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Some Remarks on Index and Generalized Loewy Length of a Gorenstein Local Ring

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1. INTRODUCTION

Throughout this paper, the word "ring" will mean commutative noetherian ring with 1.

Let R be a Gorenstein local ring. M. Auslander introduced the notion of a δ-invariant $\delta_R(M)$ for a finitely generated R-module M. It is defined as the smallest integer μ such that there exists an epimorphism $X \oplus R^{\mu} \to M$ with X a maximal Cohen–Macaulay module which has no non-zero free summand. In studying the δ-invariant, Ding [10–13] studied the notion of index. The index of R, denoted by index(R), is defined as the smallest positive integer such that $\delta_R(R/\mathfrak{m}^n) > 0$. He compared index$(R)$ with the generalized Loewy length $\ell\ell(R)$ of R, which is defined as the minimum of the Loewy lengths of $R/\mathbf{x}R$ for all systems of parameters $\mathbf{x}$ of R, and he conjectured (and proved for some special cases) that index$(R) = \ell\ell(R)$ in general (see Definition 2.7 and Remark 3.1).

In this paper, we first remark that the generalized Loewy length is not stable even under a finite étale extension, but stable under completion (Section 3). We need at least to assume that the residue field of R is infinite to study Ding's conjecture.

Next, we prove that the minimal Cohen–Macaulay approximation, which is indispensable when we study δ-invariants, of a finitely generated module M over a Cohen–Macaulay local ring R is preserved by an extension by

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a Gorenstein local homomorphism $R \to S$ under the assumption $\operatorname{Tor}_i^R(S, M) = 0$ $(i > 0)$ (Proposition 4.3). If R is Gorenstein moreover, then the higher delta invariants are also preserved (Corollary 4.6).

This unifies two known important cases: the flat case and the case $RS = R/xR$ with x both R-regular and M-regular. We note that many of the notions and the results on flat morphisms have been generalized to the situation of finite flat dimension [4, 5, 3].

Let $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ be Gorenstein local rings, and $\varphi\colon R \to S$ be a local homomorphism of finite flat dimension (i.e., the flat dimension of S as an R-module is finite). Recently, the second author proved the following.

THEOREM 1.1 [17, (3.7)]. *Let $\varphi\colon R \to S$ be as above. Then we have*

$$\operatorname{index}(R) \le \operatorname{index}(S).$$

This can be seen as a sort of flat descent (generalized to the finite flat dimension context). A good property "index $\le n$" of S is inherited by R for any n. In Section 5, assuming S is R-flat, we prove $\operatorname{index}(S) \le \operatorname{index}(R) \cdot \ell\ell(F)$ $(F := S/\mathfrak{m}S)$, which can be seen as a counterpart of the theorem above. This is our main theorem.

Furthermore, this inequality shows that if the closed fiber F is a regular local ring, we have $\operatorname{index}(R) = \operatorname{index}(S)$ as might be expected.

We cannot expect that the fiber ring $F = S/\mathfrak{m}S$ has a small index even if S has a small index. This can be seen by the following example: $R = k[[x^t]] \subset k[[x]] = S$, where t is an arbitrary positive integer. However, if S is artinian, then we have an inequality: $\ell\ell(R) + \ell\ell(F) - 1 \le \ell\ell(S) \le \ell\ell(R) \cdot \ell\ell(F)$. This will be proved in Section 6 for artinian rings which are not necessarily Gorenstein.

2. PRELIMINARIES

Unless otherwise specified, we assume that $(R, \mathfrak{m}, k)$ is a Cohen–Macaulay local ring with the canonical module K_R. By $\hat{R}$ we mean the completion of R with respect to the maximal ideal $\mathfrak{m}$. For an R-module M, $\hat{M}$ denotes $\hat{R} \otimes_R M$. For a proper ideal I of R, we denote the associated graded module $\bigoplus_{i \ge 0} I^i M / I^{i+1} M$ by $\operatorname{Gr}_I M$. Let $(S, \mathfrak{n})$ be a local ring, and $\varphi\colon R \to S$ a homomorphism. We say that φ is *local* when $\varphi(\mathfrak{m}) \subset \mathfrak{n}$.

PROPOSITION 2.1. *Let R be a Cohen–Macaulay local ring, X a maximal Cohen–Macaulay R-module, and Y (resp. Z) a finitely generated R-module of finite injective (resp. projective) dimension. Then we have* (a) $\operatorname{Ext}_R^i(X, Y) = 0$ *for* $i > 0$; (b) $\operatorname{Tor}_i^R(Z, X) = 0$ *for* $i > 0$.

