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On the Dimension of Objects and Categories I. Monoids*

BARRY MITCHELL

Department of Mathematics, Bowdoin College, Brunswick, Maine 04011

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The Yoneda definition of $\mathrm{Ext}^n(A, C)$ in terms of exact sequences of length n from C to A [*14*], [*15*], enables one to define the *homological dimension of* a nonzero object A (h.d. A) in an arbitrary Abelian category $\mathcal{A}$ as the largest integer n (or ∞) for which the one variable functor $\mathrm{Ext}^n(A, \ )$ is not zero. (We set h.d. $0 = -1$). The *global dimension* of $\mathcal{A}$ (gl.dim. $\mathcal{A}$) is then defined as the sup of the homological dimensions of all of its objects. Given a small category Π and an Abelian category $\mathcal{A}$, one can ask for the global dimension of the category $\mathcal{A}^\Pi$ of covariant functors from Π to $\mathcal{A}$ in terms of the global dimension of $\mathcal{A}$. If Π has only a finite number of objects, each such result specializes in the case where $\mathcal{A}$ is a category of modules to a statement about the global dimension of some ring [*9*, p. 147].

In this paper we shall obtain complete results (up to a mild assumption on the existence of products in $\mathcal{A}$) for the following cases:

(1) Π is a free (or free Abelian) monoid.

(2) Π is a free (or free Abelian) group.

(3) Π is a finitely generated Abelian group.

(Actually (1) and (2) will be obtained simultaneously as special instances of a more general situation.) We shall also examine the case where Π is any countable Abelian group. It turns out that the global dimension of $\mathcal{A}^\Pi$ in this case can be determined to within one. There are examples to show that there is actually this much play. Here we shall be following ideas of Balcerzyk [*1*], [*2*], who handled the torsion free case for categories of modules. In Section 4 we shall examine the category of endomorphisms in $\mathcal{A}$ satisfying a monic polynomial relation. The Ext functor of such a category turns out always to be periodic of period 2.

In a subsequent paper [*10*], we shall consider the case where Π is a finite partially-ordered set.

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For basic facts relating to the theory of adjoint functors and to the Ext functor, the reader can refer to [9], Chapters V and VII respectively.

1. Preliminaries

Throughout the paper $\mathcal{A}$ and $\mathcal{B}$ will denote Abelian categories. When there may be confusion of categories we shall use the notation $\mathrm{Ext}_{\mathcal{A}}{}^n$, or $\mathrm{Ext}_{\Pi}{}^n$ when the category concerned is a functor category $\mathcal{A}^{\Pi}$. Likewise we shall sometimes write $\mathrm{h.d.}_{\mathcal{A}}$ or $\mathrm{h.d.}_{\Pi}$. When we are working with the category of left modules over some ring R, we shall write $\mathrm{h.d.}_R$. (All rings have identities, and all modules are unitary.) The notation gl.dim. R stands for the global dimension of the category of left R-modules. All of our results will be easily obtainable for right global dimension as well.

Let $\boldsymbol{E}$ denote an exact sequence

$$0 \to C \to B_{n-1} \to \cdots \to B_1 \to B_0 \to A \to 0 \tag{1}$$

in an Abelian category $\mathcal{A}$ $(n \geqslant 1)$. Then $\boldsymbol{E}$ represents an element of $\mathrm{Ext}^n(A, C)$. A ***morphism*** $\boldsymbol{E} \to \boldsymbol{E}'$ of exact sequences of the same length is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r] \arrow[d, "\gamma"'] & B_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & B_1 \arrow[r] \arrow[d] & B_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\alpha"] & 0 \\ 0 \arrow[r] & C' \arrow[r] & B'_{n-1} \arrow[r] & \cdots & B'_1 \arrow[r] & B'_0 \arrow[r] & A' \arrow[r] & 0 \end{tikzcd} \tag{2}$$

We shall call α and γ the *covariant end* and *contravariant end* respectively of the given morphism. If $A = A'$ and $C = C'$ and γ and α are identity morphisms, then $\boldsymbol{E}$ and $\boldsymbol{E}'$ represent the same element of $\mathrm{Ext}^n(A, C)$, and we say that the morphism $\boldsymbol{E} \to \boldsymbol{E}'$ has *fixed ends*. On the other hand, if $\boldsymbol{E}$ and $\boldsymbol{E}'$ represent the same element of $\mathrm{Ext}^n(A, C)$ [notation $\boldsymbol{E} \sim \boldsymbol{E}'$], then there exists a sequence $\boldsymbol{E}''$ together with morphisms of sequences

$$\boldsymbol{E} \to \boldsymbol{E}'' \leftarrow \boldsymbol{E}'$$

with fixed ends [*15*]. (A three step process for getting from $\boldsymbol{E}$ to $\boldsymbol{E}'$ is given in [*9*, Chapter VII, Theorem 4.2]. Actually we shall be using only the fact that the number of such steps is bounded.)

Now consider a morphism (2) of sequences where γ and α are not necessarily identity morphisms. The existence of such a morphism expresses the fact that $\gamma\boldsymbol{E} \sim \boldsymbol{E}'\alpha$, where $\gamma\boldsymbol{E}$ (respectively $\boldsymbol{E}'\alpha$) represents the image under $\mathrm{Ext}^n(A, \gamma)$ [respectively $\mathrm{Ext}^n(\alpha, C')$] of the element represented by $\boldsymbol{E}$ [respectively $\boldsymbol{E}'$] in $\mathrm{Ext}^n(A, C)$ [respectively $\mathrm{Ext}^n(A', C')$].

For $n = 0$, $\mathrm{Ext}^0(A, C)$ is the group $[A, C]$ of morphisms from A to C, If $E \in \mathrm{Ext}^0(A, C)$ and $E' \in \mathrm{Ext}^0(A', C')$, then we define a morphism $E \to E'$ in this case to be a commutative diagram

$$\begin{tikzcd} A \arrow[r, "E"] \arrow[d, "\alpha"'] & C \arrow[d, "\gamma"] \\ A' \arrow[r, "E'"] & C'. \end{tikzcd}$$

Thus we have $\gamma E = E'\alpha$, and so the formal discussion of the preceding paragraph applies in the case $n = 0$ as well.

Consider an exact functor $T : \mathcal{A} \to \mathcal{B}$, so that in particular T is additive and preserves finite products. If $\boldsymbol{E}$ denotes the exact sequence (1) in $\mathcal{A}$, then we let $T(\boldsymbol{E})$ denote the exact sequence

$$0 \to T(C) \to T(B_{n-1}) \to \cdots \to T(B_1) \to T(B_0) \to T(A) \to 0$$

in $\mathcal{B}$. If $\boldsymbol{E} \sim \boldsymbol{E}'$, then clearly $T(\boldsymbol{E}) \sim T(\boldsymbol{E}')$, and consequently T induces a function

$$\rho = \rho^n_{A,C} : \mathrm{Ext}^n(A, C) \to \mathrm{Ext}^n(T(A), T(C)). \tag{3}$$

Furthermore we have the obvious relations

$$T(\gamma \boldsymbol{E}) = T(\gamma)T(\boldsymbol{E}), \qquad T(\boldsymbol{E}\alpha) = T(\boldsymbol{E})T(\alpha), \tag{4}$$

and so using the fact that T (as do all additive functors) preserves diagonal morphisms $\triangle : A \to A \oplus A$ and codiagonal morphisms $\nabla : C \oplus C \to C$, it is straightforward to show that ρ is a natural transformation of group valued bifunctors. In fact, ρ is a morphism of multiply-connected sequences of functors; that is, ρ commutes with the connecting morphisms relative to short exact sequences

$$0 \to A' \to A \to A'' \to 0 \qquad \text{and} \qquad 0 \to C' \to C \to C'' \to 0 \quad \text{in} \quad \mathcal{A}.$$

The statements h.d. $A \leqslant n$ and $\mathrm{Ext}^{n+1}(A, \ \ ) = 0$ are equivalent, as is easily seen from the fact that an exact sequence of length $>n + 1$ can be obtained by splicing an exact sequence of length $n + 1$ together with some other exact sequence. Likewise, the statements $\mathrm{Ext}^{n+1}(A, \ \ ) = 0$ and $\mathrm{Ext}^n(A, \ \ )$ is cokernel preserving are equivalent. To see this one takes advantage of the exactness of the connected sequence of functors $\mathrm{Ext}^i(A, \ \ )$, as well as the fact that any exact sequence of length $n + 1$ can be obtained by splicing an exact sequence of length n with a short exact sequence.

Let $S : \mathcal{B} \to \mathcal{A}$ be a coadjoint (i.e., a left adjoint) for $T : \mathcal{A} \to \mathcal{B}$. This means that there is a natural equivalence of group-valued bifunctors

$$\eta_{B,A} : [S(B), A] \approx [B, T(A)] \tag{5}$$

for $B \in \mathscr{B}$ and $A \in \mathcal{A}$. Then we have natural transformations

$$\psi : ST \to 1_{\mathcal{A}}\,, \qquad \varphi : 1_{\mathscr{B}} \to TS$$

which are such that

$$T(\psi_A)\varphi_{T(A)} = 1_{T(A)}\,, \qquad \psi_{S(B)}S(\varphi_B) = 1_{S(B)} \tag{6}$$

for all $A \in \mathcal{A}$ and $B \in \mathscr{B}$.

Most of our results will depend in some way or other on the following simple lemma.

