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On the Characterization of the Friedrichs Extension of Ordinary or Elliptic Differential Operators with a Strongly Singular Potential

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The second-order singular elliptic differential operator

$$T_0 u := \frac{1}{k}\left\{\sum D_s(a_{st}D_t u) + qu\right\} \qquad \text{with} \quad D_s := i\partial_s + b_s$$

is considered on $C_0^\infty(G) \subset L^2(G; k)$ where

$$G := \{x \mid x \in \mathbf{R}^n, 0 \leqslant l < |x| < m \leqslant \infty, n \geqslant 2\}.$$

(The general one-dimensional Sturm–Liouville operator is dealt with on an arbitrary interval (l, m) of the real line.) Conditions in addition to the usual ones are imposed on the coefficients which make T_0 bounded from below (but still allow strong negative singularities of q at ∂G) so that it possesses a Friedrichs extension T_F with domain

$$D(T_F) = \{u \mid u \in D(T_0^*), \text{there exists a sequence } \{u_j\} \subset C_0^\infty(G) \text{ such that } \|u_j - u\| \to 0 \text{ and } (T_0(u_j - u_{j'}), u_j - u_{j'}) \to 0 \text{ as } j, j' \to \infty\}. \quad (1)$$

Unifying and at the same time simplifying and generalizing ideas to be found in the work of Friedrichs [2, 3] and Kato [19] for the special case $a_{st} = \delta_{st}$, $b_s = 0$, $k = 1$ it is shown that (1) can be characterized by

$$D(T_F) = \left\{u \mid u \in H^2_{\text{loc}}(G) \cap L^2(G; k), \int_G \sum a_{st}(D_s u)\overline{(D_t u)}\, dx < \infty,\ T_0 u \in L^2(G; k)\right\}. \quad (2)$$

If $\int dt/t^{n-1}a(t)$ ($a(t)$ smallest eigenvalue of (a_{st})) converges at l or m the boundary condition

$$\liminf_{\substack{r \to l+ \\ m-}} \int_{|\xi|=1} |u(r\xi)|^2\, d\omega_n = 0$$

(to be imposed on the "distinguished" representatives of the equivalence classes $u \in H^2_{\text{loc}}(G)$) has to be added to (2).

Equation (2) is derived without recourse to the theory of sesquilinear forms in Hilbert space so that the condition $q_+^{1/2}u \in L^2(G)$ (q_+ positive part of q), which has to be assumed from the start when forms are considered (as Friedrichs and Kato do), can entirely be dispensed with. It is shown that it is a consequence of the other conditions to be imposed on u.

1. Notations and General Assumptions

In this paper we consider the differential expression

$$Du := \frac{1}{k}\left\{\sum_{s,t=1}^{n} D_s(a_{st}D_t u) + qu\right\} \qquad \text{where} \quad D_s := i\partial_s + b_s\,, \tag{1.1}$$

on an annular domain

$$G = S(l, m) := \{x \mid x \in \mathbf{R}^n, 0 \leqslant l < |x| < m \leqslant \infty\}, \qquad (n \geqslant 2)$$

as well as the general one-dimensional Sturm–Liouville differential expression

$$Du := \frac{1}{k}\{-(pu')' + qu\} \tag{1.2}$$

on an arbitrary interval (l, m) $(-\infty \leqslant l < m \leqslant \infty)$ of the real line. (So we shall always tacitly assume $l \geqslant -\infty$ if $n = 1$ and $l \geqslant 0$ if $n \geqslant 2$.) Our general assumptions for the coefficients of (1.1) are:

$$a_{st}(\cdot),\ b_s(\cdot),\ q(\cdot) \text{ real},\ k(\cdot) > 0 \text{ a.e. on } G; \tag{I.i}$$

$$a_{st} \in C^2(G),\ b_s \in C^1(G),\ q \in Q_{\alpha,\mathrm{loc}}(G), \tag{I.ii}$$

k and $\frac{1}{k}$ locally essentially bounded;

the matrix $(a_{st}(x))$ is symmetric and positive definite for every $x \in G$. (I.iii)

$$Q_{\alpha,\mathrm{loc}}(G) := \Big\{ f \mid \text{for every compact subset } K \subset G \text{ there exists a number } C_K(f) \text{ such that } \int_{K\cap\{y \mid |y-x|\leqslant 1\}} |x-y|^{4-n-\alpha}\, |f(y)|^2\, dy \leqslant C_K(f) \text{ for all } x \in K\Big\},$$

$\alpha > 0$ being a fixed number, denotes as usual the Stummel class.

$a^+(\cdot)$ and $a^-(\cdot)$ be the largest and smallest eigenvalue of $(a_{st}(\cdot))$, respectively. Then we define for $r \in (l, m)$

$$a(r) := \min_{|x|=r} a^-(x), \qquad A(r) := \max_{|x|=r} a^+(x).$$

The expression (1.2) will be studied under the even weaker conditions

$$p(\cdot) > 0 \text{ a.e. on } (l, m), \qquad \frac{1}{p} \in L^1_{\text{loc}}(l, m); \tag{II.i}$$

$$k(\cdot) > 0 \text{ a.e. on } (l, m), \ k \text{ and } \frac{1}{k} \text{ locally essentially bounded;} \tag{II.ii}$$

$$q \in L^1_{\text{loc}}(l, m) \text{ real.} \tag{II.iii}$$

Three assumptions in addition to (I) and one in addition to (II) will be stated in Section 4.

By

$$T_0 u = Du \qquad \text{for all} \quad u \in C_0^\infty(G)$$

a symmetric operator T_0 on $C_0^\infty(G) \subset L^2(G; k)$ can be associated with (1.1), and it is well known (Ikebe–Kato [5], Jörgens [6]) that its adjoint can be characterized by

$$T_0{}^* u = Du$$

$$\text{for all } u \in D(T_0{}^*) = \{v \mid v \in H^2_{\text{loc}}(G) \cap L^2(G; k), Du \in L^2(G; k)\}. \tag{1.3}$$

$H^2_{\text{loc}}(G)$ is the set of all functions locally belonging to $H_0{}^2(G)$ which is the completion of $C_0^\infty(G)$ in the norm

$$\|\cdot\|_2 := \Big(\sum_{s,t=1}^n \|\partial_s\partial_t \cdot\|_0^2 + \sum_{s=1}^n \|\partial_s \cdot\|_0^2 + \|\cdot\|_0^2\Big)^{1/2}$$

($\|\cdot\|_0$ is the norm of $L^2(G) := L^2(G; 1)$). $H_0{}^1(G)$ and $H^1_{\text{loc}}(G)$ are similarly defined.

