since the ultraproduct of prime rings is a prime ring as well. By Corollary 9.15 [3], E satisfies the general comparability axiom. Hence, by Corollary 8.21 [3], the ideal $\operatorname{Ker}(N)$, which contains Q, is a prime ideal. Proposition 16.25 [3] immediately implies that $\operatorname{Ker}(N)$ is in fact a maximal two-sided ideal of E. This completes the proof of the proposition. □

4. Sofic groups

In this section, we recall the notion of a sofic group from [2] and prove Theorem 1.

Definition 4.1 [2]. For a finite set V let $\mathbb{M}\mathrm{ap}(V)$ denote the monoid of self-maps of V acting on the left, the monoid operation is the composition of self-maps. We say that two elements $e, f \in \mathbb{M}\mathrm{ap}(V)$ are *ε-similar* for a real number $\varepsilon \in (0,1)$, if the number of elements $v \in V$ with $e(v) \neq f(v)$ is at most $\varepsilon|V|$. We say that e, f are $(1-\varepsilon)$-different, if the number of elements $v \in V$ with $e(v) = f(v)$ is less than $\varepsilon|V|$, i.e., if they are not $(1-\varepsilon)$-similar.

Definition 4.2 [2]. The group G is *sofic* if for each number $\varepsilon \in (0,1)$ and any finite subset $F \subseteq G$ there exists a finite set V and a function $\phi : G \to \mathbb{M}\mathrm{ap}(V)$ with the following properties:

(a) For any two elements $e, f \in F$ the map $\phi(ef)$ is ε-similar to $\phi(e)\phi(f)$.
(b) $\phi(1)$ is ε-similar to the identity map of V.
(c) For each $e \in F \setminus \{1\}$ the map $\phi(e)$ is $(1-\varepsilon)$-different from the identity map of V.

Remark. The origin of this notion is [5], where Gromov introduced the concept of initially subamenable graphs. The term "sofic group" is introduced in [9]: a finitely generated group is called sofic if its Cayley graph is initially subamenable. The above definition is taken from [2] (with right action replaced by left action, which is more appropriate here). This is the formulation which suits our need the best. In the case of finitely generated groups the two definitions in [9] and [2] are equivalent. Although we shall not use it later, for the sake of completeness we prove their equivalence. To begin with, we recall some notation, and the definition from [9].

Definition 4.3 [9]. Let G be a finitely generated group, and $B \subset G$ a fixed finite, symmetric (i.e., $B = B^{-1}$) generating set. The Cayley graph of G is a directed graph Γ, whose edges are labeled by the elements of B: the set of vertices is just G, and the edges with label $b \in B$ are the pairs (g, bg) for all $g \in G$. Let N_r denote the r-ball around $1 \in \Gamma$ (it is an edge-colored graph, and also a finite subset in G). The group G is called sofic in [9], if for each $\delta > 0$ and each $r \in \mathbb{N}$ there is a finite directed graph (V, E) edge-labeled by B, and a subset $V_0 \subset V$ with the properties, that:

(1) For each point $v \in V_0$ there is a function $\psi_v : N_r \to V$ which is an isomorphism (of labeled graphs) between N_r and the r-ball in V around v.
(2) $|V_0| \geqslant (1-\delta)|V|$.

Proposition 4.4. *For a finitely generated group G the above two notions of soficity are equivalent. In particular, Definition* 4.3 *does not depend on the choice of the generating set B.*

Proof of 4.3 ⇒ 4.2. Let $\varepsilon > 0$ and $F \subseteq G$ a finite subset. We chose $r \in \mathbb{N}$ such that the product set $F \cdot F$ is contained in N_r. Let (V, E) and $V_0 \subset V$ be the labeled directed graph, and subset corresponding to $\delta = \varepsilon$ and r. We shall use this finite set V, and define the function $\phi : G \to \mathbb{M}\mathrm{ap}(V)$ as follows. For $g \in N_r$ and $v \in V_0$ let $\phi(g)(v) = \psi_v(g)$. Otherwise, for $g \in G \setminus N_r$ and/or $v \in V \setminus V_0$, we define $\phi(g)(v)$ arbitrarily. It is an easy calculation to check conditions (a)–(c) of Definition 4.2. □