Proof. Part (a) is noted in [18]. Part (b) is a slight generalization of [16, (2.6)], and follows easily from [8]. ∎

Let M be a finitely generated R-module. A sequence of finitely generated R-modules

$$0 \to Y \xrightarrow{f} X \xrightarrow{g} M \to 0 \tag{2.2}$$

is called a *Cohen–Macaulay approximation* if X is a maximal Cohen–Macaulay R-module, Y is of finite injective dimension, and the sequence is exact. The Cohen–Macaulay approximation (2.2) is said to be *minimal* when X and Y have no non-zero direct summand in common through f. It is said to be *right minimal* when g is right minimal, that is, for any non-isomorphic map φ: $X \to X$, we have $g \circ \varphi \neq g$.

The next lemma and its corollary seem to be well known, but we state them with proofs. The complete case (which is essential) is found in [19].

LEMMA 2.3. *Let*

$$\mathbb{A} := 0 \to Y \xrightarrow{f} X \xrightarrow{g} M \to 0$$

be a sequence of finitely generated R-modules. Then, the following conditions are equivalent.

(rmin) $\mathbb{A}$ *is a right minimal Cohen–Macaulay approximation of* M.

(min) $\mathbb{A}$ *is a minimal Cohen–Macaulay approximation of* M.

$\widehat{\textbf{(rmin)}}$ $\hat{\mathbb{A}}$ *is a right minimal Cohen–Macaulay approximation of* $\hat{M}$.

$\widehat{\textbf{(min)}}$ $\hat{\mathbb{A}}$ *is a minimal Cohen–Macaulay approximation of* $\hat{M}$.

Proof. It is easy to see that $\widehat{\mathbb{A}}$ is a Cohen–Macaulay approximation if and only if $\mathbb{A}$ is a Cohen–Macaulay approximation. Thus, the problem is the minimality.

(rmin) ⇒ **(min)**. Assume that $X = X_0 \oplus X_1$ and $0 \neq X_0 \subset \operatorname{Im} f$. We define φ: $X \to X$ by $\varphi(x_0 + x_1) = x_1$ for $x_0 \in X_0$ and $x_1 \in X_1$. Then, we have $g \circ \varphi = g$, and φ is not isomorphic.

(min) ⇒ $\widehat{\textbf{(min)}}$. See Proposition 4.3 below.

$\widehat{\textbf{(min)}}$ ⇒ $\widehat{\textbf{(rmin)}}$. We may assume $\hat{R} = R$ and $\hat{\mathbb{A}} = \mathbb{A}$. Assume that $\varphi \in \operatorname{End}_R(X)$ is not an isomorphism, and $g \circ \varphi = g$. We have $X \neq 0$, and hence $M \neq 0$ by minimality of $\mathbb{A}$. We set Λ to be the sub-R-algebra of $\operatorname{End}_R(X)$ generated by R and φ. Note that Λ is commutative, and is module finite over R. As φ preserves the kernel of g, we may and shall regard Y and M as Λ-modules so that f and g are Λ-linear. Note that $\varphi \in \Lambda$ acts as an identity map on M. By Nakayama's lemma, φ is not contained in the radical of Λ, since $\varphi(M) = M$ and $M \neq 0$. As φ is not a unit in Λ (since φ is not an isomorphism on X), we have that Λ is not local. It follows that Λ has a non-trivial idempotent, say $e \neq 0, 1$, since R is henselian. With replacing e with $1 - e$ if necessary, we may assume that the image of e in $\Lambda/(1-\varphi)\Lambda$, which is local, is a unit. As e acts as an automorphism on M, it is easy to see that $X_0 := \operatorname{Im}(1-e) = \operatorname{Ker} e$ is a non-zero direct summand of X which is contained in $\operatorname{Im} f$.

$(\widehat{\mathbf{rmin}}) \Rightarrow (\mathbf{rmin})$. Assume that $\varphi \in \operatorname{End}_R(X)$ is non-isomorphic and that $g \circ \varphi = g$. Then, $\hat{\varphi}$ is non-isomorphic and $\hat{g} \circ \hat{\varphi} = \hat{g}$. ∎

COROLLARY 2.4. *Let M be a finitely generated R-module. Then, there exists a minimal Cohen–Macaulay approximation of M, uniquely up to isomorphism.*

Proof. A Cohen–Macaulay approximation of M exists, see [1]. Removing non-zero common direct summands through f if any, we obtain a minimal one from this. The uniqueness of the right minimal Cohen–Macaulay approximation is shown by a standard comparison argument in [1]. ∎

For an R-module M, we define the *f-rank* of M, denoted by $\text{f-rank}_R M$, to be the number $\max\{i \mid R^i \text{ is a direct summand of } M\}$.