In view of the weak assumptions (II) the "minimal" operator L_0' for (1.2) has to be defined by means of the restriction of the "maximal" operator L,

$$Lu = Du$$

$$\text{for all } u \in D(L) = \{v \mid v \in L^2(l, m; k), pv' \in A^1(l, m), Dv \in L^2(l, m; k)\}, \tag{1.4}$$

to functions with individual compact support,

$$L_0' u = Du$$

for all $u \in D(L_0') = \{v \mid v \in D(L), v = 0 \text{ for } l < x < l_1(v) \text{ and } m_1(v) < x < m\}$ (Naĭmark [13, p. 172f.]). In (1.4) we used as an abbreviation

$$A^1(l, m) := \{v \mid v \text{ locally absolutely continuous on } (l, m)\}.$$

Note that $pv' \in L^1_{loc}(l, m)$ [because of $pv' \in A^1(l, m)$] and (II.i) imply $v \in A^1(l, m)$. An important relation is (Naĭmark [13, pp. 177, 179])

$$L_0'^* = L. \tag{1.5}$$

The following additional notations should be noted. Any function $p(\cdot)$ with the properties (II.i) generates by

$$h_\gamma(\rho) := \left| \int_\gamma^\rho \frac{dt}{t^{n-1}p(t)} \right| \qquad (\rho \in (l, m), n \in \mathbf{N}) \tag{1.6}$$

a function defined on (l, m). $\gamma \in \{l, m\}$ is arbitrary (by this notation we imply that $\gamma = l$ or $\gamma = m$ is allowed provided (1.6) is convergent). As an abbreviation for the spherical means of a complex-valued function $u(\cdot)$ we use

$$\varphi_u(r) := \left(\int_{|\xi|=1} | u(r\xi)|^2 \, d\omega_n \right)^{1/2} \qquad (r := | x |, x \in S(l, m))$$

where $d\omega_n$ denotes the surface element of the n-dimensional unit sphere. If $n = 1$ we shall understand that $\varphi_u(| x |)$ is replaced by $| u(x)|$ (x denoting a point of the interval (l, m) on the real line then).

2. Introduction

Let X be a Hilbert space with scalar product $(\cdot, \cdot)$ and norm $\| \cdot \|$, A a symmetric operator defined on $D(A) \subset X$. In 1933, Friedrichs, developing the theory of sesquilinear forms in Hilbert space, showed that if A is bounded from below, it can be extended in a "natural way" to a self-adjoint operator A_F having the same lower bound as A [2, Part I, Satz 7, 9]. Somewhat later, Freudenthal [1] proved by a simple argument, avoiding the use of forms, that the domain of definition of A_F can be characterized by

$$D(A_F) = \{u \mid u \in D(A^*), \text{ there exists a sequence } \{u_j\} \subset D(A) \text{ such that}$$
$$\| u_j - u \| \to 0 \text{ and } (A(u_j - u_{j'}), u_j - u_{j'}) \to 0 \text{ as } j, j' \to \infty\}. \tag{2.1}$$

Moreover, he showed that in general there exist other self-adjoint extensions of A sharing with A_F the property of having the same lower bound as A (for special cases in which this property *does* determine A_F uniquely, see Poulsen [14]; for two properties distinguishing A_F from all other possible self-adjoint extensions, consult Kato [9, Theorems 2.10 and 2.11, p. 326; Problem 2.22, p. 331]; for a physical interpretation, cf. Rellich [15, p. 344 and §6]).

Taking for A the differential operators T_0 or L_0' defined in Section 1 (and imposing appropriate conditions in addition to (I) and (II) in order to secure their semiboundedness), one would, of course, like to know more about the functions belonging to (2.1). Friedrichs himself gave an explicit characterization of (2.1) for operators of the special form

$$Du := -\Delta_n u + qu \tag{2.2}$$

in Part II of his paper [2] and two years later [3] for the general Sturm–Liouville operator (1.2). Setting

$$q_\pm(\cdot) := 1/2(|\, q(\cdot)| \pm q(\cdot)),$$

his result for the latter case can be stated as follows (L_F be the Friedrichs extension of L_0'):

$$D(L_F) = \{u \mid u \in D(L);\, p^{1/2}u',\, q_+^{1/2}u \in L^2(l, m)\}. \tag{2.3}$$

($q_-^{1/2}u \in L^2(l, m)$ is a simple consequence of Lemma 1 below.) If (1.6) (with $n = 1$) converges for $\gamma = l$ or $\gamma = m$, the condition

$$u(x) \to 0 \qquad \text{for} \quad x \to l+ \quad \text{or} \quad x \to m- \tag{2.4}$$

must be added in the right-hand side of (2.3). (The requirement $p^{1/2}u' \in L^2$, meaning finiteness of the kinetic energy, is of course quite natural from the physical point of view; in the regular case, where all functions $u \in D(L)$ trivially have the properties $p^{1/2}u'$, $q_+^{1/2}u \in L^2$, it is (2.4) alone that characterizes $D(L_F)$.) Friedrichs' results for (2.2) ($n = 1$ and $n \geqslant 3$)—but under considerably weaker conditions on q—can also be found in Kato's book [9, Theorem 4.2, p. 346; Theorem 4.6, p. 349].

Our object in this paper is to give an explicit characterization of Friedrichs extension of operators of the general form (1.1) and at the same time to simplify his proof in [3] avoiding the theory of sesquilinear forms completely. It will be seen that the condition $q_+^{1/2}u \in L^2(G)$, which has to be assumed from the start when forms are considered (as Friedrichs and Kato do), can entirely be dispensed with. It is shown that it is a consequence of the other conditions to be imposed on u.

Eventually we refer to Rellich's work on the subject [15]. He treated the one-dimensional case and showed that $D(L_F)$ consists essentially of those functions behaving like the "principal solution" (for this terminology see, e.g., Hartman [4, pp. 355, 402]) of the corresponding nonoscillatory differential equation $Du = \lambda u$ [15, p. 355].[1] Moreover, he showed $L_F \geqslant \tilde{L}$ for every self-adjoint extension $\tilde{L}$ of L_0' which is bounded from below [15, §6] (cf. Kato [9, Problem 2.22, p. 331] cited above).

For a class of semibounded operators similar to those considered here the question whether there exists a *unique* self-adjoint extension is attacked in [7, 8].

Remark 1. The lack of the notion of a generalized derivative encumbered Part II of Friedrichs' paper [2] considerably. Friedrichs could rigorously deal only with the cases $n = 2, 3$ assuming $q \in C^1(G)$ [2, pp. 686, 688, and his correction p. 777] and, of course, $n = 1$.

We shall assume $q \in Q_{\alpha,\mathrm{loc}}(G)$ subsequently in order to apply (1.3). Kato, on the other hand, uses a slightly weaker condition on q; that is, the local boundedness of

$$\int_{|y-x|\leqslant 1} |x - y|^{2-n-\kappa} \, |q(y)| \, dy$$

($\kappa > 0$ fixed) but has to assume $q_+^{1/2} u \in L^2(G)$ in return.

