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Generalization of the RCGM and LSLR Pairwise Comparison Methods

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Abstract—Pairwise comparison methods are convenient procedures for predicting a sound weight vector from a set of relative comparisons between elements to be weighted. Several pairwise comparison methods exist. After a brief presentation of the least squares logarithmic regression (LSLR) method of de Graan [1] and Lootsma [2] and the recent row and column geometric mean (RCGM) of Koczkodaj and Orlowski [3], this paper proposes a common mathematical formulation for these two approaches. This common formulation leads to two generalized methods. The GLSLR is now able to process nonreciprocal comparison matrices, and the GRCGM is extended to several decision makers expressing different opinions per pairwise comparison. It also results in an explicit formulation of the weights that generalizes Koczkodaj and Orlowski's formulation of the closest consistent comparison matrix.

Keywords—Pairwise comparison, Decision making, Logarithmic least squares regression, Row column geometric mean.

1. INTRODUCTION

Rating a set of n elements $(e_1, \dots, e_n)$ under the consideration of one criterion is not readily achieved in the presence of d experts or decision makers. A convenient class of methods called "pairwise comparison methods" notably simplifies the problem by focusing the attention of decision makers on pairs of elements to be compared. The so-called comparison matrix (see Figure 1) represents all possible combinations.

	e_1	e_2	$\cdots$	e_n
e_1	(e_1, e_1)	(e_1, e_2)	$\cdots$	(e_1, e_n)
e_2	(e_2, e_1)	(e_2, e_2)	$\cdots$	(e_2, e_n)
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
e_n	(e_n, e_1)	(e_n, e_2)	$\cdots$	(e_n, e_n)

Figure 1. The comparison matrix.

Each pairwise comparison leads to a quantified value c_{ij} which is an estimate of the ratio of element weights w_i/w_j. In a general case, each decision maker may be allowed to express his/her

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own assessment of comparison c_{ij}. Let c_{ijk} denote the opinion or the vote of the decision maker k (among d) regarding elements e_i and e_j.

When considering all the comparison matrices, the votes collected for several decision makers hold in a cube. Starting from the pairwise vote cube $C = (c_{ijk})$, pairwise comparison methods consist in mapping functions that predict a suitable set of weights $W = (w_i)$.

For this problem, the answer provided by the literature is not unique and the coexistence of different methods can be justified by the following reasons.

- Each of the estimated ratios c_{ijk} leads to a specific equation linking the weights variables. Since, there are n unknown weights and up to $d \times n^2$ different equations (if each decision maker expresses n^2 comparisons), the system has great chances of being overconstrained. Errors, ambiguities, and vagueness are expected in personal judgments [4] and they result in inconsistencies. In some circumstances, inconsistencies may be explained and considered as natural for human beings [5]. They are less accepted in their ordinal and more noticeable form where they can result in cyclic preferences, i.e., for a triple of elements (i, j, k), preference of e_i over e_j and e_j over e_k coexist with preference of e_k over e_i. In case of multiple decision makers, inconsistencies also occur when different opinions are expressed for the same binary comparison. In such a case, they characterize the group as a whole and must be interpreted as the divergence of decision makers' opinions. More generally, there is consistency if and only if the following cardinal transitivity relation holds: $c_{ihx}c_{hjy} = c_{ijz}$, $i, j, h = 1, 2, \dots, n$, $x, y, z = 1, 2 \dots, d$. In case of nonrespect of this generalized transitivity relation there is no *a priori* best set of weights. This is why, according to the decision strategy, different optimization logics can be considered to yield a sound set of weights [6,7][1].
- Some simplifying hypotheses related to specific configuration of decision making allow the use of specific methods. Some examples concerning the simplification of the vote cube are listed hereafter.
  - Only precise opinions (crisp values) are considered without taking the possible imprecision of judgments into account. The corresponding pairwise comparison approaches are considered as deterministic.
  - The vote cube can be assumed to be reciprocal, i.e., ($c_{ijx} = 1/c_{jiy}$, $i, j = 1, 2 \dots, n$, $x, y = 1, 2 \dots, d$), leading to $d \times n \times (n-1)/2$ independent votes at most.
  - The number of opinions taken into account for each pairwise comparison can be fixed to exactly one (ex: common decision). In this case, the vote cube consists of a comparison matrix.

