Compactness and Collective Compactness in Spaces of Compact Operators

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This is a study of compactness in (a) spaces $K_b(X, Y)$ of compact linear operators, (b) injective tensor products $X \hat{\otimes}_e Y$, and (c) spaces $L_c(X, Y)$ of continuous linear operators, and its various relationships with equicontinuity and collective compactness. Among the applications is a result on factoring compact sets of compact operators compactly and uniformly through one and the same reflexive Banach space.

0.1. INTRODUCTION

The object of this paper is a study of compactness in various spaces of compact operators. We characterize compact subsets of (a) spaces $K_b(X, Y)$ of compact linear operators, (b) injective tensor products $X \hat{\otimes}_e Y$, and (c) spaces $L_c(X, Y)$ of continuous linear operators from $X$ into $Y$, with the topology of uniform convergence on the compact convex circled subsets of $X$ ($X$ and $Y$ locally convex spaces). Special emphasis is put on the various relationships among (i) equicontinuity, (ii) collective compactness, and (iii) compactness of sets of compact operators. A typical result in this direction is Palmer’s [27] which states that a subset $H$ of $K(X, Y)$, $X$ and $Y$ Banach, is relatively compact (in the operator norm) if and only if both $H(B_X)$ and $H'(B_{Y'})$ are relatively compact in $Y$ and $X'$, respectively. This and related results are placed in the general context of the operator space $L_c(X', Y)$ of weak*-weakly continuous linear operators from $X'$ into $Y$ which transform equicontinuous subsets of $X'$ into relatively compact subsets of $Y$, endowed with the topology of uniform convergence on the equicontinuous sets in $X'$ ($X$ and $Y$ locally convex spaces). This operator space has been introduced by Schwartz [32] as the so-called $\varepsilon$-product $X \varepsilon Y$ of $X$ and $Y$. Besides spaces of vector-valued continuous functions and vector-valued distributions, also the operator spaces $K_b(X, Y)$, $X \hat{\otimes}_e Y$, and $L_c(X, Y)$ can be represented as (linear subspaces of) a suitable $\varepsilon$-product. In this way, the general compactness results on $L_c(X', Y)$ provide a unified approach to compactness criteria for
any of the above mentioned operator spaces. This program is carried out in Section 1.

Section 2 is mainly concerned with applications of the results of Section 1 to compactness criteria in spaces $L_b(X, Y)$ of continuous linear operators from $X$ into $Y$ for the special case that $X$ (resp. $Y$) is Fréchet and $Y$ (resp. $X$) a DF space. Particular results are that a subset $H$ of $L(X, Y)$ is relatively compact in

(i) $L_c(X, Y)$ ($X$ Fréchet, $Y$ Schwartz DF), respectively in

(ii) $L_b(X, Y)$ ($X$ Schwartz DF, $Y$ Fréchet)

if and only if there exists a zero neighbourhood $U$ in $X$ such that $H(U)$ is relatively compact in $Y$ (Theorems 2.1 and 2.5). Several special cases are discussed.

In Section 3 it is shown how the results of Section 1 can be used to factor compact sets of compact operators compactly and uniformly through one and the same reflexive Banach space.

The results of this paper are based on part of Chapter IV of the author’s Habilitationsschrift [28].

0.2 TERMINOLOGY AND NOTATION

Generally, the notation and terminology are that of Horváth’s book [24], with the following exceptions: given a locally convex space $X$ (always assumed to be Hausdorff), $X'_e$, $X'_j$, and $X'_b$ denote the topological dual space of $X$ with the topology of uniform convergence on all compact convex circled, all precompact, and all bounded subsets of $X$, respectively. $X_3$, $X_e$, and $X''$ denote the spaces $X$ and $X'' = (X'_b)'$ with the topology of uniform convergence on the corresponding subsets of $X'_b$. $\mathcal{N}_X$ will denote a zero neighbourhood base in $X$. Given a bounded disk $B$ in $X$ (a disk is a convex circled set), we denote by $(X_B, B)$ the linear span of $B$ in $X$, endowed with the norm with unit ball $B$. $B$ is called completing whenever $(X_B, B)$ is a Banach space. $X$ is said to be Mackey complete if every closed bounded disk $B$ in $X$ is completing. (Note that every sequentially complete locally convex space is Mackey complete.)

Special classes of spaces. A locally convex space $X$ is called a generalized DF space (gDF) if (i) its strong dual is Fréchet, and (ii) linear operators into other locally convex spaces are continuous as soon as their restrictions to the bounded sets are [28, 29]. Besides their classical ancestors, and thus all normed spaces and strong duals of Fréchet spaces, this class includes Mackey duals and $c$-duals $Z'_c$ of Fréchet spaces $Z$, as well as all function spaces with any of the extensions of Buck’s [9, 10] strict topology.
β, cf. [29, 30]. X is called quasinormable [20] if for every equicontinuous subset \( H \) of \( X' \) there exists a zero neighbourhood \( U \) in \( X \) such that on \( H \), the strong dual topology and the topology of uniform convergence on \( U \) coincide. Every gDF space is quasinormable [30]. \( X \) is said to fulfill the countable neighbourhood condition (cnc) if for every sequence \( (U_n)_{n \in \mathbb{N}} \) of zero neighbourhoods in \( X \) there exists a sequence \( (\alpha_n)_{n \in \mathbb{N}} \) of scalars \( \alpha_n > 0 \) such that \( U = \bigcap \{ \alpha_n U_n \mid n \in \mathbb{N} \} \) again is a zero neighbourhood in \( X \). All gDF spaces (and all linear subspaces thereof) fulfill (cnc) [30, Proposition 3.1].