3. Some Inequalities of Hardy's Type

In this section we state some generalizations of an inequality due to Hardy we shall need to derive Theorems 1 and 2 of Section 4. For a proof see [7, 8] where a short history of these inequalities and their connexion with Friedrichs' work is also given.

Lemma 1. *Besides* (II.i) *assume* $u \in C^1(S(l, m))$, $p^{1/2}\nabla u \in L^2(S(l, m))$.

(*a*) *If* $h_m(\cdot) < \infty$ *and* $\liminf_{r\to m-} \varphi_u(r) = 0$, *then*

$$\lim_{r\to m-} \frac{\varphi_u^2(r)}{h_m(r)} = 0, \tag{3.1}$$

[1] Rellich showed that L_0' is bounded from below if and only if $Du = \lambda u$ is nonoscillatory for some real λ.

and for arbitrary $R_1 \in [l, m)$

$$\int_{S(R_1,m)} p(|x|)\,|\nabla u(x)|^2\,dx \geqslant \frac{1}{4}\int_{S(R_1,m)} \frac{|u(x)|^2}{p(|x|)[|x|^{n-1}h_m(|x|)]^2}\,dx$$

holds. The constant $1/4$ *is the best possible.*

(b) *If* $h_m(\cdot) = \infty$, *then*

$$\lim_{r\to m-} \frac{\varphi_u^2(r)}{h_\gamma(r)} = 0 \qquad \textit{for any} \quad \gamma \in \{l, m),$$

and for arbitrary $R_1 \in (\gamma, m)$ *we have*

$$\int_{S(R_1,m)} \frac{|u(x)|^2}{p(|x|)[|x|^{n-1}h_\gamma(|x|)]^2}\,dx < \infty.$$

LEMMA 2. *Besides* (II.i) *and* (I.ii) (*or* (II.ii)) *assume* $h_m(\cdot) = \infty$. *Then for given arbitrary numbers* r_1, R_1 *with* $l \leqslant \gamma < R_1 < r_1 < m$ *there exists a number* $C_1 > 0$ *such that*

$$\int_{S(R_1,m)} p(|x|)\,|\nabla u(x)|^2\,dx \geqslant \frac{1}{4}\int_{S(r_1,m)} \frac{|u(x)|^2}{p(|x|)[|x|^{n-1}h_\gamma(|x|)]^2}\,dx - C_1\cdot(u,u)$$

for all $u \in C^1(S(l, m)) \cap L^2(S(l, m); k)$ *with* $p^{1/2}\nabla u \in L^2(S(l, m))$. (*Of course, the choice* $\gamma = l$ *is only permitted if* $h_l(\cdot) < \infty$.)

Remark 2. It is clear that the distinction $h_l(\cdot) < \infty$, $h_l(\cdot) = \infty$ leads to results analogous to Lemmas 1 and 2.

Remark 3. Both lemmas remain valid if the assumption $u \in C^1(S(l, m))$ is relaxed to $u \in H^1_{\mathrm{loc}}(S(l, m))$. According to a theorem of Sobolev [16, p. 69f.] every equivalence class $u \in H^1_{\mathrm{loc}}(S(l, m))$ possesses a "distinguished" representative (this terminology is due to Hörmander [*Acta Math.* **94** (1954), 195]) for which the restriction to the $(n-1)$-dimensional sphere $\dot{S}_r := \{x \mid |x| = r\}$ can be defined as a square integrable function. In the extensive literature devoted to this particular problem (cf., e.g., Lions–Magenes [11, pp. 44f., 114] or Miranda [12, p. 46f.] and the references given there) this restriction is called "trace of u" and denoted by γu. Following the example of Sobolev [16] and Ladyženskaya–Ural'ceva [10] we have dispensed with a particular notation. The condition $\liminf_{r\to m-} \varphi_u(r) = 0$ has to be interpreted as a requirement on these "distinguished" representatives. Since Gauss' theorem holds for them (see, e.g., [10, p. 41f.]), the proof of Lemma 1 given in [7, 8] can be adopted without any change. In order to prove Lemma 2, use has to be made of a theorem of Kondrašev [16, p. 84] on the "L^2-continuity" of these "distinguished" representatives (for details see [7]). We note that $u \in C^1(l, m)$ can be replaced by $u \in A^1(l, m)$ in the one-dimensional case.

4. An Explicit Characterization of the Friedrichs Extension

According to the behavior of (1.6) at the endpoints we choose constants γ_0, $\gamma_1 \in \{l, m\}$ in the following way:

Case a: $h_l(\cdot) < \infty$, $h_m(\cdot) < \infty$: $\gamma_0 = l$, $\gamma_1 = m$;

Case b: $h_l(\cdot) < \infty$, $h_m(\cdot) = \infty$: $\gamma_0 = l$, $\gamma_1 \in [l, m)$ arbitrary;

Case c: $h_l(\cdot) = \infty$, $h_m(\cdot) < \infty$: $\gamma_0 \in (l, m]$ arbitrary, $\gamma_1 = m$;

Case d: $h_l(\cdot) = \infty$, $h_m(\cdot) = \infty$: γ_0, $\gamma_1 \in (l, m)$ arbitrary.

Writing

$$\mathscr{T} := \Big\{ u \mid u \in H^2_{\text{loc}}(S(l, m)) \cap L^2(S(l, m); k),$$

$$\int_{S(l,m)} \sum a_{st}(x)(D_s u(x))\overline{(D_t u(x))}\, dx < \infty,\ Du \in L^2(S(l, m); k) \Big\}$$

we associate an operator T,

$$Tu = Du \qquad \text{for all} \quad u \in D(T),$$

with (1.1) where $D(T)$ represents one of the following four subspaces of $L^2(S(l, m); k)$ according to the distinction just made:

$$\{u \mid u \in \mathscr{T},\ \liminf_{r \to l+} \varphi_u(r) = \liminf_{r \to m-} \varphi_u(r) = 0\}; \tag{4.1a}$$

$$\{u \mid u \in \mathscr{T},\ \liminf_{r \to l+} \varphi_u(r) = 0\}; \tag{4.1b}$$

$$\{u \mid u \in \mathscr{T},\ \liminf_{r \to m-} \varphi_u(r) = 0\}; \tag{4.1c}$$

$$\mathscr{T}. \tag{4.1d}$$

Remark 4. Because of (3.1)

$$\liminf_{\substack{r \to l+ \\ m-}} \varphi_u(r) = 0 \text{ implies } \lim_{\substack{r \to l+ \\ m-}} \varphi_u(r) = 0.$$

Thus $D(T)$ is a linear space.