This work addresses the research issue of extending deterministic pairwise comparison methods. Such generalization increases the methods' aptitude to tackle the variability of decision making contexts. They can also provide generalized straightforward computational formulae. This can be useful for further extensions such as, for example, extending a deterministic pairwise comparison method to take into account the possible imprecision of judgments (see [8–10] for extensions based on fuzzy sets and probability theory).

More precisely, we will extend two methods: the least squares logarithmic regression (LSLR) method of de Graan [1] and Lootsma [2] and the recent row and column geometric mean (RCGM) of Koczkodaj and Orlowski [3] by, respectively, releasing the simplifying hypotheses: reciprocity and one opinion per comparison.

In the following section, we will briefly present those two methods within a brief literature review. In Sections 3 and 4, a common formulation is proposed by releasing some restrictive

[1]Minimizing the sum of absolute errors is known to be resistant to the presence of outliers [6,7]. Such outliers can represent locally erroneous judgments or isolate opinions in a decision group. Conversely, when it is legitimate to have a solution that is representative of all the opinions (even outliers), optimization criteria of the type least squares of errors are more adequate. For example, in decision groups with mainly nonexperts, the experts themselves may be outliers!

assumptions. This formulation results in two generalized approaches, respectively, the GLSLR in Section 3 and the GRCGM in Section 4. It also yields an explicit formulation of the weights which generalizes the Koczkodaj and Orlowski's formulation of the closest consistent comparison matrix [3]. This new formulation is presented in Section 5 before concluding in Section 6.

2. DETERMINISTIC PAIRWISE COMPARISON METHODS

Each pairwise comparison method provides a mapping function which minimizes the distance between the input pairwise comparisons and the ones derived from the resulting set of weights (unknowns). Most of the pairwise comparison methods are defined by a straightforward computational formula when proposing a set of weights. They refer to built-in optimization criteria more or less easy to express and assume simplification hypothesis. Some recent approaches [6,7,11] formulate and solve the pairwise comparison problem in a more flexible but less straightforward mathematical programming way.

This literature review is restricted to few deterministic approaches. It briefly presents methods with built-in optimization criteria such as the fundamental eigenvector based method [12] and the two methods to be generalized in this paper: the geometric mean based approach of Koczkodaj and Orlowski [3] and the logarithmic least squares regression method of de Graan [1] and Lootsma [2][2]. To illustrate the existence of methods without built-in optimization criteria, this literature review also presents Bryson and Joseph's goal programming approach [6,7]. This approach has, in contrast to the previous ones, the potential for a more flexile formulation.

2.1. Eigenvector Method

This method assumes that each pairwise comparison is associated to exactly one opinion. As mentioned in the introduction, the vote cube is equivalent to a comparison matrix under this condition.

Since the comparison matrix C has positive elements, Saaty [12,13] recalls that, in such a case, the theorem of Perron and Frobenius guarantees that the largest eigenvalue $\lambda_{\max}$ is unique, real, and positive. Saaty shows that given a consistent C matrix, the eigenvector is the weight vector once normalized (its components sum to one). In case of reasonable (not too severe) inconsistencies, Saaty proposes to adopt this normalized eigenvector as an acceptable weight vector. This is the base of his priority theory and analytic hierarchy process (A.H.P). Unlike the methods presented hereafter, there is no clearly identified optimization criteria associated to the eigenvector method.

2.2. Row Geometric Mean (RGM) or Column Geometric Mean (CGM)

Other formulae often used to solve a comparison matrix are those of the row geometric mean (1) [3,8] and the column geometric mean (2) [3]. These methods require the reciprocity of the comparison matrix in order to have the least logarithmic squares as optimization criteria (Section 2.3).

$$w_i \approx \frac{r_i}{\sum\limits_{s=1}^{n} r_s}, \qquad i = 1,2,\ldots,n, \quad \text{with } r_i = \left(\prod_{j=1}^{n} c_{ij}\right)^{1/n}, \tag{1}$$

$$w_i \approx \frac{r_j}{\sum\limits_{s=1}^{n} r_s}, \qquad i = 1,2,\ldots,n, \quad \text{with } r_j = \left(\prod_{i=1}^{n} c_{ij}\right)^{-1/n}. \tag{2}$$

[2]Although based on different assumptions and different computational processes, these two popular pairwise comparison methods have very close optimization criteria (based on the logarithmic least squares) and can be considered as two different generalizations of the Row Geometric Mean approach (Section 2.2).

