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NOTE

ONE COUNTEREXAMPLE FOR TWO CONJECTURES ON THREE COLORING

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A graph is said to be *uniquely 3-colorable* if there is precisely one partition of its point set into three independent subsets. Our graph theoretic terminology is that of Harary [2].

Following Aksionov, we let $\mathfrak{A}$ be the set of all uniquely 3-colorable planar graphs. A graph G is called *u-critical* if $G \in \mathfrak{A}$, and if $G - e \notin \mathfrak{A}$ for each edge e.

Aksionov [1] proposed the following two conjectures:

Problem 1. In a uniquely 3-colorable planar graph there are two triangles having an edge in common.

Problem 2. If G is u-critical, then $q(G) = 2p(G) - 3$, where G has $p(G)$ points and $q(G)$ lines.

In this note we present a graph which refutes both conjectures.

Lemma. *Let G be the graph of Fig. 1. Any 3-coloring of G assigns different colors to vertices u_1 and u_2.*

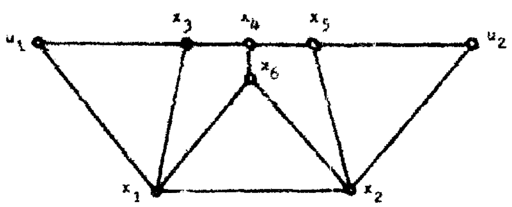

Fig. 1.

Proof. Suppose that u_1 and u_2 can obtain the same color. Then without loss of generality we color u_1 and u_2 by 1, and color x_1 and x_2 by 2 and 3, respectively. Then x_3 is colored 3 and x_5 is colored 2, hence x_4 is colored by 1. However, there is no color available for x_6.

Proposition 1. *The planar graph G of Fig. 2 is uniquely 3-colorable although no two triangles of G share an edge.*

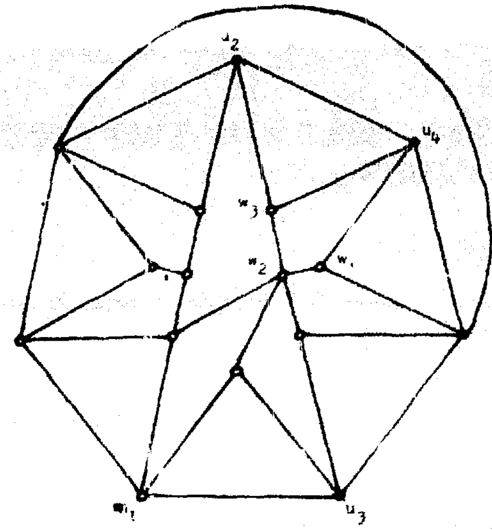

Fig. 2.

Proof. Let c be a 3-coloring of G. Applying the Lemma twice we find that $c(u_1) \neq c(u_2)$ and $c(u_2) \neq c(u_3)$. Thus without loss of generality we can let $c(u_1) = 1$, $c(u_2) = 2$, and $c(u_3) = 3$. Applying the Lemma a third time we find that $c(u_4) \neq c(u_1)$, hence $c(u_4) = 3$. The colors of the other vertices are now forced, and we obtain the unique 3-coloring shown in Fig. 3.

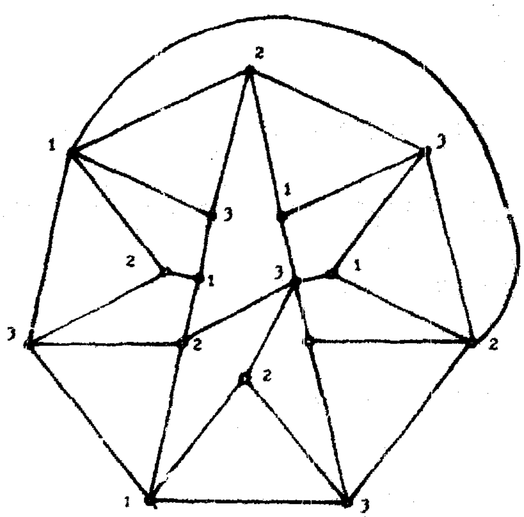

Fig. 3.