Remark 5. If $k(\cdot) \in C^0(S(l, m))$ we can define $K(r) := \min_{|x|=r} k(x)$ for $r \in (l, m)$. If $\int_l K(r)\, r^{n-1}\, dr = \infty$ $(\int^m K(r)\, r^{n-1}\, dr = \infty)$ then $u \in L^2(S(l, m); k)$ implies $\liminf_{l \to 1+} \varphi_u(r) = 0$ $(\liminf_{r \to m-} \varphi_u(r) = 0)$.

We now complete our list of conditions on the coefficients of (1.1) with three further assumptions:

(a) There exists a constant C such that $A(|x|) \leqslant C \cdot a(|x|)$ for all $x \in S(l, m)$; (I.iv)

(b) There exist constants $M \geqslant 0$, R_i, and β_i $(i = 0, 1)$ such that (I.v)

$$q(x) \geqslant \begin{cases} \dfrac{\beta_0}{a(|x|)[|x|^{n-1} h_{\gamma_0}(|x|)]^2} & \text{for } l < |x| \leqslant R_0, \\ -M & \text{for } R_0 \leqslant |x| \leqslant R_1, \\ \dfrac{\beta_1}{a(|x|)[|x|^{n-1} h_{\gamma_1}(|x|)]^2} & \text{for } R_1 \leqslant |x| < m, \end{cases}$$

[if $\gamma_0 \neq l$ $(\gamma_1 \neq m)$ γ_0 and R_0 $(\gamma_1$ and $R_1)$ have to be arranged so that $\gamma_0 > R_0$ $(\gamma_1 < R_1)$];

(c) $\dfrac{1}{k(x)} \sum a_{st}(x)\, b_s(x)\, b_t(x)$ is bounded for $l < |x| \leqslant R_0$ and $R_1 \leqslant |x| < m$. (I.vi)

Remark 6. Since

$$\sum a_{st}(\partial_s u)\overline{(\partial_t u)} \leqslant 1/(1-\epsilon) \sum a_{st}(D_s u)\overline{(D_t u)} + 1/\epsilon \sum a_{st} b_s b_t\, |u|^2$$

as well as

$$\sum a_{st}(D_s u)\overline{(D_t u)} \leqslant (1+\epsilon) \sum a_{st}(\partial_s u)\overline{(\partial_t u)} + (1 + 1/\epsilon) \sum a_{st} b_s b_t\, |u|^2$$

hold for every $\epsilon \in (0, 1)$, it is a consequence of assumption (I.vi) that for every $u \in L^2(S(l, m); k)$ the conditions

$$\int_{S(l,m)} \sum a_{st}(D_s u)\overline{(D_t u)}\, dx < \infty,$$

and

$$\int_{S(l,m)} \sum a_{st}(\partial_s u)\overline{(\partial_t u)}\, dx < \infty$$

are equivalent.

In order to deal with the one-dimensional Sturm–Liouville operator we introduce

$$\mathscr{H} := \{u \mid u \in D(L),\, p^{1/2} u' \in L^2(l, m)\}$$

instead of $\mathscr{T}$. Then we associate an operator H with (1.2) in that subspace $D(H)$ of $L^2(l, m; k)$ which originates from (4.1) if $\mathscr{T}$ is replaced by $\mathscr{H}$ and $\varphi_u(|x|)$ by $|u(x)|$. To our assumptions (II.i)–(II.iii) we add as a fourth condition (I.v) restricted to $n = 1$ (with the obvious alterations of $a(|x|)$ and $h_{\gamma_i}(|x|)$ to $p(x)$ and $h_{\gamma_i}(x)$, respectively).

We can now state our first two main results.

THEOREM 1. *Suppose* (I.i)–(I.vi) *hold with* $\beta_i > -1/4$ $(i = 0, 1)$. *Then* T_0 *is bounded from below, and its Friedrichs extension* T_F *coincides with* T.

THEOREM 2. *Suppose* (II.i)–(II.iii) *and* (I.v), *restricted to* $n = 1$, *hold with* $\beta_i > -1/4$ $(i = 0, 1)$. *Then* L_0' *is bounded from below, and its Friedrichs extension* L_F *coincides with* H.

The constant $-1/4$ occurring in both theorems is sharp.

The idea underlying our proof of these theorems is very simple. In Lemma 3 we prove the symmetry of T which implies $T \subset T^*$. Then we show $T_F \subset T$ for $\beta_i > -1/4$ (Lemma 4). Hence because of $T_F = T_F{}^* T_F = T$. In the same way, $L_F = H$ can be inferred.

5. PROOF OF THEOREMS 1 AND 2

LEMMA 3. *If the assumptions* (I.i)–(I.vi) *are satisfied, T is symmetric. It is bounded from below if* $\beta_i > -1/4$ $(i = 0, 1)$ ($\beta_i \geqslant -1/4$ *if the* $b_s(\cdot)$ *are absent*). *The constant* $-1/4$ *is sharp.*

Proof. Symmetry. Because of $T_0 \subset T$ T is densely defined. It suffices to show that (Tu, u) is real for every $u \in D(T)$. Assume $l < r < R < m$ and $u \in D(T)$. Then by Gauss' theorem (every equivalence class $u \in H^2_{\mathrm{loc}}(S(l, m))$ has a "distinguished" representative for which the restrictions of both u and $\partial_s u$ to $\dot{S}_r := \{x \mid |x| = r\}$ can be defined as square integrable functions and with which Gauss' theorem can be applied; if $n = 2, 3$, $u \in H^2_{\mathrm{loc}}(S(l, m))$ possesses representatives which are even continuous [16, p. 69f.])

$$\int_{S(r,R)} Tu\bar{u}k \, dx = f(R) - f(r) + \int_{S(r,R)} \sum a_{st}(D_s u)\overline{(D_t u)} \, dx$$
$$+ \int_{S(r,R)} q \, |u|^2 \, dx, \qquad (5.1)$$

where we have put ($dS := r^{n-1}\, d\omega_n$)

$$f(r) := i \int_{|x|=r} \sum a_{st}\nu_s(D_t u)\bar{u}\, dS \qquad \text{and} \qquad \nu_s := \frac{x_s}{|x|}.$$

Writing $q = q_+ - q_-$ we obtain, from assumption (I.v),

$$q_-(x) \leqslant \frac{|\beta_1|}{a(|x|)[|x|^{n-1}h_{\gamma_1}(x)]^2}$$

for all $|x| \geqslant R_1$. Since it is clear from Remark 6 that

$$\int_{S(l,m)} a(|x|)\, |\nabla u(x)|^2\, dx < \infty,$$

Lemma 1 yields the existence of

$$\lim_{R\to m-} \int_{S(r,R)} q_-(x)\, |u(x)|^2\, dx$$

for fixed $r > l$.

