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Letter Section

Sixth-order superstable two-step methods for second-order initial-value problems

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Abstract: For the numerical integration of general second-order initial-value problems $y''=f(x, y, y')$, $y(x_0)=y_0$, $y'(x_0)=y'_0$, we report a family of two-step sixth-order methods which are superstable for the test equation $y''+2\alpha y'+\beta^2 y=0$, $\alpha, \beta \geqslant 0$, $\alpha+\beta>0$, in the sense of Chawla [1].

Keywords: General second-order initial-value problems, region of absolute stability, interval of periodicity, interval of (weak) stability, superstable methods.

1. Introduction

A two-step method for the numerical integration of general second-order initial-value problems

$$y''=f(x, y, y'), \quad y(x_0)=y_0, \quad y'(x_0)=y'_0, \tag{1.1}$$

can be defined in the form

$$y_{n+1}-2y_n+y_{n-1}=h^2\phi(x_{n+1}, x_n, x_{n-1}, y_{n+1}, y_n, y_{n-1}). \tag{1.2}$$

When the method (1.2) is applied to the test equation

$$y''+2\alpha y'+\beta^2 y=0, \quad \alpha, \beta \geqslant 0, \quad \alpha+\beta>0, \tag{1.3}$$

we obtain the characteristic equation

$$\rho(\xi)=A(H_1, H_2)\xi^2+B(H_1, H_2)\xi+C(H_1, H_2)=0 \tag{1.4}$$

having the roots ξ_1, ξ_2 and $H_1=\alpha h$, $H_2=\beta h$.

Following Chawla [1], a method of the form (1.2) is said to be superstable, if for the method

(a) the region of absolute stability ($|\xi_{1,2}|<1$) is

$$R=\{(H_1, H_2)\colon 0<H_1, H_2<\infty\};$$

(b) the interval of periodicity (the roots $\xi_{1,2}$ are complex conjugate and each of modulus one) is

$$J = \{(H_1, H_2)\colon H_1 = 0, 0 < H_2 < \infty\};$$

(c) the interval of (weak) stability (the roots $\xi_{1,2}$ are such that $\xi_1 = 1$ and $|\xi_2| < 1$) is

$$I = \{(H_1, H_2)\colon H_2 = 0, 0 < H_1 < \infty\}.$$

By using the transformation $\xi = (1+z)/(1-z)$ and applying the Routh–Hurwitz criterion, it is known that the method is superstable if

(a) $A - B + C$, $A - C$ and $A + B + C$ have the same sign for all H_1, $H_2 > 0$;
(b) with $H_1 = 0$, $A = C$ and $2A \pm B > 0$ for all $H_2 > 0$;
(c) with $H_2 = 0$, $A + B + C = 0$ and $-2 < B/A < 0$ for all $H_1 > 0$.

Chawla [1] obtained a fourth-order superstable two-step method and Jain and Goel [3] obtained single-step superstable methods or orders four and six. No higher-order superstable methods are known so far. In this paper we report a family of two-step sixth-order methods which are shown to be superstable when applied to the test equation (1.3).

Superstable methods can be successfully used for the numerical integration of (1.1) for which $\partial f/\partial y$ and/or $\partial f/\partial y'$ are negative and large as shown in the computational results listed in [3]. Such problems occur, for example, in the numerical integration of singular perturbation problems (see [4, p.23]).

2. Derivation of sixth-order methods

Let $h > 0$ be the step length, $x_n = x_0 + nh$, $n = 0, 1, 2, \ldots$ and set $y_n = y(x_n)$, $f_n = f(x_n, y(x_n), y'(x_n))$, etc. Let

$$\bar{y}'_{n+1} = \frac{1}{2h}(3y_{n+1} - 4y_n + y_{n-1}),$$

$$\bar{y}'_n = \frac{1}{2h}(y_{n+1} - y_{n-1}),$$

$$\bar{y}'_{n-1} = \frac{1}{2h}(-y_{n+1} + 4y_n - 3y_{n-1}),$$

$$\bar{f}_n = f(x_n, y_n, \bar{y}'_n), \quad \bar{f}_{n\pm1} = f(x_{n\pm1}, y_{n\pm1}, \bar{y}'_{n\pm1}),$$

$$\bar{\bar{y}}'_{n+1} = \bar{y}'_n + \tfrac{1}{3}h(2\bar{f}_n + \bar{f}_{n+1}),$$

$$\bar{\bar{y}}'_{n-1} = \bar{y}'_n - \tfrac{1}{3}h(2\bar{f}_n + \bar{f}_{n-1}),$$

$$\bar{\bar{f}}_{n\pm1} = f(x_{n\pm1}, y_{n\pm1}, \bar{\bar{y}}'_{n\pm1}),$$

$$\bar{y}_{n\pm1/2} = \tfrac{1}{2}(y_n + y_{n\pm1}) - h^2(\alpha_1\bar{f}_n + \beta_1\bar{f}_{n\pm1}), \quad \alpha_1 + \beta_1 = \tfrac{1}{8},$$