Spaces of linear operators. \( L_s(X, Y) \), \( L_c(X, Y) \), \( L_s(X, Y) \), and \( L_b(X, Y) \) denote the space \( L(X, Y) \) of continuous linear operators from \( X \) into \( Y \) (\( X \) and \( Y \) locally convex), endowed with the topology of uniform convergence on all finite, all compact convex circled, all precompact, and all bounded subsets of \( X \), respectively.

\( K(X, Y) \) is the space of compact linear operators from \( X \) into \( Y \) (transforming a certain zero neighbourhood of \( X \) into a relatively compact subset of \( Y \)).

\( K^p(X, Y) \) is the space of all weakly continuous linear operators from \( X \) into \( Y \) which transform bounded sets into relatively compact sets. Recall from [31, Theorem 3.1] that \( K^p_s(X, Y) = K^p_s(X, Y) \) whenever \( X \) is gDF and \( Y \) is a Fréchet space.

\( L_s(X'_c, Y) = XcY \) is the space \( L(X'_c, Y) \) endowed with the topology of uniform convergence on the equicontinuous subsets of \( X' \). Given locally convex spaces \( X \) and \( Y \), \( B_{bc}(X, Y) \) and \( B_{cc}(X, Y) \) will denote the space \( B(X, Y) \) of continuous bilinear forms on \( X \times Y \) with the topology of uniform convergence on products \( M \times N \) of all bounded and of all compact subsets \( M \) and \( N \) of \( X \) and \( Y \), respectively.

For normed spaces \( X \) and \( Y \), a subset \( H \) of \( L(X, Y) \) is said to be collectively compact (respectively precompact) [3] if \( H(B_X) \) is relatively compact (respectively precompact) in \( Y \) (\( B_X = \) unit ball in \( X \)).

1. Collective Compactness and Compactness of Sets of Compact Operators

The concept of collectively precompact sets of linear operators was introduced by Anselone and Moore [3] in connection with approximate solutions of integral and operator equations. Anselone and Palmer [4-6] developed a detailed spectral approximation theory for compact operators \( h \). \( (h_n)_{n \in \mathbb{N}}, \) on a Banach space such that \( (h_n)_{n \in \mathbb{N}} \) converges strongly (i.e., in the strong operator topology) to \( h \), and the set \( \{h_n - h\}_{n \in \mathbb{N}} \) is collectively compact. (For a detailed exposition consult [2].) Since the analysis
simplifies considerably if \((h_n)_{n \in \mathbb{N}}\) converges to \(h\) in the operator norm, and since this type of convergence is equivalent to \((h_n)_{n \in \mathbb{N}}\) converging strongly to \(h\) and \(\{ h_n - h | n \in \mathbb{N} \}\) being a precompact subset of \(K_b(X)\), one was led to compare precompactness and collective precompactness for sets of compact operators. Anselone [1] conjectured that a subset \(H\) of \(K(X, Y)\), \(X\) and \(Y\) normed spaces, is precompact (in the operator norm) if both \(H\) and \(H' \subset K(Y', X')\) are collectively precompact. Positive answers were given by Anselone [1] and Anselone and Palmer [4] for special cases, and for the case of general normed spaces by Palmer [27]. Finally, Geue [16] extended the corresponding results to locally convex spaces \(X\) and \(Y\) such that \(Y\) is evaluable, and in [14] it was extended to the setting of just any locally convex spaces. The methods of proof are rather different and, mostly, quite involved. As a starting point for our results on compactness and collective compactness of subsets \(H \subset K(X, Y)\), we give now a simple proof for a result which contains all those just mentioned as special cases.

1.1 THEOREM. Let \(X\) and \(Y\) be locally convex spaces, and let \(H\) be a subset of \(L(X, Y)\) consisting of semi-precompact operators (\(h_B\) is precompact in \(Y\) for all \(h \in H\) and all \(B\) bounded in \(X\)). Then the following are equivalent:

(a) \(H\) is precompact in \(L_b(X, Y)\).

(b) (i) \(H(x)\) is precompact in \(Y\) for all \(x \in X\), and
(ii) \(H'(V^0)\) is precompact in \(X'_b\) for all zero neighbourhoods \(V\) in \(Y\).

(c) (i) \(H(B)\) is precompact in \(Y\) for all \(B\) bounded in \(X\), and
(ii) \(H'(y')\) is precompact in \(X'_b\) for all \(y' \in Y'\).

(d) (i) \(H(B)\) is precompact in \(Y\) for all \(B\) bounded in \(X\), and
(ii) \(H'(V^0)\) is precompact in \(X'_b\) for all zero neighbourhoods \(V\) in \(Y\).

Proof. Trivially, (a) implies (d) (note that \(h_B\) is precompact in \(Y\) for all \(B\) bounded in \(X\), and that \(h'(V^0)\) is precompact in \(X'_b\) for all zero neighbourhoods \(V\) in \(Y\)). Statement (d) implies any of (b) and (c). Part (b) means that \(H(x)\) is precompact in \(Y\) for all \(x \in X\), and that \(H\) is an equicontinuous subset of \(L(X, Y)\). Hence, according to a well-known version of the Arzela–Ascoli Theorem [21, 0.7, Corollary 2, p. 17], (b) implies (a) (note that the bounded subsets of \(X\) are \(\lambda\)-precompact). In just the same way, (c) implies that \(H'\) is precompact in \(L_e(Y', X'_b)\) which, by plain polarity techniques, is equivalent to \(H\) being precompact in \(L_b(X, Y)\). This completes the proof.