LEMMA 1.1. *Consider an adjoint situation* (5), *and suppose that T and S are exact functors. Then there is a natural equivalence of bifunctors*

$$\eta = \eta^n_{B,A} : \operatorname{Ext}^n(S(B), A) \approx \operatorname{Ext}^n(B, T(A))$$

which commutes with the connecting morphisms.

Proof. If $\boldsymbol{E}$ represents an element of $\operatorname{Ext}^n(S(B), A)$, we define

$$\eta(\boldsymbol{E}) = T(\boldsymbol{E})\varphi_B\,.$$

Also we define

$$\mu : \operatorname{Ext}^n(B, T(A)) \to \operatorname{Ext}^n(S(B), A)$$

by

$$\mu(F) = \psi_A S(F).$$

Now for $\boldsymbol{E} \in \operatorname{Ext}^n(S(B), A)$, the natural transformation ψ gives us a morphism $ST(\boldsymbol{E}) \to \boldsymbol{E}$ with ψ_A at the covariant end and $\psi_{S(B)}$ at the contravariant end. Hence we have

$$\psi_A ST(\boldsymbol{E}) \sim \boldsymbol{E}\psi_{S(B)}\,. \tag{7}$$

Therefore using (4), (6), and (7), we obtain

$$\mu(\eta(\boldsymbol{E})) = \mu(T(\boldsymbol{E})\varphi_B) = \psi_A ST(\boldsymbol{E})S(\varphi_B) \sim \boldsymbol{E}\psi_{S(B)}S(\varphi_B) = \boldsymbol{E}.$$

This shows that $\mu\eta$ is the identity function, and dually it follows that $\eta\mu$ is the identity. Consequently η is a 1–1 correspondence. The fact that η is actually a morphism of multiply connected sequences of group-valued bifunctors, which we shall not be using in any case, is straightforward and is left to the reader.

COROLLARY 1.2. *Let S be an exact coadjoint for the exact functor T. Then for each $B \in \mathscr{B}$ we have*

$$\text{h.d.}_{\mathcal{A}}\, S(B) \leqslant \text{h.d.}_{\mathscr{B}}\, B.$$

If φ_B is a coretraction (split monomorphism) *for all $B \in \mathscr{B}$, then equality holds, and consequently in this case we have*

$$\text{gl.dim. } \mathscr{B} \leqslant \text{gl.dim. } \mathfrak{A}.$$

Proof. The first assertion is clear from Lemma 1.1. Now if $\varphi_{B'}$ is a coretraction, then $\text{Ext}_{\mathscr{B}}{}^n(B, B')$ is a retract of $\text{Ext}_{\mathscr{B}}{}^n(B, TS(B'))$. Consequently if the former is not zero, then by Lemma 1.1, $\text{Ext}_{\mathfrak{A}}{}^n(S(B), S(B')) \neq 0$. This proves the second assertion.

Remark. If T is representative (that is, for each $B \in \mathscr{B}$, there is an $A \in \mathfrak{A}$ such that $T(A)$ is isomorphic to B), then it follows from the first of Eqs. (6) that φ_B is a coretraction. However it follows also directly from Lemma 1.1 that B and $S(B)$ have the same homological dimension in this case.

Let I be any set. We form the product category $\mathfrak{A}^I$ whose objects are I-tuples (A_i) of objects of $\mathfrak{A}$, and where a morphism from (A_i) to (B_i) is a family of morphisms $A_i \to B_i$ in $\mathfrak{A}$. (Actually, $\mathfrak{A}^I$ is just a functor category if we regard I as a category having I objects and I morphisms). Using the fact that the number of steps required to get from a representative of a member of Ext^n to another such representative is bounded, it is easy to establish an isomorphism

$$\text{Ext}^n((A_i), (C_i)) \approx \bigtimes_{i \in I} \text{Ext}^n(A_i, C_i). \tag{8}$$

Suppose now that $\mathfrak{A}$ has products indexed over I. Then we have the functor $T : \mathfrak{A}^I \to \mathfrak{A}$ such that

$$T((A_i)) = \bigtimes_{i \in I} A_i .$$

An exact coadjoint for T is the functor $S : \mathfrak{A} \to \mathfrak{A}^I$ such that $S(A) = (A_i)$ where $A_i = A$ for all $i \in I$. Using Lemma 1.1, and taking (8) into account, we therefore obtain

COROLLARY 1.3. *Suppose that $\mathfrak{A}$ has exact products over I. Then we have*

$$\text{Ext}^n \left(A, \bigtimes_{i \in I} C_i\right) \approx \bigtimes_{i \in I} \text{Ext}^n(A, C_i)$$

for all $n \geqslant 0$.

The dual of Corollary 1.3 gives us

COROLLARY 1.4. *If $\mathfrak{A}$ has exact coproducts indexed over I, then*

$$\text{h.d. } \bigoplus_{i \in I} A_i = \sup_{i \in I}(\text{h.d. } A_i).$$

LEMMA 1.5. *Let* $\boldsymbol{E}$ *represent a nonzero member of* $\mathrm{Ext}^n(A, C)$, *and let* $\boldsymbol{E}'$ *represent any member of* $\mathrm{Ext}^n(A', C')$. *Then* $\boldsymbol{E} \oplus \boldsymbol{E}'$ *represents a nonzero member of* $\mathrm{Ext}^n(A \oplus A', C \oplus C')$.

Proof. Let $u : C \to C \oplus C'$ and $v : A \to A \oplus A'$ denote the coproduct injections, and let $p : C \oplus C' \to C$ denote the projection. Then we have a morphism $\boldsymbol{E} \to \boldsymbol{E} \oplus \boldsymbol{E}'$ with u at the covariant end and v at the contravariant end, and consequently

$$u\boldsymbol{E} \sim \boldsymbol{E} \oplus \boldsymbol{E}'v.$$

Therefore

$$\boldsymbol{E} = pu\boldsymbol{E} \sim p\boldsymbol{E} \oplus \boldsymbol{E}'v,$$

and so $\boldsymbol{E} \oplus \boldsymbol{E}'$ is not zero.

COROLLARY 1.6. *If* $T : \mathscr{B} \to \mathfrak{A}$ *and* $L : \mathfrak{A} \to \mathscr{B}$ *are exact functors such that* TL *admits the identity functor on* $\mathfrak{A}$ *as a retract, and if* $\boldsymbol{E}$ *represents a nonzero element of* $\mathrm{Ext}_{\mathfrak{A}}{}^n(A, C)$, *then* $TL(\boldsymbol{E})$ *represents a nonzero element of* $\mathrm{Ext}_{\mathfrak{A}}{}^n(TL(A), TL(C))$. *Hence if* $\mathrm{h.d.}_{\mathfrak{A}}\, A \geqslant n$, *then* $\mathrm{h.d.}_{\mathscr{B}} L(A) \geqslant n$.

If Π is a small category and p is an object in Π, then we shall let $T_p : \mathfrak{A}^\Pi \to \mathfrak{A}$ denote the corresponding evaluation functor. If $D \in \mathfrak{A}^\Pi$, we shall sometimes denote $T_p(D)$ by D_p, and if $\delta : D \to E$ is a morphism in $\mathfrak{A}^\Pi$, we shall sometimes denote $T_p(\delta)$ by δ_p. If $x : p \to q$ is a morphism in Π, then x induces a natural transformation $T_p \to T_q$ whose value at D we denote by $D(x)$, or simply by x when there is no danger of confusion. Notice that T_p is always representative, for if $A \in \mathfrak{A}$, then the constant functor D defined by $D_q = A$ for all $q \in \Pi$ and $D(x) = 1_A$ for all morphisms x in Π is such that $T_p(D) = A$.

We let $C(\mathfrak{A})$ denote the class of all natural transformations from the identity functor on $\mathfrak{A}$ to itself, and we call $C(\mathfrak{A})$ the *center* of $\mathfrak{A}$. Then $C(\mathfrak{A})$ is a commutative ring with identity (neglecting the fact that it may not be a set), and in the case where $\mathfrak{A}$ is the category of (right or left) R-modules, $C(\mathfrak{A})$ is isomorphic to the center of R in the usual sense. Observe that for all $c \in C(\mathfrak{A})$ and all morphisms f in $\mathfrak{A}$ we have $fc = cf$, where the c on the left side is different from that on the right unless f is an endomorphism. An element c in $C(\mathfrak{A})$ is a unit if and only if c is a natural isomorphism from $1_{\mathfrak{A}}$ to itself. If c is not a unit, then for some $A \in \mathfrak{A}$ we can form an exact sequence

$$0 \to K \to A \xrightarrow{c} A \to K' \to 0$$

where at least one of K and K' is not zero. Since c is zero on K and K', we see that c is not a unit if and only if c is zero on some nonzero object of $\mathfrak{A}$. The value of c at an object A will be denoted by c_A.