$$\lim_{R\to m-} \int_{S(r,R)} q_+(x)\, |u(x)|^2\, dx \tag{5.2}$$

exists in a generalized sense, i.e., it exists either in the ordinary sense or it is $+\infty$. From (5.1) it follows therefore that

$$f_m := \lim_{R\to m-} |f(R)| \tag{5.3}$$

also exists in a generalized sense. Arguing by contradiction we assume $f_m > 0$. Then there are constants $R_2 \geqslant R_1$ and $K > 0$ such that

$$\int_{R_2}^{R} \frac{|f(r)|}{r^{n-1}a(r)\, h_{\gamma_1}(r)}\, dr \geqslant K \int_{R_2}^{R} \frac{1}{r^{n-1}a(r)\, h_{\gamma_1}(r)}\, dr = \pm K \log \frac{h_{\gamma_1}(R)}{h_{\gamma_1}(R_2)} \tag{5.4}$$

for all $R \in (R_2, m)$. The plus sign is valid in case b or d, the minus sign in case a or c. In any case the right-hand side of (5.4) tends to

$+\infty$ if $R \to m-$ so that the integral on the left-hand side diverges as $R \to m-$. On the other hand,

$$\left(\int_{R_2}^{R} \frac{|f(r)|}{r^{n-1}a(r)\,h_{\nu_1}(r)}\,dr\right)^2$$

$$\leqslant \int_{S(R_2,R)} \sum a_{st}(D_s u)\overline{(D_t u)}\,dx \cdot \int_{S(R_2,R)} \frac{\sum a_{st}\nu_s\nu_t\,|u|^2}{[a(|x|)\,|x|^{n-1}h_{\nu_1}(|x|)]^2}\,dx$$

$$\leqslant C \cdot \int_{S(R_2,R)} \sum a_{st}(D_s u)\overline{(D_t u)}\,dx \cdot \int_{S(R_2,R)} \frac{|u|^2}{a(|x|)[|x|^{n-1}h_{\nu_1}(|x|)]^2}\,dx, \tag{5.5}$$

using (I.iv). Since both integrals on the right-hand side of (5.5) converge as $R \to m-$ the same must be true for the integral on the left which is the desired contradiction. Hence (5.3) exists in the ordinary sense and is zero. Thus (5.2) also exists in the ordinary sense. As the same argument holds for $r \to l+$, we obtain from (5.1)

$$(Tu, u) = \int_{S(l,m)} \sum a_{st}(D_s u)\overline{(D_t u)}\,dx + \int_{S(l,m)} q\,|u|^2\,dx$$

for every $u \in D(T)$.

Semiboundedness. Suppose $\beta_i > -1/4$ so that $\epsilon_i := \beta_i + 1/4 > 0$ $(i = 0, 1)$. Fixing $\epsilon \in (0, \min(\epsilon_0, \epsilon_1, 1))$ arbitrary we obtain for every $u \in D(T)$

$$\begin{aligned}(Tu, u) \geqslant{}& (1-\epsilon)\int_{S(l,R_0)} a(|x|)\,|\nabla u|^2\,dx - (1/\epsilon - 1)\int_{S(l,R_0)} \sum a_{st}b_s b_t\,|u|^2\,dx\\
&+ \int_{S(l,r_0)} q\,|u|^2\,dx + \int_{S(R_0,R_1)} \sum a_{st}(D_s u)\overline{(D_t u)}\,dx\\
&+ \int_{S(r_0,r_1)} q\,|u|^2\,dx + (1-\epsilon)\int_{S(R_1,m)} a(|x|)\,|\nabla u|^2\,dx\\
&- (1/\epsilon - 1)\int_{S(R_1,m)} \sum a_{st}b_s b_t\,|u|^2\,dx + \int_{S(r_1,m)} q\,|u|^2\,dx.\end{aligned} \tag{5.6}$$

If $h_l(\cdot) < \infty$ $(h_m(\cdot) < \infty)$ we put $r_0 = R_0$ $(r_1 = R_1)$, if $h_l(\cdot) = \infty$ $(h_m(\cdot) = \infty)$ we choose $r_0 \in (l, R_0)$ $(r_1 \in (R_1, m))$ arbitrary. According to Lemma 1, 2 there exist constants $C_i \geqslant 0$ $(i = 0, 1)$ and according to (I.v) and (I.vi) there exist constants $M_i \geqslant 0$ $(i = 0, 1)$ and $\tilde{M} \geqslant 0$ such that

$$
\begin{aligned}
(Tu, u) \geqslant{}& [(1-\epsilon)/4 + \beta_0] \int_{S(l,r_0)} \frac{|u|^2}{a(|x|)[|x|^{n-1} h_{\gamma_0}(|x|)]^2} dx \\
&- (1-\epsilon)\, C_0 \cdot (u, u) \\
&+ [(1-\epsilon)/4 + \beta_1] \int_{S(r_1,m)} \frac{|u|^2}{a(|x|)[|x|^{n-1} h_{\gamma_1}(|x|)]^2} dx \\
&- (1-\epsilon)\, C_1 \cdot (u, u) - (1/\epsilon - 1)\, M_0 \int_{S(l,R_0)} |u|^2 k\, dx \\
&- \tilde{M} \int_{S(r_0,r_1)} |u|^2 k\, dx - (1/\epsilon - 1)\, M_1 \int_{S(R_1,m)} |u|^2 k\, dx \\
\geqslant{}& -K(\epsilon) \cdot (u, u),
\end{aligned} \tag{5.7}
$$

where $K(\epsilon) := (1-\epsilon)[C_0 + C_1 + (M_0 + M_1)/\epsilon] + \tilde{M}$. If no $b_s(\cdot)$ are present, the semiboundedness of T follows even for $\beta_i \geqslant -1/4$. Then those terms of (5.7) containing an ϵ do not occur. The fact that the constant $-1/4$ is the best possible is a consequence of Lemma 1. ∎

LEMMA 4. *Suppose the assumptions* (I.i)–(I.vi) *hold with* $\beta_i > -1/4$ $(i = 0, 1)$. *Then* $T_F \subset T$.