Proof of 4.2 ⇒ 4.3. Let $\delta > 0$ and $r \in \mathbb{N}$. We set $F = N_{2r+2}$, and choose any $\varepsilon > 0$. Let $\phi : G \to \mathbb{M}\mathrm{ap}(V)$ be the function of Definition 4.2 for this (F, ε). We use this V as the vertex set of our new graph, and for each $v \in V$ we define $\psi_v : N_{r+1} \to V$, $\psi_v(g) = \phi(g)(v)$. Let V_0 be the set of those $v \in V$ for which

(A) $\psi_v(bg) = \psi_{\psi_v(g)}(b)$ for all $g \in N_r$ and all $b \in B$,
(B) $\psi_v(1) = v$,
(C) $\psi_v(g) \neq \psi_v(h)$ whenever $g, h \in N_{r+1}$ are different elements.

Finally we build the labeled edges of V: for each $b \in B$ and $v \in V$ we add the edge $(v, \psi_v(b))$ with label b. It is easy to see that conditions (A) and (C) imply that if $v \in V_0$ then the restriction of ψ_v to N_r is an isomorphism of labeled graphs, and condition (B) implies that its image is the r-neighborhood of v in V. Hence Definition 4.3(1) is satisfied. There are $|N_r| \cdot |B| + 1$ equations to check in (A) and (B), and less than $|N_{r+1}|^2$ inequalities in (C). We know from Definition 4.2(a) that each of the equations can fail on at most $\varepsilon|V|$ exceptional v. Now we estimate the number of elements $v \in V$ such that $\psi_v(g) = \psi_v(h)$, where g, h are two fixed distinct elements of N_{r+1}. If $\psi_v(g) = \psi_v(h)$ then $\phi(g)(v) = \phi(h)(v)$, therefore

$$\phi\bigl(g^{-1}\bigr)\bigl(\phi(g)(v)\bigr) = \phi\bigl(g^{-1}\bigr)\bigl(\phi(h)(v)\bigr). \tag{1}$$

The following sets have at least $(1 - \varepsilon)|V|$ elements:

$$W_g = \bigl\{v \in V \bigm| \phi\bigl(g^{-1}\bigr)\bigl(\phi(g)(v)\bigr) = \phi(1)(v)\bigr\},$$
$$W_h = \bigl\{v \in V \bigm| \phi\bigl(g^{-1}\bigr)\bigl(\phi(h)(v)\bigr) = \phi\bigl(g^{-1}h\bigr)(v)\bigr\},$$
$$W'_{g,h} = \bigl\{v \in V \bigm| \phi\bigl(g^{-1}h\bigr)(v) \neq \phi(1)(v)\bigr\}.$$

If v satisfies (1) then $v \notin W_g \cap W_h \cap W'_{g,h}$. Hence each inequality in (C) can fail on at most $3\varepsilon|V|$ exceptional v. Hence Definition 4.3(2) holds if we choose

$$\varepsilon < \frac{\delta}{3|N_{r+1}|^2 + |N_r| \cdot |B| + 1}. \qquad \square$$

For the sake of completeness, we quote without proof some important properties of sofic groups:

Proposition 4.5 [2]. *Direct products, subgroups, inverse limits, direct limits, and free products of sofic groups are sofic. If $N \lhd G$, N is sofic and G/N is amenable, then G is also sofic.*

Proposition 4.6 [2,9]. *If G is a residually amenable group then G is sofic. In particular, amenable and residually finite groups are sofic.*

Proof of Theorem 1. Let G be a sofic group. We define the index set:

$$I = \{(F, \varepsilon) \mid F \subseteq G \text{ finite and } \varepsilon \in (0, 1)\}$$

and for each index $(H, \delta) \in I$ we define the non-empty subset

$$I_{H,\delta} = \{(F, \varepsilon) \in I \mid H \subseteq F \text{ and } \varepsilon \leqslant \delta\} \subseteq I.$$

