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Upper-embeddable Graphs and Related Topics

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In this paper, we present different results concerning the structure of upper-embeddable graphs. Various characterizations of graphs whose maximum genus satisfies additivity properties will be given.

I. PRELIMINARIES

I.1 — This note constitutes a sequel of the paper entitled "How to determine de maximum genus of a graph," previously appeared [6]. The notations and definitions will be those of [6]. For immediate references, we recall two results established in [6].

THEOREM A. *A connected graph with even (odd) Betti number is upper-embeddable if and only if it contains a cotree all (all but one) of whose components are even.*

FORMULA B. For every connected graph:

$$\gamma_M(G) = \tfrac{1}{2}(\beta(G) - \xi(G))$$

where $\xi(G)$, the Betti deficiency of G, is defined to be:

$$\xi(G) = \min_{C \subset G} \{\xi(C) \mid \xi(C) = \text{number of odd components of a cotree } C \text{ of } G\}$$

I.2 — The purpose of this paper is to examine several consequences of the preceding theorems. In particular, informations on upper-embeddable graphs, additivity of the maximum genus will be given. Many known (and unknown) results of [7], [8], [10], [11] can be deduced in a simple manner.

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II. An Interesting Known Result

In [8] it is stated that $\gamma_M(G) = 0$ if and only G is a cactus whose cycles are vertex disjoint.

This is a direct consequence of Formula B. Indeed, $\gamma_M(G) = 0$ if and only if $\xi(G) = \beta(G)$. In this case, G necessarily contains a cotree whose components are composed of an edge or a loop: G is the required cactus. The converse statement is obvious.

III. Determination of Upper-Embeddable Graphs

The matter is to find graphs with simple combinatorial structure and satisfying, moreover, the sufficient condition of theorem A.

III.1 — Jaeger [3] and Kundu [4], have, independently, proved the following proposition: Every 4-edge-connected graph contains 2 disjoint maximal trees. This implies that every 4-edge-connected graph admits a connected cotree. This is to say, $\xi(G)$ equals 0 or 1 according as $\beta(G)$ is even or odd respectively. Hence

Corollary 1. *Every 4-edge-connected graph is upper embeddable.*

The author has learned of a similar (unpublished) result by Jungerman.

III.2 — Concerning cubic graph, the preceding statement can be improved. Jaeger [3], Payan and Sakarovitch [9] have investigated vertex-induced forests in cubic graphs. For this purpose, they introduced following definitions:

III.2.1 — A subset of the vertex-set of a (cubic) graph is called a *cyclically stable set* if it induces an acyclic subgraph.

III.2.2. — Let G be a cubic graph of order n. G is called *s-maximum*, if it admits a cyclically stable set of order $[(3n - 2)/4]$. It is then proved [3] that: A cubic graph is s-maximum if and only if its vertex-set X contains a subset S such that:

(i) when $n \equiv 2 \pmod 4$: S induces a tree and $X - S$ is stable

(ii) when $n \equiv 0 \pmod 4$: either S induces a tree and $X - S$ contains exactly one edge, or S induces a forest composed of 2 trees and $X - S$ is a stable.

Besides, the theorem:

Every cyclically-4-edge-connected cubic graph is s-maximum is due to Payan and Sakarovitch [9].

By virtue of Lemma 2 (6) we have:

COROLLARY 2. *Every s-maximum cubic graph is upper-embeddable.*

COROLLARY 3. *Every cyclically-4-edge-connected cubic graph is upper-embeddable.*

III.3 — The preceding results, relatively to the connectivity, are best possible. Indeed, Bouchet [2] has discovered an infinite family of non-upper-embeddable graphs whose cyclic connectivity is 3. His result constitutes also a counter-example to a conjecture made in [10].

IV. On Additivity in Maximum Genus

IV.1 — Battle, Harary, Kodama and Youngs [1] have pointed out that the genus of a connected graph is the sum of the genera of its blocks. In general, such an additivity property is not true for maximum genus. In the sequel, necessary and sufficient conditions for maximum genus additivity will be presented.

IV.2 — Let A be an isthmus of a graph G, or, symbolically:

$$G = G_1 + \{A\} + G_2$$

Obviously:

$$\beta(G) = \beta(G_1) + \beta(G_2)$$

Besides, an isthmus never belongs to a cotree. Thus:

$$\xi(G) = \xi(G_1) + \xi(G_2)$$

By virtue of Formula B, we get:

$$\gamma_M(G) = \gamma_M(G_1) + \gamma_M(G_2).$$

This equality constitutes a theorem stated in [8].