DEFINITION 2.5. Let R be a Cohen–Macaulay local ring with K_R, and let $0 \to Y \to X \to M \to 0$ be the minimal Cohen–Macaulay approximation of M over R. We define the *δ-invariant* of M, denoted by $\delta_R(M)$, as $\text{f-rank}_R X$. And we define the *index* of a Gorenstein local ring R, denoted by $\operatorname{index}(R)$, as the integer $\min\{n \mid \delta_R(R/\mathfrak{m}^n) \neq 0\}$. For a nonnegative integer n, the δ-invariant of the nth syzygy $\Omega_R^n(M)$ of M is denoted by $\delta_R^n(M)$, and we call it the *nth δ-invariant* of M.

PROPOSITION 2.6 [10]. *Let R be a Gorenstein local ring and $M(\neq 0) \in \operatorname{mod}(R)$. Then*

(a) *If $\operatorname{pd}_R M$ is finite, then $\delta_R(M) = \mu(M)(> 0)$, where $\mu(M)$ is the minimal number of generators of M.*

(b) *If $M \to N \to 0$ is exact, then $\delta_R(M) \geq \delta_R(N)$.*

Next we define the generalized Loewy length.

DEFINITION 2.7. Let M be an R-module of finite length. The *Loewy length* of M, denoted by $\ell\ell_R(M)$, is the smallest integer n such that $\mathfrak{m}^n M = 0$. We define the *generalized Loewy length* of R, which is also denoted by $\ell\ell(R)$, as the minimum of all integers $\ell\ell_R(R/(\mathbf{x}))$, where $\mathbf{x}$ is a system of parameters of R.

3. REMARKS ON GENERALIZED LOEWY LENGTH

Remark 3.1. Ding proved that $\operatorname{index}(R) \le \ell\ell(R)$ for any Gorenstein local ring R [11]. He conjectured that $\operatorname{index}(R) = \ell\ell(R)$ for the arbitrary Gorenstein local ring R. He claimed that the conjecture is true when the associated graded ring $\operatorname{Gr}_{\mathfrak{m}} R$ of R is a Cohen–Macaulay ring [12]. He also claimed that the conjecture is true when R is gradable and $\operatorname{depth} \operatorname{Gr}_{\mathfrak{m}} R \geq \dim R - 1$ [13]. These are certainly true when $R/\mathfrak{m}$ is infinite.

If we drop the condition on the residue field, there is a counterexample to the conjecture.

EXAMPLE 3.2. Let $k = \mathbb{F}_2$, and $K = \mathbb{F}_4$, where $\mathbb{F}_q$ is the q-element field. When we set $S = k[[x, y]]$, $f = xy(x + y) \in S$, and $R = S/(f)$, then we have

$$4 = \ell\ell(R) > \ell\ell(K \otimes_k R) = \operatorname{index}(K \otimes_k R) = \operatorname{index}(R) = 3.$$

Proof. As the rings in consideration are hypersurfaces, we have that indices equal to multiplicities [10], and we have $\operatorname{index}(K \otimes_k R) = \operatorname{index}(R) = 3$. Let $\omega \in K \setminus k$, and set $z = x - \omega y$. Then, we have $\ell\ell((K \otimes_k R)/(z)) = 3$. Hence, we have $\ell\ell(K \otimes_k R) = 3$. As we have $\ell\ell(R/(x - y^2)) = 4$, we have $\ell\ell(R) \leq 4$.

It remains to show that $\ell\ell(R) > 3$. Assume the contrary. Then, there exists some R-regular element $z \in \mathfrak{m}$ such that $\mathfrak{m}^3 \subset (z)$, where $\mathfrak{m} = (x, y)R$. We set $r = \max\{i \mid z \in \mathfrak{m}^i\}$. It is easy to see that $r \leq 2$.

If $r = 2$, then $z\overline{R}$ is annihilated by $\mathfrak{m}^2$, where $\overline{R} = R/\mathfrak{m}^4$. Hence, we have

$$3 = l_R\left(R/\mathfrak{m}^2\right) \geq l_R(\overline{R}z) \geq l_R(\mathfrak{m}^3\overline{R}) = l_R\left(\mathfrak{m}^3/\mathfrak{m}^4\right) = 3.$$

This shows $z\overline{R} = \mathfrak{m}^3\overline{R}$, and hence $z \in \mathfrak{m}^3$. This contradicts $r = 2$.

Consider the case $r = 1$. Take any preimage $\zeta \in S$ of z. Letting an appropriate element in $GL_2(k)$ act on S, we may assume that $\zeta = x - g(x, y)$ with $g \in ((x, y)S)^2$ without loss of generality. As $R/(z)$ is a hypersurface, it suffices to show that $l_R(R/(z)) \geq 4$ to lead to a contradiction. When we set $R' = S/(\zeta)$, then R' is a discrete valuation ring, and $R/(z) = R'/(\bar{f})$, where $\bar{f} = \bar{x}\bar{y}(\bar{x} + \bar{y})$ is the image of f in R'. On the other hand, we have $\bar{y} \in (x, y)R'$, $\bar{x} + \bar{y} \in (x, y)R'$, and $\bar{x} = \bar{g} \in ((x, y)R')^2$. Hence, we have $\bar{f} \in ((x, y)R')^4$ and we have $l(R'/(\bar{f})) \geq 4$. ∎

As we have seen, the generalized Loewy length is not stable under a finite étale extension. However, it is stable under completion.