LEMMA 1.7. *Let Π be a small category, and let $p_1, p_2, \ldots, p_n$, q be objects in Π. Suppose that for each j, $1 \leqslant j \leqslant n$, we have a family of morphisms $x_{ij} : p_j \to q$, $1 \leqslant i \leqslant k_j$. Let $L : \mathcal{A} \to \mathcal{A}^\Pi$ be an exact functor such that the composition $T_q L : \mathcal{A} \to \mathcal{A}$ admits the identity functor on $\mathcal{A}$ as a retract. Suppose further that L takes all its values in the full subcategory of $\mathcal{A}^\Pi$ consisting of those objects for which the n expressions*

$$\Gamma_j = \sum_{i=1}^{k_j} c_{ij} x_{ij}$$

are zero, where the c_{ij} are a fixed family in $C(\mathcal{A})$. Consider an exact sequence

$$E : 0 \to L(A) \xrightarrow{\delta} D \to B \to 0 \tag{9}$$

in $\mathcal{A}^\Pi$, and suppose that there exist morphisms

$$\tau_j : L(A)_q \to D_{p_j}, \qquad 1 \leqslant j \leqslant n,$$

such that

$$\sum_{j=1}^{n} \Gamma_j \tau_j = \delta_q .$$

If E represents a nonzero element of $\mathrm{Ext}_{\mathcal{A}}{}^n(A, C)$, then $L(E)E$ represents a nonzero element of $\mathrm{Ext}_\Pi^{n+1}(B, L(C))$. Consequently if $A \neq 0$, then

$$\text{h.d.}_\Pi B \geqslant 1 + \text{h.d.}_{\mathcal{A}} A.$$

Proof. Suppose that $L(E)E \sim 0$. Using the exact sequence for Ext relative to (9), we can then write $L(E) = \mathscr{E}\delta$ for some $\mathscr{E} \in \mathrm{Ext}_\Pi{}^n(D, L(C))$. Now for each j, $1 \leqslant j \leqslant n$, the expression Γ_j induces a morphism

$$T_{p_j}(\mathscr{E}) \to T_q(\mathscr{E})$$

with 0 at the covariant end (by assumption on the values of L) and Γ_j evaluated on D at the contravariant end. Hence we can write

$$T_q(\mathscr{E})\Gamma_j \sim 0,$$

and so we have

$$T_q L(E) \sim T_q(\mathscr{E}\delta) = T_q(\mathscr{E})\, T_q(\delta) = T_q(\mathscr{E})\Big(\sum_{j=1}^{n} \Gamma_j \tau_j\Big) \sim 0.$$

But since $T_q L$ admits $1_{\mathcal{A}}$ as a retract, this contradicts Corollary 1.6.

Remarks. (i) Suppose that the exact sequence (9) and the values of L are in $\mathscr{B}$, where $\mathscr{B}$ is an Abelian subcategory of $\mathcal{A}^\Pi$ (that is, $\mathscr{B}$ is an Abelian category and the inclusion functor is exact). Then the conclusion of Lemma 1.7 is clearly true with h.d.$_{\Pi}$ B replaced by h.d.$_{\mathcal{B}}$ B.

(ii) In the present paper we shall always have $n = 1$. It will not be until [10] that we shall have $n > 1$ (and in that case we shall have each $k_j = 1$).

Let p be an object in the small category Π, and suppose that $\mathcal{A}$ has coproducts indexed over hom sets $[p, q]$ for each $q \in \Pi$. (In particular, this will always be true if each of these sets is finite.) Then the Kan construction [8] provides us with a coadjoint $S_p : \mathcal{A} \to \mathcal{A}^{\Pi}$ for the evaluation functor T_p. Explicitly, if in general we denote the coproduct of I copies of an object A by ${}^{I}A$, and if $u_i : A \to {}^{I}A$ denotes the coproduct injection corresponding to $i \in I$, then S_p is given as follows:

$$S_p(A)_q = {}^{[p,q]}A \qquad \text{for} \quad A \in \mathcal{A}, \qquad q \in \Pi$$

$$S_p(A)(x)\, u_y = u_{xy} \qquad \text{for} \quad x \in [q, q'] \quad \text{and} \quad y \in [p, q]$$

$$S_p(f)_q = {}^{[p,q]}f \qquad \text{for} \quad f : A \to A' \text{ in } \mathcal{A}.$$

If the coproducts in $\mathcal{A}$ are exact, then S_p is an exact functor.

Proposition 1.8. *Let $T : \mathcal{A} \to \mathcal{B}$ and $S : \mathcal{B} \to \mathcal{A}$ be exact functors, and suppose that S is simultaneously an adjoint and a coadjoint for T. If $A \in \mathcal{A}$, then*

$$\text{h.d.}_{\mathcal{A}}\, ST(A) \leqslant \text{h.d.}_{\mathcal{B}}\, T(A) \leqslant \text{h.d.}_{\mathcal{A}}\, A, \tag{10}$$

and consequently if A is a retract of $ST(A)$, then the inequalities are equalities in (10). *On the other hand, if T is faithful, then the existence of a single object A such that*

$$\text{h.d.}_{\mathcal{B}}\, T(A) < \text{h.d.}_{\mathcal{A}}\, A \tag{11}$$

implies that

$$\text{gl.dim. } \mathcal{A} = \infty.$$

Proof. The inequalities (10) follow from two applications of Corollary 1.2. To prove the final assertion, we shall exhibit an object $A' \in \mathcal{A}$ such that

$$\text{h.d.}_{\mathcal{A}}\, A' = 1 + \text{h.d.}_{\mathcal{A}}\, A \tag{12}$$

and such that

$$\text{h.d.}_{\mathcal{B}}\, T(A') < \text{h.d.}_{\mathcal{A}}\, A'. \tag{13}$$

The result will then follow by iteration.

From (6) [with S as adjoint] we have a coretraction (split monomorphism) $T(A) \to TST(A)$. Therefore since T is faithful, we obtain an exact sequence

$$0 \to A \to ST(A) \to A' \to 0 \tag{14}$$

which splits on application of T. Using (10) and (11) we see that

$$\text{h.d.}_{\mathcal{A}}\, ST(A) < \text{h.d.}_{\mathcal{A}}\, A, \tag{15}$$

and consequently (12) follows from (14). Now using the fact that $T(A')$ is a retract of $TST(A)$ and applying (12), (15), and Corollary 1.2, we obtain

$$\text{h.d.}_{\mathcal{B}}\, T(A') \leqslant \text{h.d.}_{\mathcal{B}}\, TST(A) \leqslant \text{h.d.}_{\mathcal{A}}\, ST(A) < \text{h.d.}_{\mathcal{A}}\, A \leqslant \text{h.d.}_{\mathcal{A}}\, A'.$$

This proves (13).

2. Free Monoids

Throughout the remainder of the paper, Π will denote a monoid, or in other words a category with one object. We shall use the membership notation $x \in \Pi$ to denote that x is a morphism in Π. An object of $\mathcal{A}^{\Pi}$ will be called a *Π-object* in $\mathcal{A}$. Thus a Π-object is an object $A \in \mathcal{A}$ together with a family of endomorphisms $\{x : A \to A\}_{x \in \Pi}$ (observe the abuse of notation) satisfying the usual functorial properties. The evaluation (forgetful) functor $T : \mathcal{A}^{\Pi} \to \mathcal{A}$ has a coadjoint $S : \mathcal{A} \to \mathcal{A}^{\Pi}$ if $\mathcal{A}$ has coproducts indexed over Π. The value of S at A will sometimes be denoted by $A(\Pi)$, and as before we shall let $u_x : A \to A(\Pi)$ denote the coproduct injection in $\mathcal{A}$ corresponding to $x \in \Pi$. If 1 is the identity element of Π, then we shall simply write u in place of u_1. The action of x on $A(\Pi)$ will be denoted by X. Thus X is given by

$$Xu_y = u_{xy}$$

for all $y \in \Pi$.

If A is a Π-object, then there is a unique morphism of Π-objects $\alpha : A(\Pi) \to A$ satisfying $\alpha u = 1_A$. Explicitly, α is given by

$$\alpha u_x = x : A \to A.$$

(Actually α is just the morphism ψ of Section 1.)

Let $\{x_i\}_{i \in I}$ be a family of elements of Π, and let I' be the subset of I consisting of those i such that x_i has a two-sided inverse x_i^{-1} in Π (or, in other words, such that x_i is an automorphism in Π). We shall say that Π is *generated* by the given family if each $x \in \Pi$ can be written (not necessarily uniquely) in the form

$$x = x_{i_1}^{n_1} x_{i_2}^{n_2} \cdots x_{i_t}^{n_t} \tag{1}$$

where n_k is a positive or negative integer if $i_k \in I'$, and is a positive integer if $i_k \in I - I'$. By contraction we may assume that no two consecutive subscripts in (1) are equal. Using this convention, if each x in Π can be written uniquely in the form (1), then we shall say that Π is a *partially-free monoid of type* (I, I').

When I' is empty, Π is just a free monoid in the ordinary sense, and when $I' = I$, Π is a free group.

Suppose that Π is generated by $\{x_i\}_{i \in I}$. Given a Π-object A, let

$$U_i : A(\Pi) \to {}^I(A(\Pi))$$

denote the coproduct injections. For each $i \in I$, let $\beta_i : A(\Pi) \to A(\Pi)$ be the unique morphism of Π-objects such that

$$\beta_i u = u_{x_i} - u x_i \,.$$

Then the β_i are the coordinates of a morphism

$$\beta : {}^I(A(\Pi)) \to A(\Pi).$$

Explicitly, β is given by

$$\beta U_i u_x = u_{x x_i} - u_x x_i \,.$$

For each $i \in I$ and $x \in \Pi$ we have

$$\alpha\beta U_i u_x = \alpha u_{x x_i} - \alpha u_x x_i = x x_i - x x_i = 0.$$

Consequently $\alpha\beta = 0$. In the following we shall adopt the notation

$$u_x = u^{n_1 \cdots n_t}_{i_1 \cdots i_t}$$

if x is given by (1).

