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Composition operators on noncommutative Hardy spaces ☆

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Abstract

In this paper we initiate the study of composition operators on the noncommutative Hardy space $H^2_{\mathbf{ball}}$, which is the Hilbert space of all free holomorphic functions of the form

$$f(X_1,\dots,X_n)=\sum_{k=0}^{\infty}\sum_{|\alpha|=k}a_\alpha X_\alpha,\qquad \sum_{\alpha\in\mathbb{F}_n^+}|a_\alpha|^2<1,$$

where the convergence is in the operator norm topology for all $(X_1,\dots,X_n)$ in the noncommutative operatorial ball $[B(\mathcal{H})^n]_1$ and $B(\mathcal{H})$ is the algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. When the symbol φ is a free holomorphic self-map of $[B(\mathcal{H})^n]_1$, we show that the composition operator

$$C_\varphi f:=f\circ\varphi,\qquad f\in H^2_{\mathbf{ball}},$$

is bounded on $H^2_{\mathbf{ball}}$. Several classical results about composition operators (boundedness, norm estimates, spectral properties, compactness, similarity) have free analogues in our noncommutative multivariable setting. The most prominent feature of this paper is the interaction between the noncommutative analytic function theory in the unit ball of $B(\mathcal{H})^n$, the operator algebras generated by the left creation operators on the full Fock space with n generators, and the classical complex function theory in the unit ball of $\mathbb{C}^n$. In a more general setting, we establish basic properties concerning the composition operators acting on Fock spaces associated with noncommutative varieties $\mathcal{V}_{\mathcal{P}_0}(\mathcal{H})\subseteq[B(\mathcal{H})^n]_1$ generated by sets $\mathcal{P}_0$ of noncommutative polynomials in n indeterminates such that $p(0)=0$, $p\in\mathcal{P}_0$. In particular, when $\mathcal{P}_0$ consists of the

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commutators $X_i X_j - X_j X_i$ for $i, j = 1, \dots, n$, we show that many of our results have commutative counterparts for composition operators on the symmetric Fock space and, consequently, on spaces of analytic functions in the unit ball of $\mathbb{C}^n$.

Keywords: Composition operator; Noncommutative Hardy space; Fock space; Creation operator; Free holomorphic function; Free pluriharmonic function; Compact operator; Spectrum; Similarity; Noncommutative variety

Contents

0. Introduction

An important consequence of Littlewood's subordination principle [12,6] is the boundedness of the composition operator C_φ on the Hardy space $H^2(\mathbb{D})$, when $\varphi : \mathbb{D} \to \mathbb{D}$ is an analytic self-map of the open unit disc $\mathbb{D} := \{z \in \mathbb{C}\colon |z| < 1\}$ and $C_\varphi f := f \circ \varphi$. This result was the starting point of the modern theory of composition operators on spaces of analytic functions, which has been developed since the 1960's through the fundamental work of Ryff [42], Nordgren [18,19], Schwartz [46], Shapiro [44], Cowen [2] and many others (see [45,3,1], and the references therein). They answered basic questions about composition operators such as boundedness, compactness, spectra, cyclicity, revealing a beautiful interaction between operator theory and complex function theory. In the multivariable setting, when φ is a holomorphic self-map of the open unit ball

$$\mathbb{B}_n := \left\{z = (z_1, \dots, z_n) \in \mathbb{C}^n\colon \|z\|_2 < 1\right\},$$

the composition operator C_φ is no longer a bounded operator on the Hardy space $H^2(\mathbb{B}_n)$. However, significant work was done concerning the spectra of automorphism-induced composition operators and compact composition operators on $H^2(\mathbb{B}_n)$ by MacCluer [13–15] and others (see [3] and its references). The study of composition operators on the Hardy space $H^2(\mathbb{B}_n)$ is close connected to the several variable function theory in the unit ball of $\mathbb{C}^n$ [41]. There is an extensive literature on composition operators on other spaces of analytic functions in several variables (see [3]).

For the interested reader, we mention two very nice books on composition operators: Shapiro's monograph [45], which is an excellent account of composition operators on $H^2(\mathbb{D})$ and the monograph [3] by Cowen and MacCluer, which is a comprehensive treatment of composition operators on spaces of analytic functions in one or several variables.

It is our hope that the present paper will open a new chapter in the theory of composition operators. The goal is to initiate the study of composition operators on the noncommutative Hardy space $H^2_{\mathbf{ball}}$ (which will be introduced shortly) and, more generally, on subspaces of the full Fock space with n generators associated to noncommutative varieties. The most prominent feature of this paper is the interplay between the noncommutative analytic function theory in the unit ball of $B(\mathcal{H})^n$, the operator algebras generated by the left creation operators $S_1, \ldots, S_n$ on the full Fock space with n generators: the Cuntz–Toeplitz algebra $C^*(S_1, \ldots, S_n)$ [4], the noncommutative disk algebra $\mathcal{A}_n$ and the analytic Toeplitz algebra F_n^∞ [26–29], as well as the classical function theory in the unit ball of $\mathbb{C}^n$ [41]. To present our results we need some notation and preliminaries on free holomorphic functions.

Initiated in [33], the theory of free holomorphic (resp. pluriharmonic) functions on the unit ball of $B(\mathcal{H})^n$, where $B(\mathcal{H})$ is the algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$, has been developed very recently (see [34–39]) in the attempt to provide a framework for the study of arbitrary n-tuples of operators on a Hilbert space. Several classical results from complex analysis and hyperbolic geometry have free analogues in this noncommutative multivariable setting. Related to our work, we mention the papers [8,16,17], and [48], where several aspects of the theory of noncommutative analytic functions are considered in various settings. We recall that the algebra $H_{\mathbf{ball}}$ of free holomorphic functions on the open operatorial n-ball of radius one is defined as the set of all power series $\sum_{\alpha\in\mathbb{F}_n^+} a_\alpha Z_\alpha$ with radius of convergence $\geqslant 1$, i.e., $\{a_\alpha\}_{\alpha\in\mathbb{F}_n^+}$ are complex numbers with $\limsup_{k\to\infty}(\sum_{|\alpha|=k}|a_\alpha|^2)^{1/2k} \leqslant 1$, where $\mathbb{F}_n^+$ is the free semigroup with n generators $g_1, \ldots, g_n$ and the identity g_0. The length of $\alpha \in \mathbb{F}_n^+$ is defined by $|\alpha| := 0$ if $\alpha = g_0$ and $|\alpha| := k$ if $\alpha = g_{i_1}\cdots g_{i_k}$, where $i_1, \ldots, i_k \in \{1, \ldots, n\}$. If $(X_1, \ldots, X_n) \in B(\mathcal{H})^n$, we denote $X_\alpha := X_{i_1}\cdots X_{i_k}$ and $X_{g_0} := I_{\mathcal{H}}$. A free holomorphic function on the open ball

$$\left[B(\mathcal{H})^n\right]_1 := \left\{(X_1, \ldots, X_n) \in B(\mathcal{H})^n \colon \left\|X_1X_n^* + \cdots + X_nX_n^*\right\|^{1/2} < 1\right\},$$

is the representation of an element $f \in H_{\mathbf{ball}}$ on the Hilbert space $\mathcal{H}$, that is, the mapping

$$\left[B(\mathcal{H})^n\right]_1 \ni (X_1, \ldots, X_n) \mapsto f(X_1, \ldots, X_n) := \sum_{k=0}^{\infty}\sum_{|\alpha|=k} a_\alpha X_\alpha \in B(\mathcal{H}),$$

where the convergence is in the operator norm topology. Due to the fact that a free holomorphic function is uniquely determined by its representation on an infinite dimensional Hilbert space, throughout this paper, we identify a free holomorphic function with its representation on a separable infinite dimensional Hilbert space.

A free holomorphic function f on $[B(\mathcal{H})^n]_1$ is bounded if $\|f\|_\infty := \sup\|f(X)\| < \infty$, where the supremum is taken over all $X \in [B(\mathcal{H})^n]_1$ and $\mathcal{H}$ is an infinite dimensional Hilbert space. Let $H^\infty_{\mathbf{ball}}$ be the set of all bounded free holomorphic functions and let $A_{\mathbf{ball}}$ be the set of all elements $f \in H^\infty_{\mathbf{ball}}$ such that the mapping

$$\left[B(\mathcal{H})^n\right]_1 \ni (X_1, \ldots, X_n) \mapsto f(X_1, \ldots, X_n) \in B(\mathcal{H})$$

has a continuous extension to the closed unit ball $[B(\mathcal{H})^n]_1^-$. We showed in [33] that $H^\infty_{\mathbf{ball}}$ and $A_{\mathbf{ball}}$ are Banach algebras under pointwise multiplication and the norm $\|\cdot\|_\infty$, which can be identified, via the noncommutative Poisson transform [30], with the noncommutative analytic Toeplitz algebra F_n^∞ and the noncommutative disc algebra $\mathcal{A}_n$, respectively.

If $f : [B(\mathcal{H})^n]_1 \to B(\mathcal{H})$ and $\varphi : [B(\mathcal{H})^n]_1 \to [B(\mathcal{H})^n]_1$ are free holomorphic functions then $f \circ \varphi$ is a free holomorphic function on $[B(\mathcal{H})^n]_1$ (see [38]), defined by

$$(f \circ \varphi)(X_1, \ldots, X_n) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} a_\alpha \varphi_\alpha(X_1, \ldots, X_n), \quad (X_1, \ldots, X_n) \in \left[B(\mathcal{H})^n\right]_1,$$

where the convergence is in the operator norm topology. The noncommutative Hardy space $H^2_{\mathbf{ball}}$ is the Hilbert space of all free holomorphic functions on $[B(\mathcal{H})^n]_1$ of the form

$$f(X_1, \ldots, X_n) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} a_\alpha X_\alpha, \qquad \sum_{\alpha \in \mathbb{F}_n^+} |a_\alpha|^2 < 1,$$

with the inner product $\langle f, g \rangle := \sum_{k=0}^{\infty} \sum_{|\alpha|=k} a_\alpha \bar{b}_\alpha$, where $g = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} b_\alpha X_\alpha$ is another free holomorphic function in $H^2_{\mathbf{ball}}$. The main question that we answer in this paper is whether $f \circ \varphi \in H^2_{\mathbf{ball}}$ for any $f \in H^2_{\mathbf{ball}}$ and whether the corresponding composition operator is bounded. This will be the starting point in our attempt to develop a theory of compositions operators on noncommutative Hardy spaces. We are interested in extracting properties of the composition operator C_φ (boundedness, spectral properties, compactness) from the operatorial or dynamical properties of the model boundary function $\widetilde{\varphi} := \text{SOT-}\lim_{r \to 1} \varphi(rS_1, \ldots, rS_n) \in F_n^\infty \otimes \mathbb{C}^n$ or the scalar representation of φ, i.e., the holomorphic function $\mathbb{B}_n \ni \lambda \mapsto \varphi(\lambda) \in \mathbb{B}_n$.

In Section 1, we characterize the free holomorphic self-maps of $[B(\mathcal{H})^n]_1$ in terms of the model boundary functions with respect to the left creation operators on the full Fock space $F^2(H_n)$. This will be used, together with the natural identification of $H^2_{\mathbf{ball}}$ with $F^2(H_n)$, to provide a noncommutative Littlewood subordination theorem for the Hardy space $H^2_{\mathbf{ball}}$. More precisely, we show that if φ is a free holomorphic self-map of the ball $[B(\mathcal{H})^n]_1$ such that $\varphi(0) = 0$ and $f \in H^2_{\mathbf{ball}}$, then $f \circ \varphi \in H^2_{\mathbf{ball}}$ and $\|f \circ \varphi\|_2 \leqslant \|f\|_2$.

Section 2 contains the core material on boundedness of compositions operators on the noncommutative Hardy space $H^2_{\mathbf{ball}}$ and estimates for their norms. An important role in our investigation will be played by the characterization of $H^2_{\mathbf{ball}}$ in terms of pluriharmonic majorants [34] and the Herglotz–Riesz type representation for positive free pluriharmonic functions [37]. The key result of this section asserts that if φ is a free holomorphic automorphism of the noncommutative ball $[B(\mathcal{H})^n]_1$ (see [38]), then

$$\left(\frac{1 - \|\varphi(0)\|}{1 + \|\varphi(0)\|}\right)^{1/2} \|f\| \leqslant \|C_\varphi f\| \leqslant \left(\frac{1 + \|\varphi(0)\|}{1 - \|\varphi(0)\|}\right)^{1/2} \|f\|$$

for any $f \in H^2_{\mathbf{ball}}$. Moreover, these inequalities are best possible and we have a formula for the norm of C_φ. Combining this result with the noncommutative Littlewood subordination theorem from the previous section, we obtain the main result which asserts that, for any free holomorphic self-map φ of $[B(\mathcal{H})^n]_1$, the composition $C_\varphi f := f \circ \varphi$ is a bounded operator on $H^2_{\mathbf{ball}}$ and

$$\frac{1}{(1 - \|\varphi(0)\|^2)^{1/2}} \leqslant \|C_\varphi\| \leqslant \left(\frac{1 + \|\varphi(0)\|}{1 - \|\varphi(0)\|}\right)^{1/2}.$$

This leads to an extension of Cowen's [2] one-variable spectral radius formula for composition operators to our noncommutative multivariable setting. More precisely, we obtain

$$r(C_\varphi) = \lim_{k\to\infty} \left(1 - \left\|\varphi^{[k]}(0)\right\|\right)^{-1/2k},$$

where $\varphi^{[k]}$ is the k-iterate of φ. Another consequence of the above-mentioned result is that C_φ is similar to a contraction if and only if there is $\xi \in \mathbb{B}_n$ such that $\varphi(\xi) = \xi$. This will also show that similarity of composition operators on $H^2_{\mathbf{ball}}$ to contractions is equivalent to power (resp. polynomial) boundedness. This is interesting in light of Pisier's [22] famous example of a polynomially bounded operator which is not similar to a contraction, and Paulsen's [20] result that every completely polynomially bounded operator is similar to a contraction. For more information on similarity problems we refer the reader to [21] and [23].

In Section 3, extending the classical result obtained by Wolff [50,51] and MacCluer's version for $\mathbb{B}_n$ (see [13]), we provide a noncommutative analogue of Wolff's theorem for free holomorphic self-maps of $[B(\mathcal{H})^n]_1$. We show that if $\varphi : [B(\mathcal{H})^n]_1 \to [B(\mathcal{H})^n]_1$ is a free holomorphic function such that its scalar representation has no fixed points in $\mathbb{B}_n$, then there is a unique point $\zeta \in \partial\mathbb{B}_n$ (the Denjoy–Wolff point of φ) such that each noncommutative ellipsoid $\mathbf{E}_c(\zeta)$ (see Section 3 for the definition) is mapped into itself by every iterate of the symbol φ. We also show that the spectral radius of a composition operator on $H^2_{\mathbf{ball}}$ is 1 when the symbol is elliptic or parabolic, which extends some of Cowen's results [2] from the single variable case.

In Section 4, we obtain a formula for the adjoint of a composition operator on $H^2_{\mathbf{ball}}$. It is shown that if $\varphi = (\varphi_1, \ldots, \varphi_n)$ is a free holomorphic self-map of the noncommutative ball $[B(\mathcal{H})^n]_1$, then

$$C_\varphi^* f = \sum_{\alpha\in\mathbb{F}_n^+} \langle f, \varphi_\alpha\rangle e_\alpha,$$

where f and $\varphi_1, \ldots, \varphi_n$ are seen as elements of the Fock space $F^2(H_n)$. As a consequence we prove that C_φ is normal if and only if

$$\varphi(X_1, \ldots, X_n) = [X_1 \ldots X_n]A$$

for some normal scalar matrix $A \in M_{n\times n}$ with $\|A\| \leqslant 1$. This leads to characterizations of self-adjoint or unitary composition operators on $H^2_{\mathbf{ball}}$. A nice connection between Fredholm composition operators on $H^2_{\mathbf{ball}}$ and the automorphisms of the open unit ball $\mathbb{B}_n$ is also presented.

In Section 5, we study compact composition operators on the noncommutative Hardy space $H^2_{\mathbf{ball}}$. Using some of Shapiro's arguments from the single variable case (see [44]) in our setting as well as some results from Section 4, we obtain a formula for the essential norm of the composition operator C_φ on $H^2_{\mathbf{ball}}$. In particular, this implies that C_φ is a compact operator if and only if

$$\lim_{k\to\infty} \sup_{f\in H^2_{\mathbf{ball}}, \|f\|_2\leqslant 1} \sum_{|\alpha|\geqslant k} \left|\langle f, \varphi_\alpha\rangle\right|^2 = 0.$$

Moreover, we show that if C_φ is a compact operator on $H^2_{\mathbf{ball}}$, then the scalar representation of φ is a holomorphic self-map of $\mathbb{B}_n$ which

(i) cannot have finite angular derivative at any point of $\partial\mathbb{B}_n$, and
(ii) has exactly one fixed point in the open ball $\mathbb{B}_n$.

As a consequence, we deduce that every compact composition operator on $H^2_{\mathbf{ball}}$ is similar to a contraction. In the end of this section, we prove that the set of compact composition operators on $H^2_{\mathbf{ball}}$ is arcwise connected in the set of all composition operators with respect to the operator norm topology.

In Section 6, we consider a noncommutative multivariable extension of Schröder equation [43] which is used to obtain results concerning the spectrum of composition operators on $H^2_{\mathbf{ball}}$ (see Theorem 6.4). Combining these results with those from Section 5, we determine the spectra of compact composition operators on $H^2_{\mathbf{ball}}$. More precisely, if φ is a free holomorphic self-map of the noncommutative ball $[B(\mathcal{H})^n]_1$ and C_φ is a compact composition operator on $H^2_{\mathbf{ball}}$, then the scalar representation of φ has a unique fix point $\xi \in \mathbb{B}_n$ and the spectrum $\sigma(C_\varphi)$ consists of 0, 1, and all possible products of the eigenvalues of the matrix

$$\big[\langle \psi_i, e_j\rangle\big]_{n\times n},$$

where $\psi = (\psi_1, \dots, \psi_n) := \Phi_\xi \circ \varphi \circ \Phi_\xi$ and Φ_ξ is the involutive free holomorphic automorphism of $[B(\mathcal{H})^n]_1$ associated with ξ, the functions $\psi_1, \dots, \psi_n$ are seen as elements of the Fock space $F^2(H_n)$, and the Hilbert space H_n has $e_1, e_2, \dots, e_n$ as orthonormal basis.

In Section 7, we consider composition operators on Fock spaces associated to noncommutative varieties in unit ball $[B(\mathcal{H})^n]_1$. Given a set $\mathcal{P}_0$ of noncommutative polynomials in n indeterminates such that $p(0) = 0$, $p \in \mathcal{P}_0$, we define a noncommutative variety $\mathcal{V}_{\mathcal{P}_0}(\mathcal{H}) \subseteq [B(\mathcal{H})^n]_1$ by setting

$$\mathcal{V}_{\mathcal{P}_0}(\mathcal{H}) := \big\{(X_1, \dots, X_n) \in \big[B(\mathcal{H})^n\big]_1 \colon\ p(X_1, \dots, X_n) = 0 \text{ for all } p \in \mathcal{P}_0\big\}.$$

According to [32], there is a universal model $(B_1, \dots, B_n)$ associated with the noncommutative variety $\mathcal{V}_{\mathcal{P}_0}(\mathcal{H})$, where $B_i = P_{\mathcal{N}_{\mathcal{P}_0}} S_i|_{\mathcal{N}_{\mathcal{P}_0}}$ and $\mathcal{N}_{\mathcal{P}_0}$ is a subspace of the full Fock space $F^2(H_n)$. Let $F_n^\infty(\mathcal{V}_{\mathcal{P}_0})$ be the w^*-closed algebra generated by $B_1, \dots, B_n$ and the identity. Using the results from Section 2 and the noncommutative commutant lifting theorem [24] (see [47] for the classical case $n = 1$), we show that given any $\widetilde{\psi} \in F_n^\infty(\mathcal{V}_{\mathcal{P}_0}) \otimes \mathbb{C}^n$ with $\|\widetilde{\psi}\| \leqslant 1$, one can define a composition operator $C_{\widetilde{\psi}} : \mathcal{N}_{\mathcal{P}_0} \to \mathcal{N}_{\mathcal{P}_0}$, which turns out to be bounded. Many results from the previous sections have analogues in this more general setting. In particular, if $\mathcal{P}_c := \{X_i X_j - X_j X_i\colon\ i, j = 1, \dots, n\}$, then $\mathcal{N}_{\mathcal{P}_c}$ coincides with the symmetric Fock space. As a consequence, many of our results have commutative counterparts for composition operators on the symmetric Fock space and on spaces of analytic functions in the unit ball of $\mathbb{C}^n$.

1. Noncommutative Littlewood subordination principle

In this section, we characterize the free holomorphic self-maps of the unit ball $[B(\mathcal{H})^n]_1$ in terms of the model boundary functions with respect to the left creation operators on the full Fock space $F^2(H_n)$. This will be used to provide a noncommutative Littlewood subordination theorem for the Hardy space $H^2_{\mathbf{ball}}$.

Let H_n be an n-dimensional complex Hilbert space with orthonormal basis $e_1, e_2, \ldots, e_n$, where $n \in \{1, 2, \ldots\}$. We consider the full Fock space of H_n defined by

$$F^2(H_n) := \mathbb{C}1 \oplus \bigoplus_{k \geqslant 1} H_n^{\otimes k},$$

where $H_n^{\otimes k}$ is the (Hilbert) tensor product of k copies of H_n. We denote $e_\alpha := e_{i_1} \otimes \cdots \otimes e_{i_k}$ if $\alpha = g_{i_1} \cdots g_{i_k}$, where $i_1, \ldots, i_k \in \{1, \ldots, n\}$, and $e_{g_0} := 1$. Note that $\{e_\alpha\}_{\alpha \in \mathbb{F}_n^+}$ is an orthonormal basis for $F^2(H_n)$. Define the left (resp. right) creation operators S_i (resp. R_i), $i = 1, \ldots, n$, acting on $F^2(H_n)$ by setting

$$S_i \varphi := e_i \otimes \varphi, \quad \varphi \in F^2(H_n),$$

(resp. $R_i \varphi := \varphi \otimes e_i$). Note that $S_i R_j = R_j S_i$ for $i, j \in \{1, \ldots, n\}$. The noncommutative disc algebra $\mathcal{A}_n$ (resp. $\mathcal{R}_n$) is the norm closed algebra generated by the left (resp. right) creation operators and the identity. The noncommutative analytic Toeplitz algebra F_n^∞ (resp. R_n^∞) is the weakly closed version of $\mathcal{A}_n$ (resp. $\mathcal{R}_n$). These algebras were introduced in [26] in connection with a noncommutative version of the classical von Neumann inequality [49].

Let $C^*(S_1, \ldots, S_n)$ be the Cuntz–Toeplitz C^*-algebra generated by the left creation operators (see [4]). The noncommutative Poisson transform at $X := (X_1, \ldots, X_n) \in [B(\mathcal{H})^n]_1^-$ is the unital completely contractive linear map $\mathbf{P}_X : C^*(S_1, \ldots, S_n) \to B(\mathcal{H})$ defined by

$$\mathbf{P}_X[f] := \lim_{r \to 1} K_{rX}^* (f \otimes I_\mathcal{H}) K_{rX}, \quad f \in C^*(S_1, \ldots, S_n),$$

where the limit exists in the operator norm topology of $B(\mathcal{H})$. Here, $K_{rX} : \mathcal{H} \to F^2(H_n) \otimes \mathcal{H}$, $0 < r \leqslant 1$, is the noncommutative Poisson kernel defined by

$$K_{rX} h := \sum_{k=0}^\infty \sum_{|\alpha|=k} e_\alpha \otimes r^{|\alpha|} \Delta_{rX} X_\alpha^* h, \quad h \in \mathcal{H},$$

where $\Delta_{rX} := (I_\mathcal{H} - r^2 X_1 X_1^* - \cdots - r^2 X_n X_n^*)^{1/2}$. We recall that

$$\mathbf{P}_X\big[S_\alpha S_\beta^*\big] = X_\alpha X_\beta^*, \quad \alpha, \beta \in \mathbb{F}_n^+.$$

When $X := (X_1, \ldots, X_n)$ is a pure row contraction, i.e. SOT-$\lim_{k \to \infty} \sum_{|\alpha|=k} X_\alpha X_\alpha^* = 0$, then we have

$$\mathbf{P}_X[f] = K_X^* (f \otimes I_\mathcal{H}) K_X, \quad f \in C^*(S_1, \ldots, S_n) \text{ or } f \in F_n^\infty.$$

Under an appropriate modification of the Poisson kernel (e_α becomes $e_{\widetilde{\alpha}}$ where $\widetilde{\alpha} = g_{i_k} \cdots g_{i_k}$ is the reverse of $\alpha = g_{i_1} \cdots g_{i_k} \in \mathbb{F}_n^+$), similar results hold for $C^*(R_1, \ldots, R_n)$ of R_n^∞. For simplicity, we use the same notation for the noncommutative Poisson transform. We refer to [30,31,35] for more on noncommutative Poisson transforms on C^*-algebras generated by isometries.

According to [33] and [37], the noncommutative Hardy space $H_{\mathbf{ball}}^\infty$ (see the introduction) can be identified with the noncommutative analytic Toeplitz algebra F_n^∞. More precisely, a bounded free holomorphic function ψ on $[B(\mathcal{H})^n]_1$ is uniquely determined by its (*model*) *boundary function* $\widetilde{\psi}(S_1,\dots,S_n)\in F_n^\infty$ defined by

$$\widetilde{\psi}=\widetilde{\psi}(S_1,\dots,S_n):=\text{SOT-}\lim_{r\to 1}\psi(rS_1,\dots,rS_n).$$

Moreover, ψ is the noncommutative Poisson transform of $\widetilde{\psi}(S_1,\dots,S_n)$ at $X:=(X_1,\dots,X_n)\in[B(\mathcal{H})^n]_1$, i.e.,

$$\psi(X_1,\dots,X_n)=\mathbf{P}_X\big[\widetilde{\psi}(S_1,\dots,S_n)\big].$$

Similar results hold for bounded free holomorphic functions on the noncommutative ball $[B(\mathcal{H})^n]_1$ with operator-valued coefficients. There are also versions of these results when the boundary function is taken with respect to the right creation operators $R_1,\dots,R_n$.

Throughout this paper, we deal with free holomorphic self-maps of the unit ball $[B(\mathcal{H})^n]_1$. The following results gives us, in particular, a characterization of these maps in terms of the model boundary functions with respect to the left creation operators on the full Fock space $F^2(H_n)$. For simplicity, $[X_1,\dots,X_n]$ denotes either the n-tuple $(X_1,\dots,X_n)\in B(\mathcal{H})^n$ or the operator row matrix $[X_1\ \dots\ X_n]$ acting from $\mathcal{H}^{(n)}$, the direct sum of n copies of a Hilbert space $\mathcal{H}$, to $\mathcal{H}$.

