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Communications through Unspecified Additive Noise

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I. INTRODUCTION

The purpose of this note is to suggest the feasibility in certain communication-through-noise situations of applying a maximin or minimax type decision criterion to the design of the receiver, to illustrate this decision criterion in two rather straightforward examples involving binary data links, and to look at the information rate which can be achieved. We are concerned here with the receiver decision problem when the signal is corrupted only by additive noise and when the additive noise has no statistical regularity or predictability of which advantage can be taken. Thus the "noise" may be natural or man-made and subject to influences unknown to the communicators. Essentially the only restriction on the noise is that its power be bounded.

The point of departure here is to use a decision criterion at the receiver which guarantees the greatest possible average reliability of reception, no matter what the noise is doing. This leads to a maximim decision criterion at the receiver and to the use of concepts from the theory of constant-sum, two-person game theory. We review briefly some of these concepts in Section II. The examples are carried out in Sections III and IV. One would expect that a receiver designed to work as well as possible against any kind of noise would not work especially well against any particular kind (say white Gaussian noise), and that there will be a resulting decrease in information rate from that achieved in systems where the noise is specified statistically. We calculate the information rate for the examples considered in Section V.

II. GAME THEORY TERMINOLOGY[1]

We shall be concerned with a special class of zero-sum, two-person games G in normal form, which may be described as follows: There are

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[1] This section consists of a very brief review of the terminology and facts from

two players, to be denoted by I and II; player I has a set of possible strategies B, player II has a set of possible strategies A; β and α will be generic elements of B and A, respectively, so β will denote a strategy of I, α a strategy of II. To avoid later confusion, these strategies will be called *pure strategies.* There is also a *payoff function* (or matrix), $P(\beta, \alpha)$, which is a real valued function of β and α. A single play of the game G consists of the following: player I chooses a pure strategy β, unknown to player II; player II chooses a pure strategy α, unknown to player I; the value of the payoff function $P(\alpha, \beta)$ is determined for this pair α, β, and player II pays player I the amount $P(\alpha, \beta)$. It is to player II's advantage to keep the payoff as small as possible, to player I's advantage to make the payoff as large as possible.

With the exception of one simple result, we shall confine ourselves entirely to cases in which B and A are either finite sets or are closed, finite intervals of the real line; thus, β and α will be real numbers bounded from above and below.

We now want to consider the *mixed extension*, Γ, of G. A *mixed strategy* (henceforth, just *strategy*) for player I is a probability distribution $F(\beta)$ on B; similarly a mixed strategy (stategy) for player II is a probability distribution $H(\alpha)$ on A. We may consider heuristically an *extended play* of the game G, or a play of Γ, to consist of the following: I has a random number generator which generates numbers β according to the distribution $F(\beta)$, II similarly can generate random numbers α according to $H(\alpha)$. A long sequence of independent random numbers $\{\beta_i\}$ is obtained from player I's random number generator, and a corresponding sequence of independent random numbers $\{\alpha_i\}$ is obtained from player II's random number generator. Then a sequence of single plays of the game G is made with I successively using the pure strategies β_i and II using α_i. The final payoff is taken to be the average of the single game payoffs; that is, the payoff of the extended game Γ is defined to be

$$\int_A \int_B P(\beta, \alpha)\, dF(\beta)\, dH(\alpha).$$

If either B or A is a finite or denumerable set, the corresponding integral may be replaced by a sum.

game theory which we need. Definitions for the most part are stated loosely. For precise statements and a general treatment of the theory, the reader is referred to Blackwell and Girshick, 1954.