LEMMA 3.3. *We have* $\ell\ell(R) = \ell\ell(\hat{R})$.

Proof. If $\mathbf{x}$ is a system of parameters of R such that $\ell\ell(R) = \ell\ell_R(R/\mathbf{x}R)$, then we have $\ell\ell(\hat{R}) \leq \ell\ell_{\hat{R}}(\hat{R}/\mathbf{x}\hat{R}) = \ell\ell_R(R/\mathbf{x}R) = \ell\ell(R)$. So it suffices to show $\ell\ell(R) \leq \ell\ell(\hat{R})$.

Let $n = \ell\ell(\hat{R})$, and take a system of parameters $(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_d) \subset \hat{R}$ such that $\mathfrak{m}^n\hat{R} \subset (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_d)$, where $d = \dim R$. Then for each $\hat{x}_i$ $(i = 1, 2, \dots, d)$, we can choose an element $x_i \in R$ such that $x_i - \hat{x}_i \in \mathfrak{m}^{n+1}\hat{R}$. We claim that $\mathfrak{m}^n \subset (x_1, x_2, \dots, x_d)$. In fact since $\mathfrak{m}^n\hat{R} \subset (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_d) \subset (x_1, x_2, \dots, x_d)\hat{R} + \mathfrak{m}^{n+1}\hat{R}$, we have

$$\mathfrak{m}^n\hat{R} \subset \bigcap_{i>0} \left((x_1, x_2, \dots, x_d)\hat{R} + \mathfrak{m}^{n+i}\hat{R}\right) = (x_1, x_2, \dots, x_d)\hat{R}.$$

Hence, we have

$$\mathfrak{m}^n = \mathfrak{m}^n\hat{R} \cap R \subset (x_1, x_2, \dots, x_d)\hat{R} \cap R = (x_1, x_2, \dots, x_d)$$

as desired. ∎

4. A GORENSTEIN HOMOMORPHISM AND MINIMAL COHEN–MACAULAY APPROXIMATIONS

Let $\varphi: R \to S$ be a flat local homomorphism of Gorenstein local rings. If $0 \to Y \to X \to M \to 0$ is a (minimal) Cohen–Macaulay approximation over R, then so is $0 \to S \otimes_R Y \to S \otimes_R X \to S \otimes_R M \to 0$. In particular, the (higher) δ-invariants are preserved by this base extension. In this section, we generalize this to the context of morphisms of finite flat dimension (Proposition 4.3, Corollary 4.6). The case $S = R/xR$ with x both R- and M-regular [2, (5.1); 20, (1.8)], as well as the flat case [20, (1.5), (1.7)], has been well known.

First, we introduce the notion of a Gorenstein local homomorphism (see [4]). Let $\varphi: (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism. We say that φ is a *Gorenstein local homomorphism* if φ is of finite flat dimension (i.e., $\mathrm{fd}_R S < \infty$), and $\mu_R^i = \mu_S^{i+s}$ for some s and arbitrary $i \geq 0$, where $\mu_R^i = \dim_k \mathrm{Ext}_R^i(k, R)$ is the ith Bass number. Next, we define a Cohen factorization which was introduced in [6].

DEFINITION 4.1. Let $\varphi: (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism of local rings. We say φ is *factorizable* if it can be decomposed as $\varphi = \sigma\tau$ with $\tau: R \to T$ flat and $T/\mathfrak{m}T$ a regular local ring, $\sigma: T \to S$ surjective, and T a complete local ring; often we shall refer to such a situation by saying that $\varphi = \sigma\tau$ is a *Cohen factorization.*

It is known that if S is a complete local ring, then there exists a Cohen factorization (see [6, (1.1)]). Let $\varphi\colon R \to S$ be a local homomorphism, and $\varphi = \sigma\tau$ its Cohen factorization. Then, φ is of finite flat dimension if and only if so is σ [6, (3.3)]. Note also that, if φ is module finite, then we have

$$\mathrm{fd}_R S < \infty \Leftrightarrow \mathrm{Tor}_i^R(S, R/\mathfrak{m}) = 0 \text{ for } i \gg 0 \Leftrightarrow \mathrm{pd}_R S < \infty.$$

By [4, (4.2), (4.6)], φ is Gorenstein if and only if so is σ.