2.3. Row and Column Geometric Mean (RCGM)

Koczkodaj and Orlowski [3] proposed recently a generalization of the RGM and CGM methods to nonreciprocal comparison matrices. Starting from an unnecessarily consistent matrix C, they provide the closest consistent C^* matrix from a least logarithmic squares point of view (see Note 1), i.e., minimizing $\sum(\log(c^*_{ij}) - \log(c_{ij}))^2$, with the following formulae:

$$c^*_{ij} = \left(\frac{R^*_i}{R^*_j}\frac{G^*_j}{G^*_i}\right)^{1/2}, \quad i,j = 1,2,\dots,n, \text{ with } R^*_i = \left(\prod_{j=1}^{n} c_{ij}\right)^{1/n} \text{ and } G^*_i = \left(\prod_{j=1}^{n} c_{ij}\right)^{1/n}. \quad (3)$$

When applied to the consistent matrix C^*, any of the previous methods yields the weight vector.

2.4. Least Squares Logarithmic Regression (LSLR)

For a consistent matrix C, all the methods listed above: EV, RGM, CGM, RCGM coincide. But they all require exactly one opinion per comparison. For dealing with multiple opinions or with no opinion per comparison, de Graan [1] and Lootsma [2] proposed a generalization of RGM through a least squares logarithmic regression approach (denoted LSLR). It consists in minimizing the distance between the logarithmic terms of the vote cubes C and C^*. This can be formulated as follows:

$$\sum_{i=1}^{n}\sum_{j=i+1}^{n}\sum_{k=1}^{d_{ij}} \alpha_{ijk}(\log(c_{ijk}) - (\log(w_i) - \log(w_j)))^2, \quad (4)$$

with c_{ijk}, $i = 1,2,\dots,n$, $j = 1,2,\dots,n$, $k = 1,2,\dots,d$ the opinion of the decision maker k for the (e_i, e_j) comparison, d the number of decision makers, and α_{ijk} $(i,j = 1,2,\dots,n,\ k = 1,2,\dots,d)$ a parameter equal to 1 when the decision maker k decides to express a personal opinion ($c_{ijk} \in\ ]0;+\infty[$) and equal to 0 otherwise. When α_{ijk} equals 0, c_{ijk} is set to an arbitrary positive nonzero value. This allows the algebraic representation of nonexpressed opinions.

The minimization of the least squares objective function (see Note 1) given by (4) leads to the resolution of the so-called normal equations

$$\theta_i \sum_{j\neq i}^{n} d_{ij} - \sum_{j\neq i}^{n-1} d_{ij}\theta_j = \sum_{j\neq i}^{n}\sum_{k=1}^{d} \alpha_{ijk} b_{ijk}, \qquad i = 1,2,\dots,n-1,$$

with

$$\begin{aligned}
\theta_i &\approx \log(w_i), && i = 1,2,\dots,n, \\
\theta_n &= 0, && \text{(weight fixed to 1)}, \\
b_{ijk} &= \log(c_{ijk}), && i,j = 1,2,\dots,n, \quad k = 1,2,\dots,d, \\
d_{ij} &= \sum_{k=1}^{d} \alpha^2_{ijk} = \sum_{k=1}^{d} \alpha_{ijk}, && i,j = 1,2,\dots,n, \text{ and } i \neq j,
\end{aligned} \quad (5)$$

d_{ij} : number of opinions for comparison (i,j).

Since the weights are defined up to a multiplicative constant, the normal equations are underconstrained. Solving the system requires setting one of the weights to an arbitrary value ($w_n = 1$).

Moreover, the possible missing opinions must not diminish the rank $(n-1)$ of the system of normal equations. This condition is satisfied when each of the n elements to be compared $(e_1,\dots,e_n)$ is involved in at least one opinion and when no pair of elements is disjoint by transitivity (elements e_i and e_j are disjoint by transitivity when $\alpha_{ihk} \times \alpha_{hjk} = 0$, $\forall\, k = 1,2,\dots,d$, $\forall\, h = 1,2,\dots,n$).