Proposition 2. *The graph G of Fig. 2 is u-critical (although* $30 = q(G) \neq 2p(G) - 3 = 29$).

Proof. From Proposition 1 we need only to establish that the removal of any line results in a graph which is not uniquely 3-colorable. According to Theorem 12.16 of [2], in any n-coloring of a uniquely n-colorable graph the subgraph induced by the union of any two color classes is connected. Hence we need only consider the deletion of lines lying on cycles colored in two colors. From Fig. 3 we see that there is only one such cycle: $u_4w_1w_2w_3$. For u_4w_1 or w_1w_2, we observe that the graph $G - w_1$ has a 3-coloring differing from that obtained from the 3-coloring of G (see Fig. 4). Similarly, for w_2w_3 or w_3u_4 we observe that the graph $G - w_3$ also has a 3-coloring differing from that obtained from G (see Fig. 5).

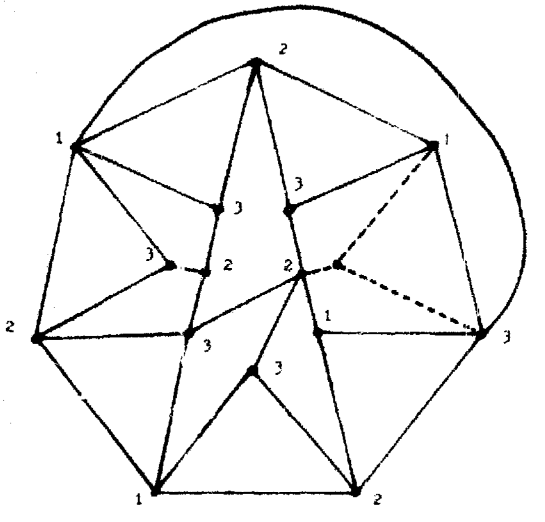

Fig. 4.

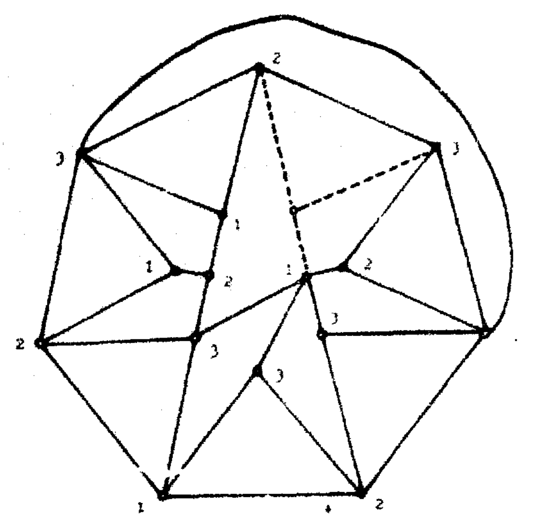

Fig. 5.

Remark 1. By piecing together more copies of the graph of Fig. 1, it is not difficult to construct an infinite family of counterexamples to Aksionov's two conjectures.

Remark 2. The following problems concerning planar uniquely 3-colorable and u-critical graphs seem to be of interest. We let d denote the *distance of the triangles* in a graph, the length of the shortest path joining vertices of different triangles.

(1) Does there exist an integer n_0 such that if a planar graph with any number of triangles has $d \geq n_0$ then the graph is not uniquely 3-colorable? It is possible that $n_0 = 1$.

(2) Find an exact upper bound for the number of lines $q(G)$ in a u-critical graph G with $p(G)$ points. Is it true that if $p(G) \geq 12$ then $q(G) \leq 9/4p(G) - 6$?

References

[1] V.A. Aksionov, On uniquely 3-colorable planar graphs, Preprint GT-2, Institute of Mathematics, Academy of Sciences of the U.S.S.R., Siberian Branch, Novosibirsk (1975); Discrete Math. 20 (1977).

[2] F. Harary, Graph Theory (Addison-Wesley, Reading, MA. 1969).