LEMMA 4.2. *Let $\varphi\colon R \to S$ be a local homomorphism of Cohen–Macaulay local rings. We assume that $\mathrm{fd}_R S < \infty$. Let X be a maximal Cohen–Macaulay R-module. Then we have $\mathrm{Tor}_i^R(S, X) = 0$ for $i > 0$. We also have $S \otimes_R X$ is a maximal Cohen–Macaulay S-module.*

Proof. We may assume that S is complete. If S is R-flat, then the assertion is obvious. Consider the general case. Take a Cohen factorization $\varphi = \sigma\tau$. As the lemma is true for τ, we may replace φ by σ, and we may assume that φ is surjective. Now, the vanishing $\mathrm{Tor}_i^R(S, X) = (i > 0)$ is Proposition 2.1(b). As for the last assertion, the same proof in [17, (3.1)] works. ∎

PROPOSITION 4.3. *Let $\varphi\colon R \to S$ be a Gorenstein local homomorphism of Cohen–Macaulay local rings, and M a finitely generated R-module. Furthermore, we assume that R has the canonical module K_R, and $\mathrm{Tor}_i^R(S, M) = 0$ for all $i > 0$. If $0 \to Y \xrightarrow{f} X \to M \to 0$ is the minimal Cohen–Macaulay approximation of M over R, then the sequence*

$$0 \to S \otimes_R Y \xrightarrow{S \otimes_R f} S \otimes_R X \to S \otimes_R M \to 0 \tag{4.4}$$

is the minimal Cohen–Macaulay approximation of $S \otimes_R M$ over S.

Remark 4.5. When φ is surjective and $S = R/I$, then, for $r \geq 0$, we have $\mathrm{Tor}_i^R(S, M) = 0$ $(i > r)$ if and only if $\mathrm{depth}_R(I, M) \geq \mathrm{pd}_R S - r$ (depth sensitivity of perfect ideals, see [9]). In particular, the Tor-independence assumption $\mathrm{Tor}_i^R(S, M) = 0$ $(i > 0)$ above is equivalent to $\mathrm{depth}_R(I, M) = \mathrm{pd}_R S$. Generalizing this to the imperfect case, the second author proved the following, which we only state the result here: Let R be a (not necessarily Cohen–Macaulay) ring, I an ideal of R of finite projective dimension, and $S = R/I$. If M is a finitely generated R-module with $M/IM \neq 0$, then we have

$$\sup\{i \mid \mathrm{Tor}_i^R(S, M) \neq 0\} = \sup\{\mathrm{pd}_{R_\mathfrak{p}} S_\mathfrak{p} - \mathrm{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{supp}(M/IM)\}.$$

COROLLARY 4.6. *In the proposition, assume moreover that R is Gorenstein. Then we have*

$$\text{f-rank}_S(S \otimes_R X) = \text{f-rank}_R X. \tag{4.7}$$

In particular, we have $\delta_S^n(S \otimes_R M) = \delta_R^n(M)$.

Proof. The equality (4.7) follows from the proposition and [17, (3.1)]. As we have $\operatorname{Tor}_i^R(S, M) = 0$ $(i > 0)$ by assumption, we have $\Omega_S^n(S \otimes_R M) \cong S \otimes_R \Omega_R^n(M)$, and we have $\delta_S^n(S \otimes_R M) = \delta_R^n(M)$. ∎

Proof of Proposition 4.3. Since it is sufficient to prove that

$$0 \to \hat{S} \otimes_R Y \xrightarrow{\hat{S} \otimes_R f} \hat{S} \otimes_R X \to \hat{S} \otimes_R M \to 0$$

is the minimal Cohen–Macaulay approximation of $\hat{S} \otimes_R M$, we may assume that S is complete.

We proceed in several steps.

Step 0. Note that $\operatorname{Tor}_i^R(S, K_R) = 0$ for $i > 0$ by Lemma 4.2. We have that $S \otimes_R K_R \cong K_S$, since φ is Gorenstein [4, (5.1)].

Step 1. Since $\operatorname{Tor}_i^R(S, M) = 0$ for $i > 0$ by assumption and $\operatorname{Tor}_i^R(S, X) = 0$ by Lemma 4.2, we have $\operatorname{Tor}_i^R(S, Y) = 0$ for $i > 0$.

Step 2. We show that $S \otimes_R Y$ is of finite injective dimension over S. First note that a module Y is of finite injective dimension if and only if there exists an exact sequence

$$0 \to I_n \overset{\varphi_n}{\to} I_{n-1} \overset{\varphi_{n-1}}{\to} \cdots \overset{\varphi_1}{\to} I_0 \overset{\varphi_0}{\to} Y \to 0 \tag{4.8}$$

such that I_j is a finite direct sum of K_R $(j = 0, 1, \dots, n)$.