The collection of non-empty subsets $\{I_{H,\delta}\}$ is closed under finite intersection, so there is an ultrafilter ω of subsets of I containing all $I_{H,\delta}$. Next, for each index $(F, \varepsilon) \in I$ we choose a finite subset $V_{F,\varepsilon}$ and a function $\phi_{F,\varepsilon} : G \to \mathbb{M}\mathrm{ap}(V_{F,\varepsilon})$ satisfying the conditions (a)–(c) of Definition 4.2. As in Section 3, for each index $\alpha \in I$ A_α denotes the D-D-bimodule with right (and left) basis V_α, $\mathrm{End}_D(A_\alpha)$ denotes the endomorphism ring of A_α as a right D-module, E denotes the product of the rings $\mathrm{End}_D(A_\alpha)$ and N is the pseudo-rank function of Proposition 3.2. Finally $R(G) = R_N$ denotes the quotient ring $E/\mathrm{Ker}(N)$. This $R(G)$ is a simple continuous von Neumann regular ring, the ring we seek in the Theorem. The left multiplication on A_α by any $d \in D$ is a right D-module endomorphism (mapping each $v \in V_\alpha$ to dv). So, D is embedded as a subring of $\mathrm{End}_D(A_\alpha)$. Then D is embedded (via the diagonal map) as a subring of E, and (composing with the quotient map) as a subring of $R(G)$. D also embeds as a subring of the group algebra (the subset $\{d \cdot 1 \mid d \in D,$ $1 \in G\} \subset D(G)$). The elements of $\mathbb{M}\mathrm{ap}(V_\alpha)$ extend D-linearly (with respect to the left D action on A_α) to endomorphisms of the right D-module A_α, hence the functions ϕ_α induce functions $G \to \mathrm{End}_D(A_\alpha)$, and can be extended to (left) D-linear functions

$$T_\alpha : D(G) \to \mathrm{End}_D(A_\alpha).$$

Taking the product of these T_α, and then composing with the quotient map $E \to R(G)$, we obtain the D-linear functions:

$$\widetilde{T} : D(G) \to E,$$
$$T : D(G) \to R(G).$$

We shall prove that this T is an injective homomorphism, completing the proof of Theorem 1. Note, that the subset $G \subset D(G)$ commutes with the subring $D \hookrightarrow D(G)$, and its image $T_\alpha(G) \subset \mathrm{End}_D(A_\alpha)$ commutes with the above subring $D \hookrightarrow \mathrm{End}_D(A_\alpha)$. Hence $T(G) \subset R(G)$ also commutes with the subring $D \hookrightarrow R(G)$. We still need to check that the restriction of T to the group $G \subset D(G)$ is a group homomorphism. We see from property (a) of Definition 4.2 that for each $g, h \in G$

$$N\big(\widetilde{T}(g)\widetilde{T}(h)-\widetilde{T}(gh)\big)=\lim_{\omega}\frac{\operatorname{rank}_D(T_\alpha(g)T_\alpha(h)-T_\alpha(gh))}{\dim_D(A_\alpha)}$$
$$\leqslant \lim_{\omega}\frac{|\{v\in V_\alpha \mid \phi_\alpha(g)(\phi_\alpha(h)(v))\neq \phi_\alpha(gh)(v)\}|}{|V_\alpha|}$$
$$\leqslant \lim_{\varepsilon\to 0}\varepsilon=0.$$

Similarly, it follows from property (b) that $N(\widetilde{T}(1)-1)=0$. Therefore the map T is a ring unital homomorphism. The only that thing remains to be shown is the injectivity of T. So let $S\subset G$ be a finite subset, and for each $s\in S$ let $k_s\in D$ be a non-zero element. We shall show that $T(\sum_{s\in S}k_s s)=\sum_{s\in S}k_s T(s)\neq 0$ in $R(G)$. For each index $\alpha=(F,\varepsilon)\in I$ we choose a maximal subset $X_\alpha\subset V_\alpha$ such that if $p,q\in X_\alpha$ and $s,t\in S$ with either $p\neq q$ or $s\neq t$ then $T_\alpha(s)(p)\neq T_\alpha(t)(q)$. Since X_α is maximal, for each $p\in V_\alpha\setminus X_\alpha$ either there exist elements $s\neq t\in S$ such that