Proof. Since $T_0 \subset T$ T_0 is bounded from below if $\beta_i > -1/4$. Therefore T_0 possesses a Friedrichs extension T_F the domain of which is given by (2.1). In view of (1.3) we only need to show

$$\int_{S(l,m)} \sum a_{st}(D_s u)\overline{(D_t u)}\, dx < \infty \tag{5.8}$$

for every $u \in D(T_F)$ (and in the corresponding cases

$$\liminf_{\substack{r \to m- \\ l+}} \varphi_u(r) = 0$$

for the "distinguished" representative of the equivalence class u) to establish the relation $T_F \subset T$. For every $u \in D(T_F)$ there exists a sequence $\{u_j\} \subset C_0^\infty(S(l, m))$ such that

$$\| u_j - u \| \to 0, \qquad (T_0(u_j - u_{j'}), u_j - u_{j'}) \to 0 \quad \text{as} \quad j, j' \to \infty.$$

With ϵ fixed as in the proof of Lemma 3 (5.7) yields ($\alpha_i(\epsilon) := (1-\epsilon)/4 + \beta_i > 0$)

$$\begin{aligned}(T_0(u_j - u_{j'}), u_j - u_{j'}) \\ \geqslant \alpha_0(\epsilon) \int_{S(l,r_0)} \frac{|u_j - u_{j'}|^2}{a(|x|)[|x|^{n-1}h_{\nu_0}(|x|)]^2} dx \\ + \alpha_1(\epsilon) \int_{S(r_1,m)} \frac{|u_j - u_{j'}|^2}{a(|x|)[|x|^{n-1}h_{\nu_1}(|x|)]^2} dx - K(\epsilon) \| u_j - u_{j'} \|^2. \qquad (5.9)\end{aligned}$$

Thus both integrals on the right-hand side of (5.9) tend to zero for $j, j' \to \infty$. From

$$\begin{aligned}\int_{S(l,m)} q_- |u_j - u_{j'}|^2 dx \leqslant |\beta_0| \int_{S(l,r_0)} \frac{|u_j - u_{j'}|^2}{a(|x|)[|x|^{n-1}h_{\nu_0}(|x|)]^2} dx \\ + |\beta_1| \int_{S(r_1,m)} \frac{|u_j - u_{j'}|^2}{a(|x|)[|x|^{n-1}h_{\nu_1}(|x|)]^2} dx \\ + \tilde{M} \int_{S(r_0,r_1)} |u_j - u_{j'}|^2 k \, dx\end{aligned}$$

we conclude

$$\int_{S(l,m)} q_- |u_j - u_{j'}|^2 dx \to 0.$$

Because of

$$\begin{aligned}(T_0(u_j - u_{j'}), u_j - u_{j'}) \\ \geqslant \int_{S(l,m)} a(|x|) |\nabla(u_j - u_{j'})|^2 dx + \int_{S(l,m)} q_+ |u_j - u_{j'}|^2 dx \\ - \int_{S(l,m)} q_- |u_j - u_{j'}|^2 dx \\ \geqslant \int_{S(l,m)} a(|x|) |\nabla(u_j - u_{j'})|^2 dx - \int_{S(l,m)} q_- |u_j - u_{j'}|^2 dx\end{aligned}$$

this implies

$$\int_{S(l,m)} a(|x|) |\nabla(u_j - u_{j'})|^2 dx \to 0.$$

Hence there exist elements $\tilde{u}_s$ $(s = 1,..., n)$ with $a^{1/2}\tilde{u}_s \in L^2(S(l, m))$ such that

$$\int_{S(l,m)} a(|x|) \, |\nabla u_j - \tilde{u}|^2 \, dx \to 0 \qquad \text{as} \quad j \to \infty$$

$(\tilde{u} := (\tilde{u}_1, ..., \tilde{u}_n))$. As a simple consequence of assumption (I.iv) we obtain

$$\int_{s(l,m)} \sum a_{st}\tilde{u}_s\bar{\tilde{u}}_t \, dx < \infty. \tag{5.10}$$

For every $v \in C_0^\infty(S(l, m))$

$$\int_{S(l,m)} u_j \partial_s \bar{v} \, dx = -\int_{S(l,m)} (\partial_s u_j)\bar{v} \, dx \qquad (s = 1,..., n)$$

holds. Therefore in the limit $j \to \infty$

$$\int_{S(l,m)} u \, \partial_s \bar{v} \, dx = -\int_{S(l,m)} \tilde{u}_s \bar{v} \, dx,$$

i.e., $\tilde{u}_s = \partial_s u$ a.e. Relation (5.8) we wished to prove now follows from (5.10) and Remark 6.

Suppose $h_l(\cdot) < \infty$. Applying

$$|\varphi_u(r) - \varphi_u(r')|^2 \leqslant \left| \int_{r'}^{r} \frac{dt}{t^{n-1}a(t)} \right| \cdot \left| \int_{r'}^{r} t^{n-1}a(t) \int_{|\xi|=1} |\nabla u(t\xi)|^2 \, d\omega_n \, dt \right|$$

(this formula can be obtained by differentiation of $\varphi_u^2(r)$ and subsequent integration) to $u_j - u_{j'} \in C_0^\infty(S(l, m))$ we find if we let r' tend to $l+$

$$\int_{|\xi|=1} |u_j(r\xi) - u_{j'}(r\xi)|^2 \, d\omega_n \leqslant h_l(r) \int_{S(l,r)} a(|x|) \, |\nabla(u_j - u_{j'})|^2 \, dx. \tag{5.11}$$

This leads to

$$\lim_{r \to l+} \int_{|\xi|=1} |\hat{u}(r\xi)|^2 \, d\omega_n = 0 \tag{5.12}$$

for the element $\hat{u}(r\xi)$ towards which $\{u_j\}$ converges in the mean on the sphere $\dot{S}_r := \{x \mid |x| = r\}$. Moreover, (5.11) shows that on every compact subset $K \subset S(l, m)$ $\{u_j\}$ converges to $\hat{u}$ in the norm of $L^2(K; k)$. Hence $\hat{u} = u$ a.e. on $\dot{S}_r$, and (5.12) also holds for the "distinguished" representative of the equivalence class $u \in D(T_F)$. This means $D(T_F) \subset D(T)$. ▮

The modifications of the proof of Lemma 3 necessary to show that H is symmetric and for $\beta_i \geqslant -1/4$ bounded from below are straightforward. (Because of $L_0' \subset H$, H is densely defined since L_0' is [13, p. 180].) Similarly the proof of Lemma 4 needs only to be changed slightly to establish $L_F \subset H$ for $\beta_i > -1/4$. It follows from (2.1) and (1.5) that it is sufficient to show $p^{1/2}u' \in L^2(l, m)$ for every $u \in D(L_F)$ (and possibly $\liminf |u(x)| = 0$ as $x \to l+$ or $x \to m-$). In order to conclude that u' coincides a.e. with its generalized derivative one has to know $u \in A^1(l, m)$ but this is trivial on account of $u \in D(L_F) \subset D(L)$. Instead of (5.11) we have

$$| u_j(x) - u_{j'}(x)|^2 \leqslant h_l(x) \int_l^x p(t) \, | u_j'(t) - u_{j'}'(t)|^2 \, dt \qquad (5.13)$$

in the one-dimensional case. The relation $\lim_{x \to l+} | u(x)| = 0$ for the limit function $u(\cdot)$ is now a direct consequence of (5.13).