The last stage consists of the normalization procedure described by the following formula:

$$w_i = \frac{\exp(\theta_i)}{\sum\limits_{j=1}^{n} \exp(\theta_j)}, \qquad i = 1, 2, \ldots, n. \tag{6}$$

Under this form (formulae (4),(5)), the symmetrical comparisons are assumed reciprocal ($\alpha_{ijk} \times c_{ijk} = \alpha_{jik}/c_{jik}$, $i, j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, d$). The previous methods EV and RCGM do not suffer from this restrictive hypothesis.

2.5. Bryson and Joseph's Goal Programming Approach

Bryson and Joseph's method [6,7] represents the mapping from the vote cube C to a suitable set of weights in the form of a flexible and always feasible logarithmic goal programming model[3] (GPM). In their formulation, they assume that each decision maker expresses exactly one opinion per binary comparison (no abstention). As detailed in formula (7), they explicitly define an objective function (optimization criteria) associated to a set of linear constraints.

$$\log(\sigma) = \min\left(\frac{1}{d} \times \sum_{k=1}^{d} \log(\sigma_k)\right),$$

with

$$\begin{aligned}
\log(\sigma_k) - \frac{1}{n \times (n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} (\log(p_{ijk}) + \log(q_{ijk})) = 0, \qquad & k = 1, 2, \ldots, d, \\
\log(w_i) - \log(w_j) + \log(p_{ijk}) - \log(q_{ijk}) = \log(c_{ijk}), \qquad & \\
i, j = 1, 2, \ldots, n, \quad & k = 1, 2, \ldots, d, \\
\min(p_{ijk}, q_{ijk}) = 1, \qquad i, j = 1, 2, \ldots, n, \quad & k = 1, 2, \ldots, d.
\end{aligned} \tag{7}$$

In these constraints, each ratio of weights w_i/w_j is multiplied, up to a logarithmic transformation, by a ratio of real numbers p_{ijk}/q_{ijk} (such as $p_{ijk} \geq 1$ and $q_{ijk} = 1$ or $p_{ijk} = 1$ and $q_{ijk} \geq 1$) in order to coincide with the vote c_{ijk} of decision maker k. The geometric mean of the products $p_{ijk}q_{ijk}$, over all the expressed opinions, constitutes the objective function to minimize. It is equivalent to the sum of absolute errors (see Note 1) and represents the average value that each entry in the vote cube would have to be multiplied or divided by in order to reach consistency.

3. GENERALIZATION OF THE LSLR APPROACH

In this section, we propose a generalization of the least squares logarithmic regression method to take into account nonreciprocal vote cubes. A short recall of the principal characteristics of this regression based approach is first presented.

A regression model may be considered as an optimized approximation of the relation between a random variable which is said to be dependent and a set of prediction variables assumed not to be random. The model is built on a set of observations of the dependent variable for different sets of values of the prediction variables. For example, if we assume that the size of an individual is related to his/her weight, a linear regression approach consists of finding the straight line

[3]Assuming that the evaluator can provide, *a priori*, interval estimates for the weights and assuming particular behavioral tendencies in the extraction of ratio estimates from these intervals, Bryson *et al.* [4] derive validity conditions for the computed weight vector. In this particular context, they demonstrate the invalidity of methods where the ratio estimates provided by the evaluator are treated with an averaging logic. All the methods previously presented in this section are based on such an averaging logic. Conversely, more flexible and detailed approaches, such as the goal programming, are more suitable for assessing such assumptions on the inputs.

modelling at best a set of (weight, size) measurements performed on a representative sample of individuals.

In the case of a linear regression model [14], an observation Y_i of the dependent variable y is related to values taken by the prediction variables $x_1, x_2, \dots, x_n$ by the way of equations of the form $Y_i = \eta_0 + \eta_1 X_{1,i} + \dots + \eta_n X_{n,i} + \varepsilon_i$. The constant coefficients $\eta_0, \eta_1, \dots, \eta_n$ are parameters which have to be estimated in order to complete the model. Coefficient ε_i is a random coefficient of error representing the difference between the linear model prediction and the observation i. In matrix notation, the equation set may be expressed as $Y = XH + E$. A least squares linear regression consists of a linear estimation model $\hat{Y} = X\Theta$. The vector Θ corresponds to the minimal value of the sum of the error squares $\sum_i (Y_i - \hat{Y}_i)^2$ between the measured values of the dependent variable (Y) and the estimated ones ($\hat{Y}$). When Θ exists it is the solution of the normal equation set $X^t Y = X^t X \Theta$ (X^t denotes the transpose of X), i.e.,

