JOURNAL OF NUMBER THEORY 36, 46–79 (1990)

A Summation Formula in Algebraic Number Fields and Applications, I

ULRICH RAUSCH

Technische Universität Clausthal, Institut für Mathematik, Erzstraße 1, D-3392 Clausthal-Zellerfeld, Federal Republic of Germany

Communicated by E. Hlawka

Received September 15, 1989

A summation formula in algebraic number fields is established which resembles Siegel's summation formula but covers a wider range of problems and yields better results. Three applications are given, one of them to the Piltz divisor problem for algebraic numbers.

INTRODUCTION

Let K be an algebraic number field of degree $[K:\mathbb{Q}]=n=r_1+2r_2$ (in the standard notation), d its discriminant, h its class number, and $r=r_1+r_2-1$ its number of fundamental units. Let $\mathscr{E}$, R, and w denote respectively the group of units, the regulator, and the number of roots of unity in K.

The conjugates of a number $v \in K$ are denoted by $v^{(p)}$ $(p=1, ..., n)$, and we write $v \succ 0$ to indicate that v is totally positive, i.e., $v \neq 0$ and $v^{(p)}>0$ for $p=1, ..., r_1$ (so $v \succ 0$ means simply $v \neq 0$ if K is totally imaginary). The numbers $e_1, ..., e_{r+1}$ are defined by

$$e_p=\begin{cases}1 & \text{for} \quad p=1, ..., r_1,\\ 2 & \text{for} \quad p=r_1+1, ..., r+1.\end{cases}$$

Throughout, x will denote a vector

$$x=(x_1, ..., x_{r+1}) \in \mathbb{R}_+^{r+1}$$

($\mathbb{R}_+$ is the set of positive real numbers), and

$$X=\prod_{p=1}^{r+1} x_p^{e_p}.$$

0022-314X/90 $3.00

In 1936, Siegel [12] gave a method for evaluating sums of the form

$$F(x)=\sum_{\substack{\nu \succ 0\\ |\nu^{(p)}|\leqslant x_p}} f(\nu), \tag{1}$$

where the summation is over the totally positive integers $\nu \in K$ satisfying $|\nu^{(p)}|\leqslant x_p$ $(p=1, ..., r+1)$, and f is an arithmetic function such that

$$f(\eta_q \nu)=f(\nu) \qquad (q=1, ..., r)$$

for some independent units $\eta_1, ..., \eta_r \in \mathscr{E}$.

Actually, Siegel confined himself to real-quadratic fields; the generalization of his result to totally real and arbitrary number fields was achieved respectively by Schaal [11] and Grotz [1].

The method consists, roughly, in passing from (1) to the smoothed sum

$$\sum_{\substack{\nu \succ 0\\ |\nu^{(p)}|\leqslant x_p}} f(\nu) \prod_{p=1}^{r+1} (1-|\nu^{(p)}|\, x_p^{-1})^{l_p-1} \qquad (1<l_p \in \mathbb{Z}) \tag{2}$$

and expressing it, via Fourier expansion "with respect to the units," as a series of complex integrals involving generating Dirichlet series of f with Grössencharacters.

Serving as a substitute for Perron's formula in algebraic number fields, Siegel's summation formula has proved to be extremely useful; nevertheless it has two drawbacks, both basically due to the smoothing procedure just mentioned.

First, it is clear from (2) that the method is appropriate only when the range of summation is the special box-like domain $|\nu^{(p)}|\leqslant x_p$ $(p=1, ..., r+1)$. (Extending the formula to other domains by approximating them by such boxes, as suggested by Siegel [12, p. 222], seems not to be practicable in general.)

Second, due to the fact that, for technical reasons, the numbers l_p in (2) have to be rational integers, the smoothing is often stronger than really necessary. This results in error estimates which are weaker than they should be.

In this paper we give a related summation formula which avoids both of these disadvantages.

To begin with, it covers sums of the form

$$F(x)=\sum_{\nu \succ 0} \Lambda(\nu)\, f(\nu)\, \Phi(|\nu^{(1)}|\, x_1^{-1}, ..., |\nu^{(r+1)}|\, x_{r+1}^{-1})$$

with generalized Grössencharacters Λ and fairly arbitrary weight functions $\Phi\colon \mathbb{R}_+^{r+1} \to \mathbb{C}$.

The essential point of our approach lies in using, instead of (2), a certain integral average $\mathscr{I}_\varepsilon(F(x))$ of $F(x)$. The degree of smoothness produced thereby is large enough to ensure the validity of the formula under rather weak conditions and to make its proof quite straightforward (in fact, all series and integrals occuring in the proof are absolutely convergent; so, using Lebesgue's integral, one finds no difficulty in justifying inversions of the order of limit processes).

On the other hand, owing to the freedom of choice regarding the parameter $\varepsilon > 0$, the error estimates obtained with the aid of this formula reflect the specific nature of the problem under consideration, without obvious losses arising from the summation formula itself.

After some basic facts about units and Grössencharacters have been given in Section 1, the summation formula is derived in Section 2. In Section 3 we establish the second ingredient of the method, viz., a kind of Tauberian theorem which allows us to infer back from the average $\mathscr{I}_\varepsilon(F(x))$ to the original function $F(x)$. Section 4 contains some lemmas that are needed for the applications.

As a first example, we consider in Section 5 the Piltz divisor problem for numbers in K, i.e., the problem of evaluating asymptotically the sum

$$D_k(x) = \sum_{\substack{\nu \succ 0 \\ |\nu^{(p)}| \leqslant x_p}} d_k(\nu), \tag{3}$$

where $d_k(\nu)$ denotes the number of representations of the principal ideal (ν) as a product of k integral ideals. (The corresponding problem for ideals was tackled by Landau [5] and, in special fields, by Landau [6, Sects. 12, 13] and Karacuba [4].) By means of Siegel's summation formula, Grotz [1] proved

$$D_k(x) = XP_{k-1}(\log X) + O(X^{1-1/(\langle k/2 \rangle r_1 + kr_2 + 1) + \delta})$$

for every $\delta > 0$; P_{k-1} is a certain polynomial of degree $k-1$ and $\langle k/2 \rangle$ denotes the least rational integer $\geqslant k/2$.

We generalize Grotz's result by inserting in (3) a factor $\Lambda(\nu)$, Λ a generalized Grössencharacter, and improve the remainder term to

$$O(X^{1-2/(nk+2)}(\log X)^{k-nk/(nk+2)}).$$

The second example, given in Section 6, deals with $\sigma(\nu)$, the sum of the norms of the ideal divisors of ν. In the simplest case (without Grössencharacter), our result is

$$\sum_{\substack{\nu \succ 0 \\ |\nu^{(p)}| \leqslant x_p}} \sigma(\nu) = \frac{2^{-r_1}\pi^{r_2}}{|\sqrt{d}|} \zeta_K(2) X^2 + O(X^{2-2/(n+2)}(\log X)^{4/(n+2)}),$$

where ζ_K denotes Dedekind's zeta function. Here, Grotz [1] obtained the error term

$$O(X^{2-1/(r+2)+\delta}).$$

The other asymptotic formulae established by Grotz in [1, 2] may be improved in a similar manner.

In the third application, just to show the capabilities of the method, we deviate from the general pattern of the preceding examples in two respects. First, the weight function Φ is no longer the characteristic function of the unit cube $(0, 1]^{r+1}$ but involves a certain norm in $\mathbb{R}^{r+1}$. Second, the summation ranges over some group of units, which is as a rule more delicate than summing over all integers since, in order to obtain any asymptotic formula whatsoever, one has to study the numbers $E_p(\tau)$ (cf. Section 1) in some detail. For the results, the reader is referred to Section 7.

In the forthcoming second part of the paper I shall give the summation formula in full generality, the weight Φ depending then on the conjugates $\nu^{(p)}$ themselves, not merely on their absolute values.

1. Units and Grössencharacters

We assume $r > 0$ and consider a free group $\mathscr{U}$ of totally positive units which has finite index $[\mathscr{E} : \mathscr{U}]$ in $\mathscr{E}$.

We fix a basis $\eta_1, \ldots, \eta_r$ of $\mathscr{U}$ and define

$$R(\mathscr{U}) = |\det(e_p \log |\eta_1^{(p)}|, \ldots, e_p \log |\eta_r^{(p)}|)_{p=1, \ldots, r}|;$$

then

$$[\mathscr{E} : \mathscr{U}] = wR(\mathscr{U})/R. \tag{1.1}$$

We further fix real numbers $\vartheta_q^{(p)}$ such that

$$\eta_q^{(p)} = |\eta_q^{(p)}|\, e^{2\pi i \vartheta_q^{(p)}} \qquad (q = 1, \ldots, r;\ p = r_1 + 1, \ldots, r+1) \tag{1.2}$$

and define for arbitrary

$$\tau = (\tau_1, \ldots, \tau_r) \in \mathbb{R}^r, \qquad a = (a_{r_1+1}, \ldots, a_{r+1}) \in \mathbb{Z}^{r_2}$$

the numbers

$$E_1(\tau, a), \ldots, E_{r+1}(\tau, a)$$

by the system of equations

$$\sum_{p=1}^{r+1} e_p E_p(\tau, a) = 0,$$
$$\sum_{p=1}^{r+1} e_p E_p(\tau, a) \log |\eta_q^{(p)}| = 2\pi \left(\tau_q - \sum_{p=r_1+1}^{r+1} a_p \vartheta_q^{(p)} \right) \qquad (q = 1, ..., r) \tag{1.3}$$

on the understanding that $a = 0$ if $r_2 = 0$.

In the case $a = 0$ we abbreviate

$$E_p(\tau, 0) =: E_p(\tau) \qquad (p = 1, ..., r+1).$$

If we take another basis of $\mathscr{U}$, $\tilde{\eta}_1, ..., \tilde{\eta}_r$ say, together with numbers $\tilde{\vartheta}_q^{(p)}$ according to (1.2), then it is easily seen that the corresponding numbers $\tilde{E}_p(\tau, a)$ satisfy

$$\tilde{E}_p(\tau, a) = E_p(\tau T + l, a) \qquad (p = 1, ..., r+1)$$

with a unimodular matrix $T \in \mathbb{Z}^{r \times r}$ and some $l \in \mathbb{Z}^r$. Hence the set of vectors

$$(E_1(m, a), ..., E_{r+1}(m, a)),$$

where $m = (m_1, ..., m_r)$ runs through $\mathbb{Z}^r$, is, as a whole, independent of the basis.

For the generalized Grössencharacter

$$\lambda_{\tau, a}(\nu) = \prod_{p=1}^{r+1} |\nu^{(p)}|^{ie_p E_p(\tau, a)} \cdot \prod_{p=r_1+1}^{r+1} \left(\frac{\nu^{(p)}}{|\nu^{(p)}|} \right)^{a_p} \qquad (0 \neq \nu \in K) \tag{1.4}$$

we have by (1.3)

$$\lambda_{\tau, a}(\eta_q) = e^{2\pi i \tau_q} \qquad (q = 1, ..., r); \tag{1.5}$$

thus $\lambda_{\tau, a}(\eta) = 1$ for every $\eta \in \mathscr{U}$ if and only if $\tau = m \in \mathbb{Z}^r$.

We conclude this section with a simple inequality used in Section 2.

For $t \in \mathbb{R}$,

$$\sum_{p=1}^{r+1} e_p |t - E_p(\tau)| \geqslant \left| \sum_{p=1}^{r+1} e_p (t - E_p(\tau)) \right| = n |t|;$$

on the other hand,

$$2\pi |\tau_q| = \left| \sum_{p=1}^{r+1} e_p (t - E_p(\tau)) \log |\eta_q^{(p)}| \right|$$
$$\leqslant \sum_{p=1}^{r+1} e_p |t - E_p(\tau)| \cdot \max_p |\log |\eta_q^{(p)}||.$$

Thus, by the Cauchy–Schwarz inequality,

$$\sum_{p=1}^{r+1} e_p^2(t-E_p(\tau))^2 \geqslant B_1\left(t^2+\sum_{q=1}^{r} \tau_q^2\right), \tag{1.6}$$

where $B_1>0$ depends only on K and $\eta_1, ..., \eta_r$.