Theorem 1.1. *Let $\varphi:[B(\mathcal{H})^n]_1\to[B(\mathcal{H})^m]_1^-$ be a free holomorphic function. Then the following statements hold.*

(i) *Either $\varphi([B(\mathcal{H})^n]_1)\subseteq[B(\mathcal{H})^m]_1$ or there exists $\zeta\in\partial\mathbb{B}_m$ such that $\varphi(X)=\zeta$ for all $X\in[B(\mathcal{H})^n]_1$.*
(ii) *φ is constant if and only if $\|\varphi(0)\|=\|\varphi\|_\infty$.*
(iii) *If φ is non-constant and $\varphi_r(X):=\varphi(rX)$, $X\in[B(\mathcal{H})^n]_1$, then the map $[0,1)\ni r\mapsto\|\varphi_r\|_\infty$ is strictly increasing.*
(iv) *If $\widetilde{\varphi}$ is the boundary function of φ with respect to $S_1,\dots,S_n$, then $\varphi([B(\mathcal{H})^n]_1)\subseteq[B(\mathcal{H})^m]_1$ if and only if either $\widetilde{\varphi}=\zeta I$ for some $\zeta\in\mathbb{B}_n$ or $\widetilde{\varphi}$ is non-constant with $\|\widetilde{\varphi}\|\leqslant 1$.*

Proof. If $\|\varphi\|_\infty<1$, then (i) holds. Assume that $\|\varphi\|_\infty=1$. In this case, if $\|\varphi(0)\|<1$ then, according to the maximum principle for free holomorphic functions (see Proposition 5.2 from [38]), we have $\|\varphi(X)\|<1$ for all $X\in[B(\mathcal{H})^n]_1$. It remains to consider the case when $\|\varphi(0)\|=1$. Set $\zeta=[\zeta_1,\dots,\zeta_m]:=\varphi(0)\in\partial\mathbb{B}_m$ and let $U\in M_{m\times m}$ be a unitary matrix such that $[\zeta_1,\dots,\zeta_m]U=\xi_1:=[1,0,\dots,0]\in\partial\mathbb{B}_m$. Let $\varphi_U(X):=[X_1,\dots,X_m]U$ and note that $g:=\varphi_U\circ\varphi:[B(\mathcal{H})^n]_1\to[B(\mathcal{H})^m]_1^-$ is a free holomorphic function with $g(0)=\xi_1$. Setting $g=(g_1,\dots,g_m)$, we deduce that g_i are free holomorphic functions with $g_1(0)=1$ and $g_i(0)=0$ if $i=2,\dots,m$. Applying Theorem 5.1 from [38] to g_1, we deduce that $g_1(X)=1$ for all $X\in[B(\mathcal{H})^n]_1$. Hence $g_2=\dots=g_m=0$. This implies that $\varphi(X)=\zeta$ for all $X\in[B(\mathcal{H})^n]_1$, and completes the proof of item (i). Since the direct implication in item (ii) is obvious, we assume that $\|\varphi(0)\|=\|\varphi\|_\infty$ and $\|\varphi\|_\infty=1$. The rest of the proof of (ii) is contained in the proof of item (i).

To prove item (iii), assume that φ is non-constant. Due to part (ii), we must have $\|\varphi(0)\|<\|\varphi\|_\infty$. Using again Proposition 5.2 from [38], we have $\|\varphi(X)\|<\|\varphi\|_\infty$ for all $X\in[B(\mathcal{H})^n]_1$. Let $0\leqslant r_1<r_2<1$. We recall that, if $r\in[0,1)$, then the boundary function $\widetilde{\varphi}_r$ is in $\mathcal{A}_n\otimes M_{1\times m}$, where $\mathcal{A}_n$ is the noncommutative disc algebra and $\|\varphi_r\|_\infty=\|\widetilde{\varphi}_r\|=\|\varphi_r(S_1,\dots,rS_n)\|$. Using the noncommutative von Neumann inequality (see [26]) and applying the above-mentioned result to φ_{r_2} and $(X_1,\ldots,X_n):=(\frac{r_1}{r_2}S_1,\ldots,\frac{r_1}{r_2}S_n)$, we obtain

$$\|\varphi_{r_1}\|_\infty=\big\|\varphi_{r_1}(S_1,\ldots,S_n)\big\|=\left\|\varphi_{r_2}\left(\frac{r_1}{r_2}S_1,\ldots,\frac{r_1}{r_2}S_n\right)\right\|<\big\|\varphi_{r_2}(S_1,\ldots,S_n)\big\|=\|\varphi_{r_2}\|_\infty,$$

which shows that (iii) holds.

Now we prove (iv). If $\varphi([B(\mathcal{H})^n]_1)\subseteq[B(\mathcal{H})^m]_1$, then $\|\widetilde{\varphi}\|=\|\varphi\|_\infty\leqslant 1$ and the result follows. Conversely, assume that $\|\widetilde{\varphi}\|\leqslant 1$ and $\widetilde{\varphi}$ is not of the form ζI for some $\zeta\in\mathbb{B}_n$. Then φ is not a constant and due to (ii) we have $\|\varphi(0)\|<\|\varphi\|_\infty$. Using now item (iii), we deduce that the map $[0,1)\ni r\mapsto\|\varphi_r\|_\infty$ is strictly increasing. If $X:=(X_1,\ldots,X_n)\in[B(\mathcal{H})^n]_1$, then there is $r\in[0,1)$ such that $\|X\|<r$. Consequently, due to the noncommutative von Neumann inequality, we have

$$\big\|\varphi(X_1,\ldots,X_n)\big\|\leqslant\big\|\varphi(rS_1,\ldots,rS_n)\big\|=\|\varphi_r\|_\infty<1.$$

The proof is complete. □

Note that if $f\in H_{\mathbf{ball}}$, then $f\in H^2_{\mathbf{ball}}$ if and only $\sup_{r\in[0,1)}\|f(rS_1,\ldots,rS_n)1\|<\infty$. Moreover, in this case, we have

$$\|f\|_2=\lim_{r\to 1}\big\|f(rS_1,\ldots,rS_n)1\big\|=\sup_{r\in[0,1)}\big\|f(rS_1,\ldots,rS_n)1\big\|.$$

If $f=\sum_{k=0}^\infty\sum_{|\alpha|=k}a_\alpha X_\alpha$ and $g=\sum_{k=0}^\infty\sum_{|\alpha|=k}b_\alpha X_\alpha$ are in $H^2_{\mathbf{ball}}$, then

$$\begin{aligned}\langle f,g\rangle&=\lim_{r\to 1}\big\langle f(rS_1,\ldots,rS_n)1,g(rS_1,\ldots,rS_n)1\big\rangle_{F^2(H_n)}\\&=\left\langle\sum_{\alpha\in\mathbb{F}_n^+}a_\alpha e_\alpha,\sum_{\alpha\in\mathbb{F}_n^+}b_\alpha e_\alpha\right\rangle_{F^2(H_n)}.\end{aligned}$$

Consequently, the noncommutative Hardy space $H^2_{\mathbf{ball}}$ can be identified with the full Fock space $F^2(H_n)$, via the unitary operator $\mathcal{U}:H^2_{\mathbf{ball}}\to F^2(H_n)$ defined by the mapping

$$H^2_{\mathbf{ball}}\ni\sum_{k=0}^\infty\sum_{|\alpha|=k}a_\alpha X_\alpha\mapsto\sum_{k=0}^\infty\sum_{|\alpha|=k}a_\alpha e_\alpha\in F^2(H_n).$$

This identification will be used throughout the paper whenever necessary. We recall from [38] that if $f:[B(\mathcal{H})^n]_1\to B(\mathcal{H})$ and $\varphi:[B(\mathcal{H})^n]_1\to[B(\mathcal{H})^n]_1$ are free holomorphic functions then $f\circ\varphi$ is a free holomorphic function on $[B(\mathcal{H})^n]_1$ defined by

$$(f\circ\varphi)(X_1,\ldots,X_n)=\sum_{k=0}^\infty\sum_{|\alpha|=k}a_\alpha\varphi_\alpha(X_1,\ldots,X_n),\quad(X_1,\ldots,X_n)\in\big[B(\mathcal{H})^n\big]_1,$$

where the convergence is in the operator norm topology.

We can prove now the following noncommutative Littlewood subordination theorem for the Hardy space $H^2_{\mathbf{ball}}$, which will play an important role in this paper.

Theorem 1.2. *Let φ be a free holomorphic self-map of the ball $[B(\mathcal{H})^n]_1$ such that $\varphi(0)=0$, and let $f\in H^2_{\mathbf{ball}}$. Then $f\circ\varphi\in H^2_{\mathbf{ball}}$ and $\|f\circ\varphi\|_2\leqslant\|f\|_2$.*

Proof. Let $\varphi:=(\varphi_1,\dots,\varphi_n)$ be a free holomorphic self-map of the ball $[B(\mathcal{H})^n]_1$ such that $\varphi(0)=0$, and let $\widetilde{\varphi}=(\widetilde{\varphi}_1,\dots,\widetilde{\varphi}_n)\in F_n^\infty\otimes\mathbb{C}^n$ be the model boundary function with respect to the left creation operators $S_1,\dots,S_n$. Thus $\widetilde{\varphi}_i:=\text{SOT-}\lim_{r\to 1}\varphi_i(rS_1,\dots,rS_n)$ for $i=1,\dots,n$. Let $\mathcal{P}_n$ be the set of all polynomials in $F^2(H_n)$ and define $C_{\widetilde{\varphi}}:\mathcal{P}_n\to F^2(H_n)$ by setting

$$C_{\widetilde{\varphi}}\biggl(\sum_{|\alpha|\leqslant m}a_\alpha e_\alpha\biggr):=\sum_{|\alpha|\leqslant m}a_\alpha\widetilde{\varphi}_\alpha(1).$$

If $q:=\sum_{|\alpha|\leqslant m}a_\alpha X_\alpha$ is a polynomial in $H^2_{\mathbf{ball}}$, then $p:=\mathcal{U}q=\sum_{|\alpha|\leqslant m}a_\alpha e_\alpha$ is a polynomial in $F^2(H_n)$. Note that $p=p(0)+\sum_{i=1}^n S_i(S_i^*p)$, where $p(0)=P_{\mathbb{C}}p=a_0:=a_{g_0}$. Hence, we deduce that

$$C_{\widetilde{\varphi}}p=a_0+\sum_{i=1}^n\widetilde{\varphi}_i C_{\widetilde{\varphi}}\bigl(S_i^*p\bigr).$$

Since $\varphi(0)=0$, the vector $\sum_{i=1}^n\widetilde{\varphi}_i C_{\widetilde{\varphi}}(S_i^*p)$ is orthogonal to the constants in $F^2(H_n)$. Consequently, using the fact that $[\widetilde{\varphi}_1,\dots,\widetilde{\varphi}_n]$ is a row contraction, we have

$$\begin{aligned}\|C_{\widetilde{\varphi}}p\|_2^2&=|a_0|^2+\Biggl\|\sum_{i=1}^n\widetilde{\varphi}_i C_{\widetilde{\varphi}}\bigl(S_i^*p\bigr)\Biggr\|^2\\&\leqslant|a_0|^2+\Biggl\|\bigoplus_{i=1}^n C_{\widetilde{\varphi}}\bigl(S_i^*p\bigr)\Biggr\|^2.\end{aligned}$$

Note that, for each $i=1,\dots,n$, we have

$$C_{\widetilde{\varphi}}\bigl(S_i^*p\bigr)=\bigl(S_i^*p\bigr)(0)+\sum_{j=1}^n\widetilde{\varphi}_j C_{\widetilde{\varphi}}\bigl(S_j^*S_i^*p\bigr).$$

Hence, using again that $\varphi(0)=0$ and that $[\widetilde{\varphi}_1,\dots,\widetilde{\varphi}_n]$ is a row contraction, we deduce that

$$\begin{aligned}\Biggl\|\bigoplus_{i=1}^n C_{\widetilde{\varphi}}\bigl(S_i^*p\bigr)\Biggr\|^2&=\Biggl\|\bigoplus_{i=1}^n\bigl(S_i^*p\bigr)(0)\Biggr\|^2+\Biggl\|\bigoplus_{i=1}^n\Biggl(\sum_{j=1}^n\widetilde{\varphi}_j C_{\widetilde{\varphi}}\bigl(S_j^*S_i^*p\bigr)\Biggr)\Biggr\|^2\\&\leqslant\sum_{|\alpha|=1}|a_\alpha|^2+\sum_{i=1}^n\Biggl\|\sum_{j=1}^n\widetilde{\varphi}_j C_{\widetilde{\varphi}}\bigl(S_j^*S_i^*p\bigr)\Biggr\|^2\\&\leqslant\sum_{|\alpha|=1}|a_\alpha|^2+\Biggl\|\bigoplus_{|\beta|=2}C_{\widetilde{\varphi}}\bigl(S_\beta^*p\bigr)\Biggr\|^2.\end{aligned}$$

Similarly, for any $k \in \{1, \dots, m+1\}$, we obtain

$$\left\| \bigoplus_{|\beta|=k-1} C_{\widetilde{\varphi}}\big(S_\beta^* p\big) \right\|^2 \leqslant \sum_{|\alpha|=k-1} |a_\alpha|^2 + \left\| \bigoplus_{|\beta|=k} C_{\widetilde{\varphi}}\big(S_\beta^* p\big) \right\|^2.$$

Using these relations and the fact that $S_\gamma^* p = 0$ for $|\gamma| \geqslant m+1$, we obtain

$$\|C_{\widetilde{\varphi}} p\|_2^2 \leqslant \sum_{|\alpha| \leqslant m} |a_\alpha|^2 = \|p\|_2^2.$$

Since $\mathcal{U} C_\varphi \mathcal{U}^{-1} p = C_{\widetilde{\varphi}} p$, we deduce that

$$\|C_\varphi q\|_2 \leqslant \|q\|_2 \quad \text{for any polynomial } q \in H^2_{\mathbf{ball}}. \tag{1.1}$$

Now, we prove that $f \circ \varphi$ is in $H^2_{\mathbf{ball}}$ for any $f \in H^2_{\mathbf{ball}}$ and $\|f \circ \varphi\|_2 \leqslant \|f\|_2$. Let $f(X_1, \dots, X_n) = \sum_{k=0}^\infty \sum_{|\alpha|=k} c_\alpha X_\alpha$ be a free holomorphic function in $H^2_{\mathbf{ball}}$. Then $f \circ \varphi$ is a free holomorphic function on $[B(\mathcal{H})^n]_1$, defined by

$$(f \circ \varphi)(X_1, \dots, X_n) = \sum_{k=0}^\infty \sum_{|\alpha|=k} c_\alpha \varphi_\alpha(X_1, \dots, X_n), \quad (X_1, \dots, X_n) \in \big[B(\mathcal{H})^n\big]_1,$$

where the convergence is in the operator norm topology. In particular, we have

$$(f \circ \varphi)(rS_1, \dots, rS_n)1 = \sum_{k=0}^\infty \sum_{|\alpha|=k} c_\alpha \varphi_\alpha(rS_1, \dots, rS_n)1, \tag{1.2}$$

where the convergence is in $F^2(H_n)$. On the other hand, setting $p_m(X_1, \dots, X_n) := \sum_{k=0}^m \sum_{|\alpha|=k} c_\alpha X_\alpha$, we have $p_m \to f$ in $H^2_{\mathbf{ball}}$ as $m \to \infty$. Therefore, $\{p_m\}$ is a Cauchy sequence in $H^2_{\mathbf{ball}}$. Due to relation (1.1), we have

$$\|p_m \circ \varphi - p_k \circ \varphi\|_2 \leqslant \|p_m - p_k\|_2, \quad m, k \in \mathbb{N}.$$

Hence, $\{p_m \circ \varphi\}$ is a Cauchy sequence in $H^2_{\mathbf{ball}}$ and, consequently, there is $g \in H^2_{\mathbf{ball}}$ such that $p_m \circ \varphi \to g$ in $H^2_{\mathbf{ball}}$. Hence, for each $r \in [0, 1)$, we have

$$\lim_{m \to \infty} (p_m \circ \varphi)(rS_1, \dots, rS_n)1 = g(rS_1, \dots, rS_n)1.$$

Combining this relation with (1.2), we get

$$g(rS_1, \dots, rS_n)1 = (f \circ \varphi)(rS_1, \dots, rS_n)1, \quad r \in [0, 1).$$

Since $f \circ \varphi$ and g are free holomorphic functions, we deduce that $f \circ \varphi = g \in H^2_{\mathbf{ball}}$. Now, since $p_m \circ \varphi \to f \circ \varphi$ in $H^2_{\mathbf{ball}}$, relation (1.1) implies $\|f \circ \varphi\|_2 \leqslant \|f\|_2$ for any $f \in H^2_{\mathbf{ball}}$. The proof is complete. □

If in addition to the hypothesis of Theorem 1.2, we assume that φ is inner, i.e. the boundary function $\widetilde{\varphi}$ is an isometry, then we can prove the following result.

Theorem 1.3. *Let φ be an inner free holomorphic self-map of the ball $[B(\mathcal{H})^n]_1$ such that $\varphi(0)=0$. Then the composition operator C_φ is an isometry on $H^2_{\mathbf{ball}}$.*

Proof. Let $\widetilde{\varphi} := [\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n]$ be the boundary function of φ with respect to the left creation opeartors. Note that due to the fact that $\varphi(0)=0$, we have $\langle 1, \widetilde{\varphi}_\alpha 1\rangle = 0$ for any $\alpha \in \mathbb{F}_n^+$ with $|\alpha| \geqslant 1$. On the other hand, since $[\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n]$ is an isometry, we have $\widetilde{\varphi}_i^* \widetilde{\varphi}_j = \delta_{ij} I_{F^2(H_n)}$ for $i, j \in \{1, \ldots, n\}$. Consequently,

$$\begin{aligned}
\langle \varphi_\alpha, \varphi_\beta \rangle_{H^2_{\mathbf{ball}}} &= \langle \widetilde{\varphi}_\alpha 1, \widetilde{\varphi}_\beta 1\rangle \\
&= \begin{cases} \langle \widetilde{\varphi}_\gamma 1, 1\rangle & \text{if } \alpha = \beta\gamma, \\ 1 & \text{if } \alpha = \beta, \\ \langle 1, \widetilde{\varphi}_\gamma 1\rangle & \text{if } \beta = \alpha\gamma \end{cases} \\
&= \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta. \end{cases}
\end{aligned}$$

This shows that $\{\varphi_\alpha\}_{\alpha\in\mathbb{F}_n^+}$ is an orthonormal set in $H^2_{\mathbf{ball}}$. If $f = \sum_{k=0}^\infty \sum_{|\alpha|=k} c_\alpha X_\alpha$ is in $H^2_{\mathbf{ball}}$, then setting $p_m(X_1, \ldots, X_n) := \sum_{k=0}^m \sum_{|\alpha|=k} c_\alpha X_\alpha$, we have $p_m \to f$ in $H^2_{\mathbf{ball}}$, as $m \to \infty$. Note that

$$\|p_m \circ \varphi\|_2^2 = \left\langle \sum_{k=0}^m \sum_{|\alpha|=k} c_\alpha \varphi_\alpha, \sum_{k=0}^m \sum_{|\beta|=k} c_\beta \varphi_\beta \right\rangle = \sum_{k=0}^m \sum_{|\alpha|=k} |c_\alpha|^2 = \|p_m\|_2^2. \tag{1.3}$$

Consequently, $\{p_m \circ \varphi\}$ is a Cauchy sequence in $H^2_{\mathbf{ball}}$ and there is $g \in H^2_{\mathbf{ball}}$ such that $p_m \circ \varphi \to g$ in $H^2_{\mathbf{ball}}$. Hence, we deduce that

$$g(rS_1, \ldots, rS_n)1 = \lim_{m\to\infty} (p_m \circ \varphi)(rS_1, \ldots, rS_n)1 = (f \circ \varphi)(rS_1, \ldots, rS_n)1, \quad r \in [0, 1).$$

Since $f \circ \varphi$ and g are free holomorphic functions, the identity theorem for free holomorphic functions implies $f \circ \varphi = g$. Therefore, relation (1.3) implies that C_φ is an isometry and the proof is complete. □

2. Composition operators on the noncommutative Hardy space $H^2_{\mathbf{ball}}$

This section contains the core material on the boundedness of compositions operators on the noncommutative Hardy space $H^2_{\mathbf{ball}}$ and the estimates of their norms. We also characterize the similarity of composition operators on $H^2_{\mathbf{ball}}$ to contractions.

Let θ be an analytic function on the open disc $\mathbb{D}$. It is well known that the map $\varphi : \mathbb{D} \to \mathbb{R}^+$ defined by $\varphi(\lambda) := |\theta(\lambda)|^2$ is subharmonic. A classical result on harmonic majorants (see Section 2.6 in [6]) states that θ is in the Hardy space $H^2(\mathbb{D})$ if and only if φ has a harmonic majorant. Moreover, the least harmonic majorant of φ is given by the Herglotz–Riesz [9,40] formula

$$h(\lambda) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+\lambda}{e^{it}-\lambda}\left|\theta\left(e^{it}\right)\right|^2 dt, \quad \lambda \in \mathbb{D}.$$

In [34], we obtained free analogues of these results. Since these results play an important role in our investigation we shall recall them.

We say that a map $h : [B(\mathcal{H})^n]_1 \to B(\mathcal{H})$ is a self-adjoint *free pluriharmonic function* on $[B(\mathcal{H})^n]_1$ if $h = \Re f := \frac{1}{2}(f^* + f)$ for some free holomorphic function f on $[B(\mathcal{H})^n]_1$. An arbitrary free pluriharmonic function is a linear combination of self-adjoint free pluriharmonic functions. A *pluriharmonic curve* in $C^*(S_1, \ldots, S_n)$ is a map $\varphi : [0, 1) \to \overline{\mathcal{A}_n + \mathcal{A}_n}^{\|\cdot\|}$ satisfying the Poisson mean value property, i.e.,

$$\varphi(r) = \mathbf{P}_{\frac{r}{t}S}\left[\varphi(t)\right] \quad \text{for } 0 \leqslant r < t < 1,$$

where $S := (S_1, \ldots, S_n)$ and $\mathbf{P}_X[u]$ is the noncommutative Poisson transform of u at X. According to [37], there exists a one-to-one correspondence $u \mapsto \varphi$ between the set of all free pluriharmonic functions on the noncommutative ball $[B(\mathcal{H})^n]_1$, and the set of all pluriharmonic curves $\varphi : [0, 1) \to \overline{\mathcal{A}_n^* + \mathcal{A}_n}^{\|\cdot\|}$. Moreover, we have

$$u(X) = \mathbf{P}_{\frac{1}{r}X}\left[\varphi(r)\right] \quad \text{for } X \in \left[B(\mathcal{H})^n\right]_r \text{ and } r \in (0, 1),$$

and $\varphi(r) = u(rS_1, \ldots, rS_n)$ if $r \in [0, 1)$. We say that a map $\psi : [0, 1) \to \overline{\mathcal{A}_n + \mathcal{A}_n}^{\|\cdot\|}$ is self-adjoint if $\psi(r) = \psi(r)^*$ for $r \in [0, 1)$. We call ψ a *sub-pluriharmonic* curve provided that for each $\gamma \in (0, 1)$ and each self-adjoint pluriharmonic curve $\varphi : [0, \gamma] \to \overline{\mathcal{A}_n + \mathcal{A}_n}^{\|\cdot\|}$, if $\psi(\gamma) \leqslant \varphi(\gamma)$, then $\psi(r) \leqslant \varphi(r)$ for any $r \in [0, \gamma]$. We proved that a self-adjoint map $g : [0, 1) \to \overline{\mathcal{A}_n^* + \mathcal{A}_n}^{\|\cdot\|}$ is a sub-pluriharmonic curve in $C^*(S_1, \ldots, S_n)$ if and only if

$$g(r) \leqslant \mathbf{P}_{\frac{r}{\gamma}S}\left[g(\gamma)\right] \quad \text{for } 0 \leqslant r < \gamma < 1.$$

We obtained a characterization for the class of all sub-pluriharmonic curves that admit free pluriharmonic majorants, and proved the existence of the least pluriharmonic majorant. We mention that all these results can be written for sub-pluriharmonic curves in $C^*(R_1, \ldots, R_n)$, where $R_1, \ldots, R_n$ are the right creation operators on the full Fock space.

In [34], we showed that, for any free holomorphic function Θ on the noncommutative ball $[B(\mathcal{H})^n]_1$, the mapping

$$\varphi : [0, 1) \to C^*(R_1, \ldots, R_n), \qquad \varphi(r) = \Theta(rR_1, \ldots, rR_n)^* \Theta(rR_1, \ldots, rR_n),$$

is a sub-pluriharmonic curve in the Cuntz–Toeplitz algebra generated by the right creation operators $R_1, \ldots, R_n$. We proved that a free holomorphic function Θ is in the noncommutative Hardy space $H^2_{\mathbf{ball}}$ if and only if φ has a pluriharmonic majorant. In this case, the least pluriharmonic majorant ψ for φ is given by $\psi(r) := \Re W(rR_1, \ldots rR_n)$, $r \in [0, 1)$, where W is the free holomorphic function having the Herglotz–Riesz type representation

$$W(X_1,\dots,X_n)=(\mu_\theta\otimes\mathrm{id})\left[\left(I+\sum_{i=1}^n R_i^*\otimes X_i\right)\left(I-\sum_{i=1}^n R_i^*\otimes X_i\right)^{-1}\right]$$

for $(X_1,\dots,X_n)\in[B(\mathcal{H})^n]_1$, where $\mu_\theta:\mathcal{R}_n^*+\mathcal{R}_n\to\mathbb{C}$ is a positive linear map uniquely determined by the function Θ.

Now, we need to recall from [38] some basic facts concerning the free holomorphic automorphisms of the noncommutative ball $[B(\mathcal{H})^n]_1$. A map $\varphi:[B(\mathcal{H})^n]_1\to[B(\mathcal{H})^n]_1$ is called free biholomorphic if φ is free homolorphic, one-to-one and onto, and has free holomorphic inverse. The automorphism group of $[B(\mathcal{H})^n]_1$, denoted by $Aut([B(\mathcal{H})^n]_1)$, consists of all free biholomorphic functions of $[B(\mathcal{H})^n]_1$. It is clear that $Aut([B(\mathcal{H})^n]_1)$ is a group with respect to the composition of free holomorphic functions. We used the theory of noncommutative characteristic functions for row contractions [25] to find all the involutive free holomorphic automorphisms of $[B(\mathcal{H})^n]_1$, which turned out to be of the form

$$\Phi_\lambda(X_1,\dots,X_n)=-\Theta_\lambda(X_1,\dots,X_n),\quad (X_1,\dots,X_n)\in\left[B(\mathcal{H})^n\right]_1,$$

for some $\lambda=[\lambda_1,\dots,\lambda_n]\in\mathbb{B}_n$, where Θ_λ is the characteristic function of the row contraction λ, acting as an operator from $\mathbb{C}^n$ to $\mathbb{C}$. We recall that the characteristic function of the row contraction λ is the boundary function (with respect to $R_1,\dots,R_n$)

$$\widetilde{\Theta}_\lambda:=\text{SOT-}\lim_{r\to 1}\Theta_\lambda(rR_1,\dots,rR_n)$$

of the free holomorphic function $\Theta_\lambda:[B(\mathcal{H})^n]_1\to[B(\mathcal{H})^n]_1$ given by

$$\Theta_\lambda(X_1,\dots,X_n):=-\lambda+\Delta_\lambda\left(I_{\mathcal{H}}-\sum_{i=1}^n\bar{\lambda}_iX_i\right)^{-1}[X_1,\dots,X_n]\Delta_{\lambda^*}$$

for $(X_1,\dots,X_n)\in[B(\mathcal{H})^n]_1$, where $\Delta_\lambda=(1-\|\lambda\|_2^2)^{1/2}I_{\mathbb{C}}$ and $\Delta_{\lambda^*}=(I_{\mathcal{K}}-\lambda^*\lambda)^{1/2}$. For simplicity, we used the notation $\lambda:=[\lambda_1 I_{\mathcal{G}},\dots,\lambda_n I_{\mathcal{G}}]$ for the row contraction acting from $\mathcal{G}^{(n)}$ to $\mathcal{G}$, where $\mathcal{G}$ is a Hilbert space.