6. Weakening of the Condition $q \geqslant -M$

Our condition (I.v) allowed "rather strong" negative singularities of the potential at the boundary but required the semiboundedness of q in the interior of $S(l, m)$ or (l, m). Using an idea of Kato [9] we relax this condition and prove that certain "weak" negative singularities in the interior (in addition to the "strong" ones at the boundary) neither destroy the semiboundedness of the operator nor affect the relation $T_F = T$ or $L_F = H$. The precise conditions to be imposed on q are formulated in Lemma 5 for the multidimensional and in Lemma 6 for the one-dimensional case.

Lemma 5. *Assume that $p(|x|)$ and $k(x)$ ($x \in S(l, m)$) satisfy* (II.i) *and* (I.ii), *respectively. Let q be a real-valued function defined on $S(l, m)$. Suppose, for every $\kappa > 0$ there exist constants $K_i(\kappa)$ ($i = 1, 2$) such that for every $x \in S(l, m)$*

$$\int_{S(l,m) \cap \{y \,|\, |y - x| \leqslant 1\}} | x - y |^{2-n-\kappa} \, | q(y)| \, dy \leqslant K_1(\kappa) \, p(|x|) \qquad (6.1)$$

and

$$\int_{S(l,m) \cap \{y \,|\, |y - x| \leqslant 1\}} | x - y |^{2-n-\kappa} \, | q(y)| \, dy \leqslant K_2(\kappa) \, k(x) \qquad (6.2)$$

*hold. **Assertion**: for every $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that*

$$\int_{S(l,m)} |q(x)|\,|u(x)|^2\,dx \leqslant \epsilon \int_{S(l,m)} p(|x|)\,|\nabla u(x)|^2\,dx + \delta(\epsilon)\cdot(u, u)$$

for all $u \in C^1(S(l, m)) \cap L^2(S(l, m); k)$ with $p^{1/2}\nabla u \in L^2(S(l, m))$.

LEMMA 6. *Assume that $p(x)$ and $k(x)$ $(x \in (l, m))$ satisfy* (II.i) *and* (II.ii), *respectively. Let q be a real-valued function defined on (l, m). Suppose there exist constants K_i $(i = 1, 2)$ such that for every $x \in (l, m)$*

$$\int_{(l,m)\cap\{t \mid t\in[x,x+1]\}} h_{t-1}(t)\,|q(t)|\,dt \leqslant K_1 \tag{6.3}$$

and

$$\int_{(l,m)\cap\{t \mid t\in[x,x+1]\}} \frac{|q(t)|}{h_{t-1}(t)}\,dt \leqslant K_2 p(x)\,k(x) \tag{6.4}$$

*hold. **Assertion**: for every $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that*

$$\int_l^m |q(x)|\,|u(x)|^2\,dx \leqslant \epsilon \int_l^m p(x)\,|u'(x)|^2\,dx + \delta(\epsilon)\cdot(u, u)$$

for all $u \in C^1(l, m) \cap L^2(l, m; k)$ with $p^{1/2}u' \in L^2(l, m)$.

As to the differentiability conditions on u, Remark 3 holds.

The special case $p = k = 1$ on $S(l, m) = S(0, \infty)$ or $(0, \infty)$ of Lemmas 5 and 6 when (6.1) and (6.2) reduce to a single and (6.3) and (6.4) to another single condition on q can be found in Kato's book [9, pp. 346,350]. The method given *loc. cit.*, p. 351, can immediately be adopted to prove Lemma 5. Lemma 6 can be proved on the lines of Example 1.8, *loc. cit.* p. 192f if one replaces the function g to be found there by

$$g(y) := \left[\frac{h_{x-1}(y)}{h_{x-1}(x)}\right]^{n-1}$$

(x and y appropriately restricted, $n \in \mathbf{N}$).

If we add to the q occurring in Theorems 1 and 2 a potential satisfying the conditions of Lemmas 5 or 6 we see that (5.6) and (5.7) in the proof of Lemma 3 remain essentially unchanged. Since the proof of Lemma 4 mainly rests on (5.7) the lemmas of this section allow us to derive the following two theorems.

THEOREM 3. *Suppose we have $q = q_1 + q_2$ in* (1.1) *and assume* *that conditions* (I.i) *to* (I.vi) *hold with* q_1. *Let* $q_2 \in Q_{\alpha,\mathrm{loc}}(S(l, m))$ *satisfy the conditions of Lemma* 5 (*with* $p(|x|)$ *replaced by the smallest eigenvalue* $a(|x|)$). *Then for* $\beta_i > -1/4$ $(i = 0, 1)$ T *is the Friedrichs extension of* T_0.

THEOREM 4. *Suppose we have* $q = q_1 + q_2$ *in* (1.2). *Assume that conditions* (II.i)–(II.iii) *and* (I.v), *restricted to* $n = 1$, *hold with* q_1 *and that* q_2 *satisfies the conditions of Lemma* 6. *Then for* $\beta_i > -1/4$ $(i = 0, 1)$ H *is the Friedrichs extension of* L_0'.

In the special case $p = k = 1$ examples for q_2 are

$$q_2(x) = \frac{c}{|x - x_0|^\sigma} \qquad (x_0 \in S(l, m))$$

with $0 < \sigma < \min(n/2, 2)$ if $n \geqslant 2$ and

$$q_2(x) = \frac{c}{|x - x_0|^\sigma} \qquad (x_0 \in (l, m))$$

with $0 < \sigma < 1$ if $n = 1$. c may take arbitrary large negative values.

7. EXAMPLES

Writing $\mathbf{R}_+^n := \mathbf{R}^n \setminus \{0\} = S(0, \infty)$ $(n \geqslant 2)$ and assuming

$$b_s(\cdot) \in C^1(\mathbf{R}_+^n) \quad \text{real} \quad (s = 1,..., n), \quad q(\cdot) \in Q_{\alpha,\mathrm{loc}}(\mathbf{R}_+^n), \tag{7.1}$$

$$Du(x) := \sum_{s=1}^{n} (i\partial_s + b_s(x))[|x|^{2\mu}(i\partial_s + b_s(x))\, u(x)] + q(x)\, u(x) \qquad (x \in \mathbf{R}_+^n,\ \mu \in \mathbf{R}^1)$$

defines an operator on $C_0^\infty(\mathbf{R}_+^n) \subset L^2(\mathbf{R}^n)$ which we call T_0. For $\mu \neq -(n-2)/2$ our condition (I.v) reads ($\beta := (n - 2 + 2\mu)^2 \beta_0 = (n - 2 + 2\mu)^2 \beta_1$)

$$q(x) \geqslant \begin{cases} \beta\,|x|^{-(2-2\mu)} & \text{for } 0 < |x| \leqslant R_0, \\ -M & \text{for } R_0 \leqslant |x| \leqslant R_1, \\ \beta\,|x|^{-(2-2\mu)} & \text{for } R_1 \leqslant |x| < \infty. \end{cases} \tag{7.2}$$