$$\Theta = \left(X^t X\right)^{-1} X^t Y. \tag{8}$$

In the pairwise comparison context, if we consider $Y_{ijk} = \alpha_{ijk} \log(c_{ijk})$, $i, j = 1, 2, \dots, n$, $i \neq j$ as the dependent variable observations, the equation $\alpha_{ijk} \log(c_{ijk}) = \alpha_{ijk}(\log(w_i) - \log(w_j) + \varepsilon_{ijk})$ relates in a linear manner an observation Y_{ijk} to the set of prediction variables ($X_1 = 0, \dots, X_i = \alpha_{ijk}$, $X_{i+1} = 0, \dots, X_j = -\alpha_{ijk}$, $X_{j+1} = 0, \dots, X_n = 0$) by the way of intermediary parameters ($\eta_0 = 0$, $\eta_1 = \log(w_1), \dots, \eta_i = \log(w_i), \dots, \eta_j = \log(w_j), \dots, \eta_n = \log(w_n)$). Let us recall that $\alpha_{ijk} = 1$ if the decision maker k expresses an opinion on comparison c_{ij} and $\alpha_{ijk} = 0$ otherwise (c_{ijk} is then set to $c_0 > 0$). With this convention it is straightforward to extend the formulae presented in Section 2.4 in order to handle both reciprocal and nonreciprocal cubes. The observation vector which was restricted to the upper triangular part of the vote cube ($j > i$) must now include all binary combinations except the reflexive ones ($i = j$).

The function to be minimized is given by

$$f = \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \sum_{k=1}^{d} \alpha_{ijk} (\log(c_{ijk}) - (\log(w_i) - \log(w_j)))^2. \tag{9}$$

This formula differs from formula (4) by the fact that now $i \neq j$ instead of $j > i$. In the same way, a generalization of formula (5) is given by

$$\theta_i \sum_{j \neq i}^{n} (d_{ij} + d_{ji}) - \sum_{j \neq i}^{n-1} (d_{ij} + d_{ji}) \theta_j = \sum_{j \neq i}^{n} \sum_{k=1}^{d} (\alpha_{ijk} b_{ijk} - \alpha_{jik} b_{jik}), \qquad i = 1, 2, \dots, n-1,$$

with

$$\begin{aligned} \theta_i &\approx \log(w_i), && i = 1, 2, \dots, n, \\ \theta_n &= 0, && \text{(weight } w_n \text{ arbitrarily fixed to 1)}, \\ b_{ijk} &= \log(c_{ijk}), && i, j = 1, 2, \dots, n, \quad k = 1, 2, \dots, d, \\ d_{ij} &= \sum_{k=1}^{d} \alpha_{ijk}, && i, j = 1, 2, \dots, n, \text{ and } i \neq j. \end{aligned} \tag{10}$$

We denote this pairwise comparison approach GLSLR for generalized least squares logarithmic regression. As for the LSLR approach, the rank of the system of normal equations must be equal to $n - 1$.

Formulae (9) and (10) do not include reflexive ($i = j$) binary comparisons (expressed for example in some blind tests). Opinions expressed on such reflexive comparisons do not influence the estimated weights. This invariance does not depend on the pairwise comparison method. Setting the reflexive votes to 1 will always improve the quality of the starting vote cube, whatever the adopted optimization logic is.

4. GENERALIZATION OF THE RCGM APPROACH

In this section, we propose a generalization of the RCGM approach of Koczkodaj and Orlowski [3] to authorize several opinions per pairwise comparison. This extension yields a more flexible and practical method. Furthermore, it opens the scope to a straightforward analytic weight formula that generalizes the formula Koczkodaj and Orlowski obtain for the closest consistent comparison matrix under the mono-opinion hypothesis. Such formulae could also be useful for further extensions to handle imprecision of judgments [8–10].

In their paper, Koczkodaj and Orlowski [3] propose to find the consistent comparison matrix C^* that is the closest to the initial C matrix containing the decision group opinions, in a logarithmic least squares sense. The problem is equivalent to minimizing the quadratic function f given by

$$f = \sum_{i,j=1}^{n} (\log(c_{ij}) - (\log(w_i) - \log(w_j)))^2. \tag{11}$$

When extending this function to multiple (or no) opinions per comparison, it leads exactly to formula (9). As noticed in the end of the previous section, it does not matter if the reflexive binary comparisons are represented or not in the objective function. The resulting set of weights will always respect the equality $c^*_{ii} = w_i/w_i = 1$, $i = 1, 2, \dots, n$.