In [38], we proved that if $\lambda:=(\lambda_1,\dots,\lambda_n)\in\mathbb{B}_n\backslash\{0\}$ and $\gamma:=\frac{1}{\|\lambda\|_2}$, then $\Phi_\lambda:=-\Theta_\lambda$ is a free holomorphic function on $[B(\mathcal{H})^n]_\gamma$ which has the following properties:

(i) $\Phi_\lambda(0)=\lambda$ and $\Phi_\lambda(\lambda)=0$;

(ii) the identities

$$\begin{aligned}I_{\mathcal{H}}-\Phi_\lambda(X)\Phi_\lambda(Y)^*&=\Delta_\lambda\big(I-X\lambda^*\big)^{-1}\big(I-XY^*\big)\big(I-\lambda Y^*\big)^{-1}\Delta_\lambda,\\ I_{\mathcal{H}\otimes\mathbb{C}^n}-\Phi_\lambda(X)^*\Phi_\lambda(Y)&=\Delta_{\lambda^*}\big(I-X^*\lambda\big)^{-1}\big(I-X^*Y\big)\big(I-\lambda^*Y\big)^{-1}\Delta_{\lambda^*},\end{aligned}\tag{2.1}$$

hold for all X and Y in $[B(\mathcal{H})^n]_\gamma$;

(iii) Φ_λ is an involution, i.e., $\Phi_\lambda(\Phi_\lambda(X))=X$ for any $X\in[B(\mathcal{H})^n]_\gamma$;

(iv) Φ_λ is a free holomorphic automorphism of the noncommutative unit ball $[B(\mathcal{H})^n]_1$;

(v) Φ_λ is a homeomorphism of $[B(\mathcal{H})^n]_1^-$ onto $[B(\mathcal{H})^n]_1^-$;

(vi) Φ_λ is inner, i.e., the boundary function $\widetilde{\Phi}_\lambda$ is an isometry.

Moreover, we determined all the free holomorphic automorphisms of the noncommutative ball $[B(\mathcal{H})^n]_1$ by showing that if $\Phi \in Aut([B(\mathcal{H})^n]_1)$ and $\lambda := \Phi(0)$, then there is a unitary operator U on $\mathbb{C}^n$ such that

$$\Phi = \Phi_\lambda \circ \Phi_U,$$

where

$$\Phi_U(X_1, \ldots X_n) := [X_1, \ldots, X_n]U, \quad (X_1, \ldots, X_n) \in \left[B(\mathcal{H})^n\right]_1.$$

We have now all the ingredients to prove the key result of this section.

Theorem 2.1. *If φ is a free holomorphic automorphism of the noncommutative ball $[B(\mathcal{H})^n]_1$, then $C_\varphi f \in H^2_{\mathbf{ball}}$ for all $f \in H^2_{\mathbf{ball}}$, and*

$$\left(\frac{1 - \|\varphi(0)\|}{1 + \|\varphi(0)\|}\right)^{1/2} \|f\| \leqslant \|C_\varphi f\| \leqslant \left(\frac{1 + \|\varphi(0)\|}{1 - \|\varphi(0)\|}\right)^{1/2} \|f\|$$

for all $f \in H^2_{\mathbf{ball}}$. Moreover, these inequalities are best possible and

$$\|C_\varphi\| = \left(\frac{1 + \|\varphi(0)\|}{1 - \|\varphi(0)\|}\right)^{1/2}.$$

Proof. Let $\varphi := (\varphi_1, \ldots, \varphi_n)$ be an inner free holomorphic self-map of the noncommutative ball $[B(\mathcal{H})^n]_1$. Then the boundary function with respect to the right creation operators $R_1, \ldots, R_n$, i.e.,

$$\widetilde{\varphi} := (\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n), \quad \text{where } \widetilde{\varphi}_i := \text{SOT-}\lim_{r \to 1} \varphi_i(rR_1, \ldots, rR_n),$$

is an isometry. Consequently, $\widetilde{\varphi}_i^* \widetilde{\varphi}_j = \delta_{ij} I_{F^2(H_n)}$ for $i, j \in \{1, \ldots, n\}$. Recall that $R_1, \ldots, R_n$ are isometries with orthogonal ranges, so $R_i^* R_j = \delta_{ij} I_{F^2(H_n)}$ for $i, j \in \{1, \ldots, n\}$. Consequently, we have

$$R_\alpha^* R_\beta = \begin{cases} R_\gamma & \text{if } \beta = \alpha\gamma, \\ I & \text{if } \alpha = \beta, \\ R_\gamma^* & \text{if } \alpha = \beta\gamma, \end{cases} \quad \text{and} \quad \widetilde{\varphi}_\alpha^* \widetilde{\varphi}_\beta = \begin{cases} \widetilde{\varphi}_\gamma & \text{if } \beta = \alpha\gamma, \\ I & \text{if } \alpha = \beta, \\ \widetilde{\varphi}_\gamma^* & \text{if } \alpha = \beta\gamma. \end{cases}$$

Fix a noncommutative polynomial $p(X_1, \ldots, X_n) := \sum_{|\alpha| \leqslant m} a_\alpha r^{|\alpha|} X_\alpha$. Note that, using the above-mentioned relations and applying the noncommutative Poisson transform (with respect to $R_1, \ldots, R_n$) at $[\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n]$, we obtain

$$\mathbf{P}_{[\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n]}\left[p(rR_1, \ldots, rR_n)^* p(rR_1, \ldots, rR_n)\right] = p(r\widetilde{\varphi}_1, \ldots, r\widetilde{\varphi}_n)^* p(r\widetilde{\varphi}_1, \ldots, r\widetilde{\varphi}_n) \tag{2.2}$$

for any $r \in [0, 1)$. Since $p \in H^2_{\mathbf{ball}}$, Theorem 2.3 from [34] shows that the map

$$\left[0, 1\right) \ni r \mapsto p(rR_1, \ldots, rR_n)^* p(rR_1, \ldots, rR_n) \in C^*(R_1, \ldots, R_n)$$

has a pluriharmonic majorant. In this case, the least pluriharmonic majorant is given by

$$[0,1)\ni r\mapsto \Re W(rR_1,\dots rR_n)\in C^*(R_1,\dots,R_n),$$

where W is the free holomorphic function on $[B(\mathcal{H})^n]_1$ having the Herglotz–Riesz type representation

$$W(X_1,\dots,X_n)=(\mu_p\otimes \mathrm{id})\left[\left(I+\sum_{i=1}^n R_i^*\otimes X_i\right)\left(I-\sum_{i=1}^n R_i^*\otimes X_i\right)^{-1}\right] \tag{2.3}$$

for $(X_1,\dots,X_n)\in[B(\mathcal{H})^n]_1$, where $\mu_p:\mathcal{R}_n^*+\mathcal{R}_n\to\mathbb{C}$ is the completely positive linear map uniquely determined by the equation

$$\mu_p\big(R_{\widetilde{\alpha}}^*\big):=\lim_{r\to 1}\big\langle p(rR_1,\dots,rR_n)^*S_{\widetilde{\alpha}}^*p(rR_1,\dots,rR_n)1,1\big\rangle \tag{2.4}$$

for $\alpha\in\mathbb{F}_n^+$, where $\widetilde{\alpha}$ is the reverse of $\alpha\in\mathbb{F}_n^+$, i.e., $\widetilde{\alpha}=g_{i_k}\cdots g_{i_k}$ if $\alpha=g_{i_1}\cdots g_{i_k}\in\mathbb{F}_n^+$. Therefore, we have

$$p(rR_1,\dots,rR_n)^*p(rR_1,\dots,rR_n)\leqslant \Re W(rR_1,\dots,rR_n)$$

for any $r\in[0,1)$. Hence, using relation (2.2) and the fact that the noncommutative Poisson transform is a completely positive map, we deduce that

$$p(r\widetilde{\varphi}_1,\dots,r\widetilde{\varphi}_n)^*p(r\widetilde{\varphi}_1,\dots,r\widetilde{\varphi}_n)\leqslant \Re W(r\widetilde{\varphi}_1,\dots,r\widetilde{\varphi}_n)$$

for any $r\in[0,1)$. The latter relation implies

$$\big\|p(r\widetilde{\varphi}_1,\dots,r\widetilde{\varphi}_n)1\big\|^2\leqslant\big\langle \operatorname{Re} W(r\widetilde{\varphi}_1,\dots,r\widetilde{\varphi}_n)1,1\big\rangle=\Re W\big(r\varphi_1(0),\dots,r\varphi_n(0)\big).$$

On the other hand, according to the Harnak type theorem for positive free pluriharmonic functions (see [36]), we have

$$\operatorname{Re} W\big(r\varphi_1(0),\dots,\varphi_n(0)\big)\leqslant \Re W(0)\frac{1+r\|\varphi(0)\|}{1-r\|\varphi(0)\|}.$$

Combining the latter two inequalities and taking $r\to 1$, we deduce that

$$\|p\circ\varphi\|_2^2=\big\|p(\widetilde{\varphi}_1,\dots,\widetilde{\varphi}_n)1\big\|^2\leqslant \Re W(0)\frac{1+\|\varphi(0)\|}{1-\|\varphi(0)\|}. \tag{2.5}$$

Using the Herglotz–Riesz representation (2.3) and relation (2.4), we obtain

$$W(0)=\mu_p(I)=\lim_{r\to 1}\big\|p(rR_1,\dots,rR_n)1\big\|^2=\|p\|_2^2.$$

Hence, and using relation (2.5), we have

$$\|p\circ\varphi\|_2 \leqslant \|p\|_2\left(\frac{1+\|\varphi(0)\|}{1-\|\varphi(0)\|}\right)^{1/2} \tag{2.6}$$

for any noncommutative polynomial $p\in H^2_{\mathbf{ball}}$. Let $f(X_1,\dots,X_n)=\sum_{k=0}^\infty\sum_{|\alpha|=k}c_\alpha X_\alpha$ be a free holomorphic function in $H^2_{\mathbf{ball}}$. Then $f\circ\varphi$ is a free holomorphic function on $[B(\mathcal{H})^n]_1$ and

$$(f\circ\varphi)(rS_1,\dots,rS_n)1=\sum_{k=0}^\infty\sum_{|\alpha|=k}c_\alpha\varphi_\alpha(rS_1,\dots,rS_n)1, \tag{2.7}$$

where the convergence is in $F^2(H_n)$. Setting $p_m(X_1,\dots,X_n):=\sum_{k=0}^m\sum_{|\alpha|=k}c_\alpha X_\alpha$, we have $p_m\to f$ in $H^2_{\mathbf{ball}}$ as $m\to\infty$. Therefore, $\{p_m\}$ is a Cauchy sequence in $H^2_{\mathbf{ball}}$. Due to relation (2.6), we have

$$\|p_m\circ\varphi-p_k\circ\varphi\|_2\leqslant\left(\frac{1+\|\varphi(0)\|}{1-\|\varphi(0)\|}\right)^{1/2}\|p_m-p_k\|_2,\quad m,k\in\mathbb{N}.$$

Consequently, $\{p_m\circ\varphi\}$ is a Cauchy sequence in $H^2_{\mathbf{ball}}$ and there is $g\in H^2_{\mathbf{ball}}$ such that $p_m\circ\varphi\to g$ in $H^2_{\mathbf{ball}}$ as $m\to\infty$. Hence, and using relation (2.7), we deduce that

$$g(rS_1,\dots,rS_n)1=\lim_{m\to\infty}(p_m\circ\varphi)(rS_1,\dots,rS_n)1=(f\circ\varphi)(rS_1,\dots,rS_n)1,\quad r\in[0,1).$$

Since $f\circ\varphi$ and g are free holomorphic functions, the identity theorem for free holomorphic functions implies $f\circ\varphi=g$. Using that fact that $p_m\circ\varphi\to f\circ\varphi$ in $H^2_{\mathbf{ball}}$ and relation (2.6), we obtain

$$\|f\circ\varphi\|_2\leqslant\left(\frac{1+\|\varphi(0)\|}{1-\|\varphi(0)\|}\right)^{1/2}\|f\|_2,\quad f\in H^2_{\mathbf{ball}}. \tag{2.8}$$

Since any free holomorphic automorphism of $[B(\mathcal{H})^n]_1$ is inner, i.e., its boundary function with respect to $R_1,\dots,R_n$ is an isometry, the result above implies the right-hand inequality of the theorem.

Now, we prove the left-hand inequality. For each $\mu:=(\mu_1,\dots,\mu_n)\in\mathbb{B}_n$, we define the vector $z_\mu:=\sum_{k=0}\sum_{|\alpha|=k}\overline{\mu}_\alpha e_\alpha$, where $\mu_\alpha:=\mu_{i_1}\cdots\mu_{i_p}$ if $\alpha=g_{i_1}\cdots g_{i_p}\in\mathbb{F}_n^+$ and $i_1,\dots,i_p\in\{1,\dots,n\}$, and $\mu_{g_0}=1$. Note that $z_\mu\in F^2(H_n)$ and $Z_\mu(X):=\sum_{k=0}\sum_{|\alpha|=k}\overline{\mu}_\alpha X_\alpha$ is in $H^2_{\mathbf{ball}}$. Since C_φ is a bounded operator on $H^2_{\mathbf{ball}}$, we have

$$\left(C_\varphi^* Z_\mu\right)(X)=\sum_{k=0}\sum_{|\alpha|=k}b_\alpha X_\alpha,\quad X\in\left[B(\mathcal{H})^n\right]_1,$$

for some coefficients $b_\alpha\in\mathbb{C}$ with $\sum_{\alpha\in\mathbb{F}_n^+}|b_\alpha|^2<\infty$. Since the monomials $\{X_\alpha\}_{\alpha\in\mathbb{F}_n^+}$ form an orthonormal basis for $H^2_{\mathbf{ball}}$, for each $\alpha\in\mathbb{F}_n^+$, we have

$$\begin{aligned} b_\alpha &= \langle C_\varphi^* Z_\mu, X_\alpha \rangle = \langle Z_\mu, C_\varphi(X_\alpha) \rangle \\ &= \langle z_\mu, \varphi_\alpha(S_1, \ldots, S_n)1 \rangle \\ &= \langle \varphi_\alpha(S_1, \ldots, S_n)^* z_\mu, 1 \rangle. \end{aligned}$$

Since $S_i^* z_\mu = \overline{\mu}_i z_\mu$, one can see that $\varphi_\alpha(S_1, \ldots, S_n)^* z_\mu = \overline{\varphi_\alpha(\mu)} z_\mu$. Consequently, we deduce that $b_\alpha = \overline{\varphi_\alpha(\mu)}$, $\alpha \in \mathbb{F}_n^+$, and

$$C_\varphi^* Z_\mu = \sum_{k=0} \sum_{|\alpha|=k} \overline{\varphi_\alpha(\mu)} X_\alpha = Z_{\varphi(\mu)}, \quad \mu := (\mu_1, \ldots, \mu_n) \in \mathbb{B}_n. \tag{2.9}$$

A straightforward computation shows that

$$\left\| C_\varphi^* Z_\mu \right\| = \| z_{\varphi(\mu)} \| = \left(\frac{1}{1 - \|\varphi(\mu)\|^2} \right)^{1/2}.$$

Now, we assume that $\varphi = \Phi_\lambda \in Aut([B(\mathcal{H})^n]_1)$. Then, using relation (2.1), we deduce that

$$\|C_{\Phi_\lambda}\| = \left\| C_{\Phi_\lambda}^* \right\| \geqslant \frac{\|C_{\Phi_\lambda}^* Z_\mu\|}{\|Z_\mu\|} = \left(\frac{1 - \|\mu\|^2}{1 - \|\Phi_\lambda(\mu)\|^2} \right)^{1/2} = \left(\frac{|1 - \langle \mu, \lambda \rangle|^2}{1 - \|\lambda\|^2} \right)^{1/2}$$

for any $\mu \in \mathbb{B}_n$. Taking $\mu \to -\frac{\bar{\lambda}}{\|\lambda\|}$ and using the fact that $\Phi_\lambda(0) = \lambda$, we obtain

$$\|C_{\Phi_\lambda}\| \geqslant \left(\frac{1 + \|\Phi_\lambda(0)\|}{1 - \|\Phi_\lambda(0)\|} \right)^{1/2}.$$

Combining this inequality with relation (2.8), we obtain

$$\|C_{\Phi_\lambda}\| = \left(\frac{1 + \|\Phi_\lambda(0)\|}{1 - \|\Phi_\lambda(0)\|} \right)^{1/2}, \tag{2.10}$$

which also shows that the right-hand inequality in the theorem is sharp.

Now, we assume that $\varphi \in Aut([B(\mathcal{H})^n]_1)$ with $\varphi(0) = \lambda$. Then, due to [38], we have $\varphi = \Phi_\lambda \circ \Phi_U$, where $U \in B(\mathbb{C}^n)$ is a unitary operator. Since Φ_U is inner and $\Phi_U(0) = 0$, Theorem 1.3 shows that C_{Φ_U} is an isometry. Consequently, using relation (2.10) and the fact that $C_\varphi = C_{\Phi_U} C_{\Phi_\lambda}$, we deduce that

$$\|C_\varphi\| = \left(\frac{1 + \|\varphi(0)\|}{1 - \|\varphi(0)\|} \right)^{1/2}.$$

Taking into account that $\Phi_\lambda \circ \Phi_\lambda = \mathrm{id}$, we deduce that

$$\|f\| \leqslant \|C_{\Phi_\lambda}\| \|C_{\Phi_\lambda} f\| \leqslant \left(\frac{1 + \|\Phi_\lambda(0)\|}{1 - \|\Phi_\lambda(0)\|} \right)^{1/2} \|C_{\Phi_\lambda} f\|$$

for any $f \in H^2_{\mathbf{ball}}$. Now, we assume that $\varphi \in Aut([B(\mathcal{H})^n]_1)$ with $\varphi(0) = \lambda$. As above, $\varphi = \Phi_\lambda \circ \Phi_U$ and $C_\varphi = C_{\Phi_U} C_{\Phi_\lambda}$. Since C_{Φ_U} is an isometry, the latter inequality implies

$$\|C_\varphi f\| = \|C_{\Phi_\lambda} C_{\Phi_U} f\| \geqslant \left(\frac{1 - \|\varphi(0)\|}{1 + \|\varphi(0)\|}\right)^{1/2} \|f\|,$$

which shows that the left-hand inequality of the theorem holds. To prove that this inequality is sharp, let $g_k \in H^2_{\mathbf{ball}}$ with $\|g_k\|_2 = 1$ and $\|C_{\Phi_\lambda}\| = \lim_{k\to\infty} \|C_{\Phi_\lambda} g_k\|$. Set $f_k := C_{\Phi_\lambda} g_k$ and note that the inequality $(\frac{1-\|\Phi_\lambda(0)\|}{1+\|\Phi_\lambda(0)\|})^{1/2} \|f_k\| \leqslant \|C_{\Phi_\lambda} f_k\|$ is equivalent to $\|C_{\Phi_\lambda} g_k\| \leqslant (\frac{1+\|\Phi_\lambda(0)\|}{1-\|\Phi_\lambda(0)\|})^{1/2}$, which is sharp due to (2.10), and proves our assertion. The proof is complete. □

Theorem 2.2. *If φ is an inner free holomorphic self-map of the noncommutative ball $[B(\mathcal{H})^n]_1$, then $C_\varphi f \in H^2_{\mathbf{ball}}$ for all $f \in H^2_{\mathbf{ball}}$, and*

$$\left(\frac{1 - \|\varphi(0)\|}{1 + \|\varphi(0)\|}\right)^{1/2} \|f\| \leqslant \|C_\varphi f\| \leqslant \left(\frac{1 + \|\varphi(0)\|}{1 - \|\varphi(0)\|}\right)^{1/2} \|f\|$$

for any $f \in H^2_{\mathbf{ball}}$. Moreover, these inequalities are best possible and

$$\|C_\varphi\| = \left(\frac{1 + \|\varphi(0)\|}{1 - \|\varphi(0)\|}\right)^{1/2}.$$

Proof. First, we consider the case when φ is an inner free holomorphic self-map of the noncommutative ball $[B(\mathcal{H})^n]_1$ with $\varphi(0) = 0$. Then Theorem 1.3 shows that the composition operator C_φ is an isometry on $H^2_{\mathbf{ball}}$ and, therefore, the theorem holds.

Now, we consider the case when $\lambda := \varphi(0) \neq 0$. Since φ is a free holomorphic self-map of the noncommutative ball $[B(\mathcal{H})^n]_1$, we must have $\|\lambda\|_2 < 1$. Let Φ_λ be the corresponding involutive free holomorphic automorphism of $[B(\mathcal{H})^n]_1$ and let $\Psi := \Phi_\lambda \circ \varphi$. Since Φ_λ is inner and the composition of inner free holomorphic functions is inner (see Theorem 1.2 from [39]), we deduce that Ψ is also inner. Since $\Psi(0) = 0$, the first part of the proof implies

$$\|C_\Psi f\| = \|f\|, \quad f \in H^2_{\mathbf{ball}}.$$

Consequently, using Theorem 2.1 and the fact that $\Phi_\lambda \circ \Phi_\lambda = \mathrm{id}$, we get

$$\begin{aligned} \|C_\varphi f\| = \|C_\Psi C_{\Phi_\lambda} f\| = \|C_{\Phi_\lambda} f\| &\leqslant \left(\frac{1 + \|\Phi_\lambda(0)\|}{1 - \|\Phi_\lambda(0)\|}\right)^{1/2} \|f\| \\ &= \left(\frac{1 + \|\varphi(0)\|}{1 - \|\varphi(0)\|}\right)^{1/2} \|f\| \end{aligned} \tag{2.11}$$

for any $f \in H^2_{\mathbf{ball}}$. Similarly, one can show that

$$\|C_\varphi f\| = \|C_{\Phi_\lambda} f\| \geqslant \left(\frac{1 - \|\Phi_\lambda(0)\|}{1 + \|\Phi_\lambda(0)\|}\right)^{1/2} \|f\| = \left(\frac{1 - \|\varphi(0)\|}{1 + \|\varphi(0)\|}\right)^{1/2} \|f\|$$

for any $f \in H^2_{\mathbf{ball}}$. Therefore, the inequalities in the theorem hold. Now, we show that they are sharp. According to Theorem 2.1, we can find $f_k \in H^2_{\mathbf{ball}}$ with $\|f_k\|_2 = 1$ such that

$$\lim_{k\to\infty} \|C_{\Phi_\lambda} f_k\| = \left(\frac{1+\|\Phi_\lambda(0)\|}{1-\|\Phi_\lambda(0)\|}\right)^{1/2}.$$

Hence, using relation (2.11) and the fact that $\Phi_\lambda(0) = \varphi(0)$, we obtain

$$\lim_{k\to\infty} \|C_\varphi f_k\| = \lim_{k\to\infty} \|C_{\Phi_\lambda} f_k\| = \left(\frac{1-\|\varphi(0)\|}{1+\|\varphi(0)\|}\right)^{1/2},$$

which shows that the right-hand inequality in the theorem is sharp. Similarly, one can show that the left-hand inequality is also sharp. The proof is complete. □

Now, we can prove the main result of this section.

Theorem 2.3. *If φ is a free holomorphic self-map of the ball $[B(\mathcal{H})^n]_1$, then the composition operator $C_\varphi f := f \circ \varphi$ is bounded on $H^2_{\mathbf{ball}}$. Moreover,*

$$\frac{1}{(1-\|\varphi(0)\|^2)^{1/2}} \leqslant \sup_{\lambda\in\mathbb{B}_n} \left(\frac{1-\|\lambda\|^2}{1-\|\varphi(\lambda)\|^2}\right)^{1/2} \leqslant \|C_\varphi\| \leqslant \left(\frac{1+\|\varphi(0)\|}{1-\|\varphi(0)\|}\right)^{1/2}.$$

Proof. If $\varphi(0) = 0$, then the right-hand inequality follows from the noncommutative Littlewood subordination principle of Theorem 1.2. Now, we consider the case when $\lambda := \varphi(0) \neq 0$. Since $\|\lambda\|_2 < 1$, let Φ_λ be the corresponding involutive free holomorphic automorphism of $[B(\mathcal{H})^n]_1$ and let $\Psi := \Phi_\lambda \circ \varphi$. Since Ψ is a free holomorphic self-map of the ball $[B(\mathcal{H})^n]_1$ with $\Psi(0) = 0$, Theorem 1.2 implies $\|C_\Psi\| \leqslant 1$. Using Theorem 2.1 and the fact that $\Phi_\lambda \circ \Phi_\lambda = \mathrm{id}$, we deduce that

$$\|C_\varphi\| = \|C_\Psi C_{\Phi_\lambda}\| \leqslant \|C_\Psi\| \|C_{\Phi_\lambda}\| \leqslant \left(\frac{1+\|\varphi(0)\|}{1-\|\varphi(0)\|}\right)^{1/2}.$$

On the other hand, as in the proof of Theorem 2.1, we have

$$\|C_\varphi\| = \|C_\varphi^*\| \geqslant \frac{\|C_\varphi^* Z_\mu\|}{\|Z_\mu\|} = \left(\frac{1-\|\mu\|^2}{1-\|\varphi(\mu)\|^2}\right)^{1/2}$$

for any $\mu \in \mathbb{B}_n$. Hence, we deduce the left-hand inequality. The proof is complete. □

Under the identification of the noncommutative Hardy space $H^2_{\mathbf{ball}}$ with the full Fock space $F^2(H_n)$, via the unitary operator $\mathcal{U} : H^2_{\mathbf{ball}} \to F^2(H_n)$ defined by

$$H^2_{\mathbf{ball}} \ni F \mapsto f := \lim_{r\to 1} F(rS_1, \ldots, rS_n)1 \in F^2(H_n),$$

the composition operator $C_\varphi : H^2_{\mathbf{ball}} \to H^2_{\mathbf{ball}}$ associated with φ, a free holomorphic self-map of $[B(\mathcal{H})^n]_1$, can be identified with the composition operator $C_{\widetilde{\varphi}} : F^2(H_n) \to F^2(H_n)$ defined by

$$C_{\widetilde{\varphi}}\left(\sum_{k=0}^{\infty}\sum_{|\alpha|=k} a_\alpha e_\alpha\right) := \lim_{r\to 1}\sum_{k=0}^{\infty}\sum_{|\alpha|=k} a_\alpha \varphi_\alpha(rS_1,\dots,rS_n)1 \tag{2.12}$$

for any $\sum_{k=0}^{\infty}\sum_{|\alpha|=k} a_\alpha e_\alpha \in F^2(H_n)$. Indeed, note that $C_{\widetilde{\varphi}} = \mathcal{U}C_\varphi\mathcal{U}^{-1}$.

A consequence of Theorem 2.3 is the following result.

Corollary 2.4. *If φ is a free holomorphic self-map of the ball $[B(\mathcal{H})^n]_1$, then the composition operator $C_{\widetilde{\varphi}} : F^2(H_n) \to F^2(H_n)$ satisfies the equation*

$$C_{\widetilde{\varphi}}\left(\sum_{k=0}^{\infty}\sum_{|\alpha|=k} a_\alpha e_\alpha\right) = \sum_{k=0}^{\infty}\sum_{|\alpha|=k} a_\alpha(\widetilde{\varphi}_\alpha 1),$$

where the convergence of the series is in $F^2(H_n)$ and $\widetilde{\varphi} :=$ SOT-$\lim_{r\to 1}\varphi(rS_1,\dots,rS_n)$ is the boundary function of φ with respect to the left creation operators $S_1,\dots,S_n$.