We now place this result in the setting of the space \(L_e(X'_c, Y)\) of weak*-weakly continuous linear operators from \(X'\) into \(Y\) which transform equicontinuous subsets of \(X'\) into relatively compact subsets of \(Y\), endowed with the
topology of uniform convergence on the equicontinuous sets in $X'$. This operator space has been introduced by Schwartz [32] as the $\varepsilon$-product $X\varepsilon Y$ of locally convex spaces $X$ and $Y$, in order to investigate spaces of vector-valued functions and of vector-valued distributions. Various other spaces of analysis can be represented as (linear subspaces of) a suitable $\varepsilon$-product. The following are the most common examples.

1.2. EXAMPLE. Spaces of compact operators

$$K_b(X, Y) \subseteq L_e(X, Y) \cong L_e(X''_c, Y) \cong (X'_b) \varepsilon Y,$$

$h \mapsto h''$.

($X, Y$ locally convex, $Y$ quasi-complete.)

1.3. EXAMPLE. Injective tensor products

$$X \hat{\otimes}_\varepsilon Y \subseteq L_e(X'_e, Y) \cong X\varepsilon Y,$$

$$x \otimes y \mapsto [(x', x) \mapsto (x, x') y].$$

($X$ and $Y$ complete locally convex spaces.)

1.4. EXAMPLE. Spaces of vector-valued continuous functions

$$C(T, X)_{co} \cong L_e(X'_c, C(T)_{co}) \cong (C(T)_{co}) \varepsilon X,$$

$$F \mapsto \{x' \mapsto x' \circ F\}.$$

($T$ completely regular Hausdorff, $X$ quasi-complete locally convex, $C(T, X)_{co}$ continuous $X$-valued functions on $T$, with the compact-open topology.)

For a more detailed discussion of these examples, see Section 1 of [12].

1.5. THEOREM. Let $X$ and $Y$ be locally convex spaces. For a subset $H$ of $L(X'_e, Y)$, the following are equivalent.

(a) $H$ is precompact in $L_e(X'_e, Y)$.
(b) (i) $H(U^0)$ is precompact in $Y$ for all zero neighbourhoods $U$ in $X$.
   (ii) $H'(V^0)$ is precompact in $X$ for all zero neighbourhoods $V$ in $Y$.
(c) (i) $H(x')$ is precompact in $Y$ for all $x' \in X'$, and
   (ii) $H'(V^0)$ is precompact in $X$ for all zero neighbourhoods $V$ in $Y$.
(d) (i) $H(U^0)$ is precompact in $Y$ for all zero neighbourhoods $U$ in $X$. 

and

(ii) \( H'(y') \) is precompact in \( X \) for all \( y' \in Y' \).

If, in addition to the assumptions, \( X \) and \( Y \) are supposed to be quasicomplete, then in any of the above equivalent conditions, the term “precompact” can be replaced by “relatively compact.”

(The equivalence of conditions (a) and (b) is a result of Schwartz [32, Sect. 1. Proposition 2, p. 22].)

1.6. Corollary. Whenever \( X \) and \( Y \) are complete locally convex spaces, then a subset \( H \) of \( X \overset{\Delta}{\rightarrow} Y \) is relatively compact in \( X \overset{\Delta}{\rightarrow} Y \) if and only if any of the equivalent conditions (b)–(d) of Theorem 1.5 holds.

This is a consequence of Theorem 1.5 and the topological linear embedding \( X \overset{\Delta}{\rightarrow} Y \overset{\sim}{\rightarrow} L_c(X'_c, Y) \) of Example 1.3. For the special case of Banach spaces \( X \) and \( Y \), the equivalence of \( H \) being relatively compact in \( X \overset{\Delta}{\rightarrow} Y \) with \( H(U^0) \) \((U \in \mathcal{U})\) and \( H'(V^0) \) \((V \in \mathcal{V})\) being relatively compact in \( Y \) and \( X \), respectively, has been proved by Holub [23, Theorem 1].

Proof of Theorem 1.5. (a) implies (b). If \( H \) is a precompact subset of \( L_c(X'_c, Y) \), and \( U \) and \( V \) are zero neighbourhoods in \( X \) and \( Y \), respectively, then there exist \( h_1, \ldots, h_n \subset H \) such that

\[
H \subseteq \bigcup \{ h_i + W(U^0, V) \mid 1 \leq i \leq n \},
\]

where \( W(U^0, V) = \{ u \in L(X'_c, Y) \mid u(U^0) \subseteq V \} \). Hence, we have:

\[
H(U^0) \subseteq \bigcup \{ h_i(U^0) + V \mid 1 \leq i \leq n \},
\]

and so \( H(U^0) \) is precompact in \( Y \), for such is the set \( \bigcup \{ h_i(U^0) \mid 1 \leq i \leq n \} \). The second condition of (b) now follows by general duality: \( L_c(X'_c, Y) \) is topologically isomorphic to \( L_c(Y'_c, X) \) \((u \mapsto u')\), so that \( H \) is precompact in \( L_c(X'_c, Y) \) if and only if the same is true for \( H' \) as a subset of \( L_c(Y'_c, X) \).

(c) implies (a). By taking adjoints, the second condition in (c) translates into \( H \) being equicontinuous from \( X'_c \) into \( Y \). Thus, according to the Arzela–Ascoli Theorem [21, 0.7, Corollary 2, p. 17], both conditions in (c) together imply that \( H \) is a precompact subset of \( L_c(X'_c, Y) \). At this point, we see that conditions (a)–(c) are equivalent. Condition (d) is equivalent to any of those, for, according to what we just proved, (d) is equivalent to \( H' \) being precompact in \( L_c(Y'_c, X) \). An appeal to the general duality argument, given at the end of the proof that (b) is being implied by (a), now completes the proof.

1.7 Theorem. Whenever \( X \) and \( Y \) are metrizable locally convex spaces, then for a subset \( H \) of \( L(X'_c, Y) \) the following can be added to the list of equivalent conditions of Theorem 1.5 (and, accordingly, to that of Corollary 1.6):
(e) There exists a precompact subset $K$ of $X$ such that $H(K^0)$ is precompact in $Y$.