The *lower value of* G, v_*, is the least upper bound of the payoffs that I can guarantee himself in the extended game Γ by properly choosing a strategy $F(\beta)$, no matter what strategy $H(\alpha)$ II employs. That is

$$v_* = \underset{F(\beta)}{\text{l.u.b.}} \left\{ \underset{H(\alpha)}{\text{g.l.b.}} \int_A \int_B P(\beta, \alpha)\, dF(\beta)\, dH(\alpha) \right\}.$$

The *upper value* v^* is

$$v^* = \underset{H(\alpha)}{\text{g.l.b.}} \left\{ \underset{F(\beta)}{\text{l.u.b.}} \int_A \int_B P(\beta, \alpha)\, dF(\beta)\, dH(\alpha) \right\}.$$

It is easy to show that always

$$v_* \leqq v^*.$$

If $v_* = v^*$, the game is said to have a *value* $v = v_*$. If there is a strategy $F_0(\beta)$ such that

$$v_* = \underset{H(\alpha)}{\text{g.l.b.}} \int_A \int_B P(\beta, \alpha)\, dF_0(\beta)\, dH(\alpha)$$

then $F_0(\beta)$ is called a *maximin strategy*. There is a corresponding definition for minimax strategy. If we ally ourselves with player I, then we are interested in finding maximin strategies, or at least strategies which guarantee a payoff close to v_*. The fundamental theorem of zero-sum, two-person games is that every finite game (both B and A finite sets) has a value and there exist maximin and minimax strategies for I and II respectively. There are various extensions of this theorem to infinite games, but we shall not need to use them explicitly.

A simple remark, but one which we shall use, is: if there exists a mixed strategy $H_0(\alpha)$ for player II with the property that

$$\int_A P(\beta, \alpha)\, dH_0(\alpha) \leqq k \qquad \text{for all } \beta$$

then $v_* \leqq k$. This follows directly from the definition of v_*.

Often there are redundant or duplicate pure strategies in games and nothing essential is changed if these are eliminated. We say G' and G'' are *equivalent* games if

1. by an elimination of duplicated pure strategies for I in G', G' becomes identical with G'', or

2. by an elimination of duplicated pure strategies for II in G', G' becomes identical with G'', or

3. there is a finite sequence of games $G' = G_1, G_2, \cdots, G_N = G''$ such that G_i and G_{i+1} are related as in 1 or 2.

Equivalent games have the same upper and lower values.

Two further remarks are needed. If in a game G a new game G' is formed by deleting some of I's pure strategies, the lower value v_*' of G' is less than or equal to v_*. This is essentially obvious, for I certainly cannot guarantee himself a larger payoff if he loses some of his possible strategies. However, if a pure strategy β_2 has the property that there is a β_1 such that

$$P(\beta_2, \alpha) \leqq P(\beta_1, \alpha)$$

for *every* α, then I can delete the strategy β_2 without decreasing the lower value of the game. Obvious parallel remarks may be made concerning the deletion of pure strategies of player II.

III. KEYED CARRIER SYSTEM WITH POWER DETECTOR

A. System Model

As a first example let us consider perhaps the simplest kind of binary data link, a synchronously-keyed carrier, in which the MARK symbol is transmitted as unmodulated carrier for an interval of duration T and the SPACE symbol is "transmitted" as an interval of silence of duration T. The system is only synchronous with respect to the modulation; either the receiver has a clock which indicates approximately the beginning and end of each symbol period, or a synchronizing signal is sent. In any event rf phase is not necessarily known. We assume the signal level at the receiver is known, so we may represent the signals at the receiver by

$$\left.\begin{array}{ll}\text{MARK:} & s(t) = b\cos(\omega_0 t + \theta) \\ \text{SPACE:} & s(t) = 0\end{array}\right\} \quad 0 \leqq t < T$$

where θ is unknown and where b is chosen so that

$$\int_0^T b^2 \cos^2 \omega_0 t \, dt = 1.$$

It is further assumed T is sufficiently greater than $1/\omega_0$ so that to a very good approximation

$$\int_0^T b^2 \cos^2(\omega_0 t + \theta)\, dt = 1$$

for all θ. Since in this example rf phase is disregarded and the modulation is a simple on or off, the receiver can recover all the information there is in the signal by just measuring the energy received during each symbol interval. Accordingly, we shall suppose that the basic signal-processing done by the receiver results in a measurement of energy in a band of arbitrary width about the carrier frequency f_0. The channel disturbance is assumed to be additive, so that the total received wave form, $y(t)$, can be written