Proof. Let $\widetilde{\varphi} := (\widetilde{\varphi}_1,\dots,\widetilde{\varphi}_n)$ be the boundary of $\widetilde{\varphi}$ and let $f = \sum_{k=0}^{\infty}\sum_{|\alpha|=k} a_\alpha X_\alpha$ be in $H^2_{\mathbf{ball}}$. Due to Theorem 2.3 and the identification of $H^2_{\mathbf{ball}}$ with $F^2(H_n)$, we have

$$\left\|\sum_{p\leqslant|\alpha|\leqslant m} a_\alpha \widetilde{\varphi}_\alpha 1\right\| \leqslant \left(\frac{1+\|\varphi(0)\|}{1-\|\varphi(0)\|}\right)^{1/2}\left(\sum_{p\leqslant|\alpha|\leqslant m}|a_\alpha|^2\right)^{1/2} \tag{2.13}$$

for any $p, m \in \mathbb{N}$, $p \leqslant m$. Consequently, since $f \in H^2_{\mathbf{ball}}$, the sequence $\{\sum_{k=0}^{m}\sum_{|\alpha|=k} a_\alpha\widetilde{\varphi}_\alpha 1\}_{m=1}^{\infty}$ is Cauchy in $F^2(H_n)$ and therefore convergent to an element in $F^2(H_n)$. Hence, and using relation (2.13), we deduce that

$$\left\|\sum_{k=0}^{\infty}\sum_{|\alpha|=k} a_\alpha \widetilde{\varphi}_\alpha 1\right\| \leqslant \left(\frac{1+\|\varphi(0)\|}{1-\|\varphi(0)\|}\right)^{1/2}\|f\|.$$

Similarly, one can show that $\sum_{k=0}^{\infty}\sum_{|\alpha|=k} a_\alpha\varphi_\alpha(rS_1,\dots,rS_n)1$ is in $F^2(H_n)$ and

$$\left\|\sum_{k=0}^{\infty}\sum_{|\alpha|=k} a_\alpha \varphi_\alpha(rS_1,\dots,rS_n)1\right\| \leqslant \left(\frac{1+\|\varphi(0)\|}{1-\|\varphi(0)\|}\right)^{1/2}\|f\|$$

for each $r \in [0,1)$. Consequently, taking into account that $\widetilde{\varphi} :=$ SOT-$\lim_{r\to 1}\varphi(rS_1,\dots,rS_n)$, a simple approximation argument shows that

$$\lim_{r\to 1}\sum_{k=0}^{\infty}\sum_{|\alpha|=k} a_\alpha\varphi_\alpha(rS_1,\dots,rS_n)1 = \sum_{k=0}^{\infty}\sum_{|\alpha|=k} a_\alpha\widetilde{\varphi}_\alpha 1$$

in $F^2(H_n)$, which together with relation (2.12) completes the proof. □

In this paper, we will use either one of the representations C_φ or $C_{\widetilde{\varphi}}$ for the composition operator with symbol φ.

Corollary 2.5. *Let $\varphi = (\varphi_1, \ldots, \varphi_n)$ be a free holomorphic self-map of the noncommutative ball $[B(\mathcal{H})^n]_1$ and let C_φ be the composition operator on $H^2_{\mathbf{ball}}$. Then the following statements hold.*

(i) $\|C_\varphi\| \geqslant 1$.
(ii) *C_φ is a contraction if and only if $\varphi(0) = 0$.*
(iii) *C_φ is an isometry if and only if $\{\varphi_\alpha\}_{\alpha\in\mathbb{F}_n}$ is an orthonormal set in $H^2_{\mathbf{ball}}$.*

Proof. Since $C_\varphi 1 = 1$, we have $\|C_\varphi\| \geqslant 1$. To prove part (ii), note that if $\|C_\varphi\| = 1$, then according to Theorem 2.3, we have

$$\frac{1}{(1 - \|\varphi(0)\|^2)^{1/2}} \leqslant \|C_\varphi\| = 1,$$

which implies $\varphi(0) = 0$. Conversely, if $\varphi(0) = 0$, the same theorem implies $\|C_\varphi\| = 1$. Now, assume that C_φ is an isometry. Then

$$\delta_{\alpha,\beta} = \big\langle C_\varphi(X_\alpha), C_\varphi(X_\beta)\big\rangle = \langle \varphi_\alpha, \varphi_\beta\rangle, \quad \alpha, \beta \in \mathbb{F}_n^+.$$

Conversely, assume that $\{\varphi_\alpha\}_{\alpha\in\mathbb{F}_n}$ is an orthonormal set in $H^2_{\mathbf{ball}}$. Then, for any

$$f = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} a_\alpha X_\alpha$$

in the Hardy space $H^2_{\mathbf{ball}}$, we have

$$\|C_\varphi f\|^2 = \left\| \sum_{k=0}^{\infty} \sum_{|\alpha|=k} a_\alpha \varphi_\alpha \right\|^2 = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} |a_\alpha|^2 = \|f\|^2.$$

The proof is complete. □

Halmos' famous similarity problem [7] asked whether any polynomially bounded operator is similar to a contraction. This long standing problem was answered by Pisier [22] in a remarkable paper where he shows that there are polynomially bounded operator which are not similar to contractions. In what follows we show that, for compositions operators on $H^2_{\mathbf{ball}}$, similarity to contractions is equivalent polynomial boundedness.

Theorem 2.6. *Let φ be a free holomorphic self-map of the noncommutative ball $[B(\mathcal{H})^n]_1$ and let C_φ be the composition operator on $H^2_{\mathbf{ball}}$. Then the following statements are equivalent*:

(i) *C_φ is similar to a contraction*;
(ii) *C_φ is polynomially bounded*;
(iii) *C_φ is power bounded*;
(iv) *there is $\xi \in \mathbb{B}_n$ such that $\varphi(\xi) = \xi$.*

Proof. The fact that an operator similar to a contraction is power bounded and polynomially bounded is a consequence of the well-known von-Neumann inequality [49]. We prove that

(iii) $\Rightarrow$ (iv). Assume that C_φ is power bounded, i.e., there is a constant $M > 0$ such that $\|C_\varphi^k\| \leqslant M$ for any $k \in \mathbb{N}$. Note that the scalar representation of φ, i.e. $\mathbb{B}_n \ni \lambda \mapsto \varphi(\lambda) \in \mathbb{B}_n$, is a holomorphic self-map of $\mathbb{B}_n$. Suppose there is no $\xi \in \mathbb{B}_n$ such that $\varphi(\xi) = \xi$. Then, due to MacCluer's result [13], there is $\gamma \in \partial\mathbb{B}_n$, called the Denjoy–Wolff point of the map $\mathbb{B}_n \ni \lambda \mapsto \varphi(\lambda) \in \mathbb{B}_n$, such that the sequence of iterates $\varphi^{[k]} := \varphi \circ \cdots \circ \varphi$ converges to γ uniformly on any compact subset of $\mathbb{B}_n$. In particular, we have $\|\varphi^{[k]}(0)\| \to 1$ as $k \to \infty$. On the other hand, Theorem 2.3 implies

$$\left\|C_\varphi^k\right\| = \|C_{\varphi^{[k]}}\| \geqslant \frac{1}{(1 - \|\varphi^{[k]}(0)\|^2)^{1/2}}.$$

Consequently, $\|C_\varphi^k\| \to \infty$ as $k \to \infty$, which contradicts the fact that C_φ is a power bounded operator. Therefore, item (iv) holds. Finally, to prove that (iv) $\Rightarrow$ (i), assume that there is $\xi \in \mathbb{B}_n$ such that $\varphi(\xi) = \xi$. Set $\Psi := \Phi_\xi \circ \varphi \circ \Phi_\xi$, where Φ_ξ is the involutive free holomorphic automorphism of $[B(\mathcal{H})^n]_1$ associated with ξ. Note that Ψ is a bounded free holomorphic function on $[B(\mathcal{H})^n]_1$ and $\Psi(0) = 0$. Due to Theorem 1.2, we have $\|C_\Psi\| \leqslant 1$. On the other hand, since $\Phi_\xi \circ \Phi_\xi = \mathrm{id}$ and $C_\varphi = C_{\Phi_\xi}^{-1} C_\Psi C_{\Phi_\xi}$, the result follows. The proof is complete. □

Corollary 2.7. *Let φ be a free holomorphic self-map of the noncommutative ball $[B(\mathcal{H})^n]_1$ and let C_φ be the composition operator on $H^2_{\mathbf{ball}}$. If there is $\xi \in \mathbb{B}_n$ such that $\varphi(\xi) = \xi$, then the spectral radius of C_φ is* 1.

Proof. According to the proof of Theorem 2.6, C_φ is similar to a composition operator C_Ψ with $\Psi(0) = 0$. Since $\Psi^{[k]}(0) = 0$, Theorem 1.2 implies $\|C_{\Psi^{[k]}}\| = 1$ for any $k \in \mathbb{N}$. Consequently, we have

$$r(C_\varphi) = r(C_\Psi) = \lim_{k\to\infty} \|C_{\Psi^{[k]}}\|^{1/k} = 1.$$

The proof is complete. □

Corollary 2.8. *Let φ be an inner free holomorphic self-map of the noncommutative ball $[B(\mathcal{H})^n]_1$ and let C_φ be the composition operator on $H^2_{\mathbf{ball}}$. Then the following statements hold.*

(i) *C_φ is an isometry if and only if $\varphi(0) = 0$.*
(ii) *C_φ is similar to an isometry if and only if there is $\xi \in \mathbb{B}_n$ such that $\varphi(\xi) = \xi$.*

Proof. Assume that C_φ is an isometry. Due to Theorem 2.2, we have

$$1 = \|C_\varphi\| = \left(\frac{1 + \|\varphi(0)\|}{1 - \|\varphi(0)\|}\right)^{1/2}.$$

Consequently, $\varphi(0) = 0$. The converse follows also from Theorem 2.2. Therefore, item (i) holds. The direct implication in item (ii) follows from Theorem 2.6. To prove the converse, assume that there is $\xi \in \mathbb{B}_n$ such that $\varphi(\xi) = \xi$ and set $\Psi := \Phi_\xi \circ \varphi \circ \Phi_\xi$, where Φ_ξ is the involutive free holomorphic automorphism of $[B(\mathcal{H})^n]_1$ associated with ξ.

According to [39], the composition of inner free holomorphic functions on $[B(\mathcal{H})^n]_1$ is inner. Consequently, Ψ is an inner free holomorphic function and $\Psi(0)=0$. Due to part (i), the composition operator C_Ψ is an isometry. Since $C_\varphi = C_{\Phi_\xi}^{-1} C_\Psi C_{\Phi_\xi}$, the result follows. □

The following result is an extension to our noncommutative multivariable setting of Cowen's [2] one-variable spectral radius formula for composition operators.

Theorem 2.9. *Let φ be a free holomorphic self-map of the noncommutative ball $[B(\mathcal{H})^n]_1$ and let C_φ be the composition operator on $H^2_{\mathbf{ball}}$. Then the spectral radius of C_φ satisfies the relation*

$$r(C_\varphi) = \lim_{k\to\infty} \left(1 - \left\|\varphi^{[k]}(0)\right\|\right)^{-1/2k}.$$

Moreover,

$$r(C_\varphi) = \lim_{k\to\infty} \left(\frac{1-\|\varphi^{[k]}(0)\|}{1-\|\varphi^{[k+1]}(0)\|}\right)^{1/2}$$

if the latter limit exists.

Proof. Note that Theorem 2.3 implies

$$\left(\frac{1}{1-\|\varphi^{[k]}(0)\|^2}\right)^{1/2k} \leqslant \left\|C_\varphi^k\right\|^{1/k} \leqslant \left(\frac{1+\|\varphi^{[k]}(0)\|}{1-\|\varphi^{[k]}(0)\|}\right)^{1/2k} \leqslant \left(\frac{2}{1-\|\varphi^{[k]}(0)\|}\right)^{1/2k}.$$

Taking $k\to\infty$, we obtain the first formula for the spectral radius of C_φ. To prove the second formula, note that

$$\begin{aligned} r(C_\varphi) &= \lim_{k\to\infty} \left(1 - \left\|\varphi^{[k]}(0)\right\|\right)^{-1/2k} \\ &= \lim_{k\to\infty} \left(\prod_{p=0}^{k-1} \frac{1-\|\varphi^{[p]}(0)\|}{1-\|\varphi^{[p+1]}(0)\|}\right)^{1/2k} \\ &= \lim_{k\to\infty} \left(\frac{1-\|\varphi^{[k]}(0)\|}{1-\|\varphi^{[k+1]}(0)\|}\right)^{1/2} \end{aligned}$$

if the latter limit exists. The proof is complete. □

3. Noncommutative Wolff theorem for free holomorphic self-maps of $[B(\mathcal{H})^n]_1$

In this section, we use Julia type lemma for free holomorphic functions [39] and the ideas from the classical result obtained by Wolff [50,51] and MacCluer's extension to $\mathbb{B}_n$ (see [13]), to provide a noncommutative analogue of Wolff's theorem for free holomorphic self-maps of $[B(\mathcal{H})^n]_1$. We also show that the spectral radius of a composition operator on $H^2_{\mathbf{ball}}$ is 1 when the symbol is elliptic or parabolic, which extends some of Cowen's results [2] from the single variable case.

Julia's lemma [10] says that if $f : \mathbb{D} \to \mathbb{D}$ is an analytic function and there is a sequence $\{z_k\} \subset \mathbb{D}$ with $z_k \to 1$, $f(z_k) \to 1$, and such that $\frac{1-|f(z_k)|}{1-|z_k|}$ is bounded, then f maps each disc in $\mathbb{D}$ tangent to $\partial\mathbb{D}$ at 1 into a disc of the same kind. Wolff [50,51] used this result to show that if f has no fixed points in $\mathbb{D}$, then there is a unique point $\xi \in \partial\mathbb{D}$ such that any closed disc in $\mathbb{D}$ which is tangent to $\partial\mathbb{D}$ at ξ is mapped into itself by every iterate of f, i.e., $f^{[1]} = f$, $f^{[k+1]} := f^{[k]} \circ f$, $k \in \mathbb{N}$. The Denjoy–Wolff theorem [50,5] asserts that, under the above-mentioned conditions, the sequence of iterates of f converges uniformly on compact subsets of $\mathbb{D}$ to the constant map $g(z) = \xi$, $z \in \mathbb{D}$. The point ξ is called the Denjoy–Wolff point of f. This result was extended to the unit ball of $\mathbb{C}^n$ by MacCluer [13].

If $A, B \in B(\mathcal{K})$ are selfadjoint operators, we say that $A < B$ if $B - A$ is positive and invertible, i.e., there exists a constant $\gamma > 0$ such that $\langle (B-A)h, h\rangle \geqslant \gamma\|h\|^2$ for any $h \in \mathcal{K}$. Note that $T \in B(\mathcal{K})$ is a strict contraction ($\|T\| < 1$) if and only if $TT^* < I$. For $0 < c < 1$ and $\xi_1 = (1, 0, \ldots, 0)$, we define the noncommutative ellipsoid

$$\mathbf{E}_c(\xi_1) := \Bigg\{(X_1, \ldots, X_n) \in B(\mathcal{H})^n: \frac{[X_1 - (1-c)I][X_1^* - (1-c)I]}{c^2} + \frac{X_2X_2^*}{c} + \cdots + \frac{X_nX_n^*}{c} < I\Bigg\}$$

with center at $(1-c)\xi_1$ and containing ξ_1 in its boundary. If $\xi \in \mathbb{B}_n$ we define the noncommutative ellipsoid $\mathbf{E}_c(\xi)$ centered at $(1-c)\xi$ and containing ξ in its boundary in a similar manner.

In [39], we obtained a Julia type lemma for free holomorphic functions. Let $F : [B(\mathcal{H})^n]_1 \to [B(\mathcal{H})^m]_1$ be a free holomorphic function and $F = (F_1, \ldots, F_m)$. Let $\{z_k\}$ be a sequence of points in $\mathbb{B}_n$ such that $\lim_{k\to\infty} z_k = (1, 0, \ldots, 0) \in \partial\mathbb{B}_n$, $\lim_{k\to\infty} F(z_k) = (1, 0, \ldots, 0) \in \partial\mathbb{B}_m$, and

$$\lim_{k\to\infty} \frac{1 - \|F(z_k)\|^2}{1 - \|z_k\|^2} = L < \infty.$$

Then $L > 0$ and

$$\big(I - F_1(X)^*\big)\big(I - F(X)F(X)^*\big)^{-1}\big(I - F_1(X)\big) \leqslant L\big(I - X_1^*\big)\big(I - XX^*\big)^{-1}(I - X_1)$$

for any $X = (X_1, \ldots, X_n) \in [B(\mathcal{H})^n]_1$. Moreover, if $0 < c < 1$, then

$$F\big(\mathbf{E}_c(\xi_1)\big) \subset \mathbf{E}_\gamma(\xi_1), \quad \text{where } \gamma := \frac{Lc}{1 + Lc - c}.$$

In what follows we provide a noncommutative analogue of Wolff's theorem for free holomorphic self-maps of $[B(\mathcal{H})^n]_1$.

Theorem 3.1. *Let $\varphi : [B(\mathcal{H})^n]_1 \to [B(\mathcal{H})^n]_1$ be a free holomorphic function such that its scalar representation has no fixed points in $\mathbb{B}_n$. Then there is a unique point $\zeta \in \partial\mathbb{B}_n$ such that each noncommutative ellipsoid $\mathbf{E}_c(\zeta)$, $c \in (0,1)$, is mapped into itself by every iterate of φ.*

Proof. Let $r_k \in (0,1)$ be a convergent sequence to 1. Define the map $\psi_k : [B(\mathcal{H})^n]^-_{r_k} \to [B(\mathcal{H})^n]^-_{r_k}$ by $\psi_k := r_k\varphi(X)$, $X \in [B(\mathcal{H})^n]^-_{r_k}$, and note that ψ_k is a free holomorphic function in $[B(\mathcal{H})^n]^-_{r_k}$. Consequently, its scalar representation $\chi_k : [\mathbb{C}^n]^-_{r_k} \to [\mathbb{C}^n]^-_{r_k}$, defined by

$\chi_k(\lambda) := \psi_k(\lambda)$, $\lambda \in [\mathbb{C}^n]^-_{r_k}$, is holomorphic in $[\mathbb{C}^n]^-_{r_k}$. According to Brouwer fixed point theorem there exists $\lambda_k \in [\mathbb{C}^n]^-_{r_k}$ such that $\chi(\lambda_k) = \lambda_k$. Hence, $\varphi(\lambda_k) = \frac{\lambda_k}{r_k}$. Passing to a subsequence and taking into account that the scalar representation of φ has no fixed point in $\mathbb{B}_n$, we may assume that $\lambda_k \to \zeta \in \partial\mathbb{B}_n$. This implies that $\varphi(\lambda_k) \to \zeta$ and

$$\frac{1 - \|\varphi(\lambda_k)\|^2}{1 - \|\lambda_k\|^2} = \frac{1 - \frac{1}{r_k^2}\|\lambda_k\|^2}{1 - \|\lambda_k\|^2} < 1.$$

Consequently, we may assume that

$$\lim_{k\to\infty} \frac{1 - \|\varphi(\lambda_k)\|^2}{1 - \|\lambda_k\|^2} = L \leqslant 1.$$

Without loss of generality, we may also assume that $\zeta = \xi_1 := (1, 0, \ldots, 0) \in \partial\mathbb{B}_n$. Using the above-mentioned Julia type lemma for free holomorphic functions, we deduce that $L > 0$ and

$$\varphi\big(\mathbf{E}_c(\xi_1)\big) \subset \mathbf{E}_\gamma(\xi_1), \quad \text{where } \gamma := \frac{Lc}{1 + Lc - c}. \tag{3.1}$$

Note that $X \in \mathbf{E}_c(\xi_1)$ if and only if

$$(I - X_1)\big(I - X_1^*\big) < \frac{c}{1-c}\big(I - XX^*\big).$$

Since $L \leqslant 1$, it is easy to see that $\gamma \leqslant c$, which implies $E_\gamma(\xi_1) \subseteq E_c(\xi_1)$. Combining this with relation (3.1), we obtain $\varphi(\mathbf{E}_c(\xi_1)) \subseteq \mathbf{E}_c(\xi_1)$ for any $c \in (0, 1)$, which proves the first part of the theorem.

To prove the uniqueness, assume that there two distinct points $\zeta, \zeta' \in \partial\mathbb{B}_n$ such that $\varphi(\mathbf{E}_c(\zeta)) \subseteq \mathbf{E}_c(\zeta)$ and $\varphi(\mathbf{E}_c(\zeta')) \subseteq \mathbf{E}_c(\zeta')$ for any $c \in (0, 1)$. Let $\mathbf{E}_c^{\mathbb{C}}(\zeta)$ be the scalar representation of the noncommutative ellipsoid $\mathbf{E}_c(\zeta)$ and let $\varphi^{\mathbb{C}}$ be the scalar representation of φ. Choose $c, c' \in (0, 1)$ such that $\mathbf{E}_c^{\mathbb{C}}(\zeta)$ and $\mathbf{E}_{c'}^{\mathbb{C}}(\zeta')$ are tangent to each other at some point $\xi \in \mathbb{B}_n$. Note that $\varphi^{\mathbb{C}}(\xi) \in \overline{\mathbf{E}_c^{\mathbb{C}}(\zeta)} \cap \overline{\mathbf{E}_{c'}^{\mathbb{C}}(\zeta')} = \{\xi\}$, which contradicts the hypothesis. The proof is complete. □

The point ζ of Theorem 3.1 is called the Denjoy–Wolff point of φ. We remark that Theorem 3.1 shows that

$$0 < \liminf_{z\to\zeta} \frac{1 - \|\varphi(z)\|^2}{1 - \|z\|^2} = \alpha \leqslant 1.$$

The number α is called the dilatation coefficient of φ. When $n = 1$, α is the angular derivative of φ at ζ.

Combining Theorem 3.1 with Julia type lemma for free holomorphic functions [39], we obtain the following result.

Theorem 3.2. *Let $\varphi : [B(\mathcal{H})^n]_1 \to [B(\mathcal{H})^n]_1$ be a free holomorphic function with Denjoy–Wolff point $\zeta \in \partial\mathbb{B}_n$ and dilatation coefficient α. Then, for any $X \in [B(\mathcal{H})^n]_1$,*

$$\big[I - \zeta\varphi(X)^*\big]\big[I - \varphi(X)\varphi(X)^*\big]^{-1}\big[I - \varphi(X)\zeta^*\big] \leqslant \alpha\big(I - \zeta X^*\big)\big(I - XX^*\big)^{-1}\big(I - X\zeta^*\big).$$

Let $\varphi : [B(\mathcal{H})^n]_1 \to [B(\mathcal{H})^n]_1$ be a free holomorphic self-map. Following the classical case, φ will be called:

(i) *elliptic* if φ fixes a point in $\mathbb{B}_n$;
(ii) *parabolic* if φ has no fixed points in $\mathbb{B}_n$ and dilatation coefficient $\alpha = 1$;
(iii) *hyperbolic* if φ has no fixed points in $\mathbb{B}_n$ and dilatation coefficient $\alpha < 1$.

In the single variable case, when $\varphi : \mathbb{D} \to \mathbb{D}$, Cowen [2] proved that the spectral radius of the composition operator C_φ on $H^2(\mathbb{D})$ is 1 if φ is elliptic or parabolic, and $\frac{1}{\sqrt{\alpha}}$ if φ is hyperbolic. We can extend his result to composition operators on $H^2_{\mathbf{ball}}$ when the symbol φ is elliptic or parabolic.

Theorem 3.3. *Let φ be a free holomorphic self-map of the noncommutative ball $[B(\mathcal{H})^n]_1$. If φ is elliptic or parabolic, then the spectral radius of the composition operator C_φ on $H^2_{\mathbf{ball}}$ is* 1.

Proof. The case when φ is elliptic was considered in Corollary 2.7. Now, we assume that φ is parabolic and let $\zeta \in \partial\mathbb{B}_n$ be the corresponding Denjoy–Wolff point. According to MacCluer version [13] of Denjoy–Wolff theorem, the iterates of the scalar representation of φ converge uniformly to ζ on compact subsets of $\mathbb{B}_n$. In particular, we have $\varphi^{[k]}(0) \to \zeta$ as $k \to \infty$. Since the dilatation coefficient of φ is 1, we must have $\liminf_{k\to\infty}(\frac{1-\|\varphi^{[k+1]}(0)\|}{1-\|\varphi^{[k]}(0)\|})^{1/2} \geqslant 1$. Consequently, as in the proof of Theorem 2.9, we deduce that

$$r(C_\varphi) \leqslant \limsup_{k\to\infty}\left(\frac{1 - \|\varphi^{[k]}(0)\|}{1 - \|\varphi^{[k+1]}(0)\|}\right)^{1/2} \leqslant 1.$$

Taking into account that $C_\varphi 1 = 1$, the result follows. $\square$

To calculate the spectral radius of a composition operator on $H^2_{\mathbf{ball}}$ when the symbol is hyperbolic remains an open problem. Another open problem is to find a Denjoy–Wolff type theorem (see [5,50]) for free holomorphic self-maps of $[B(\mathcal{H})^n]_1$.

4. Composition operators and their adjoints

In this section, we obtain a formula for the adjoint of a composition operator on $H^2_{\mathbf{ball}}$. As a consequence we characterize the normal composition operators on $H^2_{\mathbf{ball}}$. We also present a nice connection between Fredholm composition operators on $H^2_{\mathbf{ball}}$ and the automorphisms of the open unit ball $\mathbb{B}_n$.

Proposition 4.1. *Let $\varphi = (\varphi_1, \ldots, \varphi_n)$ be a free holomorphic self-map of the noncommutative ball $[B(\mathcal{H})^n]_1$. Then the adjoint of the composition C_φ on $H^2_{\mathbf{ball}}$ satisfies the relation*

$$\big(C_\varphi^* f\big)(X_1, \ldots, X_n) = \sum_{\alpha \in \mathbb{F}_n^+} \langle f, \varphi_\alpha \rangle X_\alpha, \quad f \in H^2_{\mathbf{ball}}.$$

Proof. According to Theorem 2.3, then composition operator C_φ is bounded on the Hardy space $H^2_{\mathbf{ball}}$. If $f = \sum_{k=0}^\infty \sum_{|\alpha|=k} c_\alpha X_\alpha$ is in $H^2_{\mathbf{ball}}$, then,

$$C_\varphi^* f = \sum_{k=0} \sum_{|\alpha|=k} b_\alpha X_\alpha, \quad X \in \big[B(\mathcal{H})^n\big]_1,$$

for some coefficients $b_\alpha \in \mathbb{C}$ with $\sum_{\alpha \in \mathbb{F}_n^+} |b_\alpha|^2 < \infty$. Since the monomials $\{X_\alpha\}_{\alpha \in \mathbb{F}_n^+}$ form an orthonormal basis for $H^2_{\mathbf{ball}}$, we have

$$b_\alpha = \big\langle C_\varphi^* f, X_\alpha \big\rangle = \big\langle f, C_\varphi(X_\alpha) \big\rangle = \langle f, \varphi_\alpha \rangle, \quad \alpha \in \mathbb{F}_n^+.$$

The proof is complete. □

We remark that under the identification of $H^2_{\mathbf{ball}}$ with the Fock space $F^2(H_n)$, the operator C_φ is unitarily equivalent to $C_{\widetilde{\varphi}}$ (see Corollary 2.4) and

$$C_{\widetilde{\varphi}} g = \sum_{\alpha \in \mathbb{F}_n^+} \big\langle g, \widetilde{\varphi}_\alpha(1) \big\rangle e_\alpha, \quad g \in F^2(H_n).$$

By abuse of notation, we also write $C_\varphi^* f = \sum_{\alpha \in \mathbb{F}_n^+} \langle f, \varphi_\alpha \rangle e_\alpha$, where $f, \varphi_1, \ldots, \varphi_n$ are seen as elements in the Fock space $F^2(H_n)$.