(f) (i) $H(x')$ is precompact in $Y$ for all $x' \in X'$, and

(ii) there exists a precompact subset $K$ of $X$ such that $H(K^0)$ is bounded in $Y$.

A form of condition (e) for $X$ and $Y$ Fréchet spaces is to be found in [25, (7)].

Proof for part (e). Clearly, (e) implies (b) of Theorem 1.5. Conversely, the second condition of part (b) of Theorem 1.5 translates into $H$ being equicontinuous from $X'_{\alpha}$ into $Y$. $X'_{\alpha}$ is a $gDF$ space. Hence, according to Theorem 2.2 of [31], both conditions of (b) together imply (e).

For a proof for part (f), we need the following general result which is of independent interest.

1.8. Proposition. Let $X$ and $Y$ be locally convex spaces. Whenever

(a) $X$ is metrizable, and $Y$ has a fundamental sequence of bounded sets, and either $X$ is barrelled, or $Y$ is Mackey complete, or $Y$ is sequentially evaluable (strong nullsequences in $Y'$ are equicontinuous), or

(b) $X$ fulfills the countable neighbourhood condition (particularly, if $X$ is a linear subspace of a $gDF$ space), and $Y$ is metrizable, then a subset $H$ of $L(X, Y)$ is equicontinuous if and only if there exists a zero neighbourhood $U$ in $X$ such that $H(U)$ is bounded in $Y$.

Proof. (a) Assume that $X$ is barrelled. Let $H$ be an equicontinuous subset of $L(X, Y)$, and denote by $\tilde{H}$ the associated bilinear forms $\tilde{H} : X \times Y' \to K$, $\tilde{H}(x, y') := (hx, y')$. $\tilde{H}(\cdot, y')$ is a pointwise bounded hence equicontinuous ($X$ is barrelled) subset of $X''$ for all $y' \in Y'$. and $\tilde{H}(x, \cdot)$ is an equicontinuous subset of $Y''$ as well: $H(x)$ is bounded in $Y$, and $|\tilde{H}(x, (H(x))^0)| \leq 1$. According to [34, Theorem 34.1, p. 352], $\tilde{H}$ is an equicontinuous subset of $B(X, Y''_\alpha)$. This translates into the desired assertion for $H$.

It is easy to reduce the case that $Y$ is Mackey complete to the above case, by means of the following trivial observation: whenever $X$ and $Y$ are locally convex spaces such that $X$ is metrizable and $Y$ is Mackey complete, then the continuous linear extension of any $h \in L(X, Y)$ to the completion of $X$ still maps into $Y$.

Finally, the case that $Y$ is sequentially evaluable is reduced to the Mackey complete case by the fact that under this assumption, the completion of $Y$ again has a fundamental sequence of bounded sets [30, Corollary 2.4].

Part (b) of Proposition 1.8 is an immediate consequence of the countable neighbourhood condition for $X$. 
Before applying the (pre)compactness criteria for subsets of \( L_e(X'_c, Y) \) to several particular cases, we note at this point a further consequence of Proposition 1.8, namely, the following characterization of bounded subsets of \( L_e(X'_c, Y) \).

1.9. Proposition. Let \( X \) and \( Y \) be locally convex spaces, and \( H \) a subset of \( L(X'_c, Y) \).

(a) The following are equivalent:

(i) \( H \) is bounded in \( L_e(X'_c, Y) \).

(ii) \( H(U^0) \) is bounded in \( Y \) for all zero neighbourhoods \( U \) in \( X \).

(iii) \( H \) is an equicontinuous subset of \( L(X'_b, Y) \).

(iv) \( H'(V^0) \) is bounded in \( X \) for all zero neighbourhoods \( V \) in \( Y \).

(b) Whenever

(i) \( X \) and \( Y \) are metrizable, or

(ii) \( X \) is Fréchet and \( Y \) has a fundamental sequence of bounded sets,

then \( H \) is bounded in \( L_e(X'_c, Y) \) if and only if there exists a bounded subset \( R \) of \( X \) such that \( H(R^0) \) is bounded in \( Y \).

Proof. (a) The equivalence of (i) and (ii) is a mere formality. Taking adjoints, the equivalence of (iii) and (iv) comes out to be a formality as well. Finally, according to the topological linear isomorphism \( L_e(X'_c, Y) \cong L_e(Y'_c, X) \), the set \( H \) is bounded in \( L_e(X'_c, Y) \) if and only if the set \( H' \) of adjoints is bounded in \( L_e(Y'_c, X) \), which, according to (ii), is equivalent to (iv). This completes the proof of part (a).

(b) A combination of Proposition 1.8 with condition (iii) of part (a) establishes part (b).

Using the topological isomorphism given in Example 1.2 above, Theorems 1.5 and 1.7 easily translate into (pre)compactness criteria for sets of compact operators and, more generally, for subsets of \( K_b(X, Y) \). The general case is left to the interested reader. Here, we only consider the special case of \( X \) being \( gDF \) and \( Y \) being Fréchet.

1.10. Theorem. Let \( X \) and \( Y \) be locally convex spaces such that \( X'_b \) and \( Y \) are Fréchet, and such that every nullsequence in \( X'_b \) is equicontinuous. (Both conditions on \( X \) are fulfilled if \( X \) is a \( gDF \) space.) Let \( (B_n)_{n \in \mathbb{N}} \) be an (increasing) fundamental sequence of bounded sets in \( X \), and \( (V_n)_{n \in \mathbb{N}} \), a (decreasing) zero neighbourhood base in \( Y \), all \( B_n \)'s and \( V_n \)'s closed disks.