$$y(t) = s(t) + z(t), \qquad 0 \leqq t < T$$

where $z(t)$ is the disturbing signal or noise. We shall refer to $z(t)$ as "noise," but there is no implication that it has any of the properties usually associated with noise, such as being Gaussian, stationary, or even random in any sense. In fact, the only restrictions on $z(t)$ are that it be energy limited,

$$\int_0^T z^2(t)\, dt \leqq a = \text{constant}$$

and that it be uncorrelated with $s(t)$ over the interval $0 \leqq t < T$,

$$\int_0^T s(t)\, z(t)\, dt = 0.$$

This second restriction simplifies the decision problem at the receiver a little when, as in the present case, the receiver is not sensitive to rf phase. It seems a perfectly reasonable restriction in view of the fact the symbol interval is long compared to one rf period.

Thus the output at the end of each symbol interval from what may be called the signal-processing part of the receiver—the part which we have fixed in the above discussion and which does not involve any decision making—is:

$$\begin{aligned} y &= 1 + \int_0^T z^2(t)\, dt \qquad &\text{if MARK was sent} \\ &= \int_0^T z^2(t)\, dt \qquad &\text{if SPACE was sent.} \end{aligned}$$

If we let α be the energy in the noise,

$$\alpha = \int_0^T z^2(t)\, dt,$$

then the output is either $1 + \alpha$ or α, where $0 \leqq \alpha \leqq a$, a being the maximum permissible noise-to-signal ratio.

The decision problem to be solved is to state a decision rule which will tell from the value y whether a MARK was sent or a SPACE was sent. Obviously, if the noise energy α during each symbol is known, the correct decision can always be made. However, if α is unknown and varies from symbol to symbol and $a > 1$, the decision problem is non-trivial. Let us re-emphasize that nothing is known about the noise which would permit assigning a probability distribution to α, so a Bayes' solution is out of the question. Instead, we shall specify a decision rule, which amounts, of course, to specifying a decision-making receiver, which we prove to be nearly optimum in a game-theoretic sense, now to be made precise.

B. Game Theoretic Formulation

We now use the concepts reviewed in Section II. Let the opponents in a two-person game be the person who is receiving the communication, who will be called player I, and Nature, who will be called player II. One play of the game will consist first of the choice with probability one-half of either a MARK or SPACE, unknown to both players, then the transmission of that symbol with accompanying noise introduced by player II, and then the decision at the receiver by player I as to whether a MARK or SPACE was transmitted. A pure strategy for player I is a decision rule which assigns to each output y either MARK or SPACE; we may represent MARK by 1 and SPACE by 0, so a pure strategy for player I is a zero-or-one-valued function of y. A mixed strategy for I is a randomized decision rule and can be represented by a function of y taking on values anywhere between zero and one, its value being the probability a MARK will be announced. A pure strategy for player II is a choice of α, that is, a choice of the energy of the noise in the acceptance band of the receiver. A mixed strategy for II is a probability distribution of values of α between 0 and a. We shall take as the value of the payoff function the probability that the decision made by player I is correct. In a sense, then, player II pays player I; it is to player I's advantage for the payoff to be as large as possible. This choice of payoff function is of course arbitrary, but seems quite reasonable in view of the symmetric binary nature of the transmission. The payoff function will be denoted by $P(\gamma, \alpha)$ where the variable γ corresponds to the pure strategies of I, and α to the pure strategies of II.

From what has been said above, it is clear that player II's pure strategies can be indexed by or made to correspond to the real numbers α in the interval $0 \leqq \alpha \leqq a$. Player I's pure strategies on the other hand correspond to subsets of the interval $[0, a + 1]$; in fact, any nonrandomized decision rule amounts to a specification of the set of possible values of y for which a MARK will be announced. There are too many subsets of an interval to be indexed by the numbers in the interval, so we cannot represent *all* of player I's pure strategies by real numbers in a finite interval. We could if necessary represent a quite large class of I's pure strategies by points in an interval—a class large enough to include any nonrandomized decision rule that might reasonably be implemented—by using standard mappings that carry n-dimensional and even ∞-dimensional spaces onto the line. This is not necessary, however, because we need to consider player I's full set of pure strategies only superficially in order to establish a simple upper bound on the payoff of which player I can be sure. For the rest of the discussion we are then able to restrict player I's strategies to a simple subclass which corresponds in a natural way to the points on a finite interval. We then have a two-person game in the form discussed in Section II.