As the modules in (4.8) are Tor-independent of S by Step 0 and Step 1, the sequence

$$0 \to S \otimes_R I_n \xrightarrow{S \otimes_R \varphi_n} S \otimes_R I_{n-1} \xrightarrow{S \otimes_R \varphi_{n-1}} \cdots$$
$$\xrightarrow{S \otimes_R \varphi_1} S \otimes_R I_0 \xrightarrow{X \otimes_R \varphi_0} S \otimes_R Y \longrightarrow 0$$

is exact.

Here $S \otimes_R I_j$ is a finite direct sum of K_S by Step 0. Thus, we conclude that $S \otimes_R Y$ is of finite injective dimension as an S-module.

Step 3. We know from Step 2 that $S \otimes_R Y$ is of finite injective dimension. Since $S \otimes_R X$ is a maximal Cohen–Macaulay S-module by Lemma 4.2, and (4.4) is an exact sequence by $\operatorname{Tor}_1^R(S, M) = 0$, we have that (4.4) is a Cohen–Macaulay approximation of $S \otimes_R M$ over S.

Step 4. Next, we show that the canonical homomorphism

$$S \otimes_R \operatorname{Hom}_R(K_R, Y) \to \operatorname{Hom}_S(S \otimes_R K_R, S \otimes_R Y) \tag{4.9}$$

given by $a \otimes f \mapsto a \cdot (S \otimes_R f)$ for $a \in S$ and $f \in \operatorname{Hom}_R(K_R, Y)$ is surjective. As this is obvious when S is R-flat, we may assume that φ is surjective (consider the Cohen-factorization of φ) so that $S = R/I$ for a proper ideal I of R of finite projective dimension. Note that

$$0 \to I \otimes_R Y \to Y \to S \otimes_R Y \to 0 \tag{4.10}$$

is exact since $\operatorname{Tor}_1^R(S, Y) = 0$, and that $\operatorname{Tor}_i^R(I, Y) \cong \operatorname{Tor}_{i+1}^R(S, Y) = 0$ for $i > 0$.

Let $\mathbb{F}$ be a finite free resolution of I. Then, as we have that $F \otimes_R Y \to I \otimes_R Y$ is quasi-isomorphic and that each term of $\mathbb{F} \otimes_R Y$ is of finite injective dimension, we have that the injective dimension of $I \otimes_R Y$ is finite. From the exact sequence (4.10), we obtain an exact sequence

$$\operatorname{Hom}_R(K_R, Y) \to \operatorname{Hom}_R(K_R, S \otimes Y) \to 0 = \operatorname{Ext}^1_R(K_R, I \otimes_R Y)$$

by Proposition 2.1(a). This shows that the map (4.9) is surjective.

Step 5. By an argument similar to Step 4, we also have that the canonical map

$$S \otimes_R \operatorname{Hom}_R(X, K_R) \to \operatorname{Hom}_S(S \otimes_R X, S \otimes_R K_R) \tag{4.11}$$

is surjective.

Step 6. Lastly, we prove that the sequence (4.4) is minimal. Assume the contrary. Then there exist $\sigma \in \operatorname{Hom}_S(S \otimes_R X, S \otimes_R K_R)$ and $\tau \in \operatorname{Hom}_S(S \otimes_R K_R, S \otimes_R Y)$ such that $\sigma \circ (S \otimes_R f) \circ \tau$ is the identity map of $S \otimes_R K_R$.

By Step 4 and Step 5, we can write $\sigma = \sum_i a_i(S \otimes_R \sigma_i)$ and $\tau = \sum_j b_j(S \otimes_R \tau_j)$ for some $\sigma_i \in \operatorname{Hom}_R(X, K_R)$, $\tau_j \in \operatorname{Hom}_R(K_R, Y)$, and $a_i, b_j \in S$. As we have $\sigma \circ (S \otimes_R f) \circ \tau = \mathrm{id}$, we have $S \otimes_R (\sum_i \sum_j a_i b_j (\sigma_i \circ f \circ \tau_j)) = 1$ in $\operatorname{Hom}_S(S \otimes_R K_R, S \otimes_R K_R) \cong S$. This shows that, at least for one (i, j), we have that $\sigma_i \circ f \circ \tau_j \in \operatorname{Hom}_R(K_R, K_R) = R$ is not contained in the maximal ideal $\mathfrak{m}$, because φ is local. This shows that Y and X have the common direct summand K_R through f, and this is a contradiction. ∎

5. MAIN THEOREM

Our main result in this paper is the following.

THEOREM 5.1. *Let $\varphi: (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local homomorphism of Gorenstein local rings, and $F = S/\mathfrak{m}S$ be the closed fiber. Then we have*

$$\operatorname{index}(S) \le \operatorname{index}(R) \cdot \ell\ell(F).$$

Proof. We proceed in two steps.