$$\bar{y}'_{n+1/2} = \frac{1}{4h}(5y_{n+1} - 6y_n + y_{n-1}) - \frac{h}{48}(3\bar{f}_{n+1} + 8\bar{f}_n + \bar{f}_{n-1}),$$

$$\bar{y}'_{n-1/2} = \frac{1}{4h}(-y_{n+1} + 6y_n - 5y_{n-1}) + \frac{h}{48}(\bar{f}_{n+1} + 8\bar{f}_n + 3\bar{f}_{n-1}),$$

$$\bar{f}_{n\pm1/2} = f(x_{n\pm1/2},\ \bar{y}_{n\pm1/2},\ \bar{y}'_{n\pm1/2}),$$

$$\bar{\bar{y}}_{n\pm1/2} = \tfrac{1}{2}(y_n + y_{n\pm1}) - \frac{h^2}{96}\left(\bar{f}_{n\pm1} + 10\bar{f}_{n\pm1/2} + \bar{f}_n\right),$$

$$\bar{\bar{f}}_{n\pm1/2} = f\left(x_{n\pm1/2},\ \bar{\bar{y}}_{n\pm1/2},\ \bar{y}_{n\pm1/2}\right),$$

$$\hat{y}_n = y_n + h^2 a\left[\left(\bar{f}_{n+1} + \bar{f}_{n-1}\right) - \left(\bar{\bar{f}}_{n+1} + \bar{\bar{f}}_{n-1}\right)\right], \quad a = 1/312,$$

$$\hat{y}'_n = \bar{y}'_n + \frac{h}{156}\left[2\left(\bar{f}_{n+1} - \bar{f}_{n-1}\right) - 3\left(\bar{\bar{f}}_{n+1} - \bar{\bar{f}}_{n-1}\right) - 24\left(\bar{\bar{f}}_{n+1/2} - \bar{\bar{f}}_{n-1/2}\right)\right],$$

$$\hat{f}_n = f_n(x_n,\ \hat{y}_n,\ \hat{y}'_n). \tag{2.1}$$

Then at each point x_n, $n = 1, 2, \ldots,$ the differential equation (2.1) can be discritized by

$$y_{n+1} - 2y_n + y_{n-1} = \frac{h^2}{60}\left[26\hat{f}_n + \bar{\bar{f}}_{n+1} + \bar{\bar{f}}_{n-1} + 16\left(\bar{\bar{f}}_{n+1/2} + \bar{\bar{f}}_{n-1/2}\right)\right]. \tag{2.2}$$

For $a = 0$ and the differential equation (1.1) independent of y', the method (2.2) reduces to the sixth order P-stable method discussed by Chawla [2].

To obtain the truncation error term associated with the method (2.2), we use the following expansions:

$$\bar{f}_n = f_n + \tfrac{1}{6}h^2 y_n^{(3)} G_n + \frac{h^4}{360} p_n^{(1)} + O(h^6),$$

$$\bar{f}_{n\pm1} = f_{n\pm1} - \tfrac{1}{3}h^2 y_n^{(3)} G_n \mp \frac{h^3}{12} p_n^{(2)} - \frac{h^4}{180} p_n^{(3)} \mp \frac{h^5}{360} p_n^{(4)} + O(h^6),$$

$$\bar{\bar{f}}_{n\pm1} = f_{n\pm1} + \frac{h^4}{180}\left(4y_n^{(5)} - 5p_n^{(2)}\right)G_n \pm \frac{h^5}{540} r_n^{(1)} + \frac{h^6}{7560} r_n^{(2)} \pm O(h^7),$$

$$\bar{f}_{n\pm1/2} = f_{n\pm1/2} \pm \left(\tfrac{1}{16} - \beta_1\right)h^3 y_n^{(3)} F_n + \tfrac{1}{5760} B_1 h^4 \pm O(h^5),$$

$$\bar{\bar{f}}_{n\pm1/2} = f_{n\pm1/2} + \frac{h^4}{5760}\left[10 y_n^{(3)} F_n G_n - \left(7y_n^{(5)} - 20 p_n^{(2)}\right)G_n\right]$$
$$\pm \frac{h^5}{34560} D_1 + \frac{h^6}{1935360} D_2 \pm O(h^7),$$

$$\hat{f}_n = f_n - \frac{h^4}{4680}\left[10 y_n^{(3)} F_n G_n + y_n^{(5)} + 10 p_n^{(2)}\right] + \tfrac{1}{786240} D_3 h^6 + O(h^8), \tag{2.3}$$

where

$$p_n^{(1)} = 3y_n^{(5)} G_n + 5\left(y_n^{(3)}\right)^2 H_n, \qquad p_n^{(2)} = 4y_n^{(3)} G'_n + y_n^{(4)} G_n,$$

$$p_n^{(3)} = 30 y_n^{(3)} G''_n + 15 y_n^{(4)} G'_n + 6 y_n^{(5)} G_n - 10\left(y_n^{(3)}\right)^2 H_n,$$