Near zero (infinity) this condition is, of course, interesting only if $\mu < 1$ $(\mu > 1)$. Besides (7.1) and (7.2), we have to postulate

$$|x|^{2\mu} \sum_{s=1}^{n} b_s^2(x) \quad \text{is bounded for small and large } |x|.$$

A. $\mu < -(n-2)/2$. Then we have Case b. For

$$\beta > -[(n-2+2\mu)/2]^2$$

T_0 is bounded from below, and its Friedrichs extension T_F can be characterized by

$$D(T_F) = \Big\{ u \mid u \in H^2_{\mathrm{loc}}(\mathbf{R}_+{}^n) \cap L^2(\mathbf{R}^n), \int_{\mathbf{R}^n} |x|^{2\mu} \sum_{s=1}^n |D_s u|^2 \, dx < \infty,$$

$$Du \in L^2(\mathbf{R}^n), \liminf_{r\to 0+} \int_{|\xi|=1} |u(r\xi)|^2 \, d\omega_n = 0 \Big\}.$$

B. $\mu > -(n-2)/2$. This is Case c. Since $h_\infty(\cdot) < \infty$ the boundary condition $\liminf_{r\to\infty} \varphi_u(r) = 0$ has to be imposed on u, but because of $u \in L^2(\mathbf{R}^n)$ this condition is automatically fulfilled (Remark 5). For $\beta > -[(n-2+2\mu)/2]^2$, T_0 is again bounded from below, and its Friedrichs extension T_F can be characterized by

$$D(T_F) = \Big\{ u \mid u \in H^2_{\mathrm{loc}}(\mathbf{R}_+{}^n) \cap L^2(\mathbf{R}^n),$$

$$\int_{\mathbf{R}^n} |x|^{2\mu} \sum_{s=1}^n |D_s u|^2 \, dx < \infty, Du \in L^2(\mathbf{R}^n) \Big\}.$$

For $n \geqslant 3$, the Schrödinger operator which has $\mu = 0$ falls into this category.

C. $\mu = -(n-2)/2$. This is Case d. Condition (7.2) has to be replaced by

$$q(x) \geqslant \begin{cases} \beta_0 |x|^{-n} \left(\log \dfrac{|x|}{\gamma_0}\right)^{-2} & \text{for} \quad 0 < |x| \leqslant R_0 \\ -M & \text{for} \quad R_0 \leqslant |x| < \infty \end{cases}$$

($\gamma_0 \in (R_0, \infty)$ is arbitrary). For $\beta_0 > -1/4$, T_0 is bounded from below, and its Friedrichs extension T_F can be characterized by

$$D(T_F) = \Big\{ u \mid u \in H^2_{\mathrm{loc}}(\mathbf{R}_+{}^n) \cap L^2(\mathbf{R}^n),$$

$$\int_{\mathbf{R}^n} |x|^{-(n-2)} \sum_{s=1}^n |D_s u|^2 \, dx < \infty, Du \in L^2(\mathbf{R}^n) \Big\}.$$

The two-dimensional Schrödinger operator belongs to this class of operators.

Remark 7. It is shown in [7, 8] that T_0 is essentially self-adjoint if $\mu < 1$ and $\beta > (1 - \mu)^2 - [(n - 2 + 2\mu)/2]^2$ (the constant $(1 - \mu)^2 - [(n - 2 + 2\mu)/2]^2$ is sharp). Then, of course, $T_F = \overline{T}_0$ where $\overline{T}_0$ is the closure of T_0.

To give an example for Case a we assume (7.1) and consider

$$Du(x) := \sum_{s=1}^{n} (i\partial_s + b_s(x))[|x|^{-(n-1)} e^{|x|}(i\partial_s + b_s(x))\, u(x)] + q(x)\, u(x)$$

on $C_0^\infty(\mathbf{R}_+^n) \subset L^2(\mathbf{R}^n)$. Conditions (I.v) and (I.vi) read

$$q(x) \geqslant \begin{cases} \beta_0(4\,|x|^{n-1} \sinh^2 |x|/2)^{-1} & \text{for } 0 < |x| \leqslant R_0, \\ -M & \text{for } R_0 \leqslant |x| \leqslant R_1, \\ \beta_1\, |x|^{-(n-1)}\, e^{|x|} & \text{for } R_1 \leqslant |x| < \infty, \end{cases}$$

$$|x|^{-(n-1)} e^{|x|} \sum_{s=1}^{n} b_s^2(x) \quad \text{is bounded for small and large } |x|.$$

Since Remark 5 applies, the boundary condition to be imposed at infinity is automatically satisfied. For $\beta_i > -1/4$ $(i = 0, 1)$ T_0 is bounded from below, and its Friedrichs extension T_F is given by

$$D(T_F) = \Big\{u \mid u \in H^2_{\mathrm{loc}}(\mathbf{R}_+^n) \cap L^2(\mathbf{R}^n),\ \int_{\mathbf{R}^n} |x|^{-(n-1)} e^{|x|} \sum_{s=1}^{n} |D_s u|^2\, dx < \infty,$$
$$Du \in L^2(\mathbf{R}^n),\ \liminf_{r \to 0+} \int_{|\xi|=1} |u(r\xi)|^2\, d\omega_n = 0\Big\}.$$

As to the case $n = 1$, Kato's Theorem 4.2 [9, p. 346] is contained in our Theorem 4 ($p = k = 1$ on $(0, \infty)$; we are in Case b then). Kato writes $q = q_1 + q_2 + q_3$; his q_1 corresponds to our $q_1^+ := (|q_1| + q_1)/2$, his q_3 to our $-q_1^- = (q_1 - |q_1|)/2$. (Note that we did not assume $q_1^{+1/2} u \in L^2(0, \infty)$.)

The separated Schrödinger differential expression

$$Du(x) := \frac{1}{x^2}\{-(x^2 u'(x))' + [l(l+1) - \delta x]\, u(x)\} \qquad (0 < x < \infty) \tag{7.3}$$

for the hydrogen atom (l is a nonnegative integer, $\delta > 0$) provides an example for Case c (again $h_\infty(\cdot) < \infty$, but Remark 5 applies) so that the Friedrichs extension L_F of the "minimal" operator L_0' which is associated with (7.3) can be characterized by

$$D(L_F) = \{u \mid u \in L^2(0, \infty; x^2),\ u' \in A^1(0, \infty) \cap L^2(0, \infty; x^2),\ Du \in L^2(0, \infty; x^2)\}.$$

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