For a subset \( H \) of \( K(X, Y) \), the following are equivalent:

(a) \( H \) is relatively compact in \( K_b(X, Y) \).
(b) (i) \( H(B_n) \) is relatively compact in \( Y \) for all \( n \in \mathbb{N} \), and
(ii) \( H'(V^n) \) is relatively compact in \( X'_b \) for all \( n \in \mathbb{N} \).

(c) (i) \( H(B_n) \) is relatively compact in \( Y \) for all \( n \in \mathbb{N} \), and
(ii) \( H'(y') \) is relatively compact in \( X'_b \) for all \( y' \in Y' \).

(d) (i) \( H(x) \) is relatively compact in \( Y \) for all \( x \in X \), and
(ii) \( H'(V^n) \) is relatively compact in \( X'_b \) for all \( n \in \mathbb{N} \).

(e) (i) \( H(x) \) is relatively compact in \( Y \) for all \( x \in X \), and
(ii) there exists a compact subset \( K \) of \( X'_b \) such that \( H(K^0) \) is bounded in \( Y \).

(f) There exists a compact subset \( K \) of \( X'_b \) such that \( H(K^0) \) is relatively compact in \( Y \).

More specifically:

(g) For any sequences \( 1 \leq a_n, \beta_n \leq 1 \), and any non-increasing sequence \( 0 < \beta_n \leq 1 \), the set \( U = \bigcap \{ a_n B_n + \beta_n H^{-1}(V_n) \mid n \in \mathbb{N} \} \) is a zero neighbourhood in \( X \), and its polar in \( X' \) is even compact in \( X'' \), and \( H(U) \) is relatively compact in \( Y \). In case the \( B_n \)'s have the property that for any \( m, n \in \mathbb{N} \) there exists \( k \in \mathbb{N} \) such that \( B_m + B_n \subseteq B_k \), then the sequence \( a_n \) can be chosen to be just any non-decreasing sequence \( a_n \geq 1 \).

Proof. We first note the following topological linear isomorphisms:
\( K_b(X, Y) \cong K_b^h(X, Y) \cong L_b(X''_b, Y) \). The second one is true in general for \( Y \) quasi-complete (Example 1.2), the first one is a consequence of Theorem 3.1(a) of [31].

Theorem 1.5 now tells us that propositions (a)–(c) of the result in discussion were equivalent if in the first conditions of (b) and (c) the term \( "H(B_n)" \) were replaced by \( "H''(B^n_0)" \). \( B^n_0 \) the bipolar of \( B_n \) in \( X'' = (X_b')' \). But, under the given assumptions, it is easy to see that \( H(B_n) \) is relatively compact in \( Y \) if and only if this is true for \( H''(B^n_0) \). (Note that \( B^n_0 \) is the closure of \( B_n \) in \( (X_b')' \), and that the \( h'' \)'s are \( c \)-continuous.) Hence, (a)–(c) are equivalent. Part (d) is equivalent to (e) by Proposition 1.8. Clearly, (b) implies (d). We now show that (d) implies (a): Again using Theorem 3.1(a) of [31], we first note that \( K_b(X, Y) \cong L_b(X_c, Y) \), where \( X_c \) is \( X \) viewed as a topological linear subspace of \( X'_c \). Since the second condition in (d) translates into \( H \) being equicontinuous from \( X_c \) into \( Y \), we can again use the Arzela–Ascoli Theorem to conclude that (d) implies that \( H \) is a precompact (hence relatively compact) subset of \( L_b(X_c, Y) \cong K_b(X, Y) \) (the bounded subsets of \( X \) are \( c \)-precompact). Clearly, (g) implies (f), and (f) implies (b).

We finally show that (b) implies (g): First note that, under the assumptions on \( X'_b \), the \( c \)-topology on \( X \) is coarser than the original topology of \( X \), and that \( X_c \) is a \( gDF \) space by [31, Proposition 2.6]. Together with the obser-
vation, that the second condition in (b) translates (by taking adjoints) into $H$ being equicontinuous from $X_c$ into $Y$, the gDF property of $X_c$ implies that the set $U$ specified in (g) in fact is a zero neighbourhood in $X_c$. In particular, its polar in $X'$ is strongly compact. Finally, the first condition in (b) implies that $H(U)$ is precompact (hence, relatively compact) in $Y$. This completes the proof of Theorem 1.10.

The case of normed spaces $X$ and $Y$ is noted separately.

1.11. Theorem. Let $X$ and $Y$ be normed spaces. For a subset $H$ of $K(X, Y)$, the following are equivalent:

(a) $H$ is precompact in $K_b(X, Y)$.

(b) (i) $H(B_X)$ is precompact in $Y$, and
(ii) $H'(B_{Y'})$ is relatively compact in $X'$.

(c) (i) $H(B_X)$ is precompact in $Y$, and
(ii) $H'(y')$ is relatively compact in $X'$ for all $y' \in Y'$.

(d) (i) $H(x)$ is precompact in $Y$ for all $x \in X$, and
(ii) $H'(B_{Y'})$ is relatively compact in $X'$.

(e) There exists a compact subset $K$ of $X'$ such that $H(K_w)$ is precompact in $Y$.

More specifically:

(f) For any sequences $1 \leq \alpha_n \uparrow \infty$, and $1 \geq \beta_n \downarrow 0$, the polar of the set $U = \bigcap \{\alpha_n B_X + \beta_n H^{(n)}(B_{Y'}), n \in \mathbb{N}\}$ is compact in $X'$, and $H(U)$ is precompact in $Y$.

In case $Y$ is a Banach space, the term "precompact" can everywhere be replaced by "relatively compact."