Step 1. We first prove Theorem 5.1 for $d = \dim F = 0$. Let $f = \ell\ell(F)$ and $r = \operatorname{index}(R)$. By definition of Loewy length, we have $\mathfrak{n}^f \subset \mathfrak{m}S$, and it follows $\mathfrak{n}^{fr} \subset \mathfrak{m}^r S$. Since there is an epimorphism $S/\mathfrak{n}^{fr} \to S/\mathfrak{m}^r S \cong S \otimes_R R/\mathfrak{m}^r$, we know that $\delta_S(S/\mathfrak{n}^{fr}) \ge \delta_S(S/\mathfrak{m}^r S)$ by Proposition 2.6. But Corollary 4.6 shows $\delta_S(S/\mathfrak{m}^r S) = \delta_R(R/\mathfrak{m}^r) > 0$, and this shows that $\operatorname{index}(S) \le \operatorname{index}(R) \cdot \ell\ell(F)$.

Step 2. Next we prove Theorem 5.1 in general. Let $f = \ell\ell(F)$. Then we can find a system of parameters $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_d \in F$ such that $\ell\ell(\overline{F}) = \ell\ell(F)$, where $\overline{F} = F/(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_d)F$. We set $\overline{S} = S/(x_1, x_2, \dots, x_d)S$, where x_i is a preimage of $\bar{x}_i$ $(i = 1, 2, \dots, d)$, and define the local homomorphism $\psi\colon R \to \overline{S}$ as the composite $R \xrightarrow{\varphi} S \to \overline{S}$, where the second arrow is the natural map. Then $x_1, x_2, \dots, x_d$ is an S-regular sequence and ψ is flat by Corollary of Theorem 22.5 in [15].

By Step 1, we have $\mathrm{index}(\overline{S}) \le \mathrm{index}(R) \cdot \ell\ell(\overline{F})$. Thus we know that $\mathrm{index}(S) \le \mathrm{index}(\overline{S}) \le \mathrm{index}(R) \cdot \ell\ell(\overline{F}) = \mathrm{index}(R) \cdot \ell\ell(F)$ by Theorem 1.1.

COROLLARY 5.2. *Let $\varphi\colon (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local homomorphism of Gorenstein local rings and $F = S/\mathfrak{m}S$ a regular local ring. Then we have*

$$\mathrm{index}(R) = \mathrm{index}(S).$$

Proof. This is clear because of Theorems 5.1 and 1.1 and the fact $\ell\ell(F) = 1$.

In what follows, k denotes a field.

EXAMPLE 5.3. Let r and f be positive integers, and set

$$R = k[[X]]/(X^r) \subset S = k[[X, Y]]/(X^r, Y^f).$$

Then, S is R-flat, and the closed fiber is $F = k[[Y]]/(Y^f)$. Then $\mathrm{index}(R) = r$, $\mathrm{index}(S) = r + f - 1$, and $\ell\ell(F) = f$. Therefore, we have $\mathrm{index}(S) \le \mathrm{index}(R) \cdot \ell\ell(F)$, and equality holds if and only if $r = 1$ or $f = 1$.

EXAMPLE 5.4. With the same r, f, and R as in (5.3), we set $S = k[[X, Y]]/(X^r, X - Y^f) \cong k[[Y]]/(Y^{fr})$. Then, S is flat over R, and we get $F = k[[Y]]/(Y^f)$ and $\mathrm{index}(S) = fr$. Therefore, we have $\mathrm{index}(S) = \mathrm{index}(R) \cdot \ell\ell(F)$.

We say that a $\mathbb{Z}$-graded k-algebra $R = \bigoplus_i R_i$ is *homogeneous* when R is generated by finite degree one elements over $R_0 = k$. The *Hilbert series* $H_R(t)$ of R is defined by $H_R(t) := \sum_{i \ge 0} \dim_k R_i t^i \in \mathbb{Z}[[t]]$. It is known that $Q_R(t) := (1 - t)^d H_R(t) \in \mathbb{Z}[t]$, where $d = \dim R$. The degree of $Q_R(t)$ is denoted by $s(R)$. In [14], a graded analogue of δ-invariants, indices, and generalized Loewy lengths are discussed. Herzog [14] proved that for a homogeneous Gorenstein k-algebra R, we have $\mathrm{index}(R) = s(R) + 1 = \ell\ell(R)$, provided k is infinite. This result is generalized by Ding in [13].

EXAMPLE 5.5. Let R and F be Gorenstein homogeneous k-algebras. Then, $S := R \otimes_k F$ is flat over R with the fiber F, and we have $\operatorname{index}(S) = \operatorname{index}(R) + \operatorname{index}(F) - 1$.