LEMMA 2.1. *If Π is generated by $\{x_i\}_{i \in I}$ and A is any Π-object, then*

$${}^I(A(\Pi)) \xrightarrow{\beta} A(\Pi) \xrightarrow{\alpha} A \to 0 \tag{2}$$

is an exact sequence in $\mathfrak{A}^\Pi$. If furthermore, Π is partially free of type (I, I'), then the sequence

$$0 \to {}^I(A(\Pi)) \xrightarrow{\beta} A(\Pi) \xrightarrow{\alpha} A \to 0 \tag{3}$$

is exact in $\mathfrak{A}^\Pi$ and splits in $\mathfrak{A}$.

Proof. For each $x \in \Pi$, choose a representation for x in the form (1), and define in $\mathfrak{A}$ a morphism

$$\theta : A(\Pi) \to {}^I(A(\Pi))$$

by the rule

$$\theta u_x = \sum_{k=1}^{t} \frac{n_k}{|\, n_k \,|} U_{i_k} \sum_{j=1}^{|n_k|} u^{n_1 \cdots n_{k-1} \epsilon(j)-1}_{i_1 \cdots i_{k-1} i_k} x_{i_k}^{n_k - \epsilon(j)} x_{i_{k+1}}^{n_{k+1}} \cdots x_{i_t}^{n_t} \,.$$

Here $\epsilon(j)$, which depends on k as well as j, is given by

$$\begin{aligned} \epsilon(j) &= j \quad \text{if} \quad n_k > 0 \\ &= 1 - j \quad \text{if} \quad n_k < 0. \end{aligned}$$

The definition (4) comes from the desire to have θ such that $\beta\theta + u\alpha = 1$, and indeed, by composing each side of the latter equality with the coproduct injections u_x, it is straightforward to verify that this is the case. Combining this with the relations $\alpha\beta = 0$ and $\alpha u = 1$, we obtain the exactness of (2).

Now assuming that each x has a unique representation in the form (1), the remainder of the lemma is equivalent to showing that $\theta\beta = 1$. This is done by composing each side with the coproduct injections

$$U_p u_{i_1 \ldots i_t}^{n_1 \ldots n_t}$$

and treating separately the four cases $p \neq i_t$, $p = i_t$ and $n_t > 0$, $p = i_t$ and $n_t = -1$, $p = i_t$ and $n_t < -1$.

LEMMA 2.2. *Let Π be a partially-free monoid and suppose that $\mathfrak{A}$ has exact coproducts indexed over Π. If A is any Π-object in $\mathfrak{A}$, then*

$$\text{h.d.}_\Pi A \leqslant 1 + \text{h.d.}_{\mathfrak{A}} A. \tag{5}$$

Consequently,

$$\text{gl.dim. } \mathfrak{A}^\Pi \leqslant 1 + \text{gl.dim. } \mathfrak{A}.$$

Proof. The assumption on coproducts guarantees simultaneously that the coadjoint S is exact, and that the coproducts in $\mathfrak{A}$ (and hence in $\mathfrak{A}^\Pi$) indexed over I are exact. Consequently, using Corollary 1.4 and Proposition 1.2 we obtain

$$\text{h.d.}_\Pi {}^I(A(\Pi)) = \text{h.d.}_\Pi A(\Pi) = \text{h.d.}_{\mathfrak{A}} A.$$

Hence if $\text{h.d.}_\Pi A > 1 + \text{h.d.}_{\mathfrak{A}} A$, then in the exact sequence for Ext relative to the short exact sequence (3), we would have a nonzero term flanked by two zero terms. This contradiction proves the lemma.

We shall now invoke Lemma 1.7 to exhibit objects for which equality holds in (5). We first remark that if Π is a partially-free monoid of type (I, I'), then a Π-object in $\mathfrak{A}$ is had by taking an object A in $\mathfrak{A}$, and randomly assigning automorphisms $x_i : A \to A$ for $i \in I'$, and endomorphisms $x_i : A \to A$ for $i \in I - I'$. This remark enables us to define a functor

$$L : \mathfrak{A} \to \mathfrak{A}^\Pi$$

by taking a family $\{c_i\}_{i \in I}$ where c_i is a unit in $C(\mathfrak{A})$ if x_i is in I' and c_i is an arbitrary element of $C(\mathfrak{A})$ otherwise, and converting an object $A \in \mathfrak{A}$ into a Π-object by defining $x_i = c_i$ for all i. Consider the exact sequence in $\mathfrak{A}$

$$0 \longrightarrow A \xrightarrow{u_2} A \oplus A \xrightarrow{p_1} A \longrightarrow 0 \tag{6}$$

where u_2 denotes the second coproduct injection and p_1 denotes the first projection. Convert $A \oplus A$ into a Π-object by choosing any $s \in I$, taking $x_i = c_i$ for $i \neq s$, and $x_s = c_s - u_2 p_1$. Then (6) may be regarded as an exact sequence

$$0 \to L(A) \xrightarrow{\delta} A \oplus A \to L(A) \to 0 \tag{7}$$

in $\mathfrak{A}^{\Pi}$, and we have

$$T(\delta) = u_2 = (c_s - (c_s - u_2 p_1))u_1 = (c_s - x_s)u_1 .$$

Therefore by Lemma 1.7 (with $n = 1$, $\Gamma = c_s - x_s$, and $\tau = u_1$) we obtain

$$\text{h.d.}_{\Pi} L(A) \geqslant 1 + \text{h.d.}_{\mathfrak{A}} A \tag{8}$$

providing A is not zero. Combining (8) and Lemma 2.2, we now obtain

$$\text{gl.dim. } \mathfrak{A}^{\Pi} = 1 + \text{gl.dim. } \mathfrak{A} \tag{9}$$

for any nontrivial $\mathfrak{A}$ with exact coproducts over Π. Now if $\mathfrak{A}$ has exact products instead of coproducts over Π, then Eq. (9) applies to the dual category $\mathfrak{A}^*$. Hence, using the general relations

$$\text{gl. dim. } \mathfrak{A}^* = \text{gl. dim. } \mathfrak{A}, \qquad (\mathfrak{A}^{\Pi})^* = (\mathfrak{A}^*)^{\Pi^*},$$

and also the fact that Π^* is partially free if Π is, we have

$$\begin{aligned} \text{gl. dim. } \mathfrak{A}^{\Pi} &= \text{gl. dim.}(\mathfrak{A}^{\Pi})^* = \text{gl. dim. } \mathfrak{A}^{*\Pi^*} \\ &= 1 + \text{gl. dim. } \mathfrak{A}^* = 1 + \text{gl. dim. } \mathfrak{A}. \end{aligned}$$

Thus we have proved the following theorem.

THEOREM 2.3. *Let Π be a partially-free monoid and suppose that $\mathfrak{A}$ is a nontrivial Abelian category with exact products or coproducts indexed by Π. Then*

$$\text{gl.dim. } \mathfrak{A}^{\Pi} = 1 + \text{gl.dim. } \mathfrak{A}.$$

Using the isomorphism of functor categories

$$\mathfrak{A}^{\Pi_1 \times \Pi_2} \approx (\mathfrak{A}^{\Pi_1})^{\Pi_2}$$

and induction on n, we obtain

COROLLARY 2.4. *If Π is the direct product of n partially-free monoids and $\mathfrak{A}$ has exact products or coproducts over Π, then*

$$\text{gl.dim. } \mathfrak{A}^{\Pi} = n + \text{gl.dim. } \mathfrak{A}.$$

Remark. Corollary 2.4 is valid even if n is infinite. For in this case we can take for any positive integer k a submonoid Π_k which is the direct product of k partially-free monoids, together with obvious functors $F : \mathcal{A}^{\Pi} \to \mathcal{A}^{\Pi_k}$ and $G : \mathcal{A}^{\Pi_k} \to \mathcal{A}^{\Pi}$ such that FG is the identity functor on $\mathcal{A}^{\Pi_k}$. Using Corollaries 1.6 and 2.4, we then see that gl.dim. $\mathcal{A}^{\Pi} \geqslant k +$ gl.dim. $\mathcal{A}$.

If $\mathcal{A}$ is the category of left R-modules, then $\mathcal{A}^{\Pi}$ is the category of left modules over the monoid ring $R(\Pi)$. This gives us

COROLLARY 2.5. *The global dimension of any free ring over R is* $1 +$ gl.dim. R.

COROLLARY 2.6. *The global dimension of the polynomial ring in n variables over R is* $n +$ gl.dim. R.

COROLLARY 2.7. *If Π is a free group, then*

$$\text{gl.dim. } R(\Pi) = 1 + \text{gl.dim. } R.$$

COROLLARY 2.8. *If Π is a free Abelian group of rank n, then*

$$\text{gl.dim. } R(\Pi) = n + \text{gl.dim. } R.$$

Corollary 2.6 was first proved by Eilenberg, Rosenberg, and Zelinsky [*6*]. It generalizes the Hilbert syzygy theorem, in which the notion of global dimension finds its origin (1890). Corollary 2.5 was first proved by Hochschild [7], and Corollary 2.8 was first proved by Balcerzyk [*1*].