Palmer [27, Theorems 2.1, 2.2 and 3.1] proved the equivalence of conditions (a)–(d), and Holub [23, Corollary of Theorem 1, p. 400] the equivalence of conditions (a) and (b) under the assumption that either of $X'$ and $Y$ has the approximation property.

However, the range of applicability of Theorem 1.10 goes far beyond the range of Banach spaces. Admissible domain spaces $X$ are all DF spaces in the sense of [20], and, more generally, all gDF spaces in the sense of [28, 29], in particular, all $c_\ast$, Mackey-, or strong duals of Fréchet spaces, as well as all function spaces with a strict topology: $C_b(S)_{\beta}$, $S$ locally compact Hausdorff, and $H^\infty(G)_\lambda$, $G$ a plane region (Buck [9–11]), or $C_b(T)_\xi$, $T$ completely regular Hausdorff, $\xi$ any of the substrict ($\beta_0$), strict ($\beta$), or superscript ($\beta_1$) topologies of Sentilles [33] and Fremlin et al. [15] (compare [29]).

Clearly, in any particular case, i.e., for any particular choice of domain
and range spaces $X$ and $Y$, the general criteria of Theorem 1.10 are to be translated into the language of $X$ and $Y$, and to be specified in the terms characteristic for these given spaces. In the context of compact range vector measures, this is being done in a joint publication with Graves [18]: according to results in [17] and [31], we have the following topological linear isomorphism:

$$c\mathfrak{s}(\mathcal{X}, X) \cong K_b(\mathcal{X}, X),$$

$$\Phi \mapsto \{ F \mapsto \int F d\Phi \}. $$

(Here, $c\mathfrak{s}(\mathcal{X}, X)$ denotes the space of all strongly countably additive $X$-valued measures with relatively compact range, endowed with the topology of uniform convergence on the algebra $\mathcal{X}$ of subsets of some non-empty set $\Omega$. $X$ is a Frechét space, and $\mathcal{X}(\mathcal{X})$ denotes the space of all $\mathcal{X}$-simple functions on $\Omega$, endowed with the topology of uniform convergence on the variation norm compact subsets of the space of all strongly countably additive $\mathbb{P}$-valued measures on $\mathcal{X}$.) This isomorphism allows us to translate Theorem 1.10 into compactness criteria for sets of compact range vector measures.

A further special case of Theorem 1.5 can be read from the topological linear embedding of the space $L_c(X, Y)$ into the $\varepsilon$-product $(X')_c \varepsilon Y$:

$$L_c(X, Y) \xrightarrow{\sim} L_c((X')_c, Y) \cong (X')_c \varepsilon Y$$

$$h \mapsto h$$

(For a detailed discussion, consult [32, Sect. 1, Corollary p. 36].)

1.12. **Theorem.** Let $X$ and $Y$ be locally convex spaces. For a subset $H$ of $L(X, Y)$, the following are equivalent:

(a) $H$ is precompact in $L_c(X, Y)$.

(b) (i) $H(K)$ is precompact in $Y$ for all compact disks $K$ in $X$, and
   (ii) $H'(V^0)$ is precompact in $X'_c$ for all zero neighbourhoods $V$ in $Y$.

(c) (i) $H(x)$ is precompact in $Y$ for all $x \in X$, and
   (ii) $H'(V^0)$ is precompact in $X'_c$ for all zero neighbourhoods $V$ in $Y$.

(d) (i) $H(K)$ is precompact in $Y$ for all compact disks $K$ in $X$, and
   (ii) $H'(y')$ is precompact in $X'_c$ for all $y' \in Y'$.

1.13. **Corollary.** For a barrelled locally convex space $X$, and a quasi-complete locally convex space $Y$, the space $L_c(X, Y)$ is semi-Montel if and only if $Y$ is semi-Montel.
Proof. First, observe that under the given assumptions, \( X \cong (X'_e)'_e \) and \( L_e(X, Y) \cong L_e((X'_e)'_e, Y) \) [32, Sect. 1, Corollary 36]. Let \( Y \) be semi-Montel, and \( H \) a bounded subset of \( L_e(X, Y) \cong L_e((X'_e)'_e, Y) \). Then \( H(K) \) (\( K \) a compact disk in \( X \)) is bounded, hence relatively compact in \( Y \). Moreover, \( H' \) is a bounded subset of \( L_e(Y'_c, X'_e) \), hence \( H'(V^0) \) (\( V \) a zero neighbourhood in \( Y \)) is bounded in \( X'_e \), and thus \( c \)-relatively compact, for \( X \) is barrelled. Altogether, \( H \) fulfills the conditions of part (b) of Theorem 1.12, and thus is relatively compact. For the converse, we need only observe that \( Y \) is a topological linear subspace of \( L_e((X'_e)'_e, Y) \cong L_e(X, Y) \).

2. Equicontinuity and Compactness in Spaces of Operators

In this section we study the relationship between equicontinuity and compactness of subsets \( H \) of \( L(X, Y) \) for \( X \) (respectively \( Y \)) metrizable, and \( Y \) (respectively \( X \)) with a fundamental sequence of bounded sets.

We start with the case of a metrizable domain space \( X \).

2.1. Theorem. Let \( X \) be a Fréchet space, and \( Y \) a semi-Montel locally convex space with a fundamental sequence of bounded sets. For a subset \( H \) of \( L(X, Y) \), the following are equivalent:

(a) \( H \) is equicontinuous in \( L(X, Y) \).
(b) There exists a zero neighbourhood \( U \) in \( X \) such that \( H(U) \) is relatively compact in \( Y \).
(c) \( H \) is relatively compact in \( L_s(X, Y) \).
(d) \( H \) is bounded in \( L_s(X, Y) \).