Theorem 4.2. *Let φ be a free holomorphic self-map of the noncommutative ball $[B(\mathcal{H})^n]_1$. Then the composition operator C_φ on $H^2_{\mathbf{ball}}$ is normal if and only if*

$$\varphi(X_1, \ldots, X_n) = [X_1, \ldots, X_n]A$$

for some normal scalar matrix $A \in M_{n\times n}$ with $\|A\| \leqslant 1$.

Proof. Assume that $A = [a_{ij}]_{n\times n}$ is a scalar matrix and $\|A\| \leqslant 1$. Then it is clear that the relation

$$\varphi(X_1, \ldots, X_n) = [X_1, \ldots, X_n]A, \quad (X_1, \ldots, X_n) \in \big[B(\mathcal{H})^n\big]_1,$$

defines a bounded free holomorphic function $\varphi : [B(\mathcal{H})^n]_1 \to [B(\mathcal{H})^n]_1$. According to Theorem 2.3, the composition operator C_φ is bounded on $H^2_{\mathbf{ball}}$. Setting $\varphi = (\varphi_1, \ldots, \varphi_n)$, we have the Fock representation $\varphi_j = \sum_{p=1}^n a_{pj} e_p$ for each $j = 1, \ldots, n$. Fix $\beta = g_{i_1} \cdots g_{i_k} \in \mathbb{F}_n^+$ and let $\alpha = e_{j_1} \cdots e_{j_k}$. Note that $\langle e_\beta, \varphi_\gamma \rangle = 0$ if $|\alpha| \neq |\gamma|$, $\gamma \in \mathbb{F}_n^+$, and

$$\langle e_\beta, \varphi_\alpha \rangle = \bar{a}_{i_1 j_1} \cdots \bar{a}_{i_k j_k}.$$

Consequently, using Proposition 4.1, we deduce that

$$C_\varphi^* e_\beta = \sum_{|\alpha|=k} \langle e_\beta, \varphi_\alpha\rangle e_\alpha = \sum_{\alpha=e_{j_1}\cdots e_{j_k},\, i_1,\ldots i_k\in\{1,\ldots,n\}} \bar{a}_{i_1 j_1}\cdots \bar{a}_{i_k j_k} e_\alpha.$$

Now, define

$$\psi(X_1,\ldots,X_n) = [X_1,\ldots,X_n]A^*, \quad (X_1,\ldots,X_n)\in \big[B(\mathcal{H})^n\big]_1,$$

and note that $\psi : [B(\mathcal{H})^n]_1 \to [B(\mathcal{H})^n]_1$ is a bounded free holomorphic function. Once again. Theorem 2.3 shows that the composition operator C_ψ is bounded on $H^2_{\mathbf{ball}}$. Setting $\psi = (\psi_1,\ldots,\psi_n)$, we have the Fock representation $\psi_i = \sum_{j=1}^n \bar{a}_{ij} e_j$ for each $i = 1,\ldots,n$. Hence, if $\beta = g_{i_1}\cdots g_{i_k} \in \mathbb{F}_n^+$, we have

$$C_\psi(e_\beta) = \psi_{i_1}\cdots\psi_{i_k} = \sum_{\alpha=e_{j_1}\cdots e_{j_k},\, i_1,\ldots i_k\in\{1,\ldots,n\}} \bar{a}_{i_1 j_1}\cdots \bar{a}_{i_k j_k} e_\alpha.$$

This shows that $C_\varphi^* = C_\psi$. If we assume that A is a normal matrix, then $\varphi\circ\psi = \psi\circ\varphi$. Indeed, for any $(X_1,\ldots,X_n)\in [B(\mathcal{H})^n]_1$, we have

$$(\varphi\circ\psi)(X_1,\ldots,X_n) = [X_1,\ldots,X_n]A^*A = [X_1,\ldots,X_n]AA^* = (\psi\circ\varphi)(X_1,\ldots,X_n).$$

Consequently, we deduce that

$$C_\varphi C_\varphi^* = C_\varphi C_\psi = C_{\psi\circ\varphi} = C_{\varphi\circ\psi} = C_\psi C_\varphi = C_\varphi^* C_\varphi.$$

Now we prove the direct implication. Assume that φ is a free holomorphic self-map of the noncommutative ball $[B(\mathcal{H})^n]_1$ and the composition operator C_φ is normal. Since $C_\varphi 1 = 1$, the vector $1 \in F^2(H_n)$ is also an eigenvector for C_φ^*. Since, due to Theorem 4.1, $C_\varphi^* 1 = \sum_{\alpha\in\mathbb{F}_n^+} \langle 1, \varphi_\alpha\rangle e_\alpha$, we deduce that $\langle 1, \varphi_\alpha\rangle = 0$ for all $\alpha\in\mathbb{F}_n^+$ with $|\alpha| \geqslant 1$. In particular, we have $\langle 1, \varphi_i\rangle = 0$ which implies $\varphi_i(0) = 0$ for $i = 1,\ldots,n$. Therefore $\varphi(0) = 0$ and $C_\varphi^* 1 = 1$. Consequently, we have

$$\varphi(X_1,\ldots,X_n) = [X_1,\ldots,X_n]A + (\psi_1,\ldots,\psi_n)$$

for some matrix $A\in M_{n\times n}$ and bounded free holomorphic functions $\psi_i = \sum_{|\alpha|\geqslant 2} c_\alpha^{(i)} e_\alpha$, $i = 1,\ldots,n$. Consequently, using again the Fock space representation formula for the adjoint of C_φ, we obtain

$$C_\varphi^*(e_{g_i}) = \sum_{\alpha\in\mathbb{F}_n^+} \langle e_{g_i}, \varphi_\alpha\rangle e_\alpha,$$

which implies that the subspace $\mathcal{M} := \operatorname{span}\{e_{g_i}\colon\ i = 1,\ldots,n\}$ is invariant under C_φ^*. Since $\mathcal{M}$ is finite dimensional, it is also invariant under C_φ and $C_\varphi|_{\mathcal{M}}$ is a normal operator. This implies that, for each $j = 1,\ldots,n$, $C_\varphi(e_j)$ is a linear combination of $e_1,\ldots,e_n$ and, consequently, $\varphi(X_1,\ldots,X_n) = [X_1,\ldots,X_n]A$ for $(X_1,\ldots,X_n)\in[B(\mathcal{H})^n]_1$. Since $\varphi : [B(\mathcal{H})^n]_1 \to [B(\mathcal{H})^n]_1$, we must have $\|A\| \leqslant 1$. Setting $\psi(X_1,\ldots,X_n) = [X_1,\ldots,X_n]A^*$ for $(X_1,\ldots,X_n)\in$

$[B(\mathcal{H})^n]_1$, the first part of the proof shows that C_ψ is a bounded operator on $H^2_{\mathbf{ball}}$ and $C_\varphi^* = C_\psi$. Since C_φ is normal, we have

$$C_{\psi\circ\varphi} = C_\varphi C_\psi = C_\varphi C_\varphi^* = C_\varphi^* C_\varphi = C_\psi C_\varphi = C_{\varphi\circ\psi},$$

which implies $\psi \circ \varphi(X) = \varphi \circ \psi(X)$, $X \in [B(\mathcal{H})^n]_1$. Hence, we deduce that $[X_1, \ldots, X_n]A^*A = [X_1, \ldots, X_n]AA^*$ for any $(X_1, \ldots, X_n) \in [B(\mathcal{H})^n]_1$, which implies $A^*A = AA^*$. The proof is complete. □

Due to Theorem 4.2, characterizations of self-adjoint or unitary composition operators on $H^2_{\mathbf{ball}}$ are now obvious.

Lemma 4.3. *Let φ be a free holomorphic self-map of the noncommutative ball $[B(\mathcal{H})^n]_1$ and let C_φ be the composition operator on $H^2_{\mathbf{ball}}$. If the kernel of C_φ^* is finite dimensional, then the scalar representation of φ is one-to-one.*

Proof. Let $\lambda^{(j)} = (\lambda_1^{(j)}, \ldots, \lambda_n^{(j)})$, $j = 1, \ldots, k$, be k distinct points in $\mathbb{B}_n$ and fix $p \in \{1, \ldots, k\}$. For each $j \in \{1, \ldots, k\}$ with $j \neq p$, there exists $q_j \in \{1, \ldots, n\}$ such that $\lambda_{q_j}^{(p)} \neq \lambda_{q_j}^{(j)}$. Define the free holomorphic function $\varphi_p : [B(\mathcal{H})^n]_1 \to B(\mathcal{H})$ by setting

$$\varphi_p(X_1, \ldots, X_n) = \prod_{j\in\{1,\ldots,k\},\, j\neq p} \frac{1}{\lambda_{q_j}^{(p)} - \lambda_{q_j}^{(j)}} \big(X_{q_j} - \lambda_{q_j}^{(j)} I\big).$$

Note that $\varphi_p(\lambda^{(p)}) = 1$ and $\varphi_p(\lambda^{(j)}) = 0$ for any $j \in \{1, \ldots, k\}$ with $j \neq p$.

For each $\mu := (\mu_1, \ldots, \mu_n) \in \mathbb{B}_n$, we define the vector $z_\mu := \sum_{k=0} \sum_{|\alpha|=k} \overline{\mu}_\alpha e_\alpha$, where $\mu_\alpha := \mu_{i_1} \cdots \mu_{i_p}$ if $\alpha = g_{i_1} \cdots g_{i_p} \in \mathbb{F}_n^+$ and $i_1, \ldots, i_p \in \{1, \ldots, n\}$, and $\mu_{g_0} = 1$. Since $z_\mu \in F^2(H_n)$ and $S_i^* z_\mu = \overline{\mu}_i z_\mu$, one can see that $q(S_1, \ldots, S_n)^* z_\mu = \overline{q(\mu)} z_\mu$ for any noncommutative polynomial q. Now we prove that the vectors $z_{\lambda^{(1)}}, \ldots, z_{\lambda^{(k)}}$ are linearly independent. Let $a_1, \ldots, a_k \in \mathbb{C}$ be such that $a_1 z_{\lambda^{(1)}} + \cdots + a_k z_{\lambda^{(k)}} = 0$. Due to the properties of the free holomorphic function φ_p, $p \in \{1, \ldots, k\}$, we deduce that

$$\begin{aligned}\varphi_p(S_1, \ldots, S_n)^*(a_1 z_{\lambda^{(1)}} + \cdots + a_k z_{\lambda^{(k)}}) &= a_1 \overline{\varphi_p\big(\lambda^{(1)}\big)} z_{\lambda^{(1)}} + \cdots + a_k \overline{\varphi_p\big(\lambda^{(k)}\big)} z_{\lambda^{(k)}} \\ &= a_p \overline{\varphi_p\big(\lambda^{(p)}\big)} z_{\lambda^{(p)}} = a_p z_{\lambda^{(p)}} = 0.\end{aligned}$$

Hence, we deduce that $a_1 = \cdots = a_k = 0$, which proves our assertion.

Let $\psi : \mathbb{B}_n \to \mathbb{B}_n$ be the scalar representation of φ, i.e., $\psi(\lambda) = \varphi(\lambda)$, $\lambda \in \mathbb{B}_n$. Assume that there is $\xi \in \mathbb{B}_n$ such that $\psi^{-1}(\xi)$ is an infinite set. Let $\{\lambda^{(j)}\}_{k\in\mathbb{N}} \subset \psi^{-1}(\xi)$ be a sequence of distinct points. Due to relation (2.9), we have $C_\varphi^*(z_{\lambda^{(j)}}) = C_\varphi^*(z_{\lambda^{(k)}}) = z_\xi$, which implies $z_{\lambda^{(j)}} - z_{\lambda^{(k)}} \in \ker C_\varphi^*$. As shown above, $\{z_{\lambda^{(j)}}\}_{j\in\mathbb{N}}$ is a set of linearly independent vectors. Consequently, $\ker C_\varphi^*$ is infinite dimensional, which contradicts the hypothesis. Therefore, for each $\xi \in \mathbb{B}_n$, the inverse image $\psi^{-1}(\xi)$ is a finite set. According to Rudin's result (Theorem 15.1.6 from [41]), $\psi : \mathbb{B}_n \to \mathbb{B}_n$ is an open map. Suppose that ψ is not one-to-one. Let $u, v \in \mathbb{B}_n$, $u \neq v$, be such that $\psi(u) = \psi(v)$, and let U, V be open sets in $\mathbb{B}_n$ with the property that $u \in U$, $v \in V$, and $U \cap V \neq \emptyset$. Since ψ is an open map, we deduce that $\psi(U) \cap \psi(V)$ is a nonempty open set. Consequently, we can find sequences $\{\lambda^{(j)}\}_{j\in\mathbb{N}} \subset U$ and $\{\mu^{(j)}\}_{j\in\mathbb{N}} \subset V$ of distinct points such that $\psi(\lambda^{(j)}) = \psi(\mu^{(j)})$ for all $j \in \mathbb{N}$. As above, we deduce that $z_{\lambda^{(j)}} - z_{\mu^{(j)}} \in \ker C_\varphi^*$ for $j \in \mathbb{N}$. Using the linear independence of the set $\{z_{\lambda^{(j)}}\}_{j\in\mathbb{N}} \cup \{z_{\mu^{(j)}}\}_{j\in\mathbb{N}}$, we deduce that $\ker C_\varphi^*$ is infinite dimensional, which contradicts the hypothesis. Therefore, ψ is a one-to-one map. The proof is complete. □

Note that, unlike the single variable case, if $n \geqslant 2$, then the composition operator C_φ is not one-to-one on $H^2_{\mathbf{ball}}$. For example, one can take $\varphi = (\varphi_1, \varphi_1) : [B(\mathcal{H})^2]_1 \to [B(\mathcal{H})^2]_1$ and $f = e_1e_2 - e_2e_1$, and note that $C_\varphi f = 0$.

We remark that if $\varphi \in Aut([B(\mathcal{H})^n]_1)$, then the composition operator C_φ is invertible on $H^2_{\mathbf{ball}}$ and therefore Fredholm. It will be interesting to see if the converse is true. At the moment, we can prove the following result.

Theorem 4.4. *Let φ be a free holomorphic self-map of the noncommutative ball $[B(\mathcal{H})^n]_1$. If C_φ is a Fredholm operator on $H^2_{\mathbf{ball}}$, then the scalar representation of φ is a holomorphic automorphism of $\mathbb{B}_n$.*

Proof. Let $\psi : \mathbb{B}_n \to \mathbb{B}_n$ be the scalar representation of φ, i.e., $\psi(\lambda) := \varphi(\lambda)$, $\lambda \in \mathbb{B}_n$. Due to Lemma 4.3, ψ is a one-to-one holomorphic map. We need to prove that ψ is surjective. To this end, assume that ψ is not surjective. Then there is a sequence $\{\lambda^{(k)}\} \subset \mathbb{B}_n$ and $\zeta \in \partial\mathbb{B}_n$ such that $\lambda^{(k)} \to \zeta$ as $k \to \infty$ and $\psi(\lambda^{(k)}) \to w$ for some $w \in \mathbb{B}_n$.

As we will see in the proof of Theorem 5.4 (see relation (5.2)), $\frac{z_{\lambda^{(k)}}}{\|z_{\lambda^{(k)}}\|} \to 0$ weakly as $k \to \infty$. On the other hand taking into account relation (2.9), we have

$$C_\varphi^* z_{\lambda^{(k)}} = \sum_{k=0} \sum_{|\alpha|=k} \overline{\varphi_\alpha\left(\lambda^{(k)}\right)} e_\alpha = z_{\varphi(\lambda^{(k)})}, \quad k \in \mathbb{N}.$$

Hence, we get

$$\left\| C_\varphi^* \left(\frac{z_{\lambda^{(k)}}}{\|z_{\lambda^{(k)}}\|} \right) \right\| = \frac{\|z_{\varphi(\lambda^{(k)})}\|}{\|z_{\lambda^{(k)}}\|}.$$

Since $\|z_{\varphi(\lambda^{(k)})}\| \to \|z_w\| < \infty$ and $\|z_{\lambda^{(k)}}\| \to \infty$ as $k \to \infty$, we deduce that $\|C_\varphi^*(\frac{z_{\lambda^{(k)}}}{\|z_{\lambda^{(k)}}\|})\| \to 0$ as $k \to \infty$. Denote $f_k := \frac{z_{\lambda^{(k)}}}{\|z_{\lambda^{(k)}}\|}$. Since C_φ is a Fredholm operator on $H^2_{\mathbf{ball}}$, there is an operator $\Lambda \in B(F^2(H_n))$ such that $\Lambda C_\varphi^* - I = K$ for some compact operator $K \in B(F^2(H_n))$. Consequently, we have

$$\left\| \Lambda C_\varphi^* f_k \right\|^2 = \|f_k + Kf_k\|^2 = \|f_k\|^2 + \|Kf_k\|^2 + 2\Re\langle f_k, Kf_k\rangle. \tag{4.1}$$

Since K is a compact operator, $\|f_k\| = 1$ and $f_k \to 0$ weakly as $k \to \infty$, we must have $\|Kf_k\| \to 0$. Consequently, we have $|\Re\langle f_k, Kf_k\rangle| \leqslant \|f_k\|\|Kf_k\| \to 0$ as $k \to \infty$. On the other hand, we have $\|C_\varphi^* f_k\| \to 0$. Now it is easy to see that relation (4.1) leads to a contradiction. Therefore, ψ is surjective. In conclusion ψ is an automorphism of $\mathbb{B}_n$. □

5. Compact composition operators on $H^2_{\mathbf{ball}}$

In this section we obtain a formula for the essential norm of the composition operators C_φ on $H^2_{\mathbf{ball}}$. In particular, this implies a characterization of compact composition operators. We show that if C_φ is a compact operator on $H^2_{\mathbf{ball}}$, then the scalar representation of φ is a holomorphic self-map of $\mathbb{B}_n$ which cannot have finite angular derivative at any point of $\partial\mathbb{B}_n$ and has exactly one fixed point in the open ball $\mathbb{B}_n$. As a consequence, we deduce that every compact composition operator on $H^2_{\mathbf{ball}}$ is similar to a contraction. In the end of this section, we prove that the set of compact composition operators on $H^2_{\mathbf{ball}}$ is arcwise connected in the set of all composition operators.

We recall that the essential norm of a bounded operator $T \in B(\mathcal{H})$ is defined by

$$\|T\|_e := \inf\bigl\{\|T - K\|:\ K \in B(\mathcal{H}) \text{ is compact}\bigr\}.$$

Theorem 5.1. *Let φ be a free holomorphic self-map of the noncommutative ball $[B(\mathcal{H})^n]_1$. Then the essential norm of the composition operator C_φ on $H^2_{\mathbf{ball}}$ satisfies the equality*

$$\|C_\varphi\|_e = \lim_{k\to\infty} \sup_{f\in H^2_{\mathbf{ball}},\ \|f\|_2\leqslant 1} \left(\sum_{|\alpha|\geqslant k} \bigl|\langle f, \varphi_\alpha\rangle\bigr|^2\right)^{1/2}.$$

Consequently, C_φ is a compact operator if and only if

$$\lim_{k\to\infty} \sup_{f\in H^2_{\mathbf{ball}},\ \|f\|_2\leqslant 1} \sum_{|\alpha|\geqslant k} \bigl|\langle f, \varphi_\alpha\rangle\bigr|^2 = 0.$$

Proof. Let φ be a free holomorphic self-map of the noncommutative ball $[B(\mathcal{H})^n]_1$. Since C_φ is a bounded composition operator on $H^2_{\mathbf{ball}}$ (see Theorem 2.3), one can use standard arguments (see Proposition 5.1 from [44]) to show that the essential norm of the composition operator C_φ on $H^2_{\mathbf{ball}}$ satisfies the equality

$$\|C_\varphi\|_e = \lim_{k\to\infty} \|C_\varphi P_k\|, \tag{5.1}$$

where P_k is the orthogonal projection of $F^2(H_n)$ onto the closed linear span of all e_α with $\alpha \in \mathbb{F}_n^+$ and $|\alpha| \geqslant k$. Indeed, note that the sequence $\{\|C_\varphi P_k\|\}_{k=1}^\infty$ is decreasing and, due to the fact that $I - P_k$ is a finite rank projection, we have $\|C_\varphi\|_e \leqslant \|C_\varphi P_k\|$ for any $k \in \mathbb{N}$. Hence $\|C_\varphi\|_e \leqslant \lim_{k\to\infty} \|C_\varphi P_k\|$. On the other hand, let K be a compact operator and $a := \lim_{k\to\infty} \|K P_k\|$. Assume that $a > 0$ and let $\epsilon > 0$ with $0 < a - \epsilon$. Then there is a sequence $h_k \in F^2(H_n)$ with $\|h_k\| \leqslant 1$, such that $\|P_k K^* h_k\| \geqslant a - \epsilon$ for any $k \geqslant N$ and some $N \in \mathbb{N}$. Since K^* is a compact operator, there is a subsequence $k_m \in \mathbb{N}$ such that $K^* h_{k_m} \to v$ for some $v \in F^2(H_n)$. Consequently, taking into account that $P_{k_m} v \to 0$, $\|P_k\| \leqslant 1$, and

$$\bigl\|P_{k_m} K^* h_{k_m}\bigr\| \leqslant \|P_{k_m} v\| + \|P_{k_m}\| \bigl\|v - K^* h_{k_m}\bigr\|,$$

we deduce that $P_{k_m}K^*h_{k_m} \to 0$, which is a contradiction. Therefore, $\lim_{k\to\infty} \|KP_k\| = 0$. Note also that

$$\|C_\varphi - K\| \geqslant \big\|(C_\varphi - K)P_k\big\| \geqslant \|C_\varphi P_k\| - \big\|P_k K^*\big\|.$$

Now, taking $k \to \infty$, we obtain $\|C_\varphi - K\| \geqslant \lim_{k\to\infty} \|C_\varphi P_k\|$, which proves relation (5.1).

According to Proposition 4.1 and the remarks that follow, we have

$$P_k C_\varphi^* f = \sum_{|\alpha|\geqslant k} \langle f, \varphi_\alpha\rangle e_\alpha, \quad f \in F^2(H_n),$$

where P_k is the orthogonal projection of the full Fock space $F^2(H_n)$ onto the closed span of the vectors $\{e_\alpha: \alpha \in \mathbb{F}_n^+, |\alpha| \geqslant k\}$, and $f, \varphi_1, \ldots, \varphi_n$ are seen as elements of the Fock space $F^2(H_n)$. Hence, we deduce that

$$\big\|P_k C_\varphi^*\big\| = \sup_{f\in H^2_{\mathbf{ball}},\, \|f\|\leqslant 1} \left(\sum_{|\alpha|\geqslant k} \big|\langle f, \varphi_\alpha\rangle\big|^2\right)^{1/2}.$$

Combining this result with relation (5.1), we obtain the formula for the essential norm of C_φ. The last part of the theorem is now obvious. □

Proposition 5.2. *Let $\varphi := (\varphi_1, \ldots, \varphi_n)$ be a free holomorphic self-map of the noncommutative ball $[B(\mathcal{H})^n]_1$ and let C_φ be the composition operator on $H^2_{\mathbf{ball}}$. Then the following statements hold.*

(i) *If φ is inner then C_φ is not compact.*
(ii) *If $\|\varphi\|_\infty < 1$ then C_φ is compact.*
(iii) *If $\|\varphi_1\|_\infty + \cdots + \|\varphi_n\|_\infty < 1$, then C_φ is a trace class operator.*
(iv) *If $\|\varphi_1\|_\infty^2 + \cdots + \|\varphi_n\|_\infty^2 < 1$, then C_φ is a Hilbert–Schmidt operator.*

Proof. To prove item (i), assume first that φ is an inner free holomorphic self-map of the noncommutative ball $[B(\mathcal{H})^n]_1$ with $\varphi(0) = 0$. As in the proof of Theorem 2.2, $\{\varphi_\alpha\}_{\alpha\in\mathbb{F}_n^+}$ is an orthonormal set in $H^2_{\mathbf{ball}}$. Consequently, if $\{a_\alpha\}_{|\alpha|\geqslant k} \subset \mathbb{C}$ is such that $\sum_{|\alpha|\geqslant k} |a_\alpha|^2 = 1$, then $g := \sum_{|\beta|\geqslant k} a_\beta\varphi_\beta$ is in $F^2(H_n)$ and $\|g\|_2 = 1$. Note also that

$$\sum_{|\alpha|\geqslant k} \big|\langle g, \varphi_\alpha\rangle\big|^2 = \sum_{|\alpha|\geqslant k} |a_\alpha|^2 = 1.$$

Since $\{\varphi_\alpha\}_{\alpha\in\mathbb{F}_n^+}$ is an orthonormal set in $H^2_{\mathbf{ball}}$, we have $\sum_{|\alpha|\geqslant k} |\langle f, \varphi_\alpha\rangle|^2 \leqslant \|f\|_2$ for any $f \in H^2_{\mathbf{ball}}$. Now, one can deduce that

$$\sup_{f\in H^2_{\mathbf{ball}},\, \|f\|\leqslant 1} \left(\sum_{|\alpha|\geqslant k} \big|\langle f, \varphi_\alpha\rangle\big|^2\right)^{1/2} = 1.$$

Due to Theorem 5.1, we deduce that $\|C_\varphi\|_e = 1$. Now, we consider the case when $\xi := \varphi(0) \neq 0$. Since the involutive free holomorphic automorphism Φ_ξ is inner and the composition of inner free holomorphic functions is inner (see [39]), we deduce that $\Psi := \Phi_\xi \circ \varphi$ is an inner free holomorphic self-map of $[B(\mathcal{H})^n]_1$. Since $\Psi(0) = 0$, the first part of the proof shows that C_Ψ is not compact. Taking into account that $C_\Psi = C_\varphi C_{\Phi_\xi}$, we deduce that C_φ is not compact.