It follows from Lemma 2.2 that any free group Π is cohomologically trivial; that is, that h.d.$_{\Pi}\, Z \leqslant 1$, where Z is the group of integers with trivial Π-operators. The converse is a well-known open question. (For recent developments concerning this question see [*3*].) One could ask the more general question. If Π is a cancellative monoid such that h.d.$_{\Pi}\, Z \leqslant 1$, is Π partially free? To see that the cancellative condition is necessary, consider the monoid Π whose elements are symbols $x^m y^n$, m and n being nonnegative integers, and whose multiplication is given by the rule

$$\begin{aligned} (x^m y^n)(x^r y^s) &= x^{m+r} y^s \quad \text{if} \quad r > 0, \\ &= x^m y^{n+s} \quad \text{if} \quad r = 0. \end{aligned}$$

Then $x^0 y^0$ is a two-sided identity, and it is easily verified that multiplication is associative. We construct an exact sequence

$$0 \longrightarrow Z(\Pi) \underset{\gamma}{\overset{\beta}{\rightleftarrows}} Z(\Pi) \underset{u}{\overset{\alpha}{\rightleftarrows}} Z \longrightarrow 0,$$

thereby showing that h.d.$_\Pi Z \leqslant 1$. The morphisms α and u are as usual. We take β to be the unique morphism of $Z(\Pi)$-modules such that $\beta(1) = X - 1$, and we define γ as the unique morphism (of Abelian groups) satisfying

$$\gamma(X^m Y^n) = 1 + X + \cdots + X^m - X^m Y^n.$$

Then the relations $\alpha u = 1, \beta\gamma + u\alpha = 1, \gamma\beta = 1$, and $\alpha\beta = 0$ are readily verified.

3. Groups

Let R be a ring. A left R-object structure on an object $A \in \mathfrak{A}$ is a ring homomorphism $R \to \mathrm{End}_{\mathfrak{A}}(A)$. The category of left R-objects in $\mathfrak{A}$ is denoted by ${}^R\mathfrak{A}$. We define $\mathfrak{A}^R = {}^{R^*}\mathfrak{A}$, where R^* is the opposite ring. If $\mathcal{G}$ denotes the category of Abelian groups, then $\mathcal{G}^R$ is just the category of right R-modules.

Remark. A left R-object in $\mathfrak{A}$ may be viewed as an additive, covariant functor into $\mathfrak{A}$ from the ring R viewed as an additive category with one object. For this reason, a better notation for the covariant functor category $\mathfrak{A}^\Pi$ (Π any small category) would be ${}^\Pi\mathfrak{A}$.

If the Abelian category $\mathfrak{A}$ has arbitrary coproducts, then a unique colimit preserving bifunctor $\otimes_R : \mathcal{G}^R \times {}^R\mathfrak{A} \to \mathfrak{A}$ can be constructed so as to have $R \otimes_R A = A$ as left R-objects for each $A \in {}^R\mathfrak{A}$. If R is right Noetherian, then the coproduct assumption on $\mathfrak{A}$ may be waived if we replace $\mathcal{G}^R$ by the category of all finitely generated right R-modules. (For a systematic account of all forms of tensor products of R-modules and R-objects, see [*11*, Section 3].)

Now suppose that R is commutative, and that M and A are Π-objects in $\mathcal{G}^R$ and ${}^R\mathfrak{A}$, respectively. In this case we shall consider $M \otimes_R A$ as a Π-object in $\mathfrak{A}$ with $x \in \Pi$ acting by $x \otimes x$. In the case where M is $R(\Pi)$, we have $R(\Pi) \otimes_R A = A(\Pi)$, with x acting as the endomorphism $\bar{X} : A(\Pi) \to A(\Pi)$ given by

$$\bar{X} u_y = u_{xy} x.$$

This Π-structure on $A(\Pi)$ is isomorphic to the usual one via the morphism $\varphi : A(\Pi) \to A(\Pi)$ given by $\varphi u_y = u_y\, y$.

LEMMA 3.1. *Let R be commutative, and suppose that P is a projective left $R(\Pi)$-module. If A and C are Π-objects in ${}^R\mathfrak{A}$ and $\mathfrak{A}$ respectively such that* $\mathrm{Ext}^k(A, C) = 0$, *and if $\mathfrak{A}$ has exact coproducts, then* $\mathrm{Ext}_\Pi{}^k(P \otimes_R A, C) = 0$.

Proof. Since P is a retract of a free $R(\Pi)$-module, using Corollary 1.3

we see that it suffices to consider the case where $P = R(\Pi)$. But then $R(\Pi) \otimes_R A = A(\Pi)$, and so the result follows from Lemma 1.1.

Remark. If R is Noetherian and Π is finite, and if P is a finitely-general left $R(\Pi)$-module, then Lemma 3.1 is true without the coproduct assumption on $\mathcal{A}$.

PROPOSITION 3.2. *Let $\mathcal{P}$ be a set of primes, and let Π be a finite group with a finite $\mathcal{P}$-period q [12, p. 268]. Let A and C be Π-objects in $\mathcal{A}$ such that* $\mathrm{Ext}^k(A, C) = 0$ *for $m \leqslant k \leqslant m + q$, and suppose that p_A is an isomorphism for each prime $p \in \mathcal{P}$. Then*

$$\mathrm{Ext}_\Pi^{m+q}(A, C) \approx \mathrm{Ext}_\Pi{}^m(A, C).$$

Proof. Let R denote the ring of rational numbers with denominator prime to each $p \in \mathcal{P}$. The assumption on A endows A with a (unique) R-object structure. By [12, Theorem 4.1], there is an exact sequence of left $R(\Pi)$-modules

$$0 \to R \to P_{q-1} \to \cdots \to P_1 \to P_0 \to R \to 0$$

where R has trivial Π-operators and each P_i is finitely generated and projective. Since the sequence splits as R-modules, it remains exact on tensoring over R with A, and consequently the result follows from the remark to Lemma 3.1.

If Π is a cyclic group of order n on a generator x, then (taking $\mathcal{P}$ to be the set of all primes) a periodic resolution for $R = Z$ in this case is given explicitly by

$$0 \longrightarrow Z \xrightarrow{\Delta} Z(\Pi) \xrightarrow{1-X} Z(\Pi) \xrightarrow{\nabla} Z \longrightarrow 0.$$

Take I to be a one element set in the discussion following Lemma 2.2, and define the functor L of that discussion by taking $c = 1$. If A is an object of $\mathcal{A}$ such that $n_A = 0$, then the automorphism

$$x = 1 - u_2 p_1 : A \oplus A \to A \oplus A$$

is such that $x^n = 1$, and consequently the exact sequence (7) of Section 2 is in $\mathcal{A}^\Pi$. Therefore combining Lemma 1.7 and Proposition 3.2 we obtain

COROLLARY 3.3. *Let Π be a cyclic group of order n, and let A be a nonzero object in $\mathcal{A}$ such that $n_A = 0$. If A is considered as a Π-object with trivial operators, then* $\mathrm{h.d.}_\Pi A = \infty$.

Let Π be a subgroup of group G (not necessarily finite). Let $\{x_i\Pi\}_{i\in I}$ be the distinct left cosets of Π in G, where for simplicity we assume that 1 represents its coset. For each $x \in G$ and $i \in I$ we can write

$$xx_i = x_j t \tag{1}$$

for unique elements $j \in I$ and $t \in \Pi$. Suppose that $\mathfrak{A}$ has coproducts indexed over I. For $A \in \mathfrak{A}^\Pi$ we define $S_{G\Pi}(A)$ to be the G-object ${}^IA$ where an element x in G operates on ${}^IA$ by the rule

$$xu_i = u_j t,$$

j and t being as in (1). If A is a Π-object and A' is a G-object, then a morphism $\alpha : {}^IA \to A'$ of G-objects is completely determined by a morphism $f : A \to A'$ of Π-objects if we define

$$\alpha u_i = x_i f.$$

It follows that $S_{G\Pi}$ is the coadjoint of the restriction functor $T_{\Pi G} : \mathfrak{A}^G \to \mathfrak{A}^\Pi$. If the coproducts over I are exact, then it follows from Corollary 1.2 that

$$\text{gl.dim. } \mathfrak{A}^\Pi \leqslant \text{gl.dim. } \mathfrak{A}^G. \tag{2}$$

If Π is of finite index in G (that is, if I is finite), and if A'' and A are G and Π-objects respectively, then a morphism

$$\beta : A'' \to {}^IA$$

of G-objects is completely determined by a morphism $g : A'' \to A$ of Π-objects if we define

$$p_i\beta = gx_i^{-1}$$

where p_i denotes the coproduct projection. Thus we see in this case that $S_{G\Pi}$ is an adjoint as well as a coadjoint for $T_{\Pi G}$. The composition $\alpha\beta$ can then be written

$$\alpha\beta = \alpha \sum_{i \in I} u_i p_i \beta = \sum_{i \in I} x_i f g x_i^{-1}. \tag{3}$$

In particular, let A be a G-object, and take $A' = A'' = A$. Putting $f = g = 1_A$ in (3) and letting n be the index of Π in G, we see that $n : A \to A$ admits a factorization through $S_{G\Pi}(A)$. Now if C is another G-object and if $\text{Ext}_\Pi{}^k(A, C) = 0$, then from Corollary 1.2 we have $\text{Ext}_G{}^k(S_{\Pi G}(A), C) = 0$, and consequently $n\,\text{Ext}_G{}^k(A, C) = 0$.

PROPOSITION 3.4. *If G is a group of order n and if n_A is an isomorphism, then*

$$\text{h.d.}_G\, A = \text{h.d.}_{\mathfrak{A}}\, A$$

for every G-object structure on A. Consequently, if n is a unit in $C(\mathfrak{A})$, then $\mathfrak{A}^G$ and $\mathfrak{A}$ have the same global dimension.

On the other hand, if q is a prime factor of n which is not a unit in $C(\mathfrak{A})$, *and if A is a nonzero object of $\mathfrak{A}$ such that $q_A = 0$, then considering A as a G-object with trivial operators, we have*

$$\text{h.d.}_G A = \infty. \tag{4}$$

Proof. Taking $\Pi = 1$ in the preceding discussion, the relation $n \operatorname{Ext}_G^k(A, C) = 0$ implies $\text{h.d.}_G A \leqslant \text{h.d.} A$ in the case where n_A is an isomorphism. Since the other inequality holds in any case, this gives us the first assertion.