2.2. Theorem. Every continuous linear operator from a metrizable locally convex space into a semi-reflexive (respectively semi-Montel) locally convex space with a fundamental sequence of bounded sets is weakly compact (respectively compact).

This is a special case of Proposition 1.8(a) of Section 1.

Notes. (1) The equivalence of conditions (b) and (c) of Theorem 2.1 extends corresponding results of Brauner [8, Proposition 2.3, Corollary 2.4]. Theorem 2.2 leads to a short proof of a refinement of Brauner's generalized Banach–Dieudonné Theorem:

2.3. Corollary. (Compare [8, Prop. 2.6, Cor. 2.7].) For a metrizable locally convex space \( X \), and a semi-Montel gDF space \( Y \), the space \( L_s(X, Y) \) is equal to the dual of the Fréchet space \( \hat{X} \otimes_{\gamma} Y'_b \), endowed with the topology
of uniform convergence on the compact subsets of $\hat{X} \otimes_{\pi} Y_b'$: $L_1(X, Y) = (\hat{X} \otimes_{\pi} Y_b')'$. Hence it is a semi-Montel gDF space, and the $\lambda$-topology is the finest topology on $L(X, Y)$, agreeing with the topology of pointwise convergence on the equicontinuous subsets of $L(X, Y)$.

The proof of this result is established by means of the following topological linear isomorphisms:

(a) $L_1(X, Y) \simeq L_c(\hat{X}, Y)$ (continuous linear extension of any $h \in L(X, Y)$).

(b) $L_c(\hat{X}, Y) \cong K_c(\hat{X}, Y)$ (Theorem 2.2).

(c) $K_c(\hat{X}, Y) \cong B_{cc}(X, Y_b')$, $h \mapsto \{(x, y') \mapsto (hx, y')\}$.

(d) $B_{cc}(\hat{X}, Y_b') = (\hat{X} \otimes_{\pi} Y_b')_c'$ [22, 1.2.1, Corollary 1, p. 52].

(2) For $X$ Fréchet and $Y$ DF, the result of Theorem 2.2 is that of [21, IV. 3.2, Corollary 2].

Proof of Theorem 2.1. Parts (a) and (b) are equivalent according to Proposition 1.8(a) of Section 1. Part (c) is implied by (a) by the Arzela–Ascoli Theorem. The barrelledness of $X$ yields the equivalence of (d) and (a).

In order to demonstrate the range of applicability of the above results, we consider various spaces of continuous linear operators on the space $H^\infty(G)$.

2.4. Theorem. Let $G$ be a plane region, and denote by $H^\infty(G)_x$, $H^\infty(G)_\beta$, and $H^\infty(G)_c$ the space of bounded analytic functions on $G$, endowed with the sup norm, the strict topology $\beta$, and the compact-open topology, respectively. Moreover, denote by $M_0(G) = M(G)/(H^\infty(G))'$ the dual of $H^\infty(G)_\beta$ with the total-variation-norm. Then we have the following topological linear isomorphisms:

(a) (i) $L_b(H^\infty(G)_c, H^\infty(G)_\beta) \cong K_b(H^\infty(G)_c, H^\infty(G)_\beta) \cong (H^\infty(G)_c \otimes_{\pi} M_0(G))_c'$ is a semi-Montel gDF space.

(ii) $(L_b(H^\infty(G)_c, H^\infty(G)_\beta))'_0 \cong H^\infty(G)_c \otimes_{\pi} M_0(G)$ (isomorphically).

For every $T \in (L_b(H^\infty(G)_c, H^\infty(G)_\beta))'_0$, there exist nullsequences $(f_i)_{i \in \mathbb{N}}$ and $(\mu_i)_{i \in \mathbb{N}}$ in $H^\infty(G)_c$ and in $M_0(G)$, and $(\lambda_i)_{i \in \mathbb{N}} \in l'$ such that

$$Th = \sum_{1}^{\infty} \lambda_i \int h f_i d\mu_i \quad \text{for all} \quad h \in L(H^\infty(G)_c, H^\infty(G)_\beta).$$

Moreover, whenever $G$ is simply connected, then $L_b(H^\infty(G)_c, H^\infty(G)_\beta)$ and its strong dual have the approximation property.

(b) (i) $L_c(H^\infty(G)_\infty, H^\infty(G)_\beta) \cong K_c(H^\infty(G)_\infty, H^\infty(G)_\beta) \cong (H^\infty(G)_c \otimes_{\pi} M_0(G))_c'$ is a semi-Montel gDF space.

(ii) $(L_c(H^\infty(G)_\infty, H^\infty(G)_\beta))'_0 = H^\infty(G)_c \otimes_{\pi} M_0(G)$ (isometrically).
For every $T \in (L_c(H^\infty(G)_\kappa, H^\infty(G)_\beta))'$, there exist nullsequences $(f_i)_{i \in \mathbb{N}}$ and $(\mu_i)_{i \in \mathbb{N}}$ in $H^\infty(G)^*_\kappa$ and in $M_0(G)$, and $(\lambda_i)_{i \in \mathbb{N}} \in l^1$ such that

$$Th = \sum_{i=1}^{\infty} \lambda_i (h f_i)_i \mu_i$$

for all $h \in L(H^\infty(G)_\kappa, H^\infty(G)_\beta)$.

Proof. (a) First, note that, according to the nuclearity of $H^\infty(G)_\kappa$, we have:

$$(w)_i \in M_0(G) \Leftrightarrow (w)_i \in M_0(G) \Rightarrow (w)_i \in H^\infty(G)_\kappa.$$  

Now, (i) and (ii) follow from Theorem 2.2, Corollary 2.3, and the series representation of elements of completed projective tensor products of metrizable spaces [26, Sect. 4.1, 4(6)]. Finally, whenever $G$ is simply connected, then $H^\infty(G)_\beta$ has the approximation property [7, Satz 9]. Hence, in this case, according to [26, Sect. 4.3, 4(11)] and [32, Sect. 1, Corollary p. 18], $M_0(G)$, and $H^\infty(G)_\kappa \hat{\otimes}_\pi M_0(G)$, and the $\varepsilon$-dual of the latter space have the approximation property as well.