Let us denote by GRCGM (generalized row and column geometric mean approach) the RCGM method extended to multiple opinions. We are already allowed to state that the GLSLR and the GRCGM approaches correspond to the same optimization criterion.

The weights being defined up to a multiplicative constant, Koczkodaj and Orlowski chose to minimize the Euclidean distance between matrices B and B^*, respectively, the logarithmic images of C and C^*, under the constraint $\log(w_n) = 0$ ($w_n = 1$). They used the Lagrange multipliers approach by formulating the problem as the minimization of the function $u = f + \lambda\eta_n$ relatively to the variables $\eta_1 = \log(w_1)$, $\eta_2 = \log(w_2), \dots, \eta_n = \log(w_n)$ and λ.

By applying Koczkodaj and Orlowski's approach to a variable number of opinions per comparison and setting to 0 the partial derivatives of function u relatively to η_i, $i = 1, 2, \dots, n$, one obtains the equations

$$\begin{aligned} \theta_i \sum_{j=1}^{n} Q_{ij} - \sum_{j=1}^{n} \theta_j Q_{ij} &= R_i - G_i, \qquad && i = 1, 2, \dots, n,\\ \theta_n &= 0, \end{aligned}$$

with

$$\begin{aligned} \theta_i &\approx \log(w_i), && i = 1, 2, \dots, n,\\ R_i &= \sum_{j=1}^{n} \sum_{k=1}^{d} \alpha_{ijk} b_{ijk}, && i = 1, 2, \dots, n,\\ G_i &= \sum_{j=1}^{n} \sum_{k=1}^{d} \alpha_{jik} b_{jik}, && i = 1, 2, \dots, n,\\ Q_{ij} = Q_{ji} &= \sum_{k=1}^{d} (\alpha_{ijk} + \alpha_{jik}), && i, j = 1, 2, \dots, n,\\ b_{ijk} &= \log(c_{ijk}), && i, j = 1, 2, \dots, n; \quad k = 1, 2, \dots, d. \end{aligned} \tag{12}$$

These equations are similar to those of formula (10) even if they are presented in a different way. By shifting to matrix notations we can easily retrieve the normal equations introduced in Section 3. The quantities R_i and G_i represents summations involving opinions where element i appears in the first position (i.e., c_{ihk}) for R_i and in the second position (i.e., c_{hik}) for G_i. The quantity Q_{ij} represents the number of opinions expressed to compare the elements e_i and e_j, whatever the order is, i.e., it concerns the c_{ijk} and c_{jik} comparisons, for $k = 1, 2, \dots, d$.

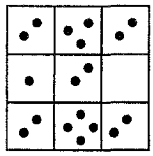

Figure 2. Example of opinion distribution in a comparison matrix for the GRCGM approach, with $n = 3$ and $q = 2$.

5. EXPLICIT WEIGHT FORMULATION UNDER A RESTRICTIVE HYPOTHESIS

Let us assume Q_{ij} (see Section 4) to be constant for all combinations of two elements. Q_{ij} is then an even integer, i.e., $Q_{ij} = 2q$, with q a constant integer. This is due to the symmetry of the comparison matrix. In particular, the reflexive opinions (for $i = j$) are naturally counted twice: $Q_{ii} = \sum_{k=1}^{d}(\alpha_{iik} + \alpha_{iik}) = 2\sum_{k=1}^{d}\alpha_{iik}$.

For example, Figure 2 illustrates the case where $q = 2$ for $n = 3$ (number of elements). Each point symbolizes one opinion.

Even if setting Q_{ij} to a constant is a restrictive assumption it is still of interest in practice. It covers the case where exactly one opinion is available per comparison. More generally, it corresponds to an equilibrated vote where a same number of opinions is attributed to each combination of two elements, i.e., to the c_{ij} and c_{ji} comparisons (including reflexive comparisons c_{ii} counted twice). When the vote cube is partially filled, it is possible to complete some of the missing entries. For instance, it is always possible to assume the missing reflexive comparisons to be equal to 1 or to consider several reflexive opinions equal to 1 so as to equilibrate the vote entries. The decision group can also duplicate (increase the importance of) some of the opinions for the comparisons $\{(e_i, e_j), (e_j, e_i)\}$ that received fewer votes than others.