Proof. We may extend the base field k if necessary, and may assume that k is infinite by Corollary 5.2. Obviously, we have $H_S(t) = H_R(t)H_F(t)$. The equality follows from this and Herzog's theorem.

6. LOEWY LENGTHS OF ARTINIAN MODULES

In this section, R is a local ring which is not necessarily Cohen–Macaulay. Let $I \subset \mathfrak{m}$ be an ideal of R. We set $G = \operatorname{Gr}_I R$, and let $\mathfrak{M}$ denote the graded maximal ideal of G. As for G-modules, we only consider graded ones. The Loewy length $\ell\ell_G(H)$ of an artinian G-module is the smallest integer n such that $\mathfrak{M}^n H = 0$ (if one prefers local rings, then he or she might consider $G_{\mathfrak{M}}$). For an R-module M and $m \in M$, we denote the initial form of m in $\operatorname{Gr}_I M$ by $\operatorname{in}_I(m)$.

LEMMA 6.1. *Let M be an artinian R-module. Then we have $\ell\ell_G(\operatorname{Gr}_I M) \leq \ell\ell_R(M)$.*

Proof. There exists some $x_1, \dots, x_r \in \mathfrak{m}$ such that $\operatorname{in}_I(x_1), \dots, \operatorname{in}_I(x_r)$ generates $\mathfrak{M}$. We set $\ell\ell(\operatorname{Gr}_I M) = s + 1$. Then, there exists some $1 \leq i_1, \dots, i_s \leq r$ and $m \in M$ such that $\operatorname{in}_I(x_{i_1}) \cdots \operatorname{in}_I(x_{i_s}) \cdot \operatorname{in}_I(m) \neq 0$. This shows

$$\operatorname{in}(x_{i_1} \cdots x_{i_s} m) = \operatorname{in}_I(x_{i_1}) \cdots \operatorname{in}_I(x_{i_s}) \cdot \operatorname{in}_I(m) \neq 0,$$

and hence $x_{i_1} \cdots x_{i_s} m \neq 0$. ∎

LEMMA 6.2. *Let M be an artinian R-module. If $I = \mathfrak{m}$, then we have $\ell\ell_G(\operatorname{Gr}_{\mathfrak{m}} M) = \ell\ell_R(M)$.*

Proof. Assume that $\mathfrak{M}^s \operatorname{Gr}_{\mathfrak{m}} M = 0$. Then we have $\mathfrak{m}^s M / \mathfrak{m}^{s+1} M = 0$, and hence $\mathfrak{m}^s M = 0$. From this and Lemma 6.1, the result follows. ∎

PROPOSITION 6.3. *Let $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ be local rings, and $R \to S$ a flat local homomorphism with the artinian closed fiber $F = S/\mathfrak{m}S$. For an artinian R-module M, we have*

$$\ell\ell_R(M) + \ell\ell(F) - 1 \leq \ell\ell_S(S \otimes_R M) \leq \ell\ell_R(M) \cdot \ell\ell(F).$$

Proof. We set $k = R/\mathfrak{m}$, $R' = \operatorname{Gr}_{\mathfrak{m}} R$, $M' = \operatorname{Gr}_{\mathfrak{m}} M$, and $S' = \operatorname{Gr}_{\mathfrak{m}S} S$. The graded maximal ideals of R' and S' are denoted by $\mathfrak{m}'$ and $\mathfrak{n}'$, respectively. Then, $S' \cong F \otimes_k R'$ is flat over R', and we have an S'-isomorphism $\operatorname{Gr}_{\mathfrak{m}S}(S \otimes_R M) \cong F \otimes_k M'$. We set $f = \ell\ell(F)$ and $r = \ell\ell_R(M)$.

Then, we have

$$(\mathfrak{n}')^{f+r-2}(F \otimes_k M') \supset \mathfrak{n}^{f-1}F \otimes_k (\mathfrak{m}')^{r-1}M' \neq 0$$

by Lemma 6.2. Hence, we have

$$\ell\ell_S(S \otimes_R M) \geq \ell\ell_{S'}(\mathrm{Gr}_{\mathfrak{m}S}(S \otimes_R M)) > f + r - 2$$

by Lemma 6.1, and the first inequality follows.

As we have

$$\mathfrak{n}^{rf}(S \otimes_R M) \subset \mathfrak{m}^r(S \otimes_R M) = 0,$$

the second inequality is obvious. ∎

COROLLARY 6.4. *Let $R \to S$ be a flat homomorphism of artinian local rings with the fiber F. Then, we have*

$$\ell\ell(R) + \ell\ell(F) - 1 \leq \ell\ell(S) \leq \ell\ell(R) \cdot \ell\ell(F).$$

Examples can be found in Section 5.

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