2. The Summation Formula

Let f be a complex-valued arithmetic function defined on the integers $\nu \neq 0$ of K and possessing the invariance property

$$f(\eta\nu)=f(\nu) \qquad (\eta \in \mathscr{U}), \tag{2.1}$$

where $\mathscr{U}$ is a group of totally positive units as in Section 1.

Let the numbers $b_p \in \mathbb{R}$ and $A_p \in \mathbb{Z}$ $(p=1, ..., r+1)$ be arbitrarily given with $A_p \in \{0, 1\}$ for $p=1, ..., r_1$. We consider the generalized Grössen-character

$$\Lambda(\nu)=\prod_{p=1}^{r+1} |\nu^{(p)}|^{ie_pb_p} \cdot \prod_{p=r_1+1}^{r+1} \left(\frac{\nu^{(p)}}{|\nu^{(p)}|}\right)^{A_p} \tag{2.2}$$

and the sign character

$$v(\nu)=\prod_{p=1}^{r_1} \left(\frac{\nu^{(p)}}{|\nu^{(p)}|}\right)^{A_p}. \tag{2.3}$$

Further suppose $\Phi\colon \mathbb{R}_+^{r+1} \to \mathbb{C}$ is a Lebesgue-measurable function and $x=(x_1, ..., x_{r+1}) \in \mathbb{R}_+^{r+1}$.

We wish to study the series

$$G(x; \Lambda v f, \Phi)=\sum_{\nu}{}' \Lambda v f(\nu)\, \Phi(|\nu|\, x), \tag{2.4}$$

summed over the integers $\nu \in K$, the prime excluding the value $\nu=0$. Here and in the sequel we abbreviate

$$\Lambda v f(\nu)=\Lambda(\nu)\, v(\nu)\, f(\nu),$$
$$|\nu|\, x=(|\nu^{(1)}|\, x_1, ..., |\nu^{(r+1)}|\, x_{r+1}).$$

We need some further notations. Let

$$\Psi(s)=2^{r_2}\int_{\mathbb{R}_+^{r+1}} \Phi(u) \prod_{p=1}^{r+1} u_p^{e_ps_p-1}\, du \qquad \text{for} \quad s=(s_1, ..., s_{r+1}) \in \mathbb{C}^{r+1}, \tag{2.5}$$

the $(r+1)$-dimensional Mellin-transform of Φ (merely formal for the time being), and, with a Grössencharacter $\lambda_{m,a}$ ($m \in \mathbb{Z}^r$) according to (1.4),

$$\Xi(s; \lambda_{m,a} vf, \mathscr{U}) = \sum_{(v)_{\mathscr{U}}}{}' \frac{\lambda_{m,a} vf(v)}{|N(v)|^s} \qquad (s \in \mathbb{C}),$$

where v runs over a complete set of non-zero integers of K which are not associated with respect to $\mathscr{U}$. The series is well-defined because of (1.5) and (2.1).

Finally, for $\varepsilon > 0$, let the integral operator $\mathscr{I}_\varepsilon$, acting on functions $F: \mathbb{R}_+^{r+1} \to \mathbb{C}$, be given by

$$\mathscr{I}_\varepsilon(F): x \mapsto (4\pi\varepsilon)^{-(r+1)/2} \int_{\mathbb{R}^{r+1}} F(xe^v)\, e^{-|v|^2/(4\varepsilon)}\, dv,$$

where $xe^v = (x_1 e^{v_1}, ..., x_{r+1} e^{v_{r+1}})$ and $|v| = (\sum_{p=1}^{r+1} v_p^2)^{1/2}$. (The double use of the letter v, as well as of the letter s above, is not likely to cause any confusion.)

For convenience we shall write $\mathscr{I}_\varepsilon(F(x))$ instead of $\mathscr{I}_\varepsilon(F)(x)$.

We now begin by investigating the special case of (2.4), where f is the characteristic function of $\mathscr{U}$, that is, we consider

$$H(x; A, \Phi, \mathscr{U}) = \sum_{\eta \in \mathscr{U}} A(\eta)\, \Phi(|\eta|\, x).$$

THEOREM 2.1. *Let $\sigma \in \mathbb{R}$ be such that*

$$B_2 := \int_{\mathbb{R}_+^{r+1}} |\Phi(u)| \prod_{p=1}^{r+1} u_p^{e_p\sigma - 1}\, du < \infty. \tag{2.6}$$

Then $H(x; A, \Phi, \mathscr{U})$ is absolutely convergent almost everywhere (that is, the set of exceptional $x \in \mathbb{R}_+^{r+1}$ has Lebesgue-measure zero). For all $x \in \mathbb{R}_+^{r+1}$ and all $\varepsilon > 0$, $\mathscr{I}_\varepsilon(H(x; A, \Phi, \mathscr{U}))$ is absolutely convergent and satisfies the equation

$$\begin{aligned}\mathscr{I}_\varepsilon(H(x; A, \Phi, \mathscr{U})) = \frac{1}{2\pi i R(\mathscr{U})} \sum_m \int_{(\sigma)} & \Psi(s + ib - iE(m, A)) \\ & \cdot \prod_{p=1}^{r+1} x_p^{-e_p(s + ib_p - iE_p(m, A))} \\ & \cdot \exp\left\{\varepsilon \sum_{p=1}^{r+1} e_p^2 (s + ib_p - iE_p(m, A))^2\right\} ds, \end{aligned} \tag{2.7}$$

where sum and integrals on the right converge absolutely, too. Here m runs through $\mathbb{Z}^r$, the path of integration is the line $\operatorname{Re} s = \sigma$, A is the system of *exponents* A_p *occuring in* (2.2), *the numbers* $E_p(m, A)$ *are defined by* (1.3) *with respect to an arbitrary basis* $\eta_1, \ldots, \eta_r$ *of* $\mathscr{U}$, *and the abbreviation*

$$s + ib - iE(m, A) = (s + ib_1 - iE_1(m, A), \ldots, s + ib_{r+1} - iE_{r+1}(m, A))$$

is used.

Proof. (a) We begin by establishing (2.7) under the assumption that $\mathscr{I}_\varepsilon(H(x; 1, |\Phi|, \mathscr{U}))$ is finite. Then $H(xe^v; 1, |\Phi|, \mathscr{U})$ converges for almost all $v \in \mathbb{R}^{r+1}$, and we may invert the order of summation and integration on the left-hand side of (2.7). Further we observe that neither side of (2.7) changes its value if we add the same real number to all of the b_p's; thus we may assume

$$\sum_{p=1}^{r+1} e_p b_p = 0$$

and consequently write, according to (1.3),

$$b_p = E_p(\tau, A) \qquad (p = 1, \ldots, r+1)$$

with suitable $\tau \in \mathbb{R}^r$, so that $A = \lambda_{\tau, A}$. Hence

$$b_p - E_p(m, A) = -E_p(m - \tau, 0) = -E_p(m - \tau)$$

and, in view of (1.5), (2.7) becomes

$$\begin{aligned} \sum_l e^{2\pi i l \cdot \tau} \mathscr{I}_\varepsilon(\Phi(|\eta_1^{l_1} \cdots \eta_r^{l_r}| x)) = \frac{1}{2\pi i R(\mathscr{U})} \sum_m \int_{(\sigma)} & \Psi(s - iE(m - \tau)) \\ & \cdot \prod_{p=1}^{r+1} x_p^{-e_p(s - iE_p(m - \tau))} \\ & \cdot \exp\left\{\varepsilon \sum_{p=1}^{r+1} e_p^2 (s - iE_p(m - \tau))^2\right\} ds, \end{aligned} \tag{2.8}$$

where $l = (l_1, \ldots, l_r)$ runs through $\mathbb{Z}^r$ and $l \cdot \tau = l_1\tau_1 + \cdots + l_r\tau_r$.

Now we regard τ as variable. In (2.8), we have on the left an absolutely converging Fourier series, while from (1.6) and (2.6) we infer that the right-hand side is majorized by

$$\frac{2^{r_2} B_2}{2\pi R(\mathscr{U}) X^\sigma} e^{\varepsilon(r_1 + 4r_2)\sigma^2} \cdot \sum_m \int_{-\infty}^{\infty} \exp\left\{-\varepsilon B_1 \left(t^2 + \sum_{q=1}^{r} (m_q - \tau_q)^2\right)\right\} dt \tag{2.9}$$

and hence converges uniformly on every bounded set $|\tau| \leqslant \tau_0$. Consequently, it represents a continuous function $h(\tau)$ which obviously has period 1 in each τ_q.

To establish (2.8), it remains only to show that $\mathscr{J}_\varepsilon(\Phi(|\eta|x))$ is the lth Fourier coefficient of $h(\tau)$ if $\eta=\eta_1^{l_1}\cdots\eta_r^{l_r}$. For then (2.8) follows by continuity simply from the fact that $h(\tau)$ is the L^2-limit of its Fourier series.

Let Q denote the cube $0\leqslant\tau_q<1$ $(q=1,...,r)$. Then, by absolute convergence,

$$\begin{aligned}
&\int_Q h(\tau)\,e^{-2\pi i l\cdot\tau}\,d\tau\\
&=\frac{1}{2\pi i R(\mathscr{U})}\sum_m\int_Q\int_{(\sigma)}\Psi(s-iE(m-\tau))\cdot\prod_{p=1}^{r+1}x_p^{-e_p(s-iE_p(m-\tau))}\\
&\quad\cdot\exp\left\{\varepsilon\sum_{p=1}^{r+1}e_p^2(s-iE_p(m-\tau))^2\right\}ds\cdot e^{-2\pi i l\cdot\tau}\,d\tau\\
&=\frac{1}{2\pi R(\mathscr{U})}\int_{\mathbb{R}^r}\int_{-\infty}^{\infty}\Psi(\sigma+it+iE(\tau))\cdot\prod_{p=1}^{r+1}x_p^{-e_p(\sigma+it+iE_p(\tau))}\\
&\quad\cdot\exp\left\{\varepsilon\sum_{p=1}^{r+1}e_p^2(\sigma+it+iE_p(\tau))^2\right\}dt\cdot e^{-2\pi i l\cdot\tau}\,d\tau
\end{aligned}$$

by the substitutions $s=\sigma+it$, $\tau\to\tau+m$, since $\mathbb{R}^r$ is the disjoint union of the translated cubes $-m+Q$, $m\in\mathbb{Z}^r$.