C. An Upper Bound on Player I's Guaranteed Payoff

As mentioned in Section II, the largest payoff player I can guarantee himself, that is, the maximum probability of making a correct decision which the communicator can guarantee himself, is the lower value, v_*, of the game. For the game being discussed we have

$$v_* \leqq \frac{1}{2} + \frac{1}{2a}.$$

The proof is as follows: Let $d_\gamma(y)$ be any nonrandomized decision function. Then if player II uses the pure strategy α and the symbol sent is MARK, the payoff is $d_\gamma(\alpha + 1)$, which is either zero or one. If the symbol sent is SPACE, the payoff is $1 - d_\gamma(\alpha)$. Since the two possible symbols are sent each with probability one-half, the payoff function is

$$P(\gamma, \alpha) = \frac{d_\gamma(\alpha + 1) + (1 - d_\gamma(\alpha))}{2}.$$

Hence, if II weights his pure strategies uniformly,

$$\frac{1}{a}\int_0^a P(\gamma, \alpha)\, d\alpha = \frac{1}{2a}\int_0^a [d_\gamma(\alpha + 1) + (1 - d_\gamma(\alpha))]\, d\alpha$$

$$= \frac{1}{2} + \frac{1}{2a}\left\{\int_a^{a+1} d_\gamma(\alpha)\, d\alpha - \int_0^1 d_\gamma(\alpha)\, d\alpha\right\} \leqq \frac{1}{2} + \frac{1}{2a}$$

for all γ and assuming $a > 1$. The conclusion follows from a remark in Section II.

D. A Near-Optimum Decision Rule

We shall now exhibit a rather simple mixed strategy for player I which guarantees a payoff that nearly realizes the upper bound on v_* just obtained, in particular, which guarantees a payoff equal to $1/2 + 1/2[a + 1]$.[2] This strategy, incidentally, is demonstrated to be a maximin strategy for player II with respect to a smaller but reasonable class of pure strategies. Some comment is required here. We do not show whether the original game has a value or not, or whether the estimate of $1/2 + 1/2a$ on the lower value is attained or attained to within arbitrary ϵ. Thus, in a mathematical sense we do not solve the game; however, the guaranteed payoff using this strategy is certainly close enough to the best possible to warrant its use in calculations about the channel.

First we replace the game G by a simpler game G' in which player II has the same set of strategies available but player I has a reduced set. Denote the lower value of G' by v_*'; as pointed out in Section II, $v_*' \leqq v_*$. In G' we allow player I precisely those pure strategies which correspond to setting a *single* threshold β and deciding SPACE if $y < \beta$, MARK if $y \geqq \beta$. Thus, player I's pure strategies may be indexed by the real numbers β, $0 \leqq \beta \leqq a + 1$ since the output y lies between 0 and $a + 1$. Furthermore, since the pure strategy $\beta = 1$ is uniformly better than any pure strategy $\beta < 1$, we can restrict β to lie in the interval $1 \leqq \beta \leqq a + 1$. Thus, the game G' has a payoff function $P(\beta, \alpha)$ which is defined on a rectangle in the plane. By enumerating the possibilities one sees easily that

$$\begin{aligned} P(\beta, \alpha) &= \tfrac{1}{2} \text{ if } \quad 0 \leqq \alpha < \beta - 1 \\ &= 1 \text{ if } \quad \beta - 1 \leqq \alpha < \beta \\ &= \tfrac{1}{2} \text{ if } \quad \beta \leqq \alpha \leqq a. \end{aligned}$$

The payoff function is shown by the graph given in Fig. 1.

[2] $[x]$ denotes the largest integer less than or equal to x.