$$T_\alpha(s)(p)=T_\alpha(t)(p) \tag{2}$$

or there is an element $q\in X_\alpha$ and elements $s,t\in S$ such that

$$T_\alpha(s)(p)=T_\alpha(t)(q). \tag{3}$$

If $S\cup S^{-1}\subset F$ then by the argument applied in Proposition 4.4, for a fixed pair $s\neq t\in S$ the number of elements p for which (2) holds is at most $3\varepsilon|V_\alpha|$. Hence (2) holds (with some $s\neq t$ in S) for at most $3\varepsilon|V_\alpha||S|^2$ possible values of p. The right hand side of (3) can take at most $|S|\cdot|X_\alpha|$ different values. On the other hand, by Definition 4.2(b) and (c), if $S\cup S^{-1}\subset F$ then the subsets $W_1=\{v\in V_\alpha \mid T_\alpha(1)(v)\neq v\}$ and $W_s=\{v\in V_\alpha \mid T_\alpha(s^{-1})(T_\alpha(s)(v))\neq T_\alpha(1)(v)\}$ have at most $\varepsilon|V_\alpha|$ elements, and the map $T_\alpha(s)$ is injective on $V_\alpha\setminus(W_1\cup W_s)$. Hence for each value of s there are at most $2\varepsilon|V_\alpha|+|S|\cdot|X_\alpha|$ possible values of p satisfying (3) for some $t\in S$. Therefore if $S\cup S^{-1}\subset F$;

$$|V_\alpha\setminus X_\alpha|\leqslant 2\varepsilon|S|\cdot|V_\alpha|+|S|^2\cdot|X_\alpha|+3\varepsilon|V_\alpha||S|^2.$$

Thus

$$|X_\alpha|\geqslant\frac{1-2\varepsilon|S|-3\varepsilon|S|^2}{|S|^2+1}|V_\alpha|.$$

If a non-zero element x of A_α is spanned by X_α then $(\sum_{s\in S}k_sT_\alpha(s))(x)\neq 0$. Hence

$$N\bigg(\sum_{s\in S}k_s\widetilde{T}(s)\bigg)\geqslant\lim_{\omega}\frac{|X_\alpha|}{|V_\alpha|}\geqslant\lim_{\varepsilon\to 0}\frac{1-2\varepsilon|S|-3\varepsilon|S|^2}{|S|^2+1}=\frac{1}{|S|^2+1}>0.$$

This proves that $T(\sum_{s\in S}k_s s)\neq 0$. Theorem 1 is proved now. □

Corollary 4.7. *If G is a sofic group and D is a division ring then $D(G)$ is stably finite.*

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References

[1] P. Ara, K. O'Meara, F. Perera, Stable finiteness of group rings in arbitrary characteristic, Adv. Math. 170 (2) (2002) 224–238.
[2] G. Elek, E. Szabó, On sofic groups, preprint, 2003.
[3] K.R. Goodearl, Von Neumann Regular Rings, in: Monogr. Stud. Math., vol. 4, Pitman, Boston, 1979.
[4] W. Gottschalk, Some general dynamical notions, in: Lecture Notes in Math., vol. 318, 1973, pp. 120–125.
[5] M. Gromov, Endomorphisms of symbolic algebraic varieties, J. Eur. Math. Soc. 1 (2) (1999) 109–197.
[6] I. Kaplansky, Fields and Rings, in: Chicago Lectures in Math., Univ. of Chicago Press, Chicago, 1969.
[7] W. Lück, L^2-Invariants: Theory and Applications to Geometry and K-Theory, in: Ergeb. Math. Grenzgeb., vol. 44, Springer-Verlag, Berlin, 2002.
[8] J. von Neumann, Continuous Geometry (foreword by Israel Halperin), in: Princeton Math. Ser., vol. 25, 1960.
[9] B. Weiss, Sofic groups and dynamical systems, in: Ergodic Theory and Harmonic Analysis, Mumbai, 1999, Sankhya Ser. A 62 (3) (2000) 350–359.