Now assuming $q_A = 0$ with $A \neq 0$, we know by Cauchy's group theorem that G has a subgroup Π of order q. But by Corollary 3.3 we have $\text{h.d.}_\Pi A = \infty$. Hence (4) follows from (2).

Remark 1. Proposition 3.4 generalizes the well-known fact that the group ring of a finite group over a field is semisimple if and only if the characteristic of the field does not divide the order of the group.

Remark 2. Eq. (4) is valid for any $A \in \mathfrak{A}$ for which q_A is a monomorphism or an epimorphism but not both. For example, if q_A is a monomorphism, then we have an exact sequence

$$0 \to A \xrightarrow{q} A \to A/qA \to 0 \tag{5}$$

where A/qA is not zero. Applying (4) to A/qA, we see from (5) that $\text{h.d.}_G A = \infty$ where A has trivial operators. In particular if $\mathfrak{A}$ is the category of Abelian groups, then $\text{h.d.}_G Z = \infty$ for all finite groups G (see [5, p. 263, Exercise 2]).

If G is any Abelian group, we take as our definition of the *rank* of G ($r(G)$) the maximum number (or ∞) of linearly-independent elements of G. Equivalently $r(G)$ is the dimension of the vector space $G \otimes_Z Q$ over the rationals. If G is a finitely-generated Abelian group, then G can be written as $\Pi \times \tau$ where τ is the torsion subgroup of G and Π is a free Abelian group on $r(G)$ generators. Combining Proposition 3.4 and Corollary 2.4 therefore yields

COROLLARY 3.5. *Let G be a finitely-generated Abelian group whose torsion subgroup has order n, and suppose that $\mathfrak{A}$ has exact, countable products or coproducts. If n is a unit in $C(\mathfrak{A})$, then*

$$\text{gl.dim. } \mathfrak{A}^G = r(G) + \text{gl.dim. } \mathfrak{A}.$$

Otherwise $\text{gl.dim. } \mathfrak{A}^G = \infty$.

THEOREM 3.6. *Let G be an Abelian group of rank r, and suppose that $\mathfrak{A}$ has exact products or coproducts indexed by G. Then*

$$r + \text{gl.dim. } \mathfrak{A} \leqslant \text{gl.dim. } \mathfrak{A}^G. \tag{6}$$

Furthermore let $\mathscr{P}$ be the set of primes p for which G has p-torsion. If p is not a unit in $C(\mathfrak{A})$ for some prime $p \in \mathscr{P}$, then gl.dim. $\mathfrak{A}^G = \infty$. *On the other hand, if p is a unit in $C(\mathfrak{A})$ for all primes $p \in \mathscr{P}$, and if further G is countable, then*

$$\text{gl.dim. } \mathfrak{A}^G \leqslant 1 + r + \text{gl.dim. } \mathfrak{A}. \tag{7}$$

Proof. Since G has a free Abelian subgroup of rank r, the inequality (6) follows from (2) and Corollary 4.4. If p is not a unit in $C(\mathfrak{A})$ for some $p \in \mathscr{P}$, then since G contains a cyclic subgroup of order p, it follows from (2) and Proposition 3.4 that $\mathfrak{A}^G$ has infinite global dimension.

Now suppose that p is a unit in $C(\mathfrak{A})$ for each prime $p \in \mathscr{P}$, and that G is countable. Let R be the ring of rational numbers with denominator prime to each $p \notin \mathscr{P}$. The assumption on $C(\mathfrak{A})$ makes $\mathfrak{A}$ isomorphic to ${}^R\mathfrak{A}$. Now G is the countable union of its finitely-generated subgroups, and so the group ring $R(G)$ is the countable direct limit of rings of the form $R(G_i)$ where G_i is finitely generated with rank $\leqslant r$. Combining Lemma 2.2 and Proposition 3.4, we see that h.d.$_{G_i}$ $R \leqslant r$ where R has trivial G_i operators. Therefore by a theorem of Berstein [4] it follows that h.d.$_G$ $R \leqslant 1 + r$, and so we have a projective $R(G)$-resolution for R of the form

$$0 \to P_{r+1} \to P_r \to \cdots \to P_1 \to P_0 \to R \to 0. \tag{8}$$

Now if A is any G-object in $\mathfrak{A}$, then we can tensor (8) over R with A and apply Lemma 3.1 to see that

$$\text{h.d.}_G\, A \leqslant 1 + r + \text{h.d.}_{\mathfrak{A}}\, A.$$

From this we obtain the inequality (7).

Balcerzyk [2] shows that if G is a torsion free, nonfinitely-generated Abelian group of rank r and R is a commutative Noetherian ring, then

$$\text{gl.dim. } R(G) = 1 + r + \text{gl.dim. } R.$$

He also points out that if R is replaced by $R(G)$, then since $R(G)(G) \approx R(G \times G)$, we must have

$$\text{gl.dim. } R(G)(G) = r + \text{gl.dim. } R(G).$$

We shall show that this phenomenon can exist in the case of a torsion group as well. In fact let $G = Z(p^\infty)$, where p is any prime. Then G has generators $\sigma_1, \sigma_2, \ldots$, where $\sigma_i^p = \sigma_{i-1}$ for $i \geqslant 2$, and each nonzero element of G can be written uniquely in the form σ_i^m where $0 < m < p^i$ and $(m, p) = 1$. Let R be any ring in which p is a unit. As above, we see that h.d.$_G$ $R \leqslant 1$, where R has trivial operators. Now if R were projective as a G-module, then we would have G-morphisms

$$R \xrightarrow{\varphi} R(G) \xrightarrow{\alpha} R$$

where α is as usual and $\alpha\varphi = 1_R$. In particular, this would mean that $\varphi \neq 0$. For some $k \geqslant 1$ we can write

$$\varphi(1) = \sum_{i=0}^{p^k-1} r_i \sigma_k{}^i. \tag{9}$$

Since φ is a G-morphism, the right side of (9) must remain unchanged on multiplication by any element of G. Multiplying by σ_{k+1}, we get a contradiction. Hence h.d.$_G\, R = 1$. In particular, if R is semisimple, this gives gl.dim. $R(G) = 1$.

Question. Is the countability condition necessary in Theorem 3.6? More specifically, does there exist a ring R and an Abelian torsion group G (necessarily noncountable) such that

$$1 + \text{gl.dim. } R < \text{gl.dim. } R(G) < \infty\,?$$

Let Π be a subgroup of the (not necessarily Abelian) group G, and let A and C be G-objects. Then the restriction functor $T_{\Pi G}$ induces a morphism

$$i(\Pi, G) : \mathrm{Ext}_G{}^n(A, C) \to \mathrm{Ext}_\Pi{}^n(A, C)$$

for all $n \geqslant 0$. Also if $x \in G$, then the obvious isomorphism $\theta : x\Pi x^{-1} \to \Pi$ induces a functor $\theta^0 : \mathcal{O}\!\mathcal{l}^\Pi \to \mathcal{O}\!\mathcal{l}^{x\Pi x^{-1}}$, and if A is a G-object, then $x : A \to A$ may be considered as a morphism $\theta^0(A) \to A$ in $\mathcal{O}\!\mathcal{l}^{x\Pi x^{-1}}$. Consequently if A and C are G-objects, then we may define morphisms

$$c_x : \mathrm{Ext}_\Pi{}^n(A, C) \to \mathrm{Ext}^n_{x\Pi x^{-1}}(A, C)$$

by taking $c_x(E) = x\theta^0(E)x^{-1}$. Finally, if Π is of finite index in G, so that $S_{G\Pi}$ is both a coadjoint and an adjoint for $T_{\Pi G}$, then relative to a G-object A we have the morphisms

$$\psi_A : S_{G\Pi}T_{\Pi G}(A) \to A,$$
$$\varphi_A : A \to S_{G\Pi}T_{\Pi G}(A)$$

described in Section 1. This enables us to define "transfer" morphisms

$$t(G, \Pi) : \mathrm{Ext}_\Pi{}^n(A, C) \to \mathrm{Ext}_G{}^n(A, C)$$

relative to G-objects A and C by taking

$$t(G, \Pi)(E) = \psi_C S_{G\Pi}(E)\varphi_A\,.$$

Without further ado, we assert that all of the results obtained in [5, Chapter XII, Sections 8, 9] are valid if $\hat{H}(G, C)$ is replaced by $\mathrm{Ext}_G(A, C)$ throughout, and if the "products" involved are the ones defined by splicing exact sequences. Furthermore, the results of [5, Chapter XII, Section 10] are also valid if $\hat{H}^n(G, C)$ is replaced by $\mathrm{Ext}_G{}^n(A, C)$ for any n satisfying $\mathrm{Ext}_{\mathcal{O}\!\mathcal{l}}{}^n(A, C) = 0$.

4. Monic Polynomial Relations

Let Π be the free monoid on a single generator x, so that $\mathfrak{A}^{\Pi}$ is just the category of endomorphisms $x : A \to A$ in $\mathfrak{A}$. If $F(x)$ is a polynomial with coefficients in $C(\mathfrak{A})$, we let $\mathfrak{A}_F$ denote the full subcategory of $\mathfrak{A}^{\Pi}$ consisting of those objects for which $F(x) = 0$. If $g(x)$ and $h(x)$ are any two polynomials with coefficients in $C(\mathfrak{A})$, then we have $g(x)h(x) = h(x)\,g(x)$, and consequently taking $h(x) = x$, we see that $g(x) : A \to A$ is a morphism in $\mathfrak{A}_F$ for any $A \in A_F$. Furthermore, it follows that if $h(x)\,g(x) : A \to A$ is an automorphism for some $A \in \mathfrak{A}_F$, then $g(x) : A \to A$ is an automorphism.