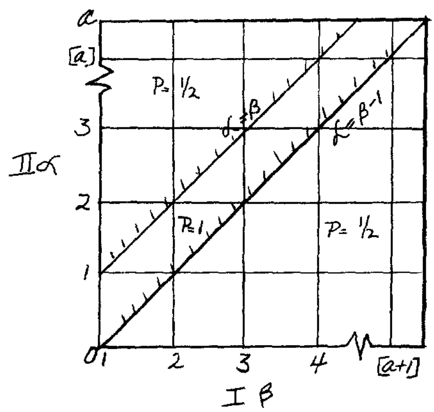

FIG. 1

We shall show that G' has a value $v' = 1/2 + 1/2[a + 1]$. First, suppose player I's pure strategies are limited to the finite set, $\beta = 1, \beta = 2, \cdots, \beta = [a + 1]$. Call the resulting game G''. Then one observes that player II's pure strategies in G'' are reduced to a finite number of equivalence classes; all pure strategies $n < \alpha \leqq n + 1$, for fixed n, $n = 0, 1, \cdots, [a]$, are equivalent. Thus, G'' is equivalent to a finite game whose payoff is a square, finite matrix with $[a + 1] \times [a + 1]$ entries, ones on a diagonal and one-halves off the diagonal, as shown in Fig. 2. Since G'' is equivalent to this finite game it has a value, which is easily seen to be

$$v'' = \frac{k-1}{k} \cdot \frac{1}{2} + \frac{1}{k} = \frac{1}{2} + \frac{1}{2k}$$

where $k = [a + 1]$. Then,

$$v_*' \geqq v'' = \frac{1}{2} + \frac{1}{2[a + 1]} .$$

A maximin strategy for player I in G'' is to choose each pure strategy randomly with probability $1/k$.

Next, suppose player II's pure strategies in G' are limited to the finite set $\alpha = 0, \alpha = 1, \cdots, \alpha = [a]$. Call the resulting game G'''. Then player I's pure strategies in G''' are reduced to a finite number of equivalence classes; $\beta = 1$ forms one class by itself; all pure strategies $n < \beta \leqq n + 1$ for fixed n, $n = 1, 2, \cdots, [a + 1]$, are equivalent. Thus, G''' is equivalent to the *same* finite game as G'', i.e., G''' is equivalent to G''.

$$
\begin{array}{cccccc}
 & \frac{1}{2} & \cdots & & \frac{1}{2} & 1 \\
 & \vdots & \cdots & & & \frac{1}{2} \\
\mathrm{II} & \frac{1}{2} & \frac{1}{2} & 1 & \cdots & \\
 & \frac{1}{2} & 1 & \frac{1}{2} & \cdots & \\
 & 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\
 & & & \mathrm{I} & &
\end{array}
$$

FIG. 2

Since G''' was formed from G' by deleting some of player II's strategies, we have

$$v_*' \leqq v''' = \frac{1}{2} + \frac{1}{2[a + 1]}$$

where $v^{*\prime}$ is the upper value of G' and v''' is the value of G'''. Thus,

$$v'_* = v^{*\prime} = v' = \frac{1}{2} + \frac{1}{2[a + 1]}.$$

We have shown that game G' has the value $1/2 + 1/2[a + 1]$ and that a good mixed strategy for player I in G' is that which weights evenly the strategies $\beta = 1, 2, \cdots, [a + 1]$. Since in passing from G' to G, player II's strategies are left unchanged, this same mixed strategy for player I will guarantee the same payoff in G. In terms of the communication problem the communicators can, then, by using the random decision rule just specified, guarantee themselves a per symbol probability of being correct of $1/2 + 1/2[a + 1]$, where a is the noise-to-signal ratio. This formula is trivially valid for $a < 1$. One should notice that a bad feature of the max-min strategy is that, although the probability of being correct cannot fall below the value stated, it cannot rise above it either, no matter what the nature of the noise, so long as the *potential* noise-to-signal ratio is a. In practice, the communicators might not know a. They would then be forced to estimate a value for a; if they estimate a too large, they can guarantee a probability of being correct of greater than $\frac{1}{2}$, but the probability is lower than need be.