Here we perform the change of variables $t_p=t+E_p(\tau)$ $(p=1,...,r+1)$, i.e.,

$$t=\frac{1}{n}\sum_{p=1}^{r+1}e_pt_p,\qquad 2\pi\tau_q=\sum_{p=1}^{r+1}e_pt_p\log|\eta_q^{(p)}|\qquad(q=1,...,r),$$

of determinant $R(\mathscr{U})/(2\pi)^r$ to obtain

$$\begin{aligned}
&\frac{1}{(2\pi)^{r+1}}\int_{\mathbb{R}^{r+1}}\Psi(\sigma+it_1,...,\sigma+it_{r+1})\cdot\prod_{p=1}^{r+1}x_p^{-e_p(\sigma+it_p)}\\
&\quad\cdot\exp\left\{\varepsilon\sum_{p=1}^{r+1}e_p^2(\sigma+it_p)^2\right\}\cdot\prod_{p=1}^{r+1}|\eta^{(p)}|^{-ie_pt_p}\,dt_1\cdots dt_{r+1}\\
&=\frac{1}{(2\pi i)^{r+1}}\int\cdots\int_{(\sigma)}\Psi(s)\cdot\prod_{p=1}^{r+1}(|\eta^{(p)}|\,x_p)^{-e_ps_p}\\
&\quad\cdot\exp\left\{\varepsilon\sum_{p=1}^{r+1}e_p^2s_p^2\right\}ds_1\cdots ds_{r+1}.
\end{aligned}$$

On substitution from the identity

$$\Psi(s) \cdot \prod_{p=1}^{r+1} (|\eta^{(p)}| \, x_p)^{-e_p s_p} = 2^{r_2} \int_{\mathbb{R}^{r+1}} \Phi(|\eta| \, xe^{v}) \prod_{p=1}^{r+1} e^{e_p s_p v_p} \, dv,$$

which is an immediate consequence of (2.5), and inversion of the order of integration, the above becomes

$$2^{r_2} \int_{\mathbb{R}^{r+1}} \Phi(|\eta| \, xe^{v}) \prod_{p=1}^{r+1} \left\{ \frac{1}{2\pi i} \int_{(\sigma)} e^{\varepsilon e_p^2 s_p^2 + e_p v_p s_p} \, ds_p \right\} dv = \mathscr{I}_\varepsilon(\Phi(|\eta| \, x))$$

by the well-known relation

$$\frac{1}{2\pi i} \int_{(\sigma)} e^{\varepsilon e_p^2 s^2 + e_p v_p s} \, ds = \frac{1}{\sqrt{4\pi\varepsilon \, e_p}} \, e^{-v^2/(4\varepsilon)}.$$

(b) In the second step we show that the above assumption holds automatically. To this end we consider for $T > 0$ the function

$$\Phi_T(u) = \begin{cases} |\Phi(u)| & \text{if } |u| \leqslant T \text{ and } |\Phi(u)| \leqslant T \\ 0 & \text{otherwise} \end{cases} \qquad (u \in \mathbb{R}_+^{r+1}).$$

Then $H(x; 1, \Phi_T, \mathscr{U})$ is a finite sum, and from the trivial estimate

$$H(x; 1, \Phi_T, \mathscr{U}) \leqslant T \cdot \sum_{\substack{|\eta^{(p)}| \, x_p \leqslant T \\ (p = 1, \ldots, r+1)}} 1 \ll T \cdot \prod_{p=1}^{r+1} \left(1 + \frac{T}{x_p}\right)$$

we infer that $\mathscr{I}_\varepsilon(H(x; 1, \Phi_T, \mathscr{U})) < \infty$ for all x and ε. Thus, by (a), an identity of the form (2.7) holds for every T; in particular, $\mathscr{I}_\varepsilon(H(x; 1, \Phi_T, \mathscr{U}))$ is bounded by the expression (2.9), uniformly in $T > 0$. Hence, by the monotone convergence theorem,

$$H(xe^{v}; 1, |\Phi|, \mathscr{U}) = \lim_{T \to \infty} H(xe^{v}; 1, \Phi_T, \mathscr{U})$$

is finite for almost all $v \in \mathbb{R}^{r+1}$ and

$$\mathscr{I}_\varepsilon(H(x; 1, |\Phi|, \mathscr{U})) = \lim_{T \to \infty} \mathscr{I}_\varepsilon(H(x; 1, \Phi_T, \mathscr{U})) < \infty.$$

This proves the assertion. ∎

THEOREM 2.2. *Let $\sigma \in \mathbb{R}$ be such that*

$$\int_{\mathbb{R}_+^{r+1}} |\Phi(u)| \prod_{p=1}^{r+1} u_p^{e_p \sigma - 1} \, du < \infty \qquad \textit{and} \qquad \sum_{(\nu)_{\mathscr{U}}}' \frac{|f(\nu)|}{|N(\nu)|^\sigma} < \infty.$$

Then $G(x; \Lambda vf, \Phi)$ is absolutely convergent almost everywhere. For all $x \in \mathbb{R}_+^{r+1}$ and all $\varepsilon > 0$, $\mathscr{I}_\varepsilon(G(x; \Lambda vf, \Phi))$ is absolutely convergent and satisfies the equation

$$\begin{aligned}
&\mathscr{I}_\varepsilon(G(x; \Lambda vf, \Phi)) \\
&\quad = \frac{1}{2\pi i R(\mathscr{U})} \sum_m \int_{(\sigma)} \Psi(s + ib - iE(m, A)) \cdot \Xi(s; \lambda_{m,A} vf, \mathscr{U}) \\
&\qquad \cdot \prod_{p=1}^{r+1} x_p^{-e_p(s + ib_p - iE_p(m, A))} \\
&\qquad \cdot \exp\left\{\varepsilon \sum_{p=1}^{r+1} e_p^2(s + ib_p - iE_p(m, A))^2\right\} ds, \qquad (2.10)
\end{aligned}$$

where sum and integrals on the right converge absolutely, too. As regards the notation, the same applies as in Theorem 2.1.

Proof. (a) Again, suppose first that $\mathscr{I}_\varepsilon(G(x; |f|, |\Phi|)) < \infty$. Collecting associated numbers and inverting the order of summation and integration yields

$$\begin{aligned}
&\mathscr{I}_\varepsilon(G(x; \Lambda vf, \Phi)) \\
&\quad = \sum_{(v)_{\mathscr{U}}}{}' \Lambda vf(v)\, \mathscr{I}_\varepsilon(H(|v|\, x; \Lambda, \Phi, \mathscr{U})) \\
&\quad = \frac{1}{2\pi i R(\mathscr{U})} \sum_{(v)_{\mathscr{U}}}{}' \sum_m \int_{(\sigma)} \Psi(s + ib - iE(m, A)) \cdot \frac{\lambda_{m,A} vf(v)}{|N(v)|^s} \\
&\qquad \cdot \prod_{p=1}^{r+1} x_p^{-e_p(s + ib_p - iE_p(m, A))} \\
&\qquad \cdot \exp\left\{\varepsilon \sum_{p=1}^{r+1} e_p^2(s + ib_p - iE_p(m, A))^2\right\} ds
\end{aligned}$$

by Theorem 2.1. By absolute convergence, we may carry out the summation over v under the integral sign to obtain (2.10).

(b) That the assumption $\mathscr{I}_\varepsilon(G(x; |f|, |\Phi|)) < \infty$ is unnecessary follows by the same argument as in the preceding theorem, viz., on consideration of $\Phi_T(u)$ and $f_T(v)$, the latter being defined as $|f(v)|$ if $|f(v)| \leqslant T$ and as zero otherwise. ∎

Remark. We have tacitly assumed that $r > 0$. The results, however, hold also in the case $r = 0$. We only have to interpret then $\mathscr{U}$ as $\{1\}$, $R(\mathscr{U})$ as

1, and to drop the summation over m except for the value $m=0$, setting $E_1(0, A)=0$. Accordingly,

$$\Lambda(v)=|v^{(1)}|^{ib_1}, \qquad v(v)=\left(\frac{v^{(1)}}{|v^{(1)}|}\right)^{A_1}, \qquad \lambda_{0,A}(v)=1$$

for $K=\mathbb{Q}$, and

$$\Lambda(v)=|v^{(1)}|^{2ib_1}\left(\frac{v^{(1)}}{|v^{(1)}|}\right)^{A_1}, \qquad v(v)=1, \qquad \lambda_{0,A}(v)=\left(\frac{v^{(1)}}{|v^{(1)}|}\right)^{A_1}$$

for imaginary-quadratic K.

The formulae (2.7) and (2.10) read then

$$\mathscr{I}_\varepsilon(\Phi(x))=\frac{1}{2\pi i}\int_{(\sigma)} \Psi(s+ib_1)\, x_1^{-e_1(s+ib_1)} e^{\varepsilon e_1^2(s+ib_1)^2}\, ds$$

and

$$\mathscr{I}_\varepsilon(G(x; \Lambda v f, \Phi))=\frac{1}{2\pi i}\int_{(\sigma)} \Psi(s+ib_1)\, \Xi(s; \lambda_{0,A} v f, \{1\}) \cdot x_1^{-e_1(s+ib_1)} e^{\varepsilon e_1^2(s+ib_1)^2}\, ds,$$

respectively. They may be proved directly by substituting from (2.5) and interchanging limit processes.

All subsequent computations will include the case $r=0$ unless otherwise indicated.

3. A Tauberian Theorem

We shall need a result that enables us to infer back from the average $\mathscr{I}_\varepsilon(F(x))$ to the function $F(x)$ itself.

In the first instance, we observe that $\mathscr{I}_\varepsilon(1)=1$ and thus

$$\mathscr{I}_\varepsilon(F(x))=F(x)+(4\pi\varepsilon)^{-(r+1)/2}\int_{\mathbb{R}^{r+1}} \{F(xe^v)-F(x)\}\, e^{-|v|^2/(4\varepsilon)}\, dv.$$

From this it is easily concluded that

$$\lim_{\varepsilon\to 0+} \mathscr{I}_\varepsilon(F(x))=F(x),$$

provided that $\mathscr{I}_\varepsilon(F(x))$ exists for sufficiently small $\varepsilon>0$ and x is a point of continuity of F.

In most applications, however, one has to make ε a function of x (or, rather, of X). The following theorem deals with this situation under the assumption that the involved functions are neither growing nor oscillating too fast:

If a function $\alpha: \mathbb{R}_+ \to \mathbb{R}_+$ satisfies

$$\alpha(\xi e^{\xi'}) \leqslant c_1 \alpha(\xi) e^{c_2|\xi'|} \qquad \text{for all} \quad \xi \in \mathbb{R}_+ \text{ and } \xi' \in \mathbb{R}, \tag{3.1}$$

where $c_1 > 0$ and $c_2 \geqslant 0$ are independent of ξ and ξ', then we say that α is (c_1, c_2)-*moderately growing.*

EXAMPLE. $\xi^c(|\log \xi| + 1)^{c'}$ $(c, c' \geqslant 0)$ is $(1, c + c')$-moderately growing.

On substituting $\xi e^{-\xi'}$ for ξ in (3.1) we immediately obtain an inequality in the opposite direction:

$$\alpha(\xi e^{\xi'}) \geqslant \frac{1}{c_1} \alpha(\xi) e^{-c_2|\xi'|} \qquad (\xi \in \mathbb{R}_+, \xi' \in \mathbb{R}). \tag{3.2}$$

THEOREM 3.1. *Suppose that α, β: $\mathbb{R}_+ \to \mathbb{R}_+$ are (c_1, c_2)-moderately growing and that the functions F, M: $\mathbb{R}_+^{r+1} \to \mathbb{R}$ satisfy the following conditions for all $x = (x_1, ..., x_{r+1}) \in \mathbb{R}_+^{r+1}$:*

$F(x) \geqslant 0$, $F(x)$ is non-decreasing with respect to each x_p;
M has continuous partial derivatives subject to

$$\left| x_p \frac{\partial}{\partial x_p} M(x) \right| \leqslant c_3 \beta(X) \qquad (p = 1, ..., r+1). \tag{3.3}$$

Moreover, let the continuous function ε: $\mathbb{R}_+ \to \mathbb{R}_+$ be such that the following hold for all $x \in \mathbb{R}_+^{r+1}$:

$$0 < \varepsilon(X) \leqslant 1; \tag{3.4}$$

$$\sqrt{\varepsilon(X)}\, \beta(X) \leqslant c_4 \alpha(X); \tag{3.5}$$

$$|\mathcal{I}_{\varepsilon(X)}(F(x) - M(x))| \leqslant c_5 \alpha(X). \tag{3.6}$$

(Here c_3, c_4, c_5 denote positive real numbers independent of x.)

Then

$$F(x) = M(x) + O(\alpha(X)) \qquad \textit{for all} \quad x \in \mathbb{R}_+^{r+1}, \tag{3.7}$$

where the O-constant depends only on $c_1, ..., c_5$ and the degree n.

Remark. The theorem remains valid if condition (3.6) is replaced by

$$|\mathcal{I}_{\varepsilon(X)}(F(x)) - M(x)| \leqslant c_5 \alpha(X). \tag{3.6$'$}$$

Proof. (a) We first establish some rough estimates which mainly serve the purpose of justifying the definition of the function we shall finally deal with. The O- and $\ll$-constants as well as $c_6, ..., c_{12} \geqslant 0$ depend only on $c_1, ..., c_5$ and n.