$$p_n^{(4)} = 20 y_n^{(3)} G'''_n + 15 y_n^{(4)} G''_n + 12 y_n^{(5)} G'_n + 2 y_n^{(6)} G_n - 20\left(y_n^{(3)}\right)^2 H'_n - 10 y_n^{(3)} y_n^{(4)} H_n,$$

$$T_1 = 4y_n^{(5)} - 5p_n^{(2)}, \qquad T_2 = 7y_n^{(5)} - 20 p_n^{(2)},$$

$$T_3 = 3y_n^{(6)} + p_n^{(1)} - p_n^{(3)}, \qquad T_4 = 57y_n^{(6)} + 16\left(p_n^{(1)} - p_n^{(3)}\right),$$
$$T_5 = p_n^{(3)} + T_1 G_n, \qquad T_6 = 9y_n^{(7)} - 56p_n^{(4)} - 56r_n^{(1)} - 7D_1,$$
$$r_n^{(1)} = 3T_1 G_n' + T_3 G_n,$$
$$r_n^{(2)} = 21T_1 G_n'' + 14T_3 G_n' + \left(12y_n^{(7)} - 7p_n^{(4)}\right)G_n,$$
$$A_1 = -8(1 - 24\beta_1)y_n^{(3)}G_n + (7 - 192\beta_1)y_n^{(4)},$$
$$B_1 = 180(1 - 16\beta_1)y_n^{(3)}F_n' - 15A_1 F_n - T_2 G_n,$$
$$C_1 = 2p_n^{(2)} - 15(1 - 16\beta_1)y_n^{(3)}G_n,$$
$$C_2 = -9y_n^{(6)} + 16p_n^{(3)} - 5\beta_1 - 8p_n^{(1)},$$
$$D_1 = 30y_n^{(3)}F_n'G_n + 15C_1F_n - 3T_2G_n' - T_4G_n,$$
$$D_2 = 420y_n^{(3)}F_n''G_n + 420C_1F_n' + 7C_2F_n - 42T_2G_n'' - 28T_4G_n' - 8\left(165y_n^{(7)} - 112p_n^{(4)}\right)G_n,$$
$$D_3 = -28T_5F_n + T_6G_n, \tag{2.4}$$
$$F = \partial f/\partial y, \qquad G = \partial f/\partial y', \qquad H = \partial^2 f/\partial y'^2, \qquad F' = \mathrm{d}F/\mathrm{d}x, \quad \text{etc.}$$

Substituting the expansions (2.3) in (2.2) and simplifying, we obtain the truncation error term as

$$t_n(h) = -\frac{h^8}{(10!)}\left[30y_n^{(8)} + 16r_n^{(2)} + D_2 + 2D_3\right] + \mathrm{O}(h^{10}). \tag{2.5}$$

When we apply the method (2.2) to the test equation (1.3), we get the characteristic equation (1.4) with

$$A = 60 + 60H_1 + 24H_1^2 + 9H_2^2 + 4H_1^3 + 9H_1H_2^2 + \tfrac{23}{6}H_1^2H_2^2 + H_2^4$$
$$+ \tfrac{5}{6}H_1^3H_2^2 + H_1H_2^4 + \tfrac{5}{36}(1 + 24\beta_1)H_1^2H_2^4 + \tfrac{5}{3}\beta_1H_2^6 + \tfrac{5}{6}\beta_1H_1H_2^6,$$
$$B = -120 - 48H_1^2 + 42H_2^2 - \tfrac{11}{3}H_1^2H_2^2 + 2H_2^4$$
$$+ \tfrac{5}{9}(1 - 12\beta_1)H_1^2H_2^4 + \tfrac{5}{12}(1 - 8\beta_1)H_2^6,$$
$$C = 60 - 60H_1 + 24H_1^2 + 9H_2^2 - 4H_1^3 - 9H_1H_2^2 + \tfrac{23}{6}H_1^2H_2^2 + H_2^4$$
$$- \tfrac{5}{6}H_1^3H_2^2 - H_1H_2^4 + \tfrac{5}{36}(1 + 24\beta_1)H_1^2H_2^4 + \tfrac{5}{3}\beta_1H_2^6 - \tfrac{5}{6}\beta_1H_1H_2^6. \tag{2.6}$$

It is easy to verify that

(a) With $H_1 = 0$, $H_2 > 0$

$$A = C, \qquad 2A + B = 60H_2^2 + 4H_2^4 + \tfrac{5}{12}H_2^6 > 0,$$
$$2A - B = 240 - 24H_2^2 + \left(\tfrac{20}{3}\beta_1 - \tfrac{5}{12}\right)H_2^6,$$

which is positive for all $\beta_1 > 407/6000$ or $\alpha_1 < 343/6000$.

(b) With $H_2 = 0$, $H_1 > 0$, $A + B + C = 0$ and $-2 < B/A < 0$.

(c) With $H_1 \neq 0$, $H_2 \neq 0$ and $\beta_1 > 407/6000$, $A - B + C$, $A - C$ and $A + B + C$ are all positive for H_1, $H_2 > 0$.

Hence the method (2.2) is superstable.

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