To prove item (ii), let $\widetilde{\varphi} := (\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n)$ be the boundary function with respect to the left creation operators $S_1, \ldots, S_n$, and set $\|\widetilde{\varphi}\| = s < 1$. It is easy to see that $\|[\widetilde{\varphi}_\alpha\colon |\alpha| = k]\| \leqslant \|[\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n]\|^k = s^k$, $k \in \mathbb{N}$. For any $g \in F^2(H_n)$ and $m \in \mathbb{N}$, we have

$$\begin{aligned}
\left\| C_{\widetilde{\varphi}} g - \sum_{k=0}^{m} \sum_{|\alpha|=k} \langle g, e_\alpha\rangle \widetilde{\varphi}_\alpha(1) \right\| &= \left\| \sum_{k=m+1} \sum_{|\alpha|=k} \langle g, e_\alpha\rangle \widetilde{\varphi}_\alpha(1) \right\| \\
&\leqslant \sum_{k=m+1} \left\| [\widetilde{\varphi}_\alpha\colon |\alpha| = k] \begin{bmatrix} \langle g, e_\alpha\rangle \\ \vdots \\ |\alpha| = k \end{bmatrix} \right\| \\
&\leqslant \sum_{k=m+1} s^k \left(\sum_{|\alpha|=k} \big|\langle g, e_\alpha\rangle\big|^2 \right)^{1/2} \\
&\leqslant \left(\sum_{k=m+1} s^{2k} \right)^{1/2} \left(\sum_{k=m+1} \sum_{|\alpha|=k} \big|\langle g, e_\alpha\rangle\big|^2 \right)^{1/2} \\
&\leqslant \|g\|_2 \frac{s^m}{\sqrt{1-s^2}}.
\end{aligned}$$

Consequently, the operator $G_m : F^2(H_n) \to F^2(H_n)$ defined by

$$G_m(g) := \sum_{k=0}^{m} \sum_{|\alpha|=k} \langle g, e_\alpha\rangle \widetilde{\varphi}_\alpha(1)$$

has finite rank and converges to the composition operator $C_{\widetilde{\varphi}}$ in the operator norm topology. Therefore, C_φ is a compact operator.

To prove item (iii), note that

$$\sum_{\alpha\in\mathbb{F}_n^+} \|C_{\widetilde{\varphi}} e_\alpha\| = \sum_{\alpha\in\mathbb{F}_n^+} \big\|\widetilde{\varphi}_\alpha(1)\big\| \leqslant \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \|\widetilde{\varphi}_\alpha\| \leqslant \sum_{k=0}^{\infty} \big(\|\widetilde{\varphi}_1\| + \cdots + \|\widetilde{\varphi}_n\|\big)^k < \infty.$$

Consequently, C_φ is a trace class operator. Finally, we prove item (iv). First, note that C_φ is a Hilbert–Schmidt operator if and only if $\sum_{\alpha\in\mathbb{F}_n^+} \|\varphi_\alpha\|_2^2 < \infty$. On the other hand, as above, one ca show that

$$\sum_{\alpha\in\mathbb{F}_n^+} \|C_{\widetilde{\varphi}} e_\alpha\|^2 \leqslant \sum_{k=0}^{\infty} \big(\|\widetilde{\varphi}_1\|^2 + \cdots + \|\widetilde{\varphi}_n\|^2\big)^k < \infty,$$

which shows that C_φ is a Hilbert–Schmidt operator. The proof is complete. □

Corollary 5.3. *If φ is an inner free holomorphic self-map of the noncommutative ball $[B(\mathcal{H})^n]_1$ such that $\varphi(0)=0$, then the essential norm of the composition operator C_φ on $H^2_{\mathbf{ball}}$ is* 1.

Theorem 5.4. *Let φ be a free holomorphic self-map of the noncommutative ball $[B(\mathcal{H})^n]_1$ and let C_φ be the composition operator on $H^2_{\mathbf{ball}}$. Then the following statements hold.*

(i) *The essential norm of C_φ on $H^2_{\mathbf{ball}}$ satisfies the inequality*

$$\|C_\varphi\|_e \geqslant \limsup_{\|\lambda\|\to 1}\left(\frac{1-\|\lambda\|^2}{1-\|\varphi(\lambda)\|^2}\right)^{1/2}.$$

(ii) *If C_φ is a compact operator on $H^2_{\mathbf{ball}}$, then the scalar representation of φ cannot have finite angular derivative at any point of $\partial\mathbb{B}_n$.*

Proof. For each $\mu := (\mu_1,\ldots,\mu_n)\in\mathbb{B}_n$, we define the vector $z_\mu := \sum_{k=0}^\infty \sum_{|\alpha|=k}\bar{\mu}_\alpha e_\alpha$, where $\mu_\alpha := \mu_{i_1}\cdots\mu_{i_p}$ if $\alpha = g_{i_1}\cdots g_{i_p}\in\mathbb{F}_n^+$ and $i_1,\ldots,i_p\in\{1,\ldots,n\}$, and $\mu_{g_0}=1$. Since $z_\mu\in F^2(H_n)$ and $S_i^* z_\mu = \bar{\mu}_i z_\mu$, one can see that $q(S_1,\ldots,S_n)^* z_\mu = \overline{q(\mu)} z_\mu$ for any noncommutative polynomial q. Let $\lambda^{(j)} := (\lambda_1^{(j)},\ldots,\lambda_n^{(j)})\in\mathbb{B}_n$ be such that $\|\lambda^{(j)}\|\to 1$ as $j\to\infty$. Since $\|z_\mu\| = \frac{1}{\sqrt{1-\|\mu\|^2}}$, we deduce that

$$\lim_{j\to\infty}\left\langle q, \frac{z_{\lambda^{(j)}}}{\|z_{\lambda^{(j)}}\|}\right\rangle = \lim_{j\to\infty}\frac{q(\lambda^{(j)})}{\|z_{\lambda^{(j)}}\|} = 0,$$

where q is seen as a noncommutative polynomial in $F^2(H_n)$. Consequently, since the unit ball of $F^2(H_n)$ is weakly compact and the polynomials are dense in $F^2(H_n)$, there is a subsequence $\frac{z_{\lambda^{(j_k)}}}{\|z_{\lambda^{(j_k)}}\|}$ which converges weakly to 0 as $j_k\to\infty$. Since this is true for any subsequence, we deduce that

$$\frac{z_{\lambda^{(j)}}}{\|z_{\lambda^{(j)}}\|}\to 0 \quad \text{weakly as } \left\|\lambda^{(j)}\right\|_2 \to 1. \tag{5.2}$$

If $K\in B(F^2(H_n))$ is an arbitrary compact operator, then $\lim_{\|\lambda^{(j)}\|\to 1}\|K^*(\frac{z_{\lambda^{(j)}}}{\|z_{\lambda^{(j)}}\|})\| = 0$. On the other hand, due to relation (2.9), we have

$$\left\|C_\varphi^* z_{\lambda^{(j)}}\right\| = \left(\frac{1}{1-\|\varphi(\lambda^{(j)})\|^2}\right)^{1/2}.$$

Using all these facts, we deduce that

$$\begin{aligned}\|C_\varphi\|_e &= \inf\bigl\{\|T-K\|:\ K\in B(\mathcal{H}) \text{ is compact}\bigr\}\\ &\geqslant \limsup_{\|\lambda^{(j)}\|\to 1}\left\|(C_\varphi - K)^*\left(\frac{z_{\lambda^{(j)}}}{\|z_{\lambda^{(j)}}\|}\right)\right\|\end{aligned}$$

$$= \limsup_{\|\lambda^{(j)}\|\to 1} \left\| C_\varphi^* \left(\frac{z_{\lambda^{(j)}}}{\|z_{\lambda^{(j)}}\|} \right) \right\|$$

$$= \limsup_{\|\lambda^{(j)}\|\to 1} \left(\frac{1 - \|\lambda^{(j)}\|^2}{1 - \|\varphi(\lambda^{(j)})\|^2} \right)^{1/2},$$

which proves item (i).

To prove part (ii), we recall that the Julia–Carathéodory theorem in $\mathbb{B}_n$ asserts that if $\psi : \mathbb{B}_n \to \mathbb{B}_n$ is analytic and $\xi \in \partial\mathbb{B}_n$, then ψ has finite angular derivative at ξ if and only if

$$\liminf_{\lambda\to\xi} \frac{1 - \|\psi(\lambda)\|}{1 - \|\lambda\|} < \infty,$$

where the limit is taking as $\lambda \to \xi$ unrestrictedly in $\mathbb{B}_n$. If C_φ is a compact operator on $H^2_{\mathbf{ball}}$, then according to part (i), we have

$$\limsup_{\lambda\to\xi} \left(\frac{1 - \|\lambda\|^2}{1 - \|\varphi(\lambda)\|^2} \right)^{1/2} = 0.$$

Now, combining these results when $\psi : \mathbb{B}_n \to \mathbb{B}_n$ is defined by $\psi(\lambda) := \varphi(\lambda)$, $\lambda \in \mathbb{B}_n$, the result in part (ii) follows. The proof is complete. □

We need the following lemma which can be extracted from [14]. We include a proof for completeness.

Lemma 5.5. *Let $\psi = (\psi_1, \ldots, \psi_n)$ be a holomorphic self-map of the open unit ball $\mathbb{B}_n$ with the property that $\psi(E(L, \zeta_1)) \subseteq E(L, \zeta_1)$ for each ellipsoid*

$$E(L, \zeta_1) := \left\{ \lambda \in \mathbb{B}_n \colon \left| 1 - \langle \lambda, \zeta_1 \rangle \right|^2 \leqslant L\left(1 - \|\lambda\|^2\right) \right\}, \quad L > 0,$$

where $\zeta_1 := (1, 0, \ldots, n) \in \mathbb{B}_n$. Then the slice function $\phi_{\zeta_1} : \mathbb{D} \to \mathbb{D}$ defined by $\phi_{\zeta_1}(z) := \psi_1(z, 0 \ldots, 0)$, $z \in \mathbb{D}$, has the property that

$$\liminf_{z\to 1} \frac{1 - |\phi_{\zeta_1}(z)|}{1 - |z|} \leqslant 1.$$

Proof. Note that when $w = (r, 0, \ldots, 0) \in \mathbb{B}_n$ with $r \in (0, 1)$ and $L := \frac{1-r}{1+r}$, the inclusion $\psi(E(L, \zeta_1)) \subseteq E(L, \zeta_1)$ implies

$$\frac{|1 - \psi_1(w)|^2}{1 - \|\psi(w)\|^2} \leqslant L.$$

Hence, and using the inequality $1 - |\psi_1(w)| \leqslant |1 - \psi_1(w)|$, we obtain

$$\frac{1 - |\psi_1(w)|}{1 + |\psi_1(w)|} \leqslant \frac{1 - r}{1 + r},$$

which implies $|\psi_1(w)| \geqslant r = \|w\|$ and, therefore,

$$\frac{1 - |\psi_1(w)|}{1 - \|w\|} \leqslant 1$$

for $w = (r, 0, \dots, 0) \in \mathbb{B}_n$. The latter inequality can be used to complete the proof. □

In what follows we also need the following lemma. Since the proof is straightforward, we shall omit it. We denote by $H^2([B(\mathcal{H})]_1)$ the Hilbert space of all free holomorphic functions on $[B(\mathcal{H})]_1$ of the form $f(X) = \sum_{k=0}^{\infty} c_k X^k$ with $\sum_{k=0}^{\infty} |a_k|^2 < \infty$. It is easy to see that $H^2([B(\mathcal{H})]_1)$ can be identified with the classical Hardy space $H^2(\mathbb{D})$.

Lemma 5.6. *Let $F : [B(\mathcal{H})^n]_1 \to B(\mathcal{H})$ be a free holomorphic function and let $\zeta_1 := (1, 0, \dots, 0) \in \partial\mathbb{B}_n$. The slice function $F_{\zeta_1} : [B(\mathcal{H})]_1 \to B(\mathcal{H})$ defined by*

$$F_{\zeta_1}(Y) := F(\zeta_1 Y), \quad Y \in \left[B(\mathcal{H})\right]_1,$$

has the following properties.

(i) *F_{ζ_1} is a free holomorphic function on $[B(\mathcal{H})]_1$.*
(ii) *If $F \in H^2_{\mathbf{ball}}$ then $F_{\zeta_1} \in H^2([B(\mathcal{H})]_1)$ and $\|F_{\zeta_1}\|_2 \leqslant \|F\|_2$.*
(iii) *The inclusion $H^2([B(\mathcal{H})]_1) \subset H^2_{\mathbf{ball}}$ is an isometry.*
(iv) *Under the identification of $H^2_{\mathbf{ball}}$ with the full Fock space $F^2(H_n)$,*

$$F_{\zeta_1} = P_{F^2(H_1)} F,$$

where $P_{F^2(H_1)}$ is the orthogonal projection of $F^2(H_n)$ onto $F^2(H_1) \subset F^2(H_n)$.
(v) *If F is bounded on $[B(\mathcal{H})^n]_1$, then F_{ζ_1} is bounded on $[B(\mathcal{H})]_1$ and $\|F_{\zeta_1}\|_\infty \leqslant \|F\|_\infty$.*

Now, we have all the ingredients to prove the following result.

Theorem 5.7. *Let $\varphi = (\varphi_1, \dots, \varphi_n)$ be a free holomorphic self-map of the noncommutative ball $[B(\mathcal{H})^n]_1$. If C_φ is a compact composition operator on $H^2_{\mathbf{ball}}$, then the scalar representation of φ is a holomorphic self-map of $\mathbb{B}_n$ which has exactly one fixed point in the open ball $\mathbb{B}_n$.*

Proof. Let $\psi = (\psi_1, \dots, \psi_n)$ be the scalar representation of φ, i.e. the map $\psi : \mathbb{B}_n \to \mathbb{B}_n$ defined by $\psi(\lambda) := \phi(\lambda)$, $\lambda \in \mathbb{B}_n$. It is clear that ψ is a holomorphic self-map of the open unit ball $\mathbb{B}_n$. Assume that ψ has no fixed points in $\mathbb{B}_n$. According to [13] (see also Theorem 3.1), there exists a unique Denjoy–Wolff point $\zeta \in \partial\mathbb{B}_n$ such that $\psi(E(L, \zeta)) \subseteq E(L, \zeta)$ for each ellipsoid $E(L, \zeta)$, $L > 0$. Without loss of generality we can assume that $\zeta = \zeta_1 := (1, 0, \dots, 0) \in \mathbb{B}_n$. Then, due to Lemma 5.5, the slice function $\phi_{\zeta_1} : \mathbb{D} \to \mathbb{D}$ defined by $\phi_{\zeta_1}(z) := \psi_1(z, 0 \dots, 0)$ has the property that

$$\liminf_{z \to 1} \frac{1 - |\phi_{\zeta_1}(z)|}{1 - |z|} \leqslant 1.$$

According to Julia–Carathéodory theorem (see [41]), ϕ_{ζ_1} has finite angular derivative at 1 which is less than or equal to 1. On the other hand, it is well known (see also Theorem 5.4 when $n = 1$)

that if a composition operator is compact on $H^2(\mathbb{D})$, then its symbol cannot have a finite angular derivative at any point. Consequently, $C_{\phi_{\zeta_1}}$ is not a compact operator on $H^2(\mathbb{D})$.

Under the identification of $H^2_{\mathbf{ball}}$ with the full Fock space $F^2(H_n)$, set

$$\Gamma = P_{F^2(H_1)}\varphi_1, \tag{5.3}$$

where $P_{F^2(H_1)}$ is the orthogonal projection of $F^2(H_n)$ onto $F^2(H_1) \subset F^2(H_n)$. According to Lemma 5.5, $\Gamma : [B(\mathcal{H})]_1 \to [B(\mathcal{H})]_1$ is a bounded free holomorphic function. Now we show that C_Γ is a compact composition operator on $F^2(H_1)$. Let $\{f^{(m)}\}_{m=1}^\infty$ be a bounded sequence in $F^2(H_1)$ such that $f^{(m)} \to 0$ weakly in $F^2(H_1)$. Since $F^2(H_1) \subset F^2(H_n)$ and $F^2(H_n) = F^2(H_1) \oplus F^2(H_1)^\perp$, it is easy to see that $f^{(m)} \to 0$ weakly in $F^2(H_n)$. Due to the compactness of C_φ on $F^2(H_n)$, we must have

$$\left\| C_\varphi f^{(m)} \right\|_{F^2(H_n)} \to 0 \quad \text{as } m \to \infty. \tag{5.4}$$

Since $f^{(m)} \in F^2(H_1)$, it has the representation $f^{(m)} = \sum_{k=0}^\infty a_k^{(m)} e_1^k$ for some coefficients $a_k^{(m)} \in \mathbb{C}$ with $\sum_{k=0}^\infty |a_k|^2 < \infty$. Hence $C_\varphi f^{(m)} = \sum_{k=0}^\infty a_k^{(m)} \varphi_1^k$, where φ_1 is seen in $F^2(H_n)$, i.e., $\varphi_1^k := \widetilde{\varphi}_1^k(1)$, and the convergence of the series is in $F^2(H_n)$. Note also that, due to (5.3), for each $k \in \mathbb{N}$, $\varphi_1^k = \Gamma^k + \chi_k$ for some $\chi_k \in F^2(H_n) \ominus F^2(H_1)$. Consequently, we have

$$C_\varphi f^{(m)} = \sum_{k=0}^\infty a_k^{(m)} \varphi_1^k = \sum_{k=0}^\infty a_k^{(m)} \Gamma^k + g = f^{(m)} \circ \Gamma + g$$

for some $g \in F^2(H_n) \ominus F^2(H_1)$. Hence, we deduce that $\|C_\Gamma f^{(m)}\|_{F^2(H_1)} \leqslant \|C_\varphi f^{(m)}\|_{F^2(H_n)}$. Using relation (5.4), we have $\|C_\Gamma f^{(m)}\|_{F^2(H_1)} \to 0$ as $m \to \infty$. This proves that the composition operator C_Γ is compact on $F^2(H_1)$. Note also that, under the natural identification of $F^2(H_1)$ with $H^2(\mathbb{D})$, i.e., $f = \sum_{k=0}^\infty c_k e_1^k \mapsto g(z) = \sum_{k=0}^\infty c_k z^k$, the composition operator C_Γ on $F^2(H_1)$ is unitarily equivalent to the composition operator C_{ϕ_ζ} on $H^2(\mathbb{D})$. Consequently, C_{ϕ_ζ} is compact, which is a contradiction. Therefore the map ψ has fixed points in $\mathbb{B}_n$.

Now we prove that ψ has only one fixed point in $\mathbb{B}_n$. Assume that there are two distinct points $\xi^{(1)}, \xi^{(2)} \in \mathbb{B}_n$ such that $\psi(\xi^{(1)}) = \xi^{(1)}$ and $\psi(\xi^{(2)}) = \xi^{(2)}$. It is well known [41] that the fixed point set of the map ψ is affine. As in the proof of Theorem 2.1, we have

$$C_\varphi^* z_\mu = \sum_{k=0} \sum_{|\alpha|=k} \overline{\varphi_\alpha(\mu)} e_\alpha = z_{\varphi(\mu)}, \quad \mu := (\mu_1, \ldots, \mu_n) \in \mathbb{B}_n,$$

where the vector $z_\mu \in F^2(H_n)$ is defined by $z_\mu := \sum_{k=0}^\infty \sum_{|\alpha|=k} \overline{\mu}_\alpha e_\alpha$. As a consequence, we deduce that $C_\varphi^* z_\xi = z_\xi$ for any ξ in the fixed point set Λ of ψ. Since Λ is infinite and according to the proof of Lemma 4.3 the vectors $\{z_\xi\}_{\xi \in \Lambda}$ are linearly independent, we deduce that $\ker(I - C_\varphi^*)$ is infinite dimensional. This contradicts the fact that C_φ is a compact operator on $H^2_{\mathbf{ball}}$. In conclusion, ψ has exactly on fixed point in $\mathbb{B}_n$. This completes the proof. □

Combining now Theorem 5.7 and Theorem 2.6, we can deduce the following similarity result.

Corollary 5.8. *Every compact composition operator on $H^2_{\mathbf{ball}}$ is similar to a contraction.*

Theorem 5.9. *The set of compact composition operators on $H^2_{\mathbf{ball}}$ is arcwise connected, with respect to the operator norm topology, in the set of all composition operators.*

Proof. Let $\varphi = (\varphi_1, \ldots, \varphi_n)$ be a non-constant free holomorphic self-map of the noncommutative ball $[B(\mathcal{H})^n]_1$ such that C_φ is a compact composition operator on $H^2_{\mathbf{ball}}$. For each $r \in [0, 1]$, consider the free holomorphic map $\varphi_r : [B(\mathcal{H})^n]_1 \to [B(\mathcal{H})^n]_1$ defined by $\varphi_r(X) = \varphi(rX)$, $X \in [B(\mathcal{H})^n]_1$. If $\|\varphi\|_\infty < 1$, then $\|\varphi_r\|_\infty < 1$ and due to Proposition 5.2, the operator C_{φ_r} is compact on $H^2_{\mathbf{ball}}$. Now assume that $\|\varphi\|_\infty = 1$. Since φ is non-constant, Theorem 1.1 implies $\|\varphi(0)\| < 1$ and the map $[0, 1) \ni r \mapsto \|\varphi_r\|_\infty$ is strictly increasing. Therefore $\|\varphi_r\|_\infty < 1$ for all $r \in [0, 1)$. Using again Proposition 5.2, we deduce that the operator C_{φ_r} is compact on $H^2_{\mathbf{ball}}$ for any $r \in [0, 1)$. Let $\mathcal{K}(H^2_{\mathbf{ball}})$ denote the algebra of all compact operators on $H^2_{\mathbf{ball}}$ and define the function $\gamma : [0, 1] \to \mathcal{K}(H^2_{\mathbf{ball}})$ by setting $\gamma(r) := C_{\varphi_r}$. Now we show that γ is a continuous map in the operator norm topology. Fix $r_0 \in [0, 1]$. For any $g(X) := \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha X_\alpha \in H^2_{\mathbf{ball}}$ set $g_r(X) := \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha r^{|\alpha|} X_\alpha \in H^2_{\mathbf{ball}}$ and note that

$$\|g_r - g_{r_0}\|_2 \to 0 \quad \text{as } r \to r_0. \tag{5.5}$$

In particular, taking $g = C_\varphi f$ where $f \in H^2_{\mathbf{ball}}$ and $\|f\|_2 \leqslant 1$, we have

$$\left\|(f \circ \varphi)_r - (f \circ \varphi)_{r_0}\right\|_2 \to 0 \quad \text{as } r \to r_0.$$

We need to show that the latter convergence is uniform with respect to $f \in H^2_{\mathbf{ball}}$ with $\|f\|_2 \leqslant 1$. Indeed, if we assume the contrary, then there is $\epsilon_0 > 0$ such that for any $n \in \mathbb{N}$ there is $r_n \in [0, 1]$ with $|r_n - r_0| < \frac{1}{n}$ and there exists $f_n \in H^2_{\mathbf{ball}}$ with $\|f_n\|_2 \leqslant 1$ such that

$$\left\|(f_n \circ \varphi)_{r_n} - (f_n \circ \varphi)_{r_0}\right\|_2 > \epsilon_0. \tag{5.6}$$

Since C_φ is a compact operator the image of the unit ball of $H^2_{\mathbf{ball}}$ under C_φ is relatively compact. Therefore there is a subsequence $\{f_{n_k}\}$ such that

$$f_{n_k} \circ \varphi \to \psi \in H^2_{\mathbf{ball}}. \tag{5.7}$$

Now, note that

$$\begin{aligned}
&\left\|(f_{n_k} \circ \varphi)_{r_{n_k}} - (f_{n_k} \circ \varphi)_{r_0}\right\|_2 \\
&\quad \leqslant \left\|(f_{n_k} \circ \varphi)_{r_{n_k}} - \psi_{r_{n_k}}\right\|_2 + \|\psi_{r_{n_k}} - \psi_{r_0}\|_2 + \left\|\psi_{r_0} - (f_{n_k} \circ \varphi)_{r_0}\right\|_2 \\
&\quad \leqslant 2\|f_{n_k} \circ \varphi - \psi\|_2 + \|\psi_{r_{n_k}} - \psi_{r_0}\|_2.
\end{aligned}$$

Due to relations (5.5) and (5.7), we deduce that

$$\left\|(f_{n_k} \circ \varphi)_{r_{n_k}} - (f_{n_k} \circ \varphi)_{r_0}\right\|_2 \to 0 \quad \text{as } r \to r_0,$$

which contradicts relation (5.6). Therefore $\|C_{\varphi_r} - C_{\varphi_{r_0}}\| \to 0$ as $r \to r_0$, which proves the continuity of the map γ. Let $\chi = (\chi_1, \ldots, \chi_n)$ be another non-constant free holomorphic self-map of the noncommutative ball $[B(\mathcal{H})^n]_1$ such that C_χ is a compact composition operator on $H^2_{\mathbf{ball}}$.

As above, the function $\ell : [0,1] \to \mathcal{K}(H^2_{\mathbf{ball}})$ given by $\ell(r) := C_{\chi_r}$ is continuous in the operator norm topology. It remains to show that there is a continuous mapping $\omega : [0,1] \to \mathcal{K}(H^2_{\mathbf{ball}})$ such that $\omega(0) = C_{\varphi_0}$ and $\omega(1) = C_{\chi_0}$. To this end, since $\|\varphi(0)\| < 1$ and $\|\chi(0)\| < 1$, we can define the map $\sigma : [0,1] \to \mathbb{B}_n$ by setting $\sigma(t) := (1-t)\varphi(0) + t\chi(0)$ for $t \in [0,1]$. Using again Proposition 5.2, we deduce that $C_{\sigma(t)I}$ is a compact composition operator on $H^2_{\mathbf{ball}}$ for any $t \in [0,1]$. Now we define $\omega : [0,1] \to \mathcal{K}(H^2_{\mathbf{ball}})$ by setting $\omega(t) := C_{\sigma(t)I}$. To prove continuity of this map in the operator norm topology, note that

$$\|C_{\sigma(t)I} f - C_{\sigma(t')I} f\| = \left|\langle f, z_{\sigma(t)} - z_{\sigma(t')}\rangle\right| \leqslant \|f\|_2 \|z_{\sigma(t)} - z_{\sigma(t')}\|_2, \tag{5.8}$$

where $z_\lambda = \sum_{\alpha \in \mathbb{F}_n^+} \bar{\lambda}_\alpha e_\alpha$ for $\lambda \in \mathbb{B}_n$. On the other hand, consider the noncommutative Cauchy kernel $\mathbf{C}_\lambda := (I - \bar{\lambda}_1 S_1 - \cdots - \bar{\lambda}_n S_n)^{-1}$, $\lambda := (\lambda_1, \ldots, \lambda_n) \in \mathbb{B}_n$. Note that $\|\bar{\lambda}_1 S_1 + \cdots + \bar{\lambda}_n S_n\| = \|\lambda\|_2 < 1$ and $\mathbf{C}_\lambda \in F_n^\infty$ for any $\lambda \in \mathbb{B}_n$. We have

$$\begin{aligned}
\|z_{\sigma(t)} - z_{\sigma(t')}\|_2 &= \left\|(\mathbf{C}_{\sigma(t)} - \mathbf{C}_{\sigma(t')})1\right\| \\
&\leqslant \|\mathbf{C}_{\sigma(t)} - \mathbf{C}_{\sigma(t')}\| \\
&\leqslant \|\mathbf{C}_{\sigma(t)}\| \|\mathbf{C}_{\sigma(t')}\| \left\|\sigma(t) - \sigma\big(t'\big)\right\|_2.
\end{aligned}$$

Consequently, since $\mathbb{B}_n \ni \lambda \mapsto \mathbf{C}_\lambda \in F_n^\infty$ is continuous, we deduce that $[0,1] \ni t \mapsto z_{\sigma(t)} \in F^2(H_n)$ is continuous as well. Combining this result with relation (5.8), we deduce the continuity of ω, which completes the proof. □

6. Schröder equation for noncommutative power series and spectra of composition operators

In this section, we consider a noncommutative multivariable Schröder type equation and use it to obtain results concerning the spectrum of composition operators on $H^2_{\mathbf{ball}}$. As a consequence, using the results from the previous section, we determine the spectra of compact composition operators on $H^2_{\mathbf{ball}}$.

First, we provide the following noncommutative Schröder [43] type result.