Since

$$M(xe^v) - M(x) = \int_0^1 \frac{d}{d\xi} M(x_1 e^{\xi v_1}, ..., x_{r+1} e^{\xi v_{r+1}})\, d\xi,$$

it follows from (3.3) and the growth property of β that

$$M(xe^v) - M(x) \ll |v|\ \beta(X) e^{c_6 |v|} \qquad (c_6 = nc_2). \tag{3.8}$$

Hence we have for $\varepsilon > 0$

$$\begin{aligned}\mathscr{I}_\varepsilon(M(x)) &= M(x) + O\left(\frac{\beta(X)}{(4\pi\varepsilon)^{(r+1)/2}} \int_{\mathbb{R}^{r+1}} |v|\ e^{c_6|v|} e^{-|v|^2/(4\varepsilon)}\, dv\right)\\ &= M(x) + O\left(\sqrt{\varepsilon}\beta(X) \int_{\mathbb{R}^{r+1}} |v|\ e^{2\sqrt{\varepsilon} c_6 |v|} e^{-|v|^2}\, dv\right)\end{aligned}$$

after the change of variables $v \to 2\sqrt{\varepsilon}\ v$. Taking $\varepsilon = \varepsilon(X)$, we obtain by (3.4) and (3.5)

$$\mathscr{I}_{\varepsilon(X)}(M(x)) = M(x) + O(\alpha(X)), \tag{3.9}$$

which shows in particular that (3.6) and (3.6′) are essentially equivalent.

In virtue of the monotonic property of F, we have for $\varepsilon > 0$

$$2^{-(r+1)} F(x) = (4\pi\varepsilon)^{-(r+1)/2} \int_{\mathbb{R}_+^{r+1}} F(x)\, e^{-|v|^2/(4\varepsilon)}\, dv \leqslant \mathscr{I}_\varepsilon(F(x));$$

thus

$$F(x) \ll |\mathscr{I}_\varepsilon(F(x) - M(x))| + |\mathscr{I}_\varepsilon(M(x))|,$$

and (3.6) and (3.9) yield for $\varepsilon = \varepsilon(X)$

$$F(x) \ll |M(x)| + \alpha(X). \tag{3.10}$$

Now let

$$R(x) = F(x) - M(x), \qquad \varphi(x) = |R(x)|/\alpha(X).$$

By (3.8) and (3.10),

$$R(xe^v) \ll |M(x)| + |v|\ \beta(X) e^{c_6|v|} + \alpha(X) e^{c_6|v|};$$

so it follows from (3.2) that the quantity

$$\psi(x) := \sup\{\varphi(xe^v)e^{-c_7|v|} : v \in \mathbb{R}^{r+1}\} \qquad (c_7 = 2c_6 + 1)$$

is always finite. Clearly, ψ has the property

$$\psi(xe^v) \leqslant \psi(x)e^{c_7|v|}, \tag{3.11}$$

and (3.7) is equivalent to the boundedness of ψ, which we proceed to show.

(b) For measurable subsets $S \subseteq \mathbb{R}^{r+1}$, let

$$\mathscr{J}(x; \varepsilon, S) = \int_S R(xe^{2\sqrt{\varepsilon}v})\, e^{-|v|^2}\, dv,$$

so that $\mathscr{J}_\varepsilon(R(x)) = \pi^{-(r+1)/2}\mathscr{J}(x; \varepsilon, \mathbb{R}^{r+1})$.

With some $\Delta \geqslant 1$, later subject to appropriate choice, we consider the cube

$$Q = \{v \in \mathbb{R}^{r+1} : -\Delta \leqslant v_p \leqslant \Delta \qquad (p = 1, ..., r+1)\}.$$

The monotonic property of F and (3.8) yield for $v \in Q$ and $0 < \varepsilon \leqslant 1$

$$\begin{aligned} R(xe^{2\sqrt{\varepsilon}v}) &\leqslant F(xe^{2\sqrt{\varepsilon}\Delta}) - \min_{v' \in Q} M(xe^{2\sqrt{\varepsilon}v'}) \\ &= R(xe^{2\sqrt{\varepsilon}\Delta}) + O(\sqrt{\varepsilon}\beta(X)\, \Delta e^{c_8\Delta}) \qquad (c_8 = 2nc_6), \end{aligned}$$

where $xe^{2\sqrt{\varepsilon}\Delta} = (x_1e^{2\sqrt{\varepsilon}\Delta}, ..., x_{r+1}e^{2\sqrt{\varepsilon}\Delta})$; hence

$$\mathscr{J}(x; \varepsilon, Q) \leqslant \{R(xe^{2\sqrt{\varepsilon}\Delta}) + O(\sqrt{\varepsilon}\beta(X)\, \Delta e^{c_8\Delta})\} \int_Q e^{-|v|^2}\, dv,$$

or, since $\int_Q e^{-|v|^2}\, dv \gg 1$,

$$R(xe^{2\sqrt{\varepsilon}\Delta}) \geqslant O(|\mathscr{J}(x; \varepsilon, Q)|) + O(\sqrt{\varepsilon}\beta(X)\, \Delta e^{c_8\Delta}). \tag{3.12}$$

Likewise,

$$R(xe^{-2\sqrt{\varepsilon}\Delta}) \leqslant O(|\mathscr{J}(x; \varepsilon, Q)|) + O(\sqrt{\varepsilon}\beta(X)\, \Delta e^{c_8\Delta}). \tag{3.13}$$

Now, by (3.1) and the definition of φ and ψ, we have

$$\begin{aligned} |R(xe^{2\sqrt{\varepsilon}v})| &= \varphi(xe^{2\sqrt{\varepsilon}v}) \cdot \alpha(Xe^{2\sqrt{\varepsilon}\sum_{p=1}^{r+1} e_p v_p}) \\ &\leqslant \psi(x)\, \alpha(X)e^{c_9|v|} \qquad (c_9 = 2c_6 + 2c_7) \end{aligned}$$

and thus

$$|\mathscr{J}(x;\varepsilon,\mathbb{R}^{r+1}\backslash Q)| \leqslant e^{-\Delta^2/2}\int_{\mathbb{R}^{r+1}} |R(xe^{2\sqrt{\varepsilon}v})|\, e^{-|v|^2/2}\, dv$$

$$\ll e^{-\Delta^2/2}\psi(x)\,\alpha(X).$$

Hence (3.12) implies

$$R(xe^{2\sqrt{\varepsilon}\Delta}) \geqslant O(|\mathscr{J}_\varepsilon(R(x))|) + O(e^{-\Delta^2/2}\psi(x)\,\alpha(X))$$
$$+ O(\sqrt{\varepsilon}\beta(X)\,\Delta e^{c_8\Delta}),$$

and the choice $\varepsilon = \varepsilon(X)$ leads to

$$R(xe^{2\sqrt{\varepsilon(X)}\Delta}) \geqslant O(\alpha(X)\cdot(e^{-\Delta^2/2}\psi(x) + \Delta e^{c_8\Delta})) \tag{3.14}$$

by (3.5) and (3.6). We write now

$$y = xe^{2\sqrt{\varepsilon(X)}\Delta}, \qquad Y = \prod_{p=1}^{r+1} y_p^{e_p} = Xe^{2n\sqrt{\varepsilon(X)}\Delta}; \tag{3.15}$$

then (3.1), (3.11), and (3.14) yield

$$R(y) \geqslant O(\alpha(Y)\cdot(e^{-\Delta^2/2}\psi(y) + \Delta)e^{c_{10}\Delta}) \qquad (c_{10} = 2c_6 + 2nc_7). \tag{3.16}$$

Here y may be looked upon as an independent variable since it runs through the whole of $\mathbb{R}_+^{r+1}$ if x does so. To see this, consider a given $y \in \mathbb{R}_+^{r+1}$ and put $x = ye^{-2\xi\Delta}$, where the number $\xi \in (0, 1]$ is chosen according to the intermediate-value theorem such that

$$\sqrt{\varepsilon(Ye^{-2n\xi\Delta})} - \xi = 0;$$

then (3.15) holds.

Combining (3.16) with the opposite inequality arising from (3.13) and writing x for y again, we obtain

$$\varphi(x) \ll (e^{-\Delta^2/2}\psi(x) + \Delta)e^{c_{10}\Delta},$$

and (3.11) shows that the same holds for ψ instead of φ.

Thus

$$\psi(x) \leqslant c_{11}(e^{-\Delta^2/2}\psi(x) + \Delta)e^{c_{10}\Delta}$$

for suitable c_{11}, and choosing $\Delta = c_{12}$ so large that $c_{11}e^{-\Delta^2/2}e^{c_{10}\Delta} \leqslant \frac{1}{2}$ we get

$$\psi(x) \leqslant 2c_{11}\Delta e^{c_{10}\Delta},$$

the desired result. ∎

4. Some Lemmas

The purpose of the first lemma is to move the supposed main term $M(x)$ under the operator $\mathcal{J}_\varepsilon$:

Lemma 4.1. *Let the function $g(s)$ be holomorphic and one-valued for $0<|s-s_0|<\varrho$ $(\varrho>0)$, let $g_1, ..., g_{r+1} \in \mathbb{R}$, and let*

$$M(x)=\operatorname*{Res}_{s=s_0}\left(g(s)\prod_{p=1}^{r+1} x_p^{e_p(s+ig_p)}\right) \qquad (x\in\mathbb{R}_+^{r+1}).$$

Then

$$\mathcal{J}_\varepsilon(M(x))=\operatorname*{Res}_{s=s_0}\left(g(s)\prod_{p=1}^{r+1} x_p^{e_p(s+ig_p)}\cdot\exp\left\{\varepsilon\sum_{p=1}^{r+1} e_p^2(s+ig_p)^2\right\}\right).$$

Proof. Expressing $M(xe^v)$ by an integral round the contour C: $|s-s_0|=\varrho/2$ and inverting the order of integration yields

$$\mathcal{J}_\varepsilon(M(x))=\frac{1}{2\pi i}\int_C g(s)\prod_{p=1}^{r+1} x_p^{e_p(s+ig_p)}$$
$$\cdot\prod_{p=1}^{r+1}\left\{\frac{1}{\sqrt{4\pi\varepsilon}}\int_{-\infty}^{\infty} e^{e_p(s+ig_p)v_p-v_p^2/(4\varepsilon)}\,dv_p\right\}ds$$

which proves the assertion since

$$\frac{1}{\sqrt{4\pi\varepsilon}}\int_{-\infty}^{\infty} e^{e_p(s+ig_p)v_p-v_p^2/(4\varepsilon)}\,dv_p=e^{\varepsilon e_p^2(s+ig_p)^2}. \qquad \blacksquare$$

The next two lemmas deal with Hecke zeta functions. Let

$$\lambda(v)=\prod_{p=1}^{r+1}|v^{(p)}|^{ie_pg_p}\cdot\prod_{p=1}^{r+1}\left(\frac{v^{(p)}}{|v^{(p)}|}\right)^{a_p},$$

where $g_p\in\mathbb{R}$, $\sum_{p=1}^{r+1} e_pg_p=0$, $a_1, ..., a_{r_1}\in\{0,1\}$, $a_{r_1+1}, ..., a_{r+1}\in\mathbb{Z}$, have the property

$$\lambda(\eta)=1 \qquad \text{for every} \quad \eta\in\mathcal{E}.$$

Then λ is a Grössencharacter for ideals, and we may define $\lambda(\mathfrak{a})$ for non-zero ideals $\mathfrak{a}$ in K by

$$\lambda(\mathfrak{a}):=\lambda(\hat{\alpha}),$$

where $\hat{\alpha}$ is a specimen out of a system of ideal numbers assigned to K such that $\mathfrak{a}=(\hat{\alpha})$ (for details, see [3]).

If $\mathfrak{K}$ is an ideal class of K (in the widest sense), $\zeta(s; \lambda, \mathfrak{K})$ is given for $\sigma = \operatorname{Re} s > 1$ by the series

$$\zeta(s; \lambda, \mathfrak{K}) = \sum_{\mathfrak{a} \in \mathfrak{K}} \frac{\lambda(\mathfrak{a})}{N(\mathfrak{a})^s},$$

extended over the integral ideals $\mathfrak{a}$ in $\mathfrak{K}$.