This result enables us to study, without loss of generality, bridgeless graphs only, in the sequel.

IV.3 — Before attempting to prove new results, we need a new definition.

DEFINITION. Suppose S is an orientable surface of genus γ upon which a graph G has a 2-cell embedding. An edge A of G is said to be *γ-regular* if there exists an embedding of G into S such that A appears in the boundary of two distinct faces. A vertex x is said to be γ-regular if it is incident with a γ-regular edge. Otherwise x constitutes a *γ-singularity*.

IV.4 — The following statement is useful in the sequel: Let G be a graph whose maximum genus is γ_M. Then, an edge A of G is γ_M-regular if and only if it belongs to some odd component P of a cotree C such that:

(i) $\xi(C) = \xi(G)$

(ii) $P - \{A\}$ is composed of even components.

Rapid justification:

The "if" part results from the proof of Theorem 3 [6].
The "only if" part is obtained by induction on the number of faces.

The following result can be derived directly from Corollary 1:

COROLLARY 4. *Let G be a 4-edge-connected graph whose Betti number $\beta(G)$ is odd. Then every edge of G is $[\beta(G)/2]$-regular.*

IV.5 — Let G be a graph with a cut-point denoted by x_0. We write, symbolically:

$$G = G_1 + \{x_0\} + G_2$$

(*Notation* In this section we replace, for briefness, $\beta(G)$, $\gamma_M(G)$, $\xi(G) \cdots$ by β, γ, $\xi \cdots$ while β_i, γ_i, $\xi_i \cdots$ $(i = 1, 2)$ stand for $\beta(G_i)$, $\gamma_M(G_i)$, $\xi(G_i) \cdots$)
Obviously:

$$\beta = \beta_1 + \beta_2$$

$$\xi_1 + \xi_2 - \xi = 0 \text{ ou } 2.$$

By virtue of Formula B, we get:

$$\gamma_1 + \gamma_2 \leqslant \gamma \leqslant \gamma_1 + \gamma_2 + 1. \qquad (C)$$

The first inequality is proved in [7]. The second inequality constitutes a new result.

From these double inequalities, it can be derived two kinds of characterization theorems. The first kind dealts with embeddability of G related two those of G_1 and G_2. The second involves additivity properties. For this purpose, the study needs to be subdivided into different cases.

IV.6 — In this paragraph, we discuss the relationship between upper embeddable blocks of a graph and the upper embeddability of the graph itself. Different eventualities may occur:

IV.6.1 — *β, β_1, β_2 are even.*
Elearly $\xi = 0 \Leftrightarrow \xi_1 = \xi_2 = 0$.
Formula B yields:

$$\gamma = \gamma_1 + \gamma_2$$

IV.6.2 — *β is even, β_1 and β_2 are odd.*
Using IV.4 and inequalities (*C*), we have:

$$\xi = 0 \Leftrightarrow \begin{cases} \text{(i)} \ \xi_1 = \xi_2 = 1 \\ \text{(ii)} \ x_0 \text{ is } \gamma_i\text{-regular for } G_i (i = 1, 2) \end{cases}$$

IV.6.3 — *β and β_1 are odd, β_2 is even.*
By applications of IV.4 and inequalities (*C*) one finds two subcases:

1) $$\xi = 1 \Leftrightarrow \begin{cases} \text{(i)} \ \xi_1 = 1 \\ \text{(ii)} \ \xi_2 = 0 \end{cases}$$

Formula *B* yields:

$$\gamma = \gamma_1 + \gamma_2$$

2) $$\xi = 1 \Leftrightarrow \begin{cases} \text{(i)} \ \xi_1 = 1 \\ \text{(ii)} \ \xi_2 = 2 \\ \text{(iii)} \ x_0 \text{ is } \gamma_i\text{-regular for } G_i (i = 1, 2) \end{cases}$$

Additivity does not hold:

$$\gamma = \gamma_1 + \gamma_2 + 1$$

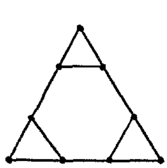
IV.6.4 — In closing, we summarize the above results in:

COROLLARY 5. *$G = G_1 + \{x_0\} + G_2$ is upper-embeddable if and only if the following hold:*