Theorem 6.1. *Let $A \in M_{n\times n}$ be a scalar matrix and let $\Lambda = (\Lambda_1, \ldots \Lambda_n)$ be an n-tuple of power series in noncommuting indeterminates $Z_1, \ldots, Z_n$, of the form*

$$\Lambda = [Z_1, \ldots, Z_n]A + [\Gamma_1, \ldots, \Gamma_n],$$

where $\Gamma_1, \ldots, \Gamma_n$ are noncommutative power series containing only monomials of degree greater than or equal to 2. If there is a noncommutative power series F which is not identically zero and satisfies the Schröder type equation

$$F \circ \Lambda = cF$$

for some $c \in \mathbb{C}$, then either $c = 1$ or c is a product of eigenvalues of the matrix A.

Proof. Since $A \in M_{n\times n}$ there is a unitary matrix $U \in M_{n\times n}$ such that $U^{-1}AU$ is an upper triangular matrix. Setting $\Phi_U = [Z_1, \ldots, Z_n]U$, the equation $F \circ \Lambda = cF$ is equivalent to $F' \circ \Lambda' = cF'$, where $F' := \Phi_U \circ F \circ \Phi_{U^{-1}}$ and

$$\Lambda' := \Phi_U \circ \Lambda \circ \Phi_{U^{-1}} = [Z_1, \ldots, Z_n]U^{-1}AU + U^{-1}[\Gamma_1, \ldots, \Gamma_n]U.$$

Therefore, we can assume that $A = [a_{ij}] \in M_{n\times n}$ is an upper triangular matrix. We introduce a total order $\leqslant$ on the free semigroup $\mathbb{F}_n^+$ as follows. If $\alpha, \beta \in \mathbb{F}_n^+$ with $|\alpha| \leqslant |\beta|$ we say that $\alpha < \beta$. If $\alpha, \beta \in \mathbb{F}_n^+$ are such that $|\alpha| = |\beta|$, then $\alpha = g_{i_1} \cdots g_{i_k}$ and $\beta = g_{j_1} \cdots g_{j_k}$ for some $i_1, \ldots, i_k, j_1, \ldots, j_k \in \{1, \ldots, k\}$. We say that $\alpha < \beta$ if either $i_1 < j_1$ or there exists $p \in \{2, \ldots, k\}$ such that $i_1 = j_1, \ldots, i_{p-1} = j_{p-1}$ and $i_p < j_p$. It is easy to see that relation $\leqslant$ is a total order on $\mathbb{F}_n^+$.

According to the hypothesis and due to the fact that A is an upper triangular matrix, we have

$$\Lambda_j = \sum_{i=1}^{j} a_{ij} X_i + \Gamma_j, \quad j = 1, \ldots, n. \tag{6.1}$$

Consequently, if $\alpha = g_{i_1} \cdots g_{i_k} \in \mathbb{F}_n^+$, $i_1, \ldots i_k \in \{1, \ldots, n\}$, then

$$\Lambda_\alpha := \Lambda_{i_1} \cdots \Lambda_{i_k} = \Psi^{<\alpha} + a_{i_1 i_1} \cdots a_{i_k i_k} X_\alpha + \chi^{(\alpha)}, \tag{6.2}$$

where $\Psi^{<\alpha}$ is a power series containing only monomials X_β such that $|\beta| = |\alpha|$ and $\beta < \alpha$, and $\chi^{(\alpha)}$ is a power series containing only monomials X_γ with $|\gamma| \geqslant |\alpha| + 1$.

Let $F = \sum_{p=0}^\infty \sum_{|\alpha|=p} c_\alpha Z_\alpha$, $c_\alpha \in \mathbb{C}$, be a noncommutative power series and assume that it satisfies the Schröder type equation $F \circ \Lambda = \lambda F$ for some $\lambda \in \mathbb{C}$ such that $\lambda \neq 1$ and λ is not a product of eigenvalues of the matrix A. We will show by induction over p, that $\sum_{|\alpha|=p} c_\alpha Z_\alpha = 0$ for any $p = 0, 1, \ldots$. Note that the above-mentioned equation is equivalent to

$$\sum_{p=0}^\infty \sum_{|\alpha|=p} c_\alpha \Lambda_\alpha = \lambda \sum_{p=0}^\infty \sum_{|\alpha|=p} c_\alpha Z_\alpha. \tag{6.3}$$

Due to relation (6.1), we have $c_0 = \lambda c_0$. Since $\lambda \neq 1$, we deduce that $c_0 = 0$. Assume that $c_\alpha = 0$ for any $\alpha \in \mathbb{F}_n^+$ with $|\alpha| < k$. According to Eqs. (6.2) and (6.3), we have

$$\sum_{|\alpha|=k} c_\alpha \bigl(\Psi^{<\alpha} + d_A(\alpha) X_\alpha + \chi^{(\alpha)}\bigr) + \sum_{p=k+1}^\infty \sum_{|\alpha|=p} c_\alpha \Lambda_\alpha = \lambda \sum_{|\alpha|=k} c_\alpha Z_\alpha + \lambda \sum_{p=k+1}^\infty \sum_{|\alpha|=p} c_\alpha Z_\alpha,$$

where $d_A(\alpha) := a_{i_1 i_1} \cdots a_{i_k i_k}$ if $\alpha = g_{i_1} \cdots g_{i_k} \in \mathbb{F}_n^+$ and $i_1, \ldots i_k \in \{1, \ldots, n\}$. Since $\chi^{(\alpha)}$ is a power series containing only monomials X_γ with $|\gamma| \geqslant |\alpha| + 1$, and the power series Λ_α, $|\alpha| \geqslant k + 1$, contains only monomials X_σ with $|\sigma| \geqslant k + 1$, we deduce that

$$\sum_{|\alpha|=k} c_\alpha \bigl(\Psi^{<\alpha} + d_A(\alpha) X_\alpha\bigr) = \lambda \sum_{|\alpha|=k} c_\alpha Z_\alpha. \tag{6.4}$$

We arrange the elements of the set $\{\alpha \in \mathbb{F}_n^+ \colon |\alpha| = k\}$ increasingly with respect to the total order, i.e., $\beta_1 < \beta_2 < \cdots < \beta_{n^k}$. Note that $\beta_1 = g_1^k$ and $\beta_{n^k} = g_n^k$. The relation (6.4) becomes

$$\sum_{j=1}^{n^k} \bigl(c_{\beta_j} \Psi^{<\beta_j} + c_{\beta_j} d(\beta_j) X_{\beta_{\beta_j}}\bigr) = \lambda \sum_{j=1}^{n^k} c_{\beta_j} X_{\beta_j}. \tag{6.5}$$

Taking into account that $\Psi^{<\alpha}$ is a power series containing only monomials X_β such that $|\beta| = |\alpha|$ and $\beta < \alpha$, one can see that the monomial $X_{\beta_{n^k}}$ occurs just once in the left-hand side of relation (6.5). Identifying the coefficients of the monomial $X_{\beta_{n^k}}$ in the equality (6.5), we deduce that

$$c_{\beta_{n^k}} d(\beta_{n^k}) = \lambda c_{\beta_{n^k}}.$$

Since $\lambda \neq a_{nn}^k = d(\beta_{n^k})$, we must have $c_{\beta_{n^k}} = 0$. Consequently, Eq. (6.5) becomes

$$\sum_{j=1}^{n^k-1} \bigl(c_{\beta_j} \Psi^{<\beta_j} + c_{\beta_j} d(\beta_j) X_{\beta_{\beta_j}}\bigr) = \lambda \sum_{j=1}^{n^k-1} c_{\beta_j} X_{\beta_j}.$$

Continuing the process, we deduce that $c_{\beta_j} = 0$ for $j = 1, \ldots, n^k$. Therefore $c_\alpha = 0$ for any $\alpha \in \mathbb{F}_n^+$ with $|\alpha| = k$, which completes our induction. The proof is complete. □

Corollary 6.2. *Let $\varphi = (\varphi_1, \ldots, \varphi_n)$ be a free holomorphic self-map of the noncommutative ball $[B(\mathcal{H})^n]_1$ such that $\varphi(\xi) = \xi$ for some $\xi \in \mathbb{B}_n$. If there is a free holomorphic function $f : [B(\mathcal{H})^n]_1 \to B(\mathcal{H})$ such that*

$$f \circ \varphi = cf$$

for some $c \in \mathbb{C}$, then either $c = 1$ or c is a product of eigenvalues of the matrix

$$\bigl[\langle \psi_i, e_j \rangle\bigr]_{n \times n},$$

where $\psi = (\psi_1, \ldots, \psi_n) := \Phi_\xi \circ \varphi \circ \Phi_\xi$ and Φ_ξ is the involutive free holomorphic automorphism of $[B(\mathcal{H})^n]_1$ associated with $\xi \in \mathbb{B}_n$, and $\psi_1, \ldots, \psi_n$ are seen as elements in the Fock space $F^2(H_n)$.

Proof. Note that $\psi(0) = 0$ and the equation $f \circ \varphi = cf$ is equivalent to the equation $f' \circ \psi = cf'$, where $f' := \Phi_\xi \circ f \circ \Phi_\xi$. Applying Theorem 6.1 to the power series associated with ψ and f' the result follows. □

Theorem 6.3. *Let $\varphi = (\varphi_1, \ldots, \varphi_n)$ be a free holomorphic self-map of the noncommutative ball $[B(\mathcal{H})^n]_1$ such that $\varphi(0) = 0$, and let C_φ be the associated composition operator on $H^2_{\mathbf{ball}}$. Then the point spectrum of C_φ^* contains the conjugates of all possible products of the eigenvalues of the matrix*

$$\bigl[\langle \varphi_i, e_j \rangle\bigr]_{n \times n},$$

where $\psi_1, \ldots, \psi_n$ are seen as elements in the Fock space $F^2(H_n)$.

Proof. For each $m = 0, 1, \ldots$, consider the subspace $\mathcal{K}_m := \operatorname{span}\{e_\alpha\colon \alpha \in \mathbb{F}_n^+,\ |\alpha| \leqslant m\}$. Since $\varphi(0) = 0$, we have $\langle C_\varphi^* e_\alpha, e_\beta\rangle = \langle e_\alpha, \varphi_\beta\rangle = 0$ for any $\alpha, \beta \in \mathbb{F}_n^+$ with $|\alpha| \leqslant m$ and $|\beta| \geqslant m+1$. This implies $C_\varphi^*(\mathcal{K}_m) \subseteq \mathcal{K}_m$ and C_φ^* has the matrix representation

$$C_\varphi^* = \begin{bmatrix} C_\varphi^*|\mathcal{K}_m & * \\ 0 & P_{F^2(H_n)\ominus\mathcal{K}_m} C_\varphi^*|_{F^2(H_n)\ominus\mathcal{K}_m} \end{bmatrix}$$

with respect to the orthogonal decomposition $F^2(H_n) = \mathcal{K}_m \oplus (F^2(H_n) \ominus \mathcal{K}_m)$, and $\sigma_p(C_\varphi^*|_{\mathcal{K}_m}) \subset \sigma_p(C_\varphi^*)$, where $\sigma_p(T)$ denotes the point spectrum of T. Moreover, since $\mathcal{K}_m$ is finite dimensional, we have

$$\sigma\big(C_\varphi^*\big) = \sigma\big(C_\varphi^*\big|_{\mathcal{K}_m}\big) \cup \sigma\big(P_{F^2(H_n)\ominus\mathcal{K}_m} C_\varphi^*\big|_{F^2(H_n)\ominus\mathcal{K}_m}\big).$$

Since $C_\varphi^*(\mathcal{K}_{m-1}) \subseteq \mathcal{K}_{m-1}$ we have the matrix decomposition

$$C_\varphi^*|\mathcal{K}_m = \begin{bmatrix} C_\varphi^*|\mathcal{K}_m & * \\ 0 & P_{\mathcal{K}_m\ominus\mathcal{K}_{m-1}} C_\varphi^*|\mathcal{K}_m\ominus\mathcal{K}_{m-1} \end{bmatrix}$$

with respect to the orthogonal decomposition $F^2(H_n) = \mathcal{K}_m \oplus (\mathcal{K}_m \ominus \mathcal{K}_{m-1})$. Consequently, we have

$$\sigma_p\big(C_\varphi^*\big|_{\mathcal{K}_m}\big) = \sigma_p\big(C_\varphi^*\big|_{\mathcal{K}_{m-1}}\big) \cup \sigma_p\big(P_{\mathcal{K}_m\ominus\mathcal{K}_{m-1}} C_\varphi^*\big|_{\mathcal{K}_m\ominus\mathcal{K}_{m-1}}\big)$$

for any $m = 1, 2 \ldots$. Iterating this formula, we get

$$\sigma_p\big(C_\varphi^*\big|_{\mathcal{K}_m}\big) = \{1\} \cup \bigcup_{j=1}^m \sigma_p\big(P_{\mathcal{K}_j\ominus\mathcal{K}_{j-1}} C_\varphi^*\big|_{\mathcal{K}_j\ominus\mathcal{K}_{j-1}}\big). \tag{6.6}$$

Now, we determine $\sigma_p(P_{\mathcal{K}_k\ominus\mathcal{K}_{k-1}} C_\varphi^*|\mathcal{K}_k\ominus\mathcal{K}_{k-1})$ for $k = 1, 2, \ldots$. As in the proof of Theorem 6.1, we can assume that

$$\varphi(X) = [X_1, \ldots, X_n]A + \big(\Gamma_1(X), \ldots, \Gamma_n(X)\big), \quad X = (X_1, \ldots, X_n) \in \big[B(\mathcal{H})^n\big]_1,$$

where $A = [a_{ij}] \in M_{n\times n}$ is an upper triangular scalar matrix and $\Gamma_1, \ldots, \Gamma_n$ are free holomorphic functions on $[B(\mathcal{H})^n]_1$ containing only monomials of degree greater than or equal to 2. Consequently, using the Fock space representation of $\varphi_1, \ldots, \varphi_n$ and $\Gamma_1, \ldots, \Gamma_n$, we have

$$\varphi_j = \sum_{i=1}^j a_{ij} e_i + \Gamma_j, \quad j = 1, \ldots, n, \tag{6.7}$$

where $\Gamma_j \in F^2(H_n) \ominus \operatorname{span}\{e_\alpha\colon |\alpha| \leqslant 1\}$. Note that the matrix $[\langle\varphi_i, e_j\rangle]_{n\times n}$ is upper triangular and its eigenvalues are $a_{11}, \ldots, a_{nn}$. Using relation (6.7), one can see that if $\alpha = g_{i_1} \cdots g_{i_k} \in \mathbb{F}_n^+$, $i_1, \ldots i_k \in \{1, \ldots, n\}$, then

$$\varphi_\alpha := \varphi_{i_1} \cdots \varphi_{i_k} = \psi^{<\alpha} + a_{i_1 i_1} \cdots a_{i_k i_k} e_\alpha + \chi^{(\alpha)}, \tag{6.8}$$

where $\psi^{<\alpha} \in \operatorname{span}\{e_\beta\colon |\beta| = |\alpha| \text{ and } \beta < \alpha\}$ and $\chi^{(\alpha)} \in \operatorname{span}\{e_\gamma\colon |\gamma| \geqslant |\alpha| + 1\}$.

We arrange the elements of the set $\{\alpha \in \mathbb{F}_n^+ \colon |\alpha| = k\}$ increasingly with respect to the total order introduced in the proof of Theorem 6.1, i.e., $\beta_1 < \beta_2 < \cdots < \beta_{n^k}$. We denote $d_A(\alpha) := a_{i_1 i_1} \cdots a_{i_k i_k}$ if $\alpha = g_{i_1} \cdots g_{i_k} \in \mathbb{F}_n^+$ and $i_1, \ldots i_k \in \{1, \ldots, n\}$. Note that $\varphi_{\beta_1} = d(\beta_1) e_{\beta_1} + \chi^{\beta_1}$ and

$$\varphi_{\beta_i} = \left(\sum_{j=1}^{i} b_{\beta_{j-1}} e_{\beta_{j-1}} \right) + d(\beta_i) e_{\beta_i} + \chi^{\beta_i} \quad \text{if } 2 \leqslant i \leqslant n^k,$$

for some $b_{\beta_{j-1}} \in \mathbb{C}$, $j = 1, \ldots, i$. Using these relations, we deduce that

$$\left\langle P_{\mathcal{K}_k \ominus \mathcal{K}_{k-1}} C_\varphi^* \big|_{\mathcal{K}_k \ominus \mathcal{K}_{k-1}} e_{\beta_j}, e_{\beta_i} \right\rangle = \overline{\langle \varphi_{\beta_i}, e_{\beta_j} \rangle} = \begin{cases} \overline{d(\beta_i)} & \text{if } i = j, \\ 0 & \text{if } i < j. \end{cases}$$

This shows that the matrix of $P_{\mathcal{K}_k \ominus \mathcal{K}_{k-1}} C_\varphi^* |_{\mathcal{K}_k \ominus \mathcal{K}_{k-1}}$ with respect to the orthonormal basis $\{e_{\beta_i}\}_{i=1}^{n^k}$ is lower triangular with the diagonal entries $\overline{d(\beta_1)}, \ldots, \overline{d(\beta_{n^k})}$. Therefore $\sigma_p(P_{\mathcal{K}_k \ominus \mathcal{K}_{k-1}} C_\varphi^* |_{\mathcal{K}_k \ominus \mathcal{K}_{k-1}})$ consists of these diagonal entries. On the other hand, due to relation (6.6), we have

$$\{1\} \cup \bigcup_{j=1}^{\infty} \sigma_p\left(P_{\mathcal{K}_j \ominus \mathcal{K}_{j-1}} C_\varphi^* \big|_{\mathcal{K}_j \ominus \mathcal{K}_{j-1}}\right) \subset \sigma_p\left(C_\varphi^*\right).$$

The proof is complete. □

Theorem 6.3 and Corollary 6.2 imply the following result concerning the spectrum of composition operators on the noncommutative Hardy space $H^2_{\mathbf{ball}}$.

Theorem 6.4. *Let φ be a free holomorphic self-map of the noncommutative ball $[B(\mathcal{H})^n]_1$ such that its scalar representation has a fixed point $\xi \in \mathbb{B}_n$, and let C_φ be the associated composition operator on $H^2_{\mathbf{ball}}$. Then*

$$\sigma_p(C_\varphi) \subseteq \{1\} \cup \mathcal{P}_{eig} \subseteq \sigma(C_\varphi),$$

where $\mathcal{P}_{eig}$ is the set of all possible products of eigenvalues of the matrix $[\langle \psi_i, e_j \rangle]_{n \times n}$, where $\psi = (\psi_1, \ldots, \psi_n) := \Phi_\xi \circ \varphi \circ \Phi_\xi$ and Φ_ξ is the involutive free holomorphic automorphism of $[B(\mathcal{H})^n]_1$ associated with $\xi \in \mathbb{B}_n$.

Proof. The first inclusion follows from Corollary 6.2. To prove the second inclusion note that $C_\varphi 1 = 1$ and $C_\psi = C_{\Phi_\xi} C_\varphi C_{\Phi_\xi}^{-1}$. Consequently, $1 \in \sigma(C_\varphi) = \sigma(C_\psi)$. Since $\psi(0) = 0$, we can apply Theorem 6.3 to the composition operator C_ψ and complete the proof. □

Now we can determine the spectra of compact composition operators on $H^2_{\mathbf{ball}}$.

Theorem 6.5. *Let φ be a free holomorphic self-map of the noncommutative ball $[B(\mathcal{H})^n]_1$. If C_φ is a compact composition operator on $H^2_{\mathbf{ball}}$, then the scalar representation of φ has a unique fix* *point* $\xi \in \mathbb{B}_n$ *and the spectrum* $\sigma(C_\varphi)$ *consists of* 0, 1, *and all possible products of the eigenvalues of the matrix*

$$\big[\langle \psi_i, e_j\rangle\big]_{n\times n},$$

where $\psi = (\psi_1, \ldots, \psi_n) := \Phi_\xi \circ \varphi \circ \Phi_\xi$ *and* Φ_ξ *is the involutive free holomorphic automorphism of* $[B(\mathcal{H})^n]_1$ *associated with* $\xi \in \mathbb{B}_n$, *and* $\psi_1, \ldots, \psi_n$ *are seen as elements in the Fock space* $F^2(H_n)$.

Proof. If C_φ is a compact composition operator on $H^2_{\mathbf{ball}}$, then, according to Theorem 5.7, the scalar representation of φ has a unique fix point $\xi \in \mathbb{B}_n$. On the other hand, it is well known that any nonzero point in the spectrum of a compact operator is an eigenvalue. Using Theorem 6.4, we deduce that

$$\sigma_p(C_\varphi) \subseteq \{1\} \cup \mathcal{P}_{eig} \subseteq \{0\} \cup \sigma_p(C_\varphi),$$

where $\mathcal{P}_{eig}$ is the set of all possible products of eigenvalues of the matrix $[\langle \psi_i, e_j\rangle]_{n\times n}$. Hence the result follows and the proof is complete. □

In [14], MacCluer determined the spectrum of composition operators on $H^2(\mathbb{B}_n)$ when the symbols are automorphisms of $\mathbb{B}_n$ which fix at least one point in $\mathbb{B}_n$. The following theorem is an extension of this result to compositions operators on $H^2_{\mathbf{ball}}$ induced by free holomorphic automorphisms of $[B(\mathcal{H})^n]_1$.

Theorem 6.6. *Let* $\varphi \in Aut(B(\mathcal{H})^n_1)$ *be such that* $\varphi(\xi) = \xi$ *for some* $\xi \in \mathbb{B}_n$. *Then the spectrum of the composition operator* C_φ *on* $H^2_{\mathbf{ball}}$ *is the closure of all possible products of the eigenvalues of the matrix*

$$\big[\langle \psi_i, e_j\rangle\big]_{n\times n},$$

where $\psi = (\psi_1, \ldots, \psi_n) := \Phi_\xi \circ \varphi \circ \Phi_\xi$ *and* Φ_ξ *is the involutive free holomorphic automorphism of* $[B(\mathcal{H})^m]_1$ *associated with* $\xi \in \mathbb{B}_n$. *Moreover,* $\sigma(C_\varphi)$ *is either the unit circle* $\mathbb{T}$, *or a finite subgroup of* $\mathbb{T}$.

Proof. Note that $\psi \in Aut(B(\mathcal{H})^n_1)$ and $\psi(0) = 0$. According to [38], the free holomorphic automorphism ψ has the form $\psi(X) = [X_1, \ldots, X_n]U$ for some unitary matrix $U \in M_{n\times n}$. It is easy to see that $U = [\langle \psi_i, e_j\rangle]_{n\times n}$. Since U is unitary there is another unitary matrix $W \in M_{n\times n}$ such that

$$W^{-1}UW = \begin{bmatrix} w_1 & 0 & \cdots & 0 \\ 0 & w_2 & \cdots & 0 \\ 0 & 0 & \cdots & w_n \end{bmatrix},$$

where $w_1, \ldots, w_n$ are the eigenvalues of U. Set $\chi := \psi_W \circ \psi \circ \psi_W^{-1}$, where $\psi_W(X) := [X_1, \ldots, X_n]W$ for $X := [X_1, \ldots, X_n] \in [B(\mathcal{H})^n]_1$. Note that $\chi(X) = [X_1, \ldots, X_n]W^{-1}UW$ and $C_\chi = C_{\psi_W}^{-1} C_{\phi_\xi}^{-1} C_\varphi C_{\phi_\xi} C_{\psi_W}$. Hence, $\sigma(C_\chi) = \sigma(\psi) = \sigma(\varphi)$. Now we determine the spectrum of C_χ. Since C_ψ is invertible and $\psi(0) = 0$, Theorem 2.3 implies $\|C_\chi\| = \|C_\psi^{-1}\| = 1$.

Therefore, $\sigma(C_\chi) \subseteq \mathbb{T}$. Using now Theorem 6.4, we deduce that $\overline{\mathcal{P}}_{eig} \subseteq \sigma(C_\psi) \subseteq \mathbb{T}$, where $\mathcal{P}_{eig}$ is the set of all possible products of eigenvalues of the matrix U. It is obvious that if $\overline{\mathcal{P}}_{eig} = \mathbb{T}$, then $\sigma(C_\psi) = \mathbb{T}$. When $\overline{\mathcal{P}}_{eig} \neq \mathbb{T}$, then $\overline{\mathcal{P}}_{eig}$ is a finite subgroup of $\mathbb{T}$. Consequently, there is $m \in \mathbb{N}$ such that $\overline{\mathcal{P}}_{eig} = \{z \in \mathbb{T}\colon z^m = 1\}$. This implies $w_j^m = 1$ for $j = 1, \ldots, n$ and $C_\chi^m = I$. Consequently, if $\lambda \in \sigma(C_\chi)$ then $\lambda^m \in \sigma(C_\chi^m) = \{1\}$. This shows that $\lambda \in \overline{\mathcal{P}}_{eig}$ and completes the proof. □

Comparing our Theorem 6.6 with MacCluer result (see Theorem 3.1 from [14]), we are led to the conclusion that if $\varphi \in Aut(B(\mathcal{H})_1^n)$ has at least one fixed point in $\mathbb{B}_n$, then the spectrum of the composition operator C_φ on $H^2_{\mathbf{ball}}$ coincides with the spectrum of the composition operator $C_{\varphi^{\mathbb{C}}}$ on $H^2(\mathbb{B}_n)$, where $\varphi^{\mathbb{C}}$ is the scalar representation of φ.

Theorem 6.7. *If $\varphi \in Aut(B(\mathcal{H})_1^n)$ and there is only one point $\zeta \in \overline{\mathbb{B}}_n$ such that $\varphi(\zeta) = \zeta$ and $\zeta \in \partial\mathbb{B}_n$, then the spectral radius of the composition operator C_φ on $H^2_{\mathbf{ball}}$ is equal to* 1 *and $\sigma(C_\varphi) \subseteq \mathbb{T}$.*

Proof. The proof that the spectral radius is 1 is similar to that of Theorem 3.3, in the parabolic case. The inclusion $\sigma(C_\varphi) \subseteq \mathbb{T}$ is due to the fact that $\varphi^{-1}(\zeta) = \zeta$ and, according to the first part of the theorem we have $r(C_\varphi^{-1}) = r(C_\varphi) = 1$. □

7. Composition operators on Fock spaces associated to noncommutative varieties

In this section, we consider composition operators on Fock spaces associated to noncommutative varieties in unit ball $[B(\mathcal{H})^n]_1$ and obtain results concerning boundedness, norm estimates, and spectral radius. In particular, we show that many of our results have commutative counterparts for composition operators on the symmetric Fock space and on spaces of analytic functions in the unit ball of $\mathbb{C}^n$. In particular, we obtain new proofs for some of Jury's [11] recent results concerning compositions operators on the unit ball $\mathbb{B}_n$.