LEMMA 4.2. *$\zeta(s; \lambda, \mathfrak{K})$ is an entire function unless $\lambda \equiv 1$ (i.e., all g_p and $a_p = 0$), in which case its only singularity in $\mathbb{C}$ is a simple pole at $s = 1$ with residue*

$$\frac{2^{r_1}(2\pi)^{r_2} R}{|\sqrt{d}|\, w}.$$

Proof. See [3] and, for $\lambda \equiv 1$, [7, Satz 154]. ∎

LEMMA 4.3. *Let $0 < \varrho \leqslant \frac{1}{2}$. Then, for $-\varrho \leqslant \sigma \leqslant 1 + \varrho$, $|s - 1| \geqslant \frac{1}{4}$,*

$$\zeta(s; \lambda, \mathfrak{K}) \ll \frac{1}{\varrho} \prod_{p=1}^{r+1} |1 + s + |a_p| - ig_p|^{e_p(1 + \varrho - \sigma)/2},$$

where the $\ll$-constant depends only on K.

The *proof* is very much the same as in [9, Sect. 8]; the factor $1/\varrho$ comes from the estimate $\zeta_K(1 + \varrho) \ll 1/\varrho$ for the Dedekind zeta function.

The following lemmas provide some estimates which will be needed in the applications.

LEMMA 4.4. *Suppose $r > 0$ and let, in the notation of Section 1,*

$$W_p(\tau, a) = E_p(\tau, a) - E_{r+1}(\tau, a) \qquad (p = 1, \dots, r).$$

Then, for any a and any $w = (w_1, \dots, w_r) \in \mathbb{R}^r$,

$$\#\{m \in \mathbb{Z}^r : w_p \leqslant W_p(m, a) \leqslant w_p + 1 \quad (p = 1, \dots, r)\} \ll R(\mathscr{U}),$$

where the $\ll$-constant depends only on K.

Proof. In view of the considerations of Section 1, we may choose the basis $\eta_1, \dots, \eta_r$ of $\mathscr{U}$ underlying the definition of the E_p's as we please. By [10, Hilfssatz 1.1], there is such a basis satisfying

$$L(\eta_1) \cdots L(\eta_r) \ll R(\mathscr{U}),$$

where $L(\eta) = \max_{1 \leqslant p \leqslant r+1} e_p \,|\log |\eta^{(p)}||$ for $\eta \in \mathscr{U}$.

By (1.3),

$$\sum_{p=1}^{r} e_p W_p(m, a) \log |\eta_q^{(p)}| = 2\pi \left(m_q - \sum_{p=r_1+1}^{r+1} a_p \vartheta_q^{(p)} \right) \qquad (q = 1, ..., r);$$

thus, if $m, m' \in \mathbb{Z}^r$ be any two points such that

$$|W_p(m, a) - W_p(m', a)| \leqslant 1 \qquad \text{for} \quad p = 1, ..., r,$$

we have

$$2\pi \, |m_q - m'_q| = \left| \sum_{p=1}^{r} e_p (W_p(m, a) - W_p(m', a)) \log |\eta_q^{(p)}| \right|$$
$$\leqslant r L(\eta_q) \qquad (q = 1, ..., r).$$

Hence the number of m's under consideration is at most

$$\prod_{q=1}^{r} \left(\frac{r}{2\pi} L(\eta_q) + 1 \right) \ll \prod_{q=1}^{r} L(\eta_q) \ll R(\mathcal{U}),$$

since $L(\eta_q) \gg 1$ by a well-known theorem of Kronecker. ∎

LEMMA 4.5. *For $p = 1, ..., r+1$, let $b_p, \varrho_p, \gamma_p \in \mathbb{R}$ such that $\varrho_p > 0$ and $\gamma^{-1} \leqslant \gamma_p \leqslant \gamma$ for some $\gamma \geqslant 1$. Then, for $\varepsilon > 0$,*

$$\frac{1}{R(\mathcal{U})} \sum_m \int_{-\infty}^{\infty} \prod_{p=1}^{r+1} \frac{(1 + |t + b_p - E_p(m, a)|)^{\gamma_p}}{\varrho_p + |t + b_p - E_p(m, a)|}$$
$$\cdot \exp\left\{ -\varepsilon \sum_{p=1}^{r+1} e_p^2 (t + b_p - E_p(m, a))^2 \right\} dt$$
$$\ll \prod_{p=1}^{r+1} (\varrho_p^{-1} + \varepsilon^{-\gamma_p/2}),$$

where the $\ll$-constant depends only on K and γ.

Proof. We denote the term to be estimated by T. Making the change of variables $t \to t - b_{r+1} + E_{r+1}(m, a)$ and then reversing the order of summation and integration, we obtain

$$T = \frac{1}{R(\mathcal{U})} \int_{-\infty}^{\infty} \frac{(1 + |t|)^{\gamma_{r+1}}}{\varrho_{r+1} + |t|} e^{-\varepsilon e_{r+1}^2 t^2}$$
$$\cdot \sum_m \prod_{p=1}^{r} \frac{(1 + |t + b'_p - W_p(m, a)|)^{\gamma_p}}{\varrho_p + |t + b'_p - W_p(m, a)|}$$
$$\cdot \exp\left\{ -\varepsilon \sum_{p=1}^{r} e_p^2 (t + b'_p - W_p(m, a))^2 \right\} dt,$$

where $b'_p = b_p - b_{r+1}$ $(p = 1, ..., r)$; the series over m has to be interpreted as 1 if $r = 0$. If $r > 0$, Lemma 4.4 tells us that, for fixed $t \in \mathbb{R}$ and any $l = (l_1, ..., l_r) \in \mathbb{Z}^r$, there are at most $O(R(\mathcal{U}))$ points $m \in \mathbb{Z}^r$ such that

$$t + b'_p - l_p - \tfrac{1}{2} \leqslant W_p(m, a) < t + b'_p - l_p + \tfrac{1}{2} \qquad (p = 1, ..., r).$$

For these,

$$|l_p| + \tfrac{1}{2} \geqslant |t + b'_p - W_p(m, a)| \geqslant \max\{0, |l_p| - \tfrac{1}{2}\} \geqslant \tfrac{1}{2}|l_p|;$$

hence

$$T \ll \int_0^\infty \frac{(1+t)^{\gamma_{r+1}}}{\varrho_{r+1} + t} e^{-\varepsilon t^2} \, dt \cdot \prod_{p=1}^r \sum_{l=0}^\infty \frac{(1+l)^{\gamma_p}}{\varrho_p + l} e^{-\varepsilon l^2/4}.$$

Here the integral is

$$\ll \varrho_{r+1}^{-1} + \int_1^\infty t^{\gamma_{r+1} - 1} e^{-\varepsilon t^2} \, dt \ll \varrho_{r+1}^{-1} + \varepsilon^{-\gamma_{r+1}/2}.$$

In order to estimate the series we put $\gamma'_p = \min\{\gamma_p, 1\}$ and observe that

$$l^{\gamma_p - \gamma'_p} e^{-\varepsilon l^2/8} \ll \varepsilon^{-(\gamma_p - \gamma'_p)/2};$$

thus

$$\begin{aligned}\sum_{l=0}^\infty \frac{(1+l)^{\gamma_p}}{\varrho_p + l} e^{-\varepsilon l^2/4} &\ll \varrho_p^{-1} + \sum_{l=1}^\infty l^{\gamma_p - 1} e^{-\varepsilon l^2/4} \\ &\ll \varrho_p^{-1} + \varepsilon^{-(\gamma_p - \gamma'_p)/2} \int_0^\infty t^{\gamma'_p - 1} e^{-\varepsilon t^2/8} \, dt \\ &\ll \varrho_p^{-1} + \varepsilon^{-\gamma_p/2}.\end{aligned}$$

This proves the assertion. ∎

LEMMA 4.6. *Let $g_1, ..., g_{r+1} \in \mathbb{R}$ such that $\sum_{p=1}^{r+1} e_p g_p = 0$. Then, in any strip $-1/4n \leqslant \sigma \leqslant \gamma$ $(s = \sigma + it)$,*

$$\frac{\prod_{p=1}^{r+1} \Gamma(e_p(s - ig_p) + 1)}{\Gamma(ns + 1)} \ll \prod_{p=1}^{r+1} (1 + |t - g_p|)^{r/(2r+2)},$$

where the $\ll$-constant depends only on K and γ.

Proof. From [7, Satz 160] we infer that

$$1 \ll |\Gamma(s)| \, e^{\pi |t|/2} (1 + |t|)^{1/2 - \sigma} \ll 1$$

uniformly in any strip $0<\sigma_1\leqslant\sigma\leqslant\sigma_2$. Hence, for $-1/4n\leqslant\sigma\leqslant\gamma$, the term under consideration is

$$\ll\frac{\prod_{p=1}^{r+1}(1+|t-g_p|)^{e_p\sigma+1/2}}{(1+|t|)^{n\sigma+1/2}}\cdot\exp\left\{\frac{\pi}{2}n|t|-\frac{\pi}{2}\sum_{p=1}^{r+1}e_p|t-g_p|\right\},$$

and it remains to show that

$$T:=\prod_{p=1}^{r+1}\left(\frac{1+|t-g_p|}{1+|t|}\right)^{e_p\sigma+1/(2r+2)}\cdot\exp\left\{\frac{\pi}{2}n|t|-\frac{\pi}{2}\sum_{p=1}^{r+1}e_p|t-g_p|\right\}$$

is bounded.

Let $g_0=(1/4n)\sum_{p=1}^{r+1}e_p|g_p|$. If $|t|<g_0$, we have

$$n|t|-\sum_{p=1}^{r+1}e_p|t-g_p|\leqslant n|t|-4ng_0+n|t|\leqslant -2ng_0$$

and consequently, since $0\leqslant e_p\sigma+1/(2r+2)\leqslant 2\gamma+1$,

$$T\ll\prod_{p=1}^{r+1}(1+g_0+|g_p|)^{2\gamma+1}\cdot e^{-\pi ng_0}\ll 1.$$

If, on the other hand, $|t|\geqslant g_0$, we get from

$$\sum_{p=1}^{r+1}e_p|t-g_p|\geqslant\left|\sum_{p=1}^{r+1}e_p(t-g_p)\right|=n|t|$$

that

$$T\ll\prod_{p=1}^{r+1}\left(1+\frac{|g_p|}{1+g_0}\right)^{2\gamma+1}\ll 1. \quad \blacksquare$$

5. The Piltz Divisor Problem for Numbers in K

Let $k\geqslant 1$ be a rational integer, and let $(\mathfrak{K})$ denote an ordered system $(\mathfrak{K}_1,\dots,\mathfrak{K}_k)$ of k ideal classes such that

$$\mathfrak{K}_1\cdots\mathfrak{K}_k=\mathfrak{K}_0,\qquad\text{the principal class.}\tag{5.1}$$

For non-zero integers $\nu\in K$, we define

$$d_k(\nu;(\mathfrak{K}))=\sum_{\substack{(\mathfrak{a}_1,\dots,\mathfrak{a}_k)\\ \mathfrak{a}_j\in\mathfrak{K}_j\,(j=1,\dots,k)\\ \mathfrak{a}_1\cdots\mathfrak{a}_k=(\nu)}}1,$$

the number of representations of the principal ideal (ν) as a product of k integral ideals, the jth one lying in the class $\mathfrak{K}_j$ $(j=1, ..., k)$, and

$$d_k(\nu)=\sum_{\substack{(\mathfrak{a}_1, ..., \mathfrak{a}_k) \\ \mathfrak{a}_1 \cdots \mathfrak{a}_k=(\nu)}} 1,$$

the number of such representations without any restrictions regarding the classes. (So, if $k=1$, $\mathfrak{K}_1=\mathfrak{K}_0$ and $d_k(\nu; (\mathfrak{K}))=d_k(\nu)=1$.)

With Λ and v given by (2.2) and (2.3), we consider for $x \in \mathbb{R}_+^{r+1}$ the summatory functions

$$D_k(x; \Lambda v, (\mathfrak{K}))=\sum_{0<|\nu^{(p)}| \leqslant x_p} \Lambda v(\nu)\, d_k(\nu; (\mathfrak{K}))$$

and

$$D_k(x; \Lambda)=\sum_{\substack{\nu \succ 0 \\ |\nu^{(p)}| \leqslant x_p}} \Lambda(\nu)\, d_k(\nu),$$

the former being extended over all non-zero integers $\nu \in K$ contained in the box $|\nu^{(p)}| \leqslant x_p$ $(p=1, ..., r+1)$, the latter over the totally positive ones among these only.