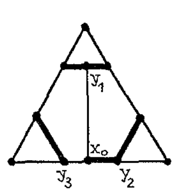
(i) *G_1 is upper embeddable (i.e. $\xi_1 \leqslant 1$)*
(ii) $0 \leqslant \xi_2 \leqslant 2$
(iii) *x_0 is γ_i-regular for G_i when $\xi_1 \cdot \xi_2 \geqslant 1$ ($i = 1, 2$).*

COROLLARY 6. *Let $G = G_1 + \{x_0\} + G_2$ be an upper-embeddable graph. Then $\gamma = \gamma_1 + \gamma_2$ if and only if $0 \leqslant \xi_1 + \xi_2 \leqslant 1$.*

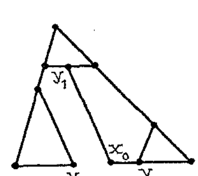
IV.6.5 — *Remark.* In [10], it is conjectured that a bridgeless graph each of whose blocks is upper-embeddable is upper embeddable. Corollary 5 points out that a counter example to this conjecture can be constructed.
Indeed, in [8], it is noted that the graph G^* is not upper-embeddable.



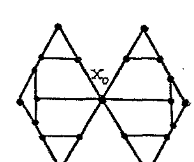
G^*



G_1



$\tilde{G}$



G

On the contrary, the graph G_1 is upper embeddable with 2 faces (precisely, the heavy lines constitute a cotree with exactly one odd component). Its maximum genus is 2. We then proved that x_0 is a 2-singularity, i.e. the edges (x_0, y_i) $(i = 1, 2, 3)$ are 2-singular. Indeed, the regularity of (x_0, y_1) implies that $G_1 - \{(x_0, y_1)\} \simeq G^*$ would be upper-embeddable.

In the same manner, regularity of (x_0, y_2) (or (x_0, y_3)) implies that $G_1 - \{(x_0, y_2)\} \simeq \tilde{G}$ would be upper embeddable. This is impossible by IV.2.

Finally, the coalescence, by means of x_0, of two copies of G_1 provides the required counter-example.

IV.7 — In this paragraph we treat the relation between $\gamma, \gamma_1, \gamma_2$.

Applications of Theorem A, Formula B, inequalities (C), and IV.4 yield the following corollaries.

COROLLARY 7. *Let G_1, G_2 be two graphs such that:*

$$\xi_1 \cdot \xi_2 > 0$$

Then

$$\gamma = \gamma_1 + \gamma_2$$

if and only if x_0 is γ_i-singular $(i = 1, 2)$ for at least one value of i.

COROLLARY 8. *Let G_1, G_2 be two graphs such that:*

$$\xi_1 = 0$$

Then

$$\gamma = \gamma_1 + \gamma_2$$

V. CONCLUDING REMARK

We make no claims for the completeness of the above list of consequences. In particular, the exploration of graphs satisfying the sufficient condition of theorem A seems well worth further refinement.

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REFERENCES

1. J. BATTLE, F. HARARY, Y. KODAMA, AND J. W. T. YOUNGS, Additivity of the genus of a graph, *Bull. Amer. Math. Soc.* **68** (1962), 565–568.

2. A. Bouchet, Genre maximum d'un Δ-graphe, Colloque international CNRS – Orsay, 1976.
3. F. Jaeger, Thesis, Université de Grenoble, June 1976.
4. S. Kundu, Bounds on the number of disjoint spanning trees, *J. Combinatorial Theory Ser. B.* **17** (1974), 199–203.
5. Nguyen huy Xuong, How to determine the maximum genus of a graph, *J. Combinatorial Theory Ser. B.* **26** (1979).
6. E. A. Nordhaus, B. M. Stewart, and A. T. White, On the maximum genus of a graph, *J. Combinatorial Theory Ser. B.* **11** (1971), 258–267.
7. E. A. Nordhaus, R. D. Ringeisen, B. M. Stewart, and A. T. White, A Kuratowski-type theorem for the maximum genus of a graph, *J. Combinatorial Theory Ser. B.* **12** (1972), 260–267.
8. C. Payan and M. Sakarovitch, Ensembles cycliquement stables et graphes cubiques, *in* "Actes des journées franco-belges sur la théorie des graphes, 1974."
9. R. D. Ringeisen, Upper and lower imbeddable graphs, *in* "Graph Theory and Applications," Springer-Verlag, New York/Berlin, 1972.
10. R. D. Ringeisen, On additivity in maximum genus, Colloque international CNRS – Orsay, 1976.