Our assumption ($Q_{ij} = 2q$) allows us to transform formula (12) into

$$\theta_i \sum_{j=1}^{n} 2q - \sum_{j=1}^{n} 2q\theta_j = R_i - G_i,\ i = 1, 2, \ldots, n \Leftrightarrow \theta_i = \frac{R_i - G_i}{2nq} + \frac{\sum_{j=1}^{n}\theta_j}{n},\ i = 1, 2, \ldots, n. \quad (13)$$

The weight w_n being arbitrarily fixed to 1 leads to

$$\begin{aligned} \theta_n &= \log(w_n) = 0 = \frac{R_n - G_n}{2nq} + \frac{\sum_{j=1}^{n}\theta_j}{n}, \qquad \text{hence,} \\ \theta_i &= \frac{R_i - G_i - (R_n - G_n)}{2nq}, \qquad i = 1, 2, \ldots, n. \end{aligned} \quad (14)$$

The not yet normalized weights may now be expressed as

$$w_i \approx \exp(\theta_i),\ i = 1, 2, \ldots, n \Leftrightarrow w_i \sqrt[2q]{\frac{R_i^*}{G_i^*}\frac{G_n^*}{R_n^*}}, \qquad i = 1, 2, \ldots, n. \quad (15)$$

The terms R_s^* and G_s^* are homogeneous with geometric means of opinions of row s and column s, respectively. They are given by

$$\begin{aligned} R_s^* &= \sqrt[n]{\exp(R_s)} = \sqrt[n]{\exp\left(\sum_{r=1}^{n}\sum_{k=1}^{d}\alpha_{srk}\log(c_{srk})\right)} = \sqrt[n]{\prod_{r=1}^{n}\prod_{k=1}^{d} c_{srk}^{\alpha_{srk}}}, \\ G_s^* &= \sqrt[n]{\exp(R_s)} = \sqrt[n]{\exp\left(\sum_{r=1}^{n}\sum_{k=1}^{d}\alpha_{rsk}\log(c_{rsk})\right)} = \sqrt[n]{\prod_{r=1}^{n}\prod_{k=1}^{d} c_{rsk}^{\alpha_{rsk}}}, \end{aligned} \qquad s = 1, 2, \ldots, n. \quad (16)$$

The two constant terms R_n^* and G_n^* disappear when the weights are normalized. This normalization leads to the definitive weight expression

$$
\begin{aligned}
w_{iN} &\approx \frac{\sqrt[2q]{R_i^*/G_i^*}}{\sum_{i=1}^{n} \sqrt[2q]{R_i^*/G_i^*}}, \qquad i = 1,2,\dots,n, \quad \text{with} \\
R_s^* &= \sqrt[n]{\prod_{r=1}^{n} \prod_{k=1}^{d} c_{srk}^{\alpha_{srk}}}, \\
G_s^* &= \sqrt[n]{\prod_{r=1}^{n} \prod_{k=1}^{d} c_{rsk}^{\alpha_{rsk}}}, \qquad s = 1,2,\dots,n.
\end{aligned} \tag{17}
$$

Finally, it is possible to give an explicit formula for the comparisons c_{ij}^* composing the resulting consistent matrix C^* in the form

$$
\begin{aligned}
c_{ij}^* &= \frac{w_i}{w_j} \approx \sqrt[2q]{\frac{R_i^*}{G_i^*}\frac{G_j^*}{R_j^*}}, \qquad i,j = 1,2,\dots,n, \quad \text{with} \\
R_s^* &= \sqrt[n]{\prod_{r=1}^{n} \prod_{k=1}^{d} c_{srk}^{\alpha_{srk}}}, \\
G_s^* &= \sqrt[n]{\prod_{r=1}^{n} \prod_{k=1}^{d} c_{rsk}^{\alpha_{rsk}}}, \qquad s = 1,2,\dots,n.
\end{aligned} \tag{18}
$$