Clearly,

$$D_k(x; \Lambda)=2^{-r_1} \sum_{v} \sum_{(\mathfrak{K})} D_k(x; \Lambda v, (\mathfrak{K})), \tag{5.2}$$

where the summation ranges over the 2^{r_1} possible sign characters v and the h^{k-1} systems $(\mathfrak{K}_1, ..., \mathfrak{K}_k)$ satisfying (5.1).

THEOREM 5.1. *For $X \geqslant 2$,*

$$D_k(x; \Lambda v, (\mathfrak{K}))=\Delta(\Lambda v)\, C_k^* \prod_{p=1}^{r+1} \frac{x_p^{e_p(1+ib_p)}}{1+ib_p} \cdot P_{k-1}^*(\log X) + O(X^{1-2/(nk+2)}(\log X)^{k-nk/(nk+2)}) \tag{5.3}$$

and

$$D_k(x; \Lambda)=\Delta(\Lambda)\, C_k \prod_{p=1}^{r+1} \frac{x_p^{e_p(1+ib_p)}}{1+ib_p} \cdot P_{k-1}(\log X) + O(X^{1-2/(nk+2)}(\log X)^{k-nk/(nk+2)}), \tag{5.4}$$

where

$$\Delta(\Lambda v)=\begin{cases}1 & \text{if } A_1=\cdots=A_{r+1}=0,\\ 0 & \text{else,}\end{cases}$$

$$C_k^*=\frac{1}{(k-1)!}\left(\frac{2^{r_1}(2\pi)^{r_2}}{|\sqrt{d}|}\right)^k\left(\frac{R}{w}\right)^{k-1}, \qquad C_k=2^{-r_1}h^{k-1}C_k^*.$$

P_{k-1}^* and P_{k-1} denote polynomials of degree $k-1$ with leading coefficient 1, which depend on K, k, Λ and, as regards P_{k-1}^*, on $(\mathfrak{K})$.

The O-constants depend only on K, k and Λ.

Remark. Evidently, the part of Λ involving the b_p's does not affect the order of magnitude of D_k. This is a quite common phenomenon which occurs, e.g., also in Section 6 of this paper and in [10, Sects. 6 and 7]. But, on the other hand, compare Section 7 of this paper and [10, Sect. 4], especially Vorbemerkung 1 following [10, Satz 4.1].

Proof. Due to (5.2), (5.4) is an immediate consequence of (5.3); so we concentrate on (5.3). Obviously,

$$d_k(\eta v; (\mathfrak{K}))=d_k(v; (\mathfrak{K})) \qquad \text{for every} \quad \eta\in\mathscr{E}. \tag{5.5}$$

Thus, in order to apply Theorem 2.2, we may take for $\mathscr{U}$ any group that meets the specifications of Section 1, e.g., one with minimal $R(\mathscr{U})$.

The weight function Φ fitting our problem is the characteristic function of the unit cube $(0, 1]^{r+1}$; so

$$\Psi(s)=2^{r_2}\int_0^1\cdots\int\prod_{p=1}^{r+1}u_p^{e_p s_p-1}\,du=\frac{1}{s_1\cdots s_{r+1}} \qquad (\operatorname{Re} s_p>0).$$

As to the generating Dirichlet series Ξ, we observe that $\lambda_{m,A}v$ may be regarded as a character of the factor group $\mathscr{E}/\mathscr{U}$ of order $[\mathscr{E}:\mathscr{U}]=wR(\mathscr{U})/R$. Hence, if $\lambda_{m,A}v$ is a Grössencharacter for ideals, i.e., $\lambda_{m,A}v(\eta)=1$ for every $\eta\in\mathscr{E}$, we may, in view of (5.5), collect terms belonging to the same principal ideal (v) to obtain

$$\begin{aligned}\frac{1}{R(\mathscr{U})}\,\Xi(s;\lambda_{m,A}v\,d_k(\cdot;(\mathfrak{K})),\mathscr{U})&=\frac{w}{R}\sum_{(v)}{}'\,\frac{\lambda_{m,A}v(v)\,d_k(v;(\mathfrak{K}))}{|N(v)|^s}\\&=\frac{w}{R}\prod_{j=1}^{k}\zeta(s;\lambda_{m,A}v,\mathfrak{K}_j),\end{aligned}$$

provided $\operatorname{Re} s>1$. If, on the contrary, $\lambda_{m,A}v(\eta)\neq 1$ for some $\eta\in\mathscr{E}$, then

$$\Xi(s;\lambda_{m,A}v\,d_k(\cdot;(\mathfrak{K})),\mathscr{U})=0.$$

Thus, by Theorem 2.2 (with x_p^{-1} in place of x_p; $p=1, ..., r+1$), we have for $\sigma>1$, $\varepsilon>0$

$$\begin{aligned}\mathscr{I}_\varepsilon(D_k(x; \Lambda v, (\mathfrak{R}))) = \frac{w}{2\pi i R} \sum_m{}^* \int_{(\sigma)} \prod_{p=1}^{r+1} \frac{1}{s+ib_p-iE_p(m, A)} \\ \cdot \prod_{j=1}^{k} \zeta(s; \lambda_{m,A} v, \mathfrak{R}_j) \cdot \prod_{p=1}^{r+1} x_p^{e_p(s+ib_p-iE_p(m,A))} \\ \cdot \exp\left\{\varepsilon \sum_{p=1}^{r+1} e_p^2(s+ib_p-iE_p(m, A))^2\right\} ds, \quad (5.6)\end{aligned}$$

where $\sum_m^*$ is over those m's only which make $\lambda_{m,A} v$ a Grössencharacter for ideals.

Now let $0<\varrho\leqslant\frac{1}{2}$. By Lemmas 4.2 and 4.3, we can move the path of integration in (5.6) to the line $\operatorname{Re} s=\varrho$ if we take into account the possible k-fold pole of the ζ-product at $s=1$, which occurs only in the case $m=0$, $A=0$.

According to Lemma 4.1, the contribution of the point $s=1$ may be written as $\mathscr{I}_\varepsilon(M(x; \Lambda v))$, where

$$\begin{aligned}M(x; \Lambda v) &= \Delta(\Lambda v) \frac{w}{R} \operatorname*{Res}_{s=1} \left(\prod_{p=1}^{r+1} \frac{x_p^{e_p(s+ib_p)}}{s+ib_p} \cdot \prod_{j=1}^{k} \zeta(s; 1, \mathfrak{R}_j)\right) \\ &= \Delta(\Delta v) C_k^* \prod_{p=1}^{r+1} \frac{x_p^{e_p(1+ib_p)}}{1+ib_p} \cdot P_{k-1}^*(\log X).\end{aligned}$$

(The dependence of M on k and $(\mathfrak{R})$ is not emphasized because these remain fixed throughout.)

From Lemma 4.3 we obtain that, for $0<\varepsilon\leqslant 1$, the sum of the integrals along $\operatorname{Re} s=\varrho$ is

$$\begin{aligned}&\ll X^\varrho \varrho^{-k} \sum_m \int_{-\infty}^{\infty} \prod_{p=1}^{r+1} \frac{|1+\varrho+|A_p|+i(t-E_p(m, A))|^{e_pk/2}}{|\varrho+i(t+b_p-E_p(m, A))|} \\ &\quad \cdot \exp\left\{-\varepsilon \sum_{p=1}^{r+1} e_p^2(t+b_p-E_p(m, A))^2\right\} dt \\ &\ll X^\varrho \varrho^{-k} \sum_m \int_{-\infty}^{\infty} \prod_{p=1}^{r+1} \frac{(1+|t+b_p-E_p(m, A)|)^{e_pk/2}}{\varrho+|t+b_p-E_p(m, A)|} \\ &\quad \cdot \exp\left\{-\varepsilon \sum_{p=1}^{r+1} e_p^2\, (t+b_p-E_p(m, A))^2\right\} dt \\ &\ll X^\varrho \varrho^{-k} \prod_{p=1}^{r+1} (\varrho^{-1}+\varepsilon^{-e_pk/4})\end{aligned}$$

by Lemma 4.5 (since, in the present context, $R(\mathscr{U})$ depends only on the field).

Thus, for $0<\varrho\leqslant\frac{1}{2}$ and $0<\varepsilon\leqslant 1$, we have

$$\begin{aligned}
&\mathscr{I}_\varepsilon(D_k(x;\Lambda v,(\mathfrak{R}))-M(x;\Lambda v))\\
&\qquad\ll X^\varrho\varrho^{-k}\prod_{p=1}^{r+1}(\varrho^{-1}+\varepsilon^{-e_pk/4}).
\end{aligned}\tag{5.7}$$

Our next aim is, of course, to apply Theorem 3.1. We have to bear in mind that this demands estimates valid for all $x\in\mathbb{R}_+^{r+1}$, not merely for large X; so some care regarding the logarithm is required. We choose

$$\begin{aligned}
\varrho&=(2\log(X+3))^{-1}<\tfrac{1}{2},\\
\varepsilon=\varepsilon(X)&=\left(\frac{\log(X+3)}{X+3}\right)^{4/(nk+2)}<1,\\
\alpha(X)&=(X+3)^{1-2/(nk+2)}(|\log X|+1)^{k-nk/(nk+2)},\\
\beta(X)&=X(|\log X|+1)^{k-1}.
\end{aligned}$$

Then

$$x_p\frac{\partial}{\partial x_p}M(x;\Lambda v)\ll\beta(X)\qquad(p=1,\ldots,r+1),$$

$\sqrt{\varepsilon(X)}\,\beta(X)\ll\alpha(X)$, and the right-hand side of (5.7) becomes $\ll\alpha(X)$ as well.

It is plain that direct application of Theorem 3.1 is only possible in the case $\Lambda v\equiv 1$; it yields

$$D_k(x;1,(\mathfrak{R}))=M(x;1)+O(\alpha(X)).\tag{5.8}$$

But, once we have this, we apply Theorem 3.1 to both of the functions

$$\begin{aligned}
&D_k(x;1,(\mathfrak{R}))+\operatorname{Re}D_k(x;\Lambda v,(\mathfrak{R})),\\
&D_k(x;1,(\mathfrak{R}))+\operatorname{Im}D_k(x;\Lambda v,(\mathfrak{R}));
\end{aligned}$$

these are sums of non-negative terms and thus monotonic. From the resulting asymptotic formulae we subtract (5.8) and obtain

$$D_k(x;\Lambda v,(\mathfrak{R}))=M(x;\Lambda v)+O(\alpha(X))$$

which, for $X\geqslant 2$, implies (5.3). ∎

6. THE AVERAGE OF $\sigma(\nu)$

For non-zero integers $\nu \in K$, let

$$\sigma(\nu; \mathfrak{K}) = \sum_{\substack{\mathfrak{a} \mid \nu \\ \mathfrak{a} \in \mathfrak{K}}} N(\mathfrak{a}) \qquad \text{and} \qquad \sigma(\nu) = \sum_{\mathfrak{a} \mid \nu} N(\mathfrak{a})$$

denote, respectively, the sum of the norms of the ideal divisors of ν lying in the class $\mathfrak{K}$, and of the norms of all divisors of ν.