Notice that when $\mathfrak{A}$ is the category of right R-modules, $\mathfrak{A}_F$ is the category of right modules over the polynomial ring $R[x]$ reduced modulo the ideal generated by $F(x)$.

Suppose now that $F(x) = G(f(x))$, where $F(x)$ and $G(x)$ have coefficients in $C(\mathfrak{A})$, and $f(x)$ is a monic polynomial given explicitly by

$$f(x) = x^n + \sum_{j=0}^{n-1} c_j x^j,\ c_j \in C(\mathfrak{A}). \tag{1}$$

Then we have a functor $T_{FG} : \mathfrak{A}_F \to \mathfrak{A}_G$ which assigns to the endomorphism $x : A \to A$ the endomorphism $f(x) : A \to A$. We construct a functor $S_{GF} : \mathfrak{A}_G \to \mathfrak{A}_F$ by assigning to the endomorphism $y : A \to A$ the endomorphism $X : A^n \to A^n$ defined by

$$\begin{aligned} Xu_i &= u_{i+1} \qquad \text{for} \quad 0 \leqslant i < n-1 \\ &= u_0 y - \sum_{j=0}^{n-1} c_j u_j \qquad \text{for} \quad i = n-1. \end{aligned}$$

Then $X^i u_0 = u_i$ for $0 \leqslant i \leqslant n-1$, and consequently we can write

$$\begin{aligned} f(X)\,u_0 &= Xu_{n-1} + \sum_{j=0}^{n-1} c_j u_j \\ &= u_0 y. \end{aligned}$$

Then we have for $1 \leqslant i \leqslant n-1$,

$$f(X)u_i = f(X)X^i u_0 = X^i f(X)u_0 = X^i u_0 y = u_i y.$$

It follows that

$$f(X) = \sum_{j=0}^{n-1} u_j y p_j$$

where the p_j are the coproduct projections, and since $G(y) : A \to A$ is zero, we see that $F(X) = G(f(X)) : A^n \to A^n$ is zero. Thus $S_{GF}(A)$ is indeed an object of $\mathcal{A}_F$.

Now consider a morphism $\alpha : A^n \to A'$, where $A \in \mathcal{A}_G$ and $A' \in \mathcal{A}_F$. Such a morphism is determined by its coordinates $\alpha_i = \alpha u_i$, $0 \leqslant i \leqslant n-1$. Composing each side of the relation $x\alpha = \alpha X$ on the right with the u_i, we see that α is a morphism in $\mathcal{A}_F$ if and only if

$$\begin{aligned} \alpha_1 &= x\alpha_0 \\ \alpha_2 &= x\alpha_1 \\ &\vdots \\ \alpha_{n-1} &= x\alpha_{n-2} \\ \alpha_0 y - \sum_{j=0}^{n-1} c_j\alpha_j &= x\alpha_{n-1}. \end{aligned} \tag{2}$$

Given α_0, α_i is thus determined from the first i of these equations by

$$\alpha_i = x^i\alpha_0 . \tag{3}$$

Using (3), we then see that the last of Eq. (2) is satisfied if and only if $\alpha_0 y = f(x)\alpha_0$. In other words, we have shown that the morphisms $\alpha : S_{GF}(A) \to A'$ in $\mathcal{A}_F$ are in one to one correspondence with the morphisms $\alpha_0 : A \to T_{FG}(A')$ in $\mathcal{A}_G$, and it follows that S_{GF} is a coadjoint for T_{FG}.

Now relative to $A \in \mathcal{A}_G$, write

$$p_iX = p_iX \sum_{j=0}^{n-1} u_jp_j .$$

Then we find that

$$\begin{aligned} p_iX &= p_{i-1} - c_i p_{n-1} \quad \text{for} \quad 1 \leqslant i \leqslant n-1 \\ &= yp_{n-1} - c_0 p_{n-1} \quad \text{for} \quad i = 0. \end{aligned} \tag{4}$$

Consider a morphism $\beta : A'' \to A^n$ where $A \in \mathcal{A}_G$ and $A'' \in \mathcal{A}_F$. Such a morphism is determined by its coordinates $\beta_i = p_i\beta$, $0 \leqslant i \leqslant n-1$. Composing each side of $\beta x = X\beta$ on the left with the p_i and using (4), we see that β is a morphism in $\mathcal{A}_F$ if and only if

$$\begin{aligned} \beta_{n-1}x &= \beta_{n-2} - c_{n-1}\beta_{n-1} \\ \beta_{n-2}x &= \beta_{n-3} - c_{n-2}\beta_{n-1} \\ &\vdots \\ \beta_1 x &= \beta_0 - c_1\beta_{n-1} \\ \beta_0 x &= y\beta_{n-1} - c_0\beta_{n-1} . \end{aligned} \tag{5}$$

Given β_{n-1}, β_{n-i} is determined from the first $i-1$ of these equations $(2 \leqslant i \leqslant n)$ by the rule

$$\beta_{n-i} = \beta_{n-1}(x^{i-1} + c_{n-1}x^{i-2} + \cdots + c_{n-i+1}). \tag{6}$$

In particular, setting $i = n$ and composing (6) on the right with x, we see that if the first $n-1$ of Eq. (5) are satisfied, then the last of Eq. (5) is satisfied if and only if $y\beta_{n-1} = \beta_{n-1}f(x)$. In other words, we have shown that the morphisms $\beta : A'' \to S_{GF}(A)$ in $\mathcal{O}\!\ell_F$ are in one to one correspondence with the morphisms $\beta_{n-1} : T_{FG}(A'') \to A$ in $\mathcal{O}\!\ell_G$, and it follows that S_{GF} is also an adjoint for T_{FG}.

Let $A \in \mathcal{O}\!\ell_F$, and consider A as an object of $\mathcal{O}\!\ell_G$ via T_{FG}. Take $A' = A'' = A$ in the above, with $\alpha_0 = \beta_{n-1} = 1_A$. Then writing

$$\alpha\beta = \alpha \sum_{j=0}^{n-1} u_j p_j \beta$$

and using (3) and (6), a straigthforward computation shows that

$$\alpha\beta = f'(x) : A \to A$$

where $f'(x)$ denotes the derivative of $f(x)$. Thus if $f'(x)$ is an isomorphism on A, then A is a retract of $S_{FG}(A)$ in $\mathcal{O}\!\ell_F$.

We remark that the only condition on $\mathcal{O}\!\ell$ needed thus far is that it be an additive category with finite products.

If $G(x) = x$, then $F(x) = f(x)$ and $\mathcal{O}\!\ell_G = \mathcal{O}\!\ell$. In this case we shall write T_f and S_f in place of T_{FG} and S_{GF}. Given $A \in \mathcal{O}\!\ell_f$, consider the sequence

$$0 \to A \xrightarrow{\gamma} A(\Pi) \xrightarrow{\beta} A(\Pi) \xrightarrow{\alpha} A \to 0 \tag{7}$$

where α and β are the unique morphisms in $\mathcal{O}\!\ell_f$ satisfying $\alpha u_0 = 1_A$ and $\beta u_0 = u_1 - u_0 x$ respectively, and

$$\gamma = \sum_{k=0}^{n-1} u_k(x^{n-k-1} + c_{n-1}x^{n-k-2} + \cdots + c_{k+1}).$$

Then it is straightforward to verify that $\alpha\beta = 0$ and $\beta\gamma = 0$, and that γ is a morphism in $\mathcal{O}\!\ell_f$. Furthermore, a contracting homotopy for (7) is given by

$$A \xleftarrow{\rho} A(\Pi) \xleftarrow{\theta} A(\Pi) \xleftarrow{u} A$$

where

$$\begin{aligned} \theta u_k &= \sum_{i=0}^{k-1} u_i x^{k-1-i} \qquad \text{for} \quad k \neq 0 \\ &= 0 \qquad \text{for} \quad k = 0, \\ \rho u_k &= 0 \qquad \text{for} \quad k \neq n-1 \\ &= 1_A \qquad \text{for} \quad k = n-1. \end{aligned}$$

Consequently (7) is an exact sequence in $\mathcal{A}_f$, and we obtain

LEMMA 4.1. *If $f(x)$ is any monic polynomial with coefficients in $C(\mathcal{A})$ and A and C are any objects of $\mathcal{A}_f$ such that* $\operatorname{Ext}_{\mathcal{A}}{}^k(A, C) = 0$ *for* $m \leqslant k \leqslant m + 2$, *then*

$$\operatorname{Ext}_f^{m+2}(A, C) \approx \operatorname{Ext}_f{}^m(A, C).$$

If $A \in \mathcal{A}$, then by Corollary 1.2 we have

$$\text{h.d.}_f\, S_f(A) = \text{h.d.}_{\mathcal{A}}\, A. \tag{8}$$

On the other hand if $A \in \mathcal{A}_f$, then again by Corollary 1.2 we can write

$$\text{h.d. } T_f(A) \leqslant \text{h.d.}_f\, A. \tag{9}$$

If (9) is always an equality, then combining this with (8) we obtain

$$\text{gl.dim. } \mathcal{A}_f = \text{gl.dim. } \mathcal{A}.$$

On the other hand if (9) is a strict inequality for some $A \in \mathcal{A}_f$, then from Lemma 4.1 we see that

$$\text{h.d.}_f\, A = \infty.$$

We are now going to sharpen this result. In the sequel we shall write R in place of $C(\mathcal{A})$.