IV. PHASE REVERSAL MODULATION WITH CORRELATION DETECTION

A. SYSTEM MODEL

Again we consider a binary data link, this time one in which rf phase must be known at the receiver. As before, we assume the signal level at the receiver is known, but we allow the MARK signal to be any waveform at all. The SPACE signal is its negative. Thus, with the normalization imposed, the signals are

$$\begin{aligned} &\text{MARK:} && s(t) = f(t) \\ &\text{SPACE:} && s(t) = -f(t) \end{aligned} \Bigg\} \quad 0 \leqq t < T$$

where

$$\int_0^T f^2(t)\, dt = 1.$$

The signal at the receiver terminals is

$$y(t) = s(t) + z(t), \qquad 0 \leqq t < T$$

where the noise $z(t)$ is any waveform whatever subject to the restriction

$$\int_0^T z^2(t)\, dt \leqq a.$$

We postulate[3] that the basic detector is a correlator which computes

$$y = \int_0^T f(t) y(t)\, dt.$$

It is desired to find a decision rule which will tell from the value y whether a MARK or SPACE was sent.

B. Game Theoretic Formulation

The two-person game is set up essentially as before with the payoff again the probability of being correct on each decision. Since any two noise waveforms which have the same inner product with $f(t)$ must produce exactly the same effect, a pure strategy of player II (Nature) can be indexed by a real number α,

$$\alpha = \int_0^T f(t) z(t)\, dt$$

where $-\sqrt{a} \leqq \alpha \leqq \sqrt{a}$. Player I's strategies are as before.

[3] By being more precise about formulating the model, one can *prove* in this context that a correlation detector provides as good pre-decision data processing as possible.

Again, we can obtain a simple upper bound on the payoff player I can guarantee himself. In fact

$$P(\gamma, \alpha) = \frac{d_\gamma(\alpha + 1) + 1 - d_\gamma(\alpha - 1)}{2}.$$

Hence, if II weights his strategies uniformly,

$$\frac{1}{2\sqrt{a}} \int_{-\sqrt{a}}^{\sqrt{a}} P(\gamma, \alpha)\, d\alpha \leqq \frac{1}{2} + \frac{1}{2\sqrt{a}}$$

for all γ, from which it follows that

$$v_* \leqq \frac{1}{2} + \frac{1}{2\sqrt{a}}.$$

We can also show, by an argument paralleling the previous one step-by-step that player I can guarantee himself a payoff of $1/2 + 1/2[\sqrt{a} + 1]$ by adopting the following mixed strategy; let $d_\beta(y)$ be a single-threshold decision function defined by

$$\begin{aligned} d_\beta(y) &= 0, \qquad y < \beta, \\ &= 1, \qquad y \geqq \beta. \end{aligned}$$

Each such decision function corresponds to a pure strategy which we call β (as before). Consider the $[\sqrt{a} + 1]$ pure strategies $\{\beta_n\}$, $n = -[\sqrt{a}], -[\sqrt{a}] + 2, -[\sqrt{a}] + 4, \cdots, [\sqrt{a}]$. The mixed strategy for player I which guarantees the stated payoff is to choose the β_n randomly with equal probability weighting. The results here are much as in the first example, of course, the essential difference being that making use of the rf phase allows the communicators to work against voltage noise-to-signal ratio instead of power noise-to-signal ratio.

V. INFORMATION RATE

Since, for even a moderately large noise-to-signal ratio, the probability of error per symbol in the systems we have described is large, it would, of course, be necessary to code the messages into the systems with codes using a great deal of redundancy. One must pay the price of a lower information rate in order to gain the relative invulnerability against all kinds of noise. It is of interest then to calculate the information rates in the sense of Shannon and see how they depend on the various parameters.

A. Keyed Carrier System

Since the over-all system constitutes a symmetric binary channel, the information rate per symbol duration is

$$r = 1 + (1 - p)\log(1 - p) + p \log p$$

where r is the rate, p is the probability of a correct choice for each symbol, and the log is to the base 2. We have from Section II,

$$p = \frac{1}{2} + \frac{1}{2[a + 1]}$$

but for the rate calculations we shall use the approximate value,

$$p \cong \frac{1}{2} + \frac{1}{2a}.$$

We want to find the rate per second, R, rather than the rate per symbol duration, T, so it is necessary to relate the required symbol duration to other parameters. Let us assume that T is inversely proportional to the bandwidth, B, of the receiver—this constraint is plausible because for a fixed level of performance, the ringing time of the receiver predetection filter should be kept in constant ratio with the symbol duration T. Thus, we have

$$T = \frac{\rho}{B}, \qquad \rho = \text{constant.}$$

Now we consider two cases:

1. The maximum possible noise power density is constant over the whole frequency spectrum and is equal to N watts/cycle. The signal power is fixed at S watts. Then

$$a = \frac{NBT}{ST} = \frac{NB}{S}.$$

2. The maximum possible noise power and the signal power are both independent of bandwidth, so a = constant.

For case 1 we have:

$$R = \frac{B}{\rho}\left\{1 + \left(\frac{1}{2} - \frac{S}{2NB}\right)\log\left(\frac{1}{2} - \frac{S}{2NB}\right) + \left(\frac{1}{2} + \frac{S}{2NB}\right)\log\left(\frac{1}{2} + \frac{S}{2NB}\right)\right\}$$

$$= \frac{B}{2\rho}\left\{\left(1 - \frac{S}{NB}\right)\log\left(1 - \frac{S}{NB}\right) + \left(1 + \frac{S}{NB}\right)\log\left(1 + \frac{S}{NB}\right)\right\},$$

$$\frac{NB}{S} \geqq 1.$$

For large bandwidth, NB/S large, this gives, by expanding the logarithms:

$$R = \frac{\log_2 e}{4\rho} \cdot \frac{1}{B}\left(\frac{S}{N}\right)^2 + O\left(\frac{1}{B^2}\right)$$

where $O(1/B^2)$ is a quantity such that $B^2 O(1/B^2)$ is bounded as $B \to \infty$. Thus, the information rate goes to zero linearly with the bandwidth for fixed signal power, which is not surprising since the noise-to-signal ratio goes to infinity. However, one notices that the signal-to-noise power per cycle ratio enters as the square, whereas the bandwidth enters linearly.

For case 2 we have:

$$R = \frac{B}{\rho}\left\{\left(1 - \frac{1}{a}\right)\log\left(1 - \frac{1}{a}\right) + \left(1 + \frac{1}{a}\right)\log\left(1 + \frac{1}{a}\right)\right\} \qquad a \geqq 1$$

and for large a (noise-to-signal ratio)

$$R = \frac{B}{4\rho}\left[\frac{\log_2 e}{a^2} + O\left(\frac{1}{a^3}\right)\right].$$

These two cases presumably set limits on the sort of noise behavior one might encounter in practice. It may be noted in passing that it is possible to let the noise energy grow with bandwidth in such fashion that as $B \to \infty$, the noise-to-signal ratio and the information rate both become infinite.

B. Correlation Detection System

Again, in this example, the channel is symmetric and binary, so the information rate per symbol is, as before:

$$r = 1 + (1 - p)\log(1 - p) + (1 + p)\log(1 + p).$$

The probability of making a correct decision per symbol is now

$$p = \frac{1}{2} + \frac{1}{2[\sqrt{a} + 1]}.$$

Thus, we need only replace a by $\sqrt{a}$ in the expressions of the preceding paragraph to obtain the information rates for the phase reversal system with correlation detection. Under the same assumptions as in case 1 above,

$$R = \frac{B}{2\rho}\left\{\left(1 - \sqrt{\frac{S}{NB}}\right)\log\left(1 - \sqrt{\frac{S}{NB}}\right) + \left(1 + \sqrt{\frac{S}{NB}}\right)\log\left(1 + \sqrt{\frac{S}{NB}}\right)\right\}, \qquad \frac{NB}{S} \geqq 1.$$

For large bandwidth (small S/NB)

$$R = \frac{\log_2 e}{4\rho}\left(\frac{S}{N}\right) + O\left(\frac{1}{B^{1/2}}\right).$$

Thus, for large bandwidth, under the assumption of constant *potential* noise power per cycle, the rate does not much depend on the bandwidth, but is nearly proportional to the signal-power-to-noise-power-per-cycle ratio. It is interesting to compare this with the well-known result of Shannon's, that for a channel with additive white noise and limited average power the channel capacity is asymptotically proportional to the signal-to-noise ratio, S/N.

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