Adopting the notation of Section 5, we consider

$$S(x; \Lambda v, \mathfrak{K}) = \sum_{0 < |\nu^{(p)}| \leqslant x_p} \Lambda v(\nu)\, \sigma(\nu; \mathfrak{K})$$

and

$$S(x; \Lambda) = \sum_{\substack{\nu \succ 0 \\ |\nu^{(p)}| \leqslant x_p}} \Lambda(\nu)\, \sigma(\nu).$$

Plainly,

$$S(x; \Lambda) = 2^{-r_1} \sum_{v} \sum_{\mathfrak{K}} S(x; \Lambda v, \mathfrak{K}). \tag{6.1}$$

THEOREM 6.1. *For $X \geqslant 2$,*

$$\begin{aligned} S(x; \Lambda v, \mathfrak{K}) = {} & \Delta(\Lambda v) \frac{2^{r_1}(2\pi)^{r_2}}{|\sqrt{d}|} \zeta(2; 1, \mathfrak{K}^{-1}) \prod_{p=1}^{r+1} \frac{x_p^{e_p(2+ib_p)}}{2+ib_p} \\ & + O(X^{2-2/(n+2)}(\log X)^{4/(n+2)}) \end{aligned} \tag{6.2}$$

and

$$\begin{aligned} S(x; \Lambda) = {} & \Delta(\Lambda) \frac{(2\pi)^{r_2}}{|\sqrt{d}|} \zeta_K(2) \prod_{p=1}^{r+1} \frac{x_p^{e_p(2+ib_p)}}{2+ib_p} \\ & + O(X^{2-2/(n+2)}(\log X)^{4/(n+2)}), \end{aligned} \tag{6.3}$$

where $\Delta(\Lambda v)$ has the meaning of Theorem 5.1, $\mathfrak{K}^{-1}\mathfrak{K} = \mathfrak{K}_0$, and ζ_K denotes the Dedekind zeta function.

The O-constants depend only on K and Λ.

Proof. The reasoning is very similar to that of Section 5, so we shall be brief. In view of (6.1), it is sufficient to show (6.2). $\mathcal{U}$, Φ, and Ψ remain the same as before, and for Ξ we have, provided $\operatorname{Re} s > 2$,

$$\frac{1}{R(\mathscr{U})}\,\Xi(s;\lambda_{m,A}v\sigma(\cdot;\mathfrak{K}),\mathscr{U})=\frac{w}{R}\,\zeta(s-1;\lambda_{m,A}v,\mathfrak{K})\,\zeta(s;\lambda_{m,A}v,\mathfrak{K}^{-1})$$

if $\lambda_{m,A}v$ is a Grössencharacter for ideals, while otherwise

$$\Xi(s;\lambda_{m,A}v\sigma(\cdot;\mathfrak{K}),\mathscr{U})=0.$$

Hence, for $\sigma>2$ and $\varepsilon>0$,

$$\begin{aligned}\mathscr{J}_\varepsilon(S(x;Av,\mathfrak{K}))=\frac{w}{2\pi iR}\sum_m{}^*\int_{(\sigma)}\prod_{p=1}^{r+1}&\frac{x_p^{e_p(s+ib_p-iE_p(m,A))}}{s+ib_p-iE_p(m,A)}\\ &\cdot\zeta(s-1;\lambda_{m,A}v,\mathfrak{K})\,\zeta(s;\lambda_{m,A}v,\mathfrak{K}^{-1})\\ &\cdot\exp\left\{\varepsilon\sum_{p=1}^{r+1}e_p^2(s+ib_p-iE_p(m,A))^2\right\}ds.\end{aligned}$$

We move the contour of integration to the left, across the possible pole at $s=2$, up to the line $\operatorname{Re}s=1+\varrho$ $(0<\varrho\leqslant\frac{1}{2})$. The contribution of the point $s=2$ is $\mathscr{J}_\varepsilon(M(x;Av))$, where

$$M(x;Av)=\Delta(Av)\,\frac{2^{r_1}(2\pi)^{r_2}}{|\sqrt{d}|}\,\zeta(2;1,\mathfrak{K}^{-1})\prod_{p=1}^{r+1}\frac{x_p^{e_p(2+ib_p)}}{2+ib_p},$$

and the sum of the integrals along $\operatorname{Re}s=1+\varrho$ can, for $0<\varepsilon\leqslant1$, be estimated as

$$\begin{aligned}&\ll X^{1+\varrho}\varrho^{-2}\sum_m\int_{-\infty}^{\infty}\prod_{p=1}^{r+1}\frac{|1+\varrho+|A_p|+i(t-E_p(m,A))|^{e_p/2}}{|1+\varrho+i(t+b_p-E_p(m,A))|}\\ &\quad\cdot\exp\left\{-\varepsilon\sum_{p=1}^{r+1}e_p^2(t+b_p-E_p(m,A))^2\right\}dt\\ &\ll X^{1+\varrho}\varrho^{-2}\prod_{p=1}^{r+1}(1+\varepsilon^{-e_p/4})\\ &\ll X^{1+\varrho}\varrho^{-2}\varepsilon^{-n/4}.\end{aligned}$$

The assertion follows now on our choosing

$$\begin{aligned}\varrho&=(2\log(X+3))^{-1}<\tfrac{1}{2}\\ \varepsilon=\varepsilon(X)&=\left(\frac{\log^2(X+3)}{X+3}\right)^{4/(n+2)}<1,\\ \alpha(X)&=(X+3)^{2-2/(n+2)}(\log(X+3))^{4/(n+2)},\\ \beta(X)&=X^2.\quad\blacksquare\end{aligned}$$

7. A Weighted Sum over Units

Throughout this section we assume $r > 0$.

Let φ: $[0, 1] \to \mathbb{C}$ be a continuous function, continuously differentiable on $[0, 1)$, such that

$$\int_0^1 |\varphi'(u)| \, du < \infty$$

(for example, $\varphi(u) = (1-u)^c$, $c \geqslant 0$).

Moreover, for $\alpha > 0$, let $\|\cdot\|_\alpha$ denote the "α-norm"

$$\|z\|_\alpha = \left(\sum_{p=1}^{r+1} |z_p|^\alpha \right)^{1/\alpha} \qquad (z \in \mathbb{C}^{r+1})$$

(which of course is a norm only if $\alpha \geqslant 1$).

With Λ given by (2.2) and a group $\mathscr{U}$ according to Section 1, we consider for $x \in \mathbb{R}_+^{r+1}$ the sum

$$U(x; \Lambda) = \sum_{\substack{\eta \in \mathscr{U} \\ \|\eta x^{-1}\|_\alpha \leqslant 1}} \Lambda(\eta) \, \varphi(\|\eta x^{-1}\|_\alpha),$$

where $\eta x^{-1} = (\eta^{(1)} x_1^{-1}, ..., \eta^{(r+1)} x_{r+1}^{-1})$.

Theorem 7.1. *For $X \geqslant 2$,*

$$U(x; \Lambda) = \delta(\Lambda) \frac{\varphi(0)}{r! \, R(\mathscr{U})} \log^r X + O(\log^{r-1} X), \tag{7.1}$$

where

$$\delta(\Lambda) = \begin{cases} 1 & \text{if } \Lambda(\eta) = 1 \text{ for every } \eta \in \mathscr{U}, \\ 0 & \text{else.} \end{cases}$$

The O-constant depends on K, Λ, φ, α, and $\mathscr{U}$.

Proof. Since both sides of (7.1) are linear with respect to φ, it suffices to prove the assertion for real-valued φ. For the same reason, we may assume that φ is non-increasing and $\geqslant 0$; otherwise split φ into $\varphi = (\varphi_- - \varphi_+)/2$, where

$$\varphi_\pm(u) = \int_u^1 (|\varphi'(v)| \pm \varphi'(v)) \, dv + |\varphi(1)| \mp \varphi(1).$$

Further, as was pointed out already in the proof of Theorem 2.1, there is no loss of generality in assuming that

$$\sum_{p=1}^{r+1} e_p b_p = 0$$

and hence writing, whenever convenient,

$$b_p = E_p(\tau, A), \qquad b_p - E_p(m, A) = -E_p(m-\tau) \qquad (p = 1, ..., r+1);$$

the underlying basis $\eta_1, ..., \eta_r$ of $\mathscr{U}$ is from now on regarded as fixed. From (1.5) we infer that only the value of τ_q mod 1 is relevant; so we suppose

$$|\tau_q| \leqslant \tfrac{1}{2} \qquad (q = 1, ..., r) \tag{7.2}$$

and observe that then $\delta(A) = 1$ or 0 according as $\tau = 0$ or $\neq 0$.

Regarding the function Ψ of our problem, we learn from [8, Sect. 67] that

$$\begin{aligned}
\Psi(s) &= 2^{r_2} \int_{\substack{u \in \mathbb{R}_+^{r+1} \\ \|u\|_\alpha \leqslant 1}} \varphi(\|u\|_\alpha) \prod_{p=1}^{r+1} u_p^{e_p s_p - 1}\, du \\
&= 2^{r_2}\alpha^{-r} \frac{\prod_{p=1}^{r+1} \Gamma((1/\alpha)\, e_p s_p)}{\Gamma((1/\alpha) \sum_{p=1}^{r+1} e_p s_p)} \int_0^1 \varphi(u) \cdot u^{\sum_{p=1}^{r+1} e_p s_p - 1}\, du \\
&= \frac{1}{s_1 \cdots s_{r+1}} \cdot \frac{\prod_{p=1}^{r+1} \Gamma((1/\alpha)\, e_p s_p + 1)}{\Gamma((1/\alpha) \sum_{p=1}^{r+1} e_p s_p + 1)} \\
&\quad \cdot \left(\sum_{p=1}^{r+1} e_p s_p\right) \int_0^1 \varphi(u) \cdot u^{\sum_{p=1}^{r+1} e_p s_p - 1}\, du,
\end{aligned}$$

provided $\operatorname{Re} s_p > 0$ $(p = 1,, ..., r+1)$.

Hence, Theorem 2.1 yields for $\sigma > 0$, $\varepsilon > 0$

$$\mathscr{I}_\varepsilon(U(x; A)) = \frac{1}{2\pi i R(\mathscr{U})} \sum_m \int_{(\sigma)} \psi_1(s; m-\tau)\, \psi_2(s; m-\tau)\, \psi_3(s)\, ds, \tag{7.3}$$

where, for $\tau' \in \mathbb{R}^r$, we put

$$\begin{aligned}
\psi_1(s; \tau') &= \prod_{p=1}^{r+1} \frac{1}{s - iE_p(\tau')}, \\
\psi_2(s; \tau') &= \frac{\prod_{p=1}^{r+1} \Gamma((e_p/\alpha)(s - iE_p(\tau')) + 1)}{\Gamma((n/\alpha)s + 1)} \cdot \prod_{p=1}^{r+1} x_p^{e_p(s - iE_p(\tau'))} \\
&\quad \cdot \exp\left\{\varepsilon \sum_{p=1}^{r+1} e_p^2 (s - iE_p(\tau'))^2\right\}, \\
\psi_3(s) &= ns \int_0^1 \varphi(u)\, u^{ns-1}\, du.
\end{aligned}$$

Now let $0<\varepsilon\leqslant 1$, $s=\sigma+it$. ψ_2 is holomorphic for $\sigma>-\alpha/n$, and by Lemma 4.6 we have

$$\psi_2(s;\tau')\ll X^\sigma \prod_{p=1}^{r+1} (1+|t-E_p(\tau')|)^{r/(2r+2)}\cdot\exp\left\{-\varepsilon\sum_{p=1}^{r+1} e_p^2(t-E_p(\tau'))^2\right\}, \tag{7.4}$$

uniformly in the strip $-\alpha/4n\leqslant\sigma\leqslant 2$, say.

The function ψ_3 is capable of analytic continuation beyond the half-plane $\sigma>0$, since integration by parts yields for $\sigma>0$

$$\psi_3(s)=\varphi(1)-\int_0^1 \varphi'(u)\,u^{ns}\,du, \tag{7.5}$$

and the right-hand side is holomorphic throughout the half-plane $\sigma>-1/n$ (note that, by hypothesis, φ' is bounded near $u=0$). From (7.5) it is easily deduced that, uniformly for $\sigma>-1/2n$,

$$\psi_3(s)\ll 1. \tag{7.6}$$

Let $\sigma_0=\min\{\alpha/4n, 1/2n\}$. We move the path of integration in (7.3) across the poles on the imaginary axis arising from ψ_1 to the line $\operatorname{Re} s=-\sigma_0$.