Let us consider again our assumption $Q_{ij} = 2q$. It means that the number of opinions expressed on two elements (i.e., on the comparisons c_{ij} and c_{ji}) is constant and even. The case $q = 1$ corresponds to exactly one opinion per comparison. Formula (18) is then equivalent to the formula given hereafter which, in return, is identical to formula (3) proposed by Koczkodaj and Orlowski

$$
\begin{aligned}
c_{ij}^* &= \frac{w_i}{w_j} \approx \sqrt{\frac{R_i^*}{G_i^*}\frac{G_j^*}{R_j^*}} = \sqrt[2n]{\prod_{r=1}^{n} \left(\frac{c_{ir}}{c_{ri}}\frac{c_{rj}}{c_{jr}}\right)}, \qquad i,j = 1,2,\dots,n, \quad \text{with} \\
R_s^* &= \sqrt[n]{\prod_{r=1}^{n} c_{sr}}, \qquad G_s^* = \sqrt[n]{\prod_{r=1}^{n} c_{rs}}.
\end{aligned} \tag{19}
$$

6. CONCLUSION

Pairwise comparison methods are convenient procedures that decompose the weighting of n elements into pairwise relative comparisons so as to help decision makers in predicting the soundest weights.

Two popular approaches exist in the literature: the least squares logarithmic regression method (LSLR) by de Graan [1] and Lootsma [2] and the recent row and column geometric mean (RCGM) by Koczkodaj and Orlowski [3]. Both approaches are based on the minimization of logarithmic least squares. However, they consider a different hypothesis: the LSLR approach assumes the reciprocity property for the vote cube while the RCGM approach requires the presence of exactly one opinion per comparison.

In this paper, each of these approaches has been extended by releasing the corresponding restrictive assumption. Both extensions lead to the same generalized objective function. With the assumption of a constant opinion number for all the pairwise comparisons, new algebraic expressions have been established for the predicted weights. We have verified that for one opinion per comparison, our formula matches that of Koczkodaj and Orlowski. Such an explicit formulation can for instance be useful when extending the deterministic pairwise comparison approach to a modeling of judgemental imprecision [8–10].

REFERENCES

1. J.G. de Graan, Extensions to the multiple criteria of T.L. Saaty, Report National Institute of Water Supply, (1980).
2. F.A. Lootsma, Performance evaluation of nonlinear optimization methods via multi-criteria decision analysis and via linear model analysis, In *Nonlinear Optimization*, Vol. 1, (Edited by M.J.D. Powell), pp. 419–453, Academic Press, London, (1982).
3. W.W. Koczkodaj and M. Orlowski, Computing a consistent approximation to a generalized pairwise comparisons matrix, *Computers Math. Applic.* **37**, 79–85, (1999).
4. N. Bryson, A. Mobolurin and O. Ngwenyama, Modelling pairwise comparisons on ratio scales, *European Journal of Operational Research* **83**, 639–654, (1995).
5. A. Tversky, Intransitivity of preferences, *Psychological Review* **76** (1), 31–48, (1969).
6. N. Bryson, A goal programming method for generating priority vectors, *Journal of the Operational Research Society* **46**, 641–648, (1995).
7. N.K.-M. Bryson and A. Joseph, Generating consensus priority point vectors: A logarithmic goal programming approach, *Computers & Operations Research* **26**, 637–643, (1999).
8. C.G.E. Boender, J.G. de Graan and F.A. Lootsma, Multi-criteria decision analysis with fuzzy pairwise comparisons, *Fuzzy Sets and Systems* **29**, 133–143, (1989).
9. J.J. Buckley, Fuzzy hierarchical analysis, *Fuzzy Sets and Systems* **17**, 233–247, (1985).
10. F. Limayem, Modèles de pondération par les méthodes de tri croisé pour l'aide à la décision collaborative en projet, Thèse de doctorat, 23 Novembre 2001, Ecole Centrale Paris (2001).
11. A. Arbel, Approximate articulation of preference and priority derivation, *European Journal of Operational Research* **43**, 317–326, (1989).
12. T.L. Saaty, A scaling method for priorities in hierarchical structures, *Journal of Mathematical Psychology* **15** (3), 234–281, (1977).
13. F.A. Lootsma, Saaty's priority theory and the nomination of a senior professor in operations research, *European Journal of Operational Research* **4**, 380–388, (1980).
14. N.R. Draper and H. Smith, *Applied Regression Analysis*, Second edition, Wiley Series in Probability and Mathematical Statistics, (Edited by S.W. Sons), New York, (1980).