THEOREM 4.2. *If $f(x)$ is a monic polynomial with coefficients in R, and if there are polynomials $\lambda(x)$ and $\mu(x)$ such that*

$$\lambda(x) f(x) + \mu(x) f'(x) = 1 \tag{10}$$

in $R[x]$, then

$$\text{gl.dim. } \mathcal{A}_f = \text{gl.dim. } \mathcal{A}.$$

Proof. Since $f(x) = 0$ for all A in $\mathcal{A}_f$, it follows from (10) that $f'(x)$ is always an isomorphism. Consequently A is a retract of $S_f(A)$, and so (9) is always an equality.

THEOREM 4.3. *Let $f(x) = g(x)h(x)$ in $R[x]$ where $g(x)$ and $h(x)$ are monic polynomials and degree $g(x) = k > 0$. Let A be a nonzero object of $\mathcal{A}$ for which there is a polynomial $k(x) \in R[x]$ such that $c_A = 0$ for all coefficients c of $h(x) - g(x)k(x)$. Then*

$$\text{h.d.}_f\, S_h(A) = \infty = \text{h.d.}_f\, S_g(A). \tag{11}$$

Proof. The exact sequence of $R[x]$ modules

$$0 \longrightarrow R[x]/(g(x)) \xrightarrow{h(x)} R[x]/(f(x)) \longrightarrow R[x]/(h(x)) \longrightarrow 0$$

splits as a sequence of R-modules. Consequently if we tensor with A over R, we obtain an exact sequence in $\mathfrak{A}_f$

$$0 \to S_g(A) \xrightarrow{\delta} S_f(A) \to S_h(A) \to 0. \tag{12}$$

If we let v_i $(0 \leqslant i \leqslant k-1)$ denote the coproduct injections for $S_g(A)$ and u_i $(0 \leqslant i \leqslant n-1)$ those for $S_f(A)$, the morphism δ is given by

$$\delta v_i = h(X)u_i, \qquad 0 \leqslant i \leqslant k-1.$$

Also we have a morphism $\tau : S_g(A) \to S_f(A)$ in $\mathfrak{A}$ (but not in $\mathfrak{A}_f$) defined by

$$\tau v_i = k(X)u_i, \qquad 0 \leqslant i \leqslant k-1,$$

and by the assumption on A it follows that $g(X)\tau = \delta$. Therefore using Remark 1 following Lemma 1.7 (with $L = S_g$), we obtain

$$\text{h.d.}_f\, S_h(A) \geqslant 1 + \text{h.d.}_{\mathfrak{A}}\, A.$$

The first equality in (11) now follows from Lemma 4.1, and the second equality follows from the first and from the exact sequence (12).

Remark. The tensor product involved in the proof of Theorem 4.3 is the functor $\otimes_R A$ defined on the category of free R-modules with a finite base. In general it is not defined on the category of all R-modules unless $\mathfrak{A}$ has infinite coproducts. If R is not a set, then we can replace it by the subring generated by $g(x)$, $h(x)$, and $k(x)$.

EXAMPLE 1. Let F be a field, and let $\mathfrak{A}$ be the category of modules over F. If $f(x) \in F[x]$ and $f(x)$ is separable and has no repeated factor over F, then by Theorem 4.2 we have

$$\text{gl.dim.}\, F[x]/(f(x)) = 0.$$

On the other hand, if $f(x)$ has a repeated factor over F, then by Theorem 4.3,

$$\text{gl.dim.}\, F[x]/(f(x)) = \infty.$$

This is just a special case of the following general fact: If R is a principal ideal domain and a is not a unit in R, then gl.dim. $R/(a) = \infty$ or 0 depending on whether or not a has a repeated factor in its prime decomposition (see [5, p. 122, Exercise 1]).

EXAMPLE 2. Let $f(x) = x^n$ with $n > 1$. Taking $g(x) = x$ and letting A by any nonzero object of $\mathfrak{A}_f$ with $x = 0$, we have by Theorem 4.3,

$$\text{h.d.}_f\, A = \infty.$$

EXAMPLE 3. Let $f(x) = x^n - 1$. Over the integers we can write

$$(-n)(x^n - 1) + x(nx^{n-1}) = n.$$

Consequently, if n is a unit in $C(\mathfrak{A})$, then by Theorem 4.2 we have

$$\text{gl.dim. } \mathfrak{A}_f = \text{gl.dim. } \mathfrak{A}.$$

On the other hand, if n is not a unit in $C(\mathfrak{A})$, then since we have over the integers

$$x^n - 1 = (x - 1) \sum_{i=0}^{n-1} x^i,$$

and mod n we can write

$$\sum_{i=0}^{n-1} x^i \equiv (x - 1) \sum_{i=1}^{n-1} (n - i)\, x^{i-1},$$

it follows from Theorem 4.3, taking $g(x) = x - 1$, that

$$\text{h.d.}_f\, A = \infty$$

where A is any nonzero object with $n_A = 0$ and $x = 1$. This example also illustrates Proposition 3.4, since $\mathfrak{A}_f$ in this case is the same as $\mathfrak{A}^G$ where G is a cyclic group of order n.

EXAMPLE 4. Suppose that Π is a monoid generated by a single element x. If Π is neither a free monoid nor a free group, let n be the first positive integer such that $x^n = x^{n-d}$ for some d satisfying $0 < d \leqslant n$. Then $\mathfrak{A}^\Pi$ is the same as $\mathfrak{A}_f$ where $f(x) = x^n - x^{n-d}$. If $d = n$, this reduces to Example 3. If $d < n - 1$, then taking $h(x) = x$, we see again by Theorem 4.3 that $\text{h.d.}_\Pi\, A = \infty$ where $A \neq 0$ and $x = 0$. If $d = n - 1$ and $n - 1$ is a unit in $C(\mathfrak{A})$, then since we can write

$$(-n^2x^{n-2})(x^n - x) + (nx^{n-1} - (n - 1))(nx^{n-1} - 1) = n - 1,$$

by Theorem 4.2 we obtain

$$\text{gl.dim. } \mathfrak{A}^\Pi = \text{gl.dim. } \mathfrak{A}.$$

On the other hand, if $n - 1$ is not a unit in $C(\mathfrak{A})$, then we have over the integers

$$x^n - x = (x - 1) \sum_{i=1}^{n-1} x^i,$$

and mod$(n - 1)$ we can write

$$\sum_{i=1}^{n-1} x^i \equiv (x - 1) \sum_{i=2}^{n-1} (n - i)\, x^{i-1}.$$

Therefore taking $g(x) = x - 1$, by Theorem 4.3 we have $\text{h.d.}_\Pi A = \infty$, where A is any nonzero object with $(n-1)_A = 0$ and $x = 1$.

Remark. Theorems 4.2 and 4.3 do not tell the whole story, since Example 1 breaks down in the case where $f(x)$ is not separable. Also if we take $\mathcal{A}$ to be the category of Abelian groups and $f(x) = x^2 + 1$, then the hypothesis of Theorem 4.2 is not satisfied. Nevertheless, since $\mathcal{A}_f$ in this case is just the category of modules over the Gaussian integers, and since the latter is a principal ideal domain, we have

$$\text{gl.dim. } \mathcal{A}_f = 1 = \text{gl.dim. } \mathcal{A}.$$

COROLLARY 4.4. *Let $f(x)$ and A be as in Theorem 4.3, and let Π be the free monoid on a set of generators $\{x_i\}_{i \in I}$. Suppose that $\mathcal{B}$ is a full Abelian subcategory of $\mathcal{A}^\Pi$, and that $s \in I$ is such that $f(x_s) = 0$ for all objects in $\mathcal{B}$. Suppose also that there is a family $c_i \in C(\mathcal{A})$ for $i \neq s$ such that $\mathcal{B}$ contains all objects of $\mathcal{A}^\Pi$ satisfying $x_i = c_i$ for $i \neq s$ and $f(x_s) = 0$. Then considering $S_g(A)$ as an object of $\mathcal{B}$ by taking $x_i = c_i$ for $i \neq s$ and $x_s = X$, we have*

$$\text{h.d.}_\mathcal{B}\, S_g(A) = \infty.$$

Proof. Let $F : \mathcal{B} \to \mathcal{A}_f$ be the functor which forgets all the x_i except x_s, and let $L : \mathcal{A}_f \to \mathcal{B}$ be the functor which extends an endomorphism x_s to a family of endomorphisms x_i by defining $x_i = c_i$ for $i \neq s$. Then F and L are exact functors and the composition FL is the identity functor on $\mathcal{A}_f$. Therefore the result follows from Corollary 1.6 and Theorem 4.3.

EXAMPLE 5. Let $\mathcal{B}$ be the Grassmann category on I generators over $\mathcal{A}$, or in other words the full subcategory of $\mathcal{A}^\Pi$ satisfying

$$x_i^2 = 0, \; x_i x_j + x_j x_i = 0$$

for all $i, j \in I$, Π being the free monoid on I generators. Then taking s to be any member of I with $f(x) = x^2$ and $c_i = 0$ for $i \neq s$, it follows from Corollary 4.4 that

$$\text{h.d.}_\mathcal{B}\, A = \infty$$

where A is any nonzero object of $\mathcal{A}$ with endomorphisms $x_i = 0$ for all $i \in I$.

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