Let $\mathcal{P}_0$ be a set on noncommutative polynomials in n indeterminates such that $p(0) = 0$ for all $p \in \mathcal{P}_0$. Consider the noncomutative variety $\mathcal{V}_{\mathcal{P}_0}(\mathcal{H}) \subseteq [B(\mathcal{H})^n]_1$ defined by

$$\mathcal{V}_{\mathcal{P}_0}(\mathcal{H}) := \left\{(X_1, \ldots, X_n) \in \left[B(\mathcal{H})^n\right]_1\colon\ p(X_1, \ldots, X_n) = 0 \text{ for all } p \in \mathcal{P}_0\right\}.$$

Let

$$\mathcal{M}_{\mathcal{P}_0} := \overline{\operatorname{span}}\left\{S_\alpha p(S_1, \ldots, S_n) S_\beta 1\colon\ p \in \mathcal{P}_0,\ \alpha, \beta \in \mathbb{F}_n^+\right\}$$

and $\mathcal{N}_{\mathcal{P}_0} := F^2(H_n) \ominus \mathcal{M}_{\mathcal{P}_0}$. We remark that $1 \in \mathcal{N}_{\mathcal{P}_0}$ and the subspace $\mathcal{N}_{\mathcal{P}_0}$ is invariant under $S_1^*, \ldots, S_n^*$ and $R_1^*, \ldots, R_n^*$. Define the *constrained left* (resp. *right*) *creation operators* by setting

$$B_i := P_{\mathcal{N}_{\mathcal{P}_0}} S_i|_{\mathcal{N}_{\mathcal{P}_0}} \quad \text{and} \quad W_i := P_{\mathcal{N}_{\mathcal{P}_0}} R_i|_{\mathcal{N}_{\mathcal{P}_0}}, \quad i = 1, \ldots, n.$$

We proved in [32] that the n-tuple $(B_1, \ldots, B_n) \in \mathcal{V}_{\mathcal{P}_0}(\mathcal{N}_{\mathcal{P}_0})$ is the universal model associated with the noncommutative variety $\mathcal{V}_{\mathcal{P}_0}(\mathcal{H})$. Let $F_n^\infty(\mathcal{V}_{\mathcal{P}_0})$ be the w^*-closed algebra generated by $B_1, \ldots, B_n$ and the identity. The w^* and WOT topologies coincide on this algebra and

$$F_n^\infty(\mathcal{V}_{\mathcal{P}_0}) = P_{\mathcal{N}_{\mathcal{P}_0}} F_n^\infty\big|_{\mathcal{N}_{\mathcal{P}_0}} = \left\{f(B_1, \ldots, B_n)\colon\ f \in F_n^\infty\right\},$$

where if f has the Fourier representation $\sum_{\alpha\in\mathbb{F}_n^+} a_\alpha S_\alpha$ then

$$f(B_1,\ldots,B_n)=\text{SOT-}\lim_{r\to 1}\sum_{k=0}^{\infty}\sum_{|\alpha|=k} r^{|\alpha|}a_\alpha B_\alpha.$$

The latter limit exists due to the F_n^∞-functional calculus for row contractions [27]. Similar results hold for $R_n^\infty(\mathcal{V}_{\mathcal{P}_0})$, the w^*-closed algebra generated by $W_1,\ldots,W_n$ and the identity. Moreover,

$$F_n^\infty(\mathcal{V}_{\mathcal{P}_0})'=R_n^\infty(\mathcal{V}_{\mathcal{P}_0}) \quad \text{and} \quad R_n^\infty(\mathcal{V}_{\mathcal{P}_0})'=F_n^\infty(\mathcal{V}_{\mathcal{P}_0}),$$

where $'$ stands for the commutant. According to [32], each $\widetilde{\chi}\in F_n^\infty(\mathcal{V}_{\mathcal{P}_0})$ generates a mapping $\chi:\mathcal{V}_{\mathcal{P}_0}(\mathcal{H})\to B(\mathcal{H})$ given by

$$\chi(X_1,\ldots,X_n):=\mathbf{P}_X[\widetilde{\chi}], \qquad X:=(X_1,\ldots,X_n)\in\mathcal{V}_{\mathcal{P}_0}(\mathcal{H}),$$

where $\mathbf{P}_X$ is the noncommutative Poisson transform associated with $\mathcal{V}_{\mathcal{P}_0}(\mathcal{H})$. On the other hand, since $\widetilde{\chi}=P_{\mathcal{N}_{\mathcal{P}_0}}\widetilde{\phi}|_{\mathcal{N}_{\mathcal{P}_0}}$ for some $\widetilde{\phi}=\sum_{\alpha\in\mathbb{F}_n^+} a_\alpha S_\alpha$ in F_n^∞, we have

$$\chi(X_1,\ldots,X_n)=\sum_{k=0}^{\infty}\sum_{|\alpha|=k} a_\alpha X_\alpha, \quad (X_1,\ldots,X_n)\in\mathcal{V}_{\mathcal{P}_0}(\mathcal{H}),$$

where the convergence is in the operator norm topology. This shows that χ is the restriction to $\mathcal{V}_{\mathcal{P}_0}(\mathcal{H})$ of a bounded free holomorphic function on $[B(\mathcal{H})^n]_1$, namely $X\mapsto\phi(X)=\mathbf{P}_X[\widetilde{\psi}]$. We remark that the map χ does not depend on the choice of $\widetilde{\phi}\in F_n^\infty$ with the property that $\widetilde{\chi}=P_{\mathcal{N}_{\mathcal{P}_0}}\widetilde{\phi}|_{\mathcal{N}_{\mathcal{P}_0}}$. Note also that $\chi(0)=\langle\widetilde{\chi}1,1\rangle$.

We remark that when $f\in F^2(H_n)$ and $f=\sum_{k=0}^{\infty}\sum_{|\alpha|=k} a_\alpha e_\alpha$, then $f\in\mathcal{N}_{\mathcal{P}_0}$ if and only if

$$\sum_{k=0}^{\infty}\sum_{|\alpha|=k} a_\alpha e_\alpha=\sum_{k=0}^{\infty}\sum_{|\alpha|=k} a_\alpha B_\alpha 1.$$

We say that $\widetilde{\psi}\in F_n^\infty(\mathcal{V}_{\mathcal{P}_0})\otimes\mathbb{C}^n$ is non-scalar operator if it does not have the form $(a_1 I_{\mathcal{N}_{\mathcal{P}_0}},\ldots, a_n I_{\mathcal{N}_{\mathcal{P}_0}})$ for some $a_i\in\mathbb{C}$. The main result of this section is the following.

Theorem 7.1. *Let $\widetilde{\psi}=(\widetilde{\psi}_1,\ldots,\widetilde{\psi}_n)\in F_n^\infty(\mathcal{V}_{\mathcal{P}_0})\otimes\mathbb{C}^n$ be a non-scalar operator with $\|\widetilde{\psi}\|\leqslant 1$. Then the following statements hold.*

(i) *If $g\in\mathcal{N}_{\mathcal{P}_0}$ has the representation $\sum_{k=0}^{\infty}\sum_{|\alpha|=k} c_\alpha e_\alpha$ then*

$$g\circ\widetilde{\psi}:=\sum_{k=0}^{\infty}\sum_{|\alpha|=k} c_\alpha\widetilde{\psi}_\alpha 1\in\mathcal{N}_{\mathcal{P}_0},$$

where the convergence of the series is in $F^2(H_n)$.

(ii) *The composition operator* $C_{\widetilde{\psi}} : \mathcal{N}_{\mathcal{P}_0} \to \mathcal{N}_{\mathcal{P}_0}$ *defined by*

$$C_{\widetilde{\psi}} g := g \circ \widetilde{\psi}, \quad g \in \mathcal{N}_{\mathcal{P}_0},$$

is bounded. Moreover,

$$\|P_{\mathcal{N}_{\mathcal{P}_0}} z_{\psi(0)}\| \leqslant \sup_{\lambda \in \mathcal{V}_{\mathcal{P}_0}(\mathbb{C})} \frac{\|P_{\mathcal{N}_{\mathcal{P}_0}} z_{\psi(\mu)}\|}{\|z_\mu\|} \leqslant \|C_{\widetilde{\psi}}\| \leqslant \left(\frac{1 + \|\psi(0)\|}{1 - \|\psi(0)\|}\right)^{1/2}.$$

(iii) *The adjoint of the composition operator* $C_{\widetilde{\psi}} : \mathcal{N}_{\mathcal{P}_0} \to \mathcal{N}_{\mathcal{P}_0}$ *satisfies the formula*

$$C_{\widetilde{\psi}}^* g = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \langle g, \widetilde{\psi}_\alpha(1)\rangle P_{\mathcal{N}_{\mathcal{P}_0}} e_\alpha, \quad g \in \mathcal{N}_{\mathcal{P}_0}.$$

Proof. Since $R_n^\infty(\mathcal{V}_{\mathcal{P}_0})' = F_n^\infty(\mathcal{V}_{\mathcal{P}_0})$, the operator $\widetilde{\psi} : \mathcal{N}_{\mathcal{P}_0} \otimes \mathbb{C}^n \to \mathcal{N}_{\mathcal{P}_0}$ satisfies the commutation relations

$$\widetilde{\psi}(W_i \otimes I_{\mathbb{C}^n}) = W_i \widetilde{\psi}, \quad i = 1, \ldots, n.$$

Since $W_i := P_{\mathcal{N}_{\mathcal{P}_0}} R_i|_{\mathcal{N}_{\mathcal{P}_0}}$, $i = 1, \ldots, n$, it is clear that $[R_1 \otimes I_{\mathbb{C}_n}, \ldots, R_1 \otimes I_{\mathbb{C}_n}]$ is an isometric dilation of the row contraction $[W_1 \otimes I_{\mathbb{C}_n}, \ldots, W_1 \otimes I_{\mathbb{C}_n}]$. According to the noncommutative commutant theorem [24], there exists $\widetilde{\varphi} = [\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n] : F^2(H_n) \otimes \mathbb{C}^n \to F^2(H_n)$ with the properties $\|\widetilde{\varphi}\| \leqslant 1$, $\widetilde{\varphi}^*|_{\mathcal{N}_{\mathcal{P}_0}} = \widetilde{\psi}^*$, and $\widetilde{\varphi}(R_i \otimes I_{\mathbb{C}^n}) = R_i \widetilde{\varphi}$ for $i = 1, \ldots, n$. Hence, we deduce that $\widetilde{\varphi}_j^*|_{\mathcal{N}_{\mathcal{P}_0}} = \widetilde{\psi}_j^*$ and $\widetilde{\varphi}_j R_i = R_i \widetilde{\varphi}_j$ for $i, j = 1, \ldots, n$. Since, due to [28], the commutant of the right creation operators $R_1, \ldots, R_n$ coincides with the noncommutative analytic Toeplitz algebra F_n^∞, we deduce that $\widetilde{\varphi}_j \in F_n^\infty$, $j = 1, \ldots, n$. Since $\widetilde{\varphi}^*|_{\mathcal{N}_{\mathcal{P}_0}} = \widetilde{\psi}^*$ and $\widetilde{\psi}$ is a non-scalar operator, so is $\widetilde{\varphi}$. According to Theorem 2.3 and Corollary 2.4, the composition operator $C_{\widetilde{\varphi}} : F^2(H_n) \to F^2(H_n)$ satisfies the equation

$$C_{\widetilde{\varphi}}\left(\sum_{k=0}^{\infty} \sum_{|\alpha|=k} a_\alpha e_\alpha\right) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} a_\alpha (\widetilde{\varphi}_\alpha 1) \tag{7.1}$$

for any $f = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} a_\alpha e_\alpha$ in $F^2(H_n)$. Since $\widetilde{\varphi}_j^*|_{\mathcal{N}_{\mathcal{P}_0}} = \widetilde{\psi}_j^*$, $j = 1, \ldots, n$, we have $P_{\mathcal{N}_{\mathcal{P}_0}} \widetilde{\varphi}_\alpha|_{\mathcal{N}_{\mathcal{P}_0}} = \widetilde{\psi}_\alpha$ for all $\alpha \in \mathbb{F}_n^+$. Since $1 \in \mathcal{N}_{\mathcal{P}_0}$, we assume that $f \in \mathcal{N}_{\mathcal{P}_0}$ in relation (7.1) and, taking the projection on $\mathcal{N}_{\mathcal{P}_0}$, we complete the proof of part (i).

Now, to prove item (ii), note that part (i) implies $C_{\widetilde{\psi}} = P_{\mathcal{N}_{\mathcal{P}_0}} C_{\widetilde{\varphi}}|_{\mathcal{N}_{\mathcal{P}_0}}$. Using this relation and Theorem 2.3, we deduce that $\|C_{\widetilde{\psi}}\| \leqslant (\frac{1+\|\psi(0)\|}{1-\|\psi(0)\|})^{1/2}$. Recall that $z_\lambda := \sum_{\alpha \in \mathbb{F}_n^+} \bar{\lambda}_\alpha e_\alpha$, $\lambda \in \mathbb{B}_n$. Note that if $\lambda = (\lambda_1, \ldots, \lambda_n)$ is in the scalar representation of the noncommutative variety $\mathcal{V}_{\mathcal{P}_0}$, i.e.,

$$\mathcal{V}_{\mathcal{P}_0}(\mathbb{C}) := \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{B}_n : \ p(\lambda_1, \ldots, \lambda_n) = 0, \ p \in \mathcal{P}_0\},$$

then we have

$$\langle [S_\alpha p(S_1, \ldots, S_n) S_\beta](1), z_\lambda \rangle = \lambda_\alpha p(\lambda) \lambda_\beta = 0,$$

for any $p \in \mathcal{P}_0$ and $\alpha, \beta \in \mathbb{F}_n^+$. Hence $z_\lambda \in \mathcal{N}_{\mathcal{P}_0}$ for any $\lambda \in \mathcal{V}_{\mathcal{P}_0}(\mathbb{C})$. As in the proof of Theorem 2.1, we have

$$C_{\widetilde{\varphi}}^* z_\mu = \sum_{k=0} \sum_{|\alpha|=k} \overline{\varphi_\alpha(\mu)} e_\alpha = z_{\varphi(\mu)}, \quad \mu := (\mu_1, \dots, \mu_n) \in \mathbb{B}_n.$$

Now, note that

$$\|C_{\widetilde{\psi}}\| = \|C_{\widetilde{\psi}}^*\| \geqslant \frac{\|C_{\widetilde{\psi}}^* z_\mu\|}{\|z_\mu\|} = \frac{\|P_{\mathcal{N}_{\mathcal{P}_0}} C_{\widetilde{\varphi}}^* z_\mu\|}{\|z_\mu\|} = \frac{\|P_{\mathcal{N}_{\mathcal{P}_0}} z_{\psi(\mu)}\|}{\|z_\mu\|}$$

for any $\lambda \in \mathcal{V}_{\mathcal{P}_0}(\mathbb{C})$. Since $0 \in \mathcal{V}_{\mathcal{P}_0}(\mathbb{C})$ the first two inequalities in part (ii) follow.

Now, it remains to prove part (iii). According to Proposition 4.1, we have

$$C_{\widetilde{\psi}}^* g = P_{\mathcal{N}_{\mathcal{P}_0}} C_{\widetilde{\varphi}}^* g = \sum_{\alpha \in \mathbb{F}_n^+} \langle g, \widetilde{\varphi}_\alpha 1 \rangle P_{\mathcal{N}_{\mathcal{P}_0}} e_\alpha, \quad g \in F^2(H_n).$$

Since $P_{\mathcal{N}_{\mathcal{P}_0}} \widetilde{\varphi}_\alpha|_{\mathcal{N}_{\mathcal{P}_0}} = \widetilde{\psi}_\alpha$ for all $\alpha \in \mathbb{F}_n^+$ and $1 \in \mathcal{N}_{\mathcal{P}_0}$, we deduce part (iii). The proof is complete. □

We remark that under the conditions of Theorem 7.1, we can use Theorem 1.1 to show that

$$\|\psi(X_1, \dots, X_n)\| < 1, \quad (X_1, \dots, X_n) \in \mathcal{V}_{\mathcal{P}_0}(\mathcal{H}).$$

Consequently, $g \circ \widetilde{\psi}$ induces the map

$$(g \circ \psi)(X) := \sum_{k=0}^{\infty} \sum_{|\alpha|=k} c_\alpha \psi_\alpha(X), \quad X \in \mathcal{V}_{\mathcal{P}_0}(\mathcal{H}),$$

where the convergence is in the operator norm topology. Using Corollary 2.4, we deduce that

$$\lim_{r \to 1} (g \circ \psi)(rB_1, \dots, rB_n)1 = g \circ \widetilde{\psi}.$$

Moreover, the map $g \circ \psi$ is the restriction to $\mathcal{V}_{\mathcal{P}_0}(\mathcal{H})$ of the free holomorphic function $g \circ \varphi$ on $[B(\mathcal{H})^n]_1$, where φ was introduced in the proof of Theorem 7.1.

Corollary 7.2. *Let $\widetilde{\psi} = (\widetilde{\psi}_1, \dots, \widetilde{\psi}_n) \in F_n^\infty(\mathcal{V}_{\mathcal{P}_0}) \otimes \mathbb{C}^n$ be a non-scalar operator with $\|\widetilde{\psi}\| \leqslant 1$ and $p(\psi(0)) = 0$ for all $p \in \mathcal{P}_0$. Then the norm of composition operator $C_{\widetilde{\psi}} : \mathcal{N}_{\mathcal{P}_0} \to \mathcal{N}_{\mathcal{P}_0}$ satisfies the inequalities*

$$\frac{1}{(1 - \|\psi(0)\|^2)^{1/2}} \leqslant \|C_{\widetilde{\psi}}\| \leqslant \left(\frac{1 + \|\psi(0)\|}{1 - \|\psi(0)\|} \right)^{1/2}.$$

Moreover, the spectral radius of $C_{\widetilde{\psi}}$ satisfies the relation

$$r(C_{\widetilde{\psi}}) = \lim_{k \to \infty} \left(1 - \|\varphi^{[k]}(0)\|\right)^{-1/2k}.$$

Proof. Since $p(\psi(0)) = 0$ for all $p \in \mathcal{P}_0$, we have $\psi(0) \in \mathcal{V}_{\mathcal{P}_o}(\mathbb{C})$ and, as in the proof of Theorem 7.1, we deduce that $z_{\psi(0)} \in \mathcal{N}_{\mathcal{P}_0}$. Consequently,

$$\|P_{\mathcal{N}_{\mathcal{P}_0}} z_{\psi(0)}\| = \|z_{\psi(0)}\| = \frac{1}{(1 - \|\psi(0)\|^2)^{1/2}}.$$

Combining this relation with part (ii) of Theorem 7.1, we deduce the inequalities above. The proof of the last part of this corollary is similar to the proof of Theorem 2.9. □

Now we consider an important particular case. If $\mathcal{P}_c := \{X_i X_j - X_j X_i\colon i, j = 1, \ldots, n\}$, then $\mathcal{N}_{\mathcal{P}_c} = \overline{\operatorname{span}}\{z_\lambda\colon \lambda \in \mathbb{B}_n\} = F_s^2$, the symmetric Fock space. For each $\lambda = (\lambda_1, \ldots, \lambda_n)$ and each n-tuple $\mathbf{k} := (k_1, \ldots, k_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0 := \{0, 1, \ldots\}$, let $\lambda^{\mathbf{k}} := \lambda_1^{k_1} \cdots \lambda_n^{k_n}$. For each $\mathbf{k} \in \mathbb{N}_0^n$, we denote

$$\Lambda_{\mathbf{k}} := \left\{\alpha \in \mathbb{F}_n^+\colon \lambda_\alpha = \lambda^{\mathbf{k}} \text{ for all } \lambda \in \mathbb{C}^n\right\}$$

and define the vector

$$w^{\mathbf{k}} := \frac{1}{\gamma_{\mathbf{k}}} \sum_{\alpha \in \Lambda_{\mathbf{k}}} e_\alpha \in F^2(H_n), \qquad \text{where } \gamma_{\mathbf{k}} := \operatorname{card} \Lambda_{\mathbf{k}}.$$

The set $\{w^{\mathbf{k}}\colon \mathbf{k} \in \mathbb{N}_0^n\}$ consists of orthogonal vectors in $F^2(H_n)$ which span the symmetric Fock space F_s^2 and $\|w^{\mathbf{k}}\| = \frac{1}{\sqrt{\gamma_{\mathbf{k}}}}$. The symmetric Fock space F_s^2 can be identified with the Drury–Arveson space $\mathbf{H}_n^2$ of all functions $\varphi : \mathbb{B}_n \to \mathbb{C}$ which admit a power series representation $\varphi(\lambda) = \sum_{\mathbf{k} \in \mathbb{N}_0} c_{\mathbf{k}} \lambda^{\mathbf{k}}$ with

$$\|\varphi\|_2 = \sum_{\mathbf{k} \in \mathbb{N}_0} |c_{\mathbf{k}}|^2 \frac{1}{\gamma_{\mathbf{k}}} < \infty.$$

More precisely, every element $\varphi = \sum_{\mathbf{k} \in \mathbb{N}_0} c_{\mathbf{k}} w^{\mathbf{k}}$ in F_s^2 has a functional representation on $\mathbb{B}_n$ given by

$$\varphi(\lambda) := \langle \varphi, z_\lambda \rangle = \sum_{\mathbf{k} \in \mathbb{N}_0} c_{\mathbf{k}} \lambda^{\mathbf{k}}, \quad \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{B}_n, \tag{7.2}$$

and

$$|\varphi(\lambda)| \leqslant \frac{\|\varphi\|_2}{\sqrt{1 - \|\lambda\|^2}}, \quad \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{B}_n.$$

Arveson showed that the algebra $F_n^\infty(\mathcal{V}_{\mathcal{P}_c})$ can be identified with the algebra of all multipliers of $\mathbf{H}_n^2$. Under this identification the creation operators $L_i := P_{F_s^2} S_i|_{F_s^2}$, $i = 1, \ldots, n$, on the symmetric Fock space become the multiplication operators $M_{z_1}, \ldots, M_{z_n}$ by the coordinate functions $z_1, \ldots, z_n$ of $\mathbb{C}^n$.

Theorem 7.3. *Let $\widetilde{\psi} = (\widetilde{\psi}_1, \ldots, \widetilde{\psi}_n) \in F_n^\infty(\mathcal{V}_{\mathcal{P}_c}) \otimes \mathbb{C}^n$ be a non-scalar operator with $\|\widetilde{\psi}\| \leqslant 1$. Under the identification of the symmetric Fock space F_s^2 with the Drury–Arveson space $\mathbf{H}_n^2$, the composition operator $C_{\widetilde{\psi}} : F_s^2 \to F_s^2$ has the functional representation*

$$(C_{\widetilde{\psi}} f)(\lambda) = f\big(\psi(\lambda)\big), \quad \lambda \in \mathbb{B}_n.$$

Moreover, if $f \in F_s^2$, then

$$\big(C_{\widetilde{\psi}}^* f\big)(\lambda) = \langle f, z_\lambda \circ \widetilde{\psi}\rangle, \quad \lambda \in \mathbb{B}_n,$$

where $z_\lambda := \sum_{\alpha \in \mathbb{F}_n^+} \bar{\lambda}_\alpha e_\alpha$.

Proof. As in the proof of Theorem 7.1, due to the noncommutative commutant lifting theorem, there is $\widetilde{\varphi} = (\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n) \in F_n^\infty \otimes \mathbb{C}^n$ a non-scalar operator with $\|\widetilde{\varphi}\| \leqslant 1$, such that $\widetilde{\varphi}_i^*|_{F_s^2} = \widetilde{\psi}_i^*$, $i = 1, \ldots, n$. In particular, due to (7.2), we have $\varphi(\lambda) = \psi(\lambda)$, $\lambda \in \mathbb{B}_n$. Fix $f = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha e_\alpha \in F_s^2$ and $\lambda \in \mathbb{B}_n$. Since $z_\lambda \in F_s^2$ and $P_{F_s^2} \widetilde{\varphi}_\alpha|_{F_s^2} = \widetilde{\psi}_\alpha$ for all $\alpha \in \mathbb{F}_n^+$, we can use relations (7.2), (2.9), as well as Corollary 2.4 and Theorem 7.1, to obtain

$$\begin{aligned} f\big(\psi(\lambda)\big) &= \langle f, z_{\psi(\lambda)}\rangle = \langle f, z_{\varphi(\lambda)}\rangle = \big\langle f, C_{\widetilde{\varphi}}^* z_\lambda\big\rangle = \langle C_{\widetilde{\varphi}} f, z_\lambda\rangle \\ &= \Big\langle \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha \widetilde{\varphi}_\alpha 1, z_\lambda \Big\rangle = \Big\langle \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha P_{F_s^2} \widetilde{\varphi}_\alpha 1, z_\lambda \Big\rangle = \Big\langle \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha \widetilde{\psi}_\alpha 1, z_\lambda \Big\rangle \\ &= \langle C_{\widetilde{\psi}} f, z_\lambda\rangle = (C_{\widetilde{\psi}} f)(\lambda). \end{aligned}$$

Therefore, the first part of the theorem holds. To prove the second part, note that according to item (iii) of Theorem 7.1, we have

$$C_{\widetilde{\psi}}^* f = \sum_{k=0}^\infty \sum_{|\alpha|=k} \big\langle f, \widetilde{\psi}_\alpha(1)\big\rangle P_{F_s^2} e_\alpha, \quad f \in F_s^2. \tag{7.3}$$

On the other hand, since $z_\lambda \in F_s^2$, part (i) of Theorem 7.1 implies $z_\lambda \circ \widetilde{\psi} \in F_s^2$ and

$$z_\lambda \circ \widetilde{\psi} = \sum_{k=0}^\infty \sum_{|\alpha|=k} \bar{\lambda}_\alpha \widetilde{\psi}_\alpha 1,$$

where the convergence is in $F^2(H_n)$. Consequently, using relations (7.3) and (7.2), we deduce that

$$\begin{aligned} \langle f, z_\lambda \circ \widetilde{\psi}\rangle &= \Big\langle f, \sum_{k=0}^\infty \sum_{|\alpha|=k} \bar{\lambda}_\alpha \widetilde{\psi}_\alpha 1 \Big\rangle = \sum_{k=0}^\infty \sum_{|\alpha|=k} \langle f, \widetilde{\psi}_\alpha 1\rangle \lambda_\alpha \\ &= \Big\langle \sum_{k=0}^\infty \sum_{|\alpha|=k} \big\langle f, \widetilde{\psi}_\alpha(1)\big\rangle e_\alpha, z_\lambda \Big\rangle = \big(C_{\widetilde{\psi}}^* f\big)(\lambda) \end{aligned}$$

for any $\lambda \in \mathbb{B}_n$. The proof is complete. □

Since $\psi(\lambda) \in \mathcal{V}_{\mathcal{P}_c}$ for all $\lambda \in \mathbb{B}_n$ part (ii) of Theorem 7.1 implies the following result concerning the composition operators on the symmetric Fock space F_s^2 and, consequently, on the Drury–Arveson space $\mathbf{H}_n^2$. The next result was obtained by Jury [11] using different methods.

Corollary 7.4. *Let $\widetilde{\psi} = (\widetilde{\psi}_1, \ldots, \widetilde{\psi}_n) \in F_n^\infty(\mathcal{V}_{\mathcal{P}_c}) \otimes \mathbb{C}^n$ be a non-scalar operator with $\|\widetilde{\psi}\| \leqslant 1$. Then the composition operator $C_{\widetilde{\psi}} : F_s^2 \to F_s^2$ is bounded and*

$$\frac{1}{(1 - \|\psi(0)\|^2)^{1/2}} \leqslant \sup_{\lambda \in \mathbb{B}_n} \left(\frac{1 - \|\lambda\|^2}{1 - \|\psi(\lambda)\|^2} \right)^{1/2} \leqslant \|C_{\widetilde{\psi}}\| \leqslant \left(\frac{1 + \|\psi(0)\|}{1 - \|\psi(0)\|} \right)^{1/2}.$$

It is obvious now that the formula for the spectral radius of $C_{\widetilde{\psi}}$ (see Corollary 7.2) holds. We also remark that one can deduce commutative versions of Corollary 2.5, Theorem 2.6, and Corollary 2.7. We leave this task to the reader.

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