By (7.4), (7.6), and Lemma 4.5, the integrals along $\operatorname{Re} s=-\sigma_0$ contribute at most

$$\begin{aligned}&\ll X^{-\sigma_0}\sum_m\int_{-\infty}^{\infty}\prod_{p=1}^{r+1}\frac{(1+|t+b_p-E_p(m,A)|)^{r/(2r+2)}}{\sigma_0+|t+b_p-E_p(m,A)|}\\&\quad\cdot\exp\left\{-\varepsilon\sum_{p=1}^{r+1} e_p^2(t+b_p-E_p(m,A))^2\right\}dt\\&\ll X^{-\sigma_0}\prod_{p=1}^{r+1}(\sigma_0^{-1}+\varepsilon^{-r/(4r+4)})\\&\ll X^{-\sigma_0}\varepsilon^{-r/4}.\end{aligned}$$

Our next task is to evaluate

$$\frac{1}{R(\mathscr{U})}\sum_m T(m-\tau), \tag{7.7}$$

where $T(\tau')$ denotes the sum of the residues of $\psi_1(s;\tau')\,\psi_2(s;\tau')\,\psi_3(s)$ on $\operatorname{Re} s=0$. It is clear from (1.3) that $\psi_1(s;m-\tau)$ has a pole of order $r+1$

only when $m-\tau=0$, which, because of (7.2), means $m=\tau=0$. So the contribution of the case $m=\tau$, whether it occurs or not, is, by Lemma 4.1,

$$\delta(\Lambda)\, T(0)/R(\mathscr{U})=\mathscr{J}_\varepsilon(\delta(\Lambda)\, P(\log X)),$$

where

$$P(\log X)=\frac{1}{R(\mathscr{U})}\operatorname*{Res}_{s=0}\left(\frac{1}{s^{r+1}}\,\frac{\prod_{p=1}^{r+1}\Gamma((e_p/\alpha)s+1)}{\Gamma((n/\alpha)s+1)}\,\psi_3(s)\,X^s\right)$$

is a polynomial in $\log X$ of degree r. Its highest coefficients are easily calculated from (7.5):

$$P(\log X)$$
$$=\frac{1}{r!\,R(\mathscr{U})}\cdot\left(\varphi(0)\log^r X-nr\left(\int_0^1\varphi'(u)\log u\,du\right)\log^{r-1}X+\cdots\right).$$

In order to estimate the remaining terms of (7.7) we proceed as follows. Given $\tau'\in\mathbb{R}^r$ and $\varrho\in(0,\sigma_0]$, we centre at each of the points $iE_p(\tau')$ $(p=1,...,r+1)$ a disc of radius ϱ. The boundary $C(\tau',\varrho)$ of the union of these discs consists of at most $r+1$ closed curves with total length $\ll\varrho$. For any $s\in C(\tau',\varrho)$ we have

$$|s-iE_p(\tau')|\geqslant\varrho\qquad\text{for } p=1,...,r+1;\tag{7.8}$$

$$|s-iE_p(\tau')|=\varrho\qquad\text{for at least one } p=p(s)\in\{1,...,r+1\}.\tag{7.9}$$

Integrating round $C(\tau',\varrho)$ we obtain

$$T(\tau')=\frac{1}{2\pi i}\int_{C(\tau',\varrho)}\psi_1(s;\tau')\,\psi_2(s;\tau')\,\psi_3(s)\,ds.$$

Now it follows immediately from (7.4) that, for $s\in C(\tau',\varrho)$,

$$\psi_2(s;\tau')\ll e^{\varrho\,|\log X|}\varepsilon^{-r/4}\exp\left\{-\frac{\varepsilon}{2}\sum_{p=1}^{r+1}e_p^2(t-E_p(\tau'))^2\right\},$$

and (1.6) implies

$$\sum_{p=1}^{r+1}e_p^2(t-E_p(\tau'))^2\geqslant B_1\,|\tau'|^2,\tag{7.10}$$

$|\tau'|$ denoting the Euclidean norm. Hence

$$T(\tau') \ll \mu(\tau', \varrho)\, \varrho\, e^{\varrho\, |\log X|} \varepsilon^{-r/4} e^{-\varepsilon B_3 |\tau'|^2}, \tag{7.11}$$

where $B_3 = B_1/2$ and $\mu(\tau', \varrho) = \max\{|\psi_1(s; \tau')| : s \in C(\tau', \varrho)\}$.

The crucial point is thus to estimate $\mu(m-\tau, \varrho)$ when $m \in \mathbb{Z}^r$, $m \neq \tau$. Since then $|m-\tau| \gg 1$, we infer from (7.10) that at least one factor of $\psi_1(s; m-\tau)$ is $\ll 1$; the other factors are $\ll \varrho^{-1}$ by (7.8). So $\mu(m-\tau, \varrho) \ll \varrho^{-r}$, and (7.11) yields

$$\begin{aligned} \sum_{m \neq \tau} T(m-\tau) &\ll \varrho^{-r+1} e^{\varrho\, |\log X|} \varepsilon^{-r/4} \sum_m e^{-\varepsilon B_3 |m-\tau|^2} \\ &\ll \varrho^{-r+1} e^{\varrho\, |\log X|} \varepsilon^{-3r/4}. \end{aligned}$$

Putting here $\varrho = \sigma_0(|\log X| + 1)^{-1}$ and collecting our results, we find that, for $0 < \varepsilon \leqslant 1$,

$$\mathscr{I}_\varepsilon(U(x; \Lambda) - \delta(\Lambda)\, P(\log X)) \ll X^{-\sigma_0} \varepsilon^{-r/4} + (|\log X| + 1)^{r-1} \varepsilon^{-3r/4}.$$

Since

$$x_p \frac{\partial}{\partial x_p} P(\log X) = e_p P'(\log X) \ll (|\log X| + 1)^{r-1} =: \beta(X)$$

for $p = 1, ..., r+1$, we may choose in Theorem 3.1

$$\varepsilon(X) = 1, \qquad \alpha(X) = X^{-\sigma_0} + (|\log X| + 1)^{r-1}.$$

In order to deduce

$$U(x; \Lambda) = \delta(\Lambda)\, P(\log X) + O(\alpha(X)),$$

which for $X \geqslant 2$ implies (7.1), we proceed now as in Section 5, observing that, due to our assumptions on φ, $U(x; 1)$ as well as $U(x; 1) + \operatorname{Re} U(x; \Lambda)$ and $U(x; 1) + \operatorname{Im} U(x; \Lambda)$ are non-decreasing with respect to each x_p. ∎

For a certain class of number fields we can do better, at any rate in the case $\Lambda \equiv 1$. The fields in question are those having the property

$$r \geqslant 2, \textit{ and } |\eta^{(p)}| \neq 1 \ (p = 1, ..., n) \textit{ for every } \eta \in \mathscr{E} \textit{ which is not a root of unity.} \tag{7.12}$$

Clearly, the totally real fields of degree $n \geqslant 3$ lie in this class, and it is easy to see that the fields of odd degree (with $r \geqslant 2$) belong to it, too; cf. [10, p. 318].

THEOREM 7.2. *Suppose that K has the property (7.12). Then there is a positive constant c_K depending only on K such that, for $X \geqslant 2$,*

$$U(x; 1) = \frac{1}{r!\, R(\mathscr{U})} \left(\varphi(0) \log^r X - nr \left(\int_0^1 \varphi'(u) \log u \, du \right) \log^{r-1} X \right) + O((\log X)^{r-1-c_K}).$$

The O-constant depends on K, φ, α and $\mathscr{U}$.

Proof. We adopt notation and results from the preceding proof; the difference lies in the treatment of $\mu(m, \varrho)$.

Since $\Lambda \equiv 1$ we have $\tau = 0$. Let $m \in \mathbb{Z}^r$, $m \neq 0$. On solving the equations

$$\begin{aligned} &\sum_{p=1}^{r+1} e_p E_p(m) = 0, \\ &\sum_{p=1}^{r+1} e_p E_p(m) \log |\eta_q^{(p)}| = 2\pi m_q \qquad (q = 1, ..., r) \end{aligned} \tag{7.13}$$

for the numbers $E_p(m)$ we find that

$$E_p(m) \ll |m| \qquad (p = 1, ..., r+1).$$

Now let $s \in C(m, \varrho)$ and put

$$w_p = \frac{1}{2\pi i} (s - iE_p(m)) \qquad (p = 1, ..., r+1).$$

Then, as a consequence of (7.9), we have also

$$w_p \ll |m| \qquad (q = 1, ..., r+1).$$

Moreover, we infer from (7.13) that

$$\sum_{p=1}^{r+1} e_p w_p \log |\eta^{(p)}|$$

is a rational integer when $\eta \in \mathscr{U}$, and is not always zero. Hence [10, Hilfssatz 2.4] (which is based upon a result of A. Baker on linear forms in the logarithms of algebraic numbers) tells us that

$$|w_p| \gg |m|^{-c}$$

for at least two indices $p \in \{1, ..., r+1\}$, where $c > 0$ depends only on K. So at least two of the factors in $\psi_1(s; m)$ are $\ll |m|^c$, while the others are

$\ll \varrho^{-1}$ by (7.8). Consequently, $\mu(m, \varrho) \ll \varrho^{-r+1} |m|^{2c}$, and (7.11) yields for $0 < \varepsilon \leqslant 1$

$$\sum_{m \neq 0} T(m) \ll \varrho^{-r+2} e^{\varrho |\log X|} \varepsilon^{-r/4} \sum_{m} |m|^{2c} e^{-\varepsilon B_3 |m|^2}$$
$$\ll \varrho^{-r+2} e^{\varrho |\log X|} \varepsilon^{-c'} \qquad (c' = c + \tfrac{3}{4} r).$$

Putting again $\varrho = \sigma_0(|\log X| + 1)^{-1}$, we have thus

$$\mathscr{J}_\varepsilon(U(x; 1) - P(\log X)) \ll X^{-\sigma_0} \varepsilon^{-r/4} + (|\log X| + 1)^{r-2} \varepsilon^{-c'},$$

and the assertion follows by Theorem 3.1 on our choosing

$$\beta(X) = (|\log X| + 1)^{r-1}, \qquad \varepsilon(X) = (|\log X| + 1)^{-2c_K},$$
$$\alpha(X) = X^{-\sigma_0}(|\log X| + 1)^{rc_K/2} + (|\log X| + 1)^{r-1-c_K},$$

where $c_K = (2c' + 1)^{-1}$. ∎

References

1. W. Grotz, Einige Anwendungen der Siegelschen Summenformel, *Acta Arith.* **38** (1980), 69–95.
2. W. Grotz, Mittelwert der Eulerschen φ-Funktion und des Quadrates der Dirichletschen Teilerfunktion in algebraischen Zahlkörpern, *Monatsh. Math.* **88** (1979), 219–228.
3. E. Hecke, Eine neue Art von Zetafunktionen und ihre Beziehungen zur Verteilung der Primzahlen (Zweite Mitteilung), *Math. Z.* **6** (1920), 11–51.
4. A. A. Karacuba, The Dirichlet divisor problem in number fields, *Soviet Math. Dokl.* **13** (1972), 697–698.
5. E. Landau, Über eine idealtheoretische Funktion, *Trans. Amer. Math. Soc.* **13** (1912), 1–21.
6. E. Landau, Über die Anzahl der Gitterpunkte in gewissen Bereichen, *Nachr. Königl. Ges. Wiss. Göttingen, Math.-Phys. Kl.* **6** (1912), 687–771.
7. E. Landau, "Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale," Reprint, Chelsea, New York, 1949.
8. N. Nielsen, "Handbuch der Theorie der Gammafunktion," Reprint, Chelsea, New York, 1965.
9. H. Rademacher, On the Phragmén–Lindelöf theorem and some applications, *Math. Z.* **72** (1959), 192–204.
10. U. Rausch, Geometrische Reihen in algebraischen Zahlkörpern, *Acta Arith.* **47** (1986), 313–345.
11. W. Schaal, On the expression of a number as the sum of two squares in totally real algebraic number fields, *Proc. Amer. Math. Soc.* **16** (1965), 529–537.
12. C. L. Siegel, Mittelwerte arithmetischer Funktionen in Zahlkörpern, *Trans. Amer. Math. Soc.* **39** (1936), 219–224.