Part (b) is a direct consequence of Corollary 2.3.

We turn now to the dual situation of a domain space $X$ with a fundamental sequence of bounded sets.

2.5. Theorem. Let $X$ be a Schwartz gDF space, and $Y$ a Fréchet space. For a subset $H$ of $L(X, Y)$, the following are equivalent:

(a) (i) $H$ is equicontinuous, and

(ii) $H(x)$ is relatively compact in $Y$ for all $x \in X$.

(b) There exists a zero neighbourhood $U$ in $X$ such that $H(U)$ is relatively compact in $Y$.

(c) $H$ is relatively compact in $L_b(X, Y)$.

2.6. Theorem. Let $X$ be a locally convex space with the countable neighbourhood condition (e.g., $X$ a gDF space).

Then every continuous linear operator from $X$ into a reflexive (respectively Montel) Fréchet space is weakly compact (respectively compact).

This is a special case of Proposition 1.8(b).

Notes. (1) The equivalence of conditions (b) and (c) of Theorem 2.5 extends corresponding results of Brauner [8, Proposition 2.3, Corollary 2.4].

(2) In connection with the solution of Grothendieck's "Problème des topologies" for gDF spaces [31, Theorem 1.9], Theorem 2.6 implies a kind of dual result to the one of Corollary 2.3.
2.7. Corollary. For a $gDF$ space $X$ and a Fréchet-Montel space $Y$, the space $L_b(X, Y)$ is topologically isomorphic to the strong dual of the $gDF$ space $X \otimes Y_b$:

$$L_b(X, Y) = (X \otimes Y_b)_b.$$ 

If, in addition to the assumptions, $X$ is semi-reflexive, then the dual of $L_b(X, Y)$ is algebraically isomorphic to $X \otimes Y_b$.

The proof of this result is established by means of the following topological linear isomorphisms:

(a) $L_b(X, Y) \cong K_b(X, Y)$ (Theorem 2.6).
(b) $K_b(X, Y) \cong B_{bb}(X, Y_b), h \mapsto \{(x, y') \mapsto (hx, y')\}$.
(c) $B_{bb}(X, Y_b) \cong (X \otimes Y_b)_b$ [31. Theorem 1.9].

(3) For a quasinormable locally convex space with the countable neighbourhood condition, and a metrizable locally convex space $Y$, it has been shown in [31] that

(a) an equicontinuous subset $H$ of $L(X, Y)$ transforms a certain zero neighbourhood into a precompact set whenever $H(B)$ is precompact for all $B$ bounded in $X$ [31, Theorem 2.2], and that

(b) any $h \in L(X, Y)$, which transforms bounded sets into weakly relatively compact (respectively precompact) sets, is weakly compact (respectively precompact) [31, Theorem 2.3].

Theorems 2.5 and 2.6 reveal that for the special case of a Schwartz $gDF$ space $X$, the conclusion of result (a) already holds if $H$ is equicontinuous and only $H(x)$ is precompact for all $x \in X$, and that for the special case of a reflexive (respectively Montel) Fréchet space $Y$, the conclusion of result (b) holds without the assumption of quasinormability for $X$. This is worth noticing, for the countable neighbourhood condition is inherited by every linear subspace, whereas this is not true for quasinormability.

Proof of Theorem 2.5. For a semi-Montel $gDF$ space $X$, Theorem 2.5 is a special case of Theorems 1.5 and 1.7 of Section 1: $L_b(X, Y) \cong L_c((X'_c)^c, Y)$. The general case of a Schwartz $gDF$ space $X$ is easily reduced to this special case by completing $X$ and continuously extending any $h \in L(X, Y)$. Note that $L_b(X, Y) \cong L_b(X, Y)$ according to [30, Corollary 2.4].

3. Factoring Compact Sets of Compact Operators

According to the Davis/FIGiel/Johnson/Pelczynski factorization theorem [13], weakly compact operators factor through reflexive Banach spaces.
Combining this result with Theorem 1.10 of Section 1, it will now be shown that compact sets of compact operators can be factored compactly and uniformly through one and the same reflexive Banach space.

3.1. Theorem. Let $X$ be a gDF space, $Y$ a Fréchet space, and let $H$ be a compact subset of $K_b(X, Y)$.

(a) $H$ can be factored compactly and uniformly through one and the same reflexive Banach space: there exists a linear subspace $Y_r$ of $Y$ together with a norm $r$ on $Y_r$ such that

(i) $(Y_r, r)$ is a reflexive Banach space, and the embedding of $(Y_r, r)$ into $Y$ is compact, and

(ii) the ranges of all $h \in H$ are contained in $Y_r$, and $H$ is a compact subset of $K_b(X, (Y_r, r))$.

(b) If the set $H$ is formed by a nullsequence $(h_n)_{n \in \mathbb{N}}$ in $K_b(X, Y)$, then the construction of part (a) is such that $(h_n)_{n \in \mathbb{N}}$ is a nullsequence in $K_b(X, (Y_r, r))$.

Proof: If $H$ is a compact subset of $K_b(X, Y)$, then, by Theorem 1.10 part (f), there exist a compact disk $K$ in $X'$ and a compact disk $C$ in $Y$ such that $H(K') \subset C$. Since $Y$ is a Fréchet space, a well-known consequence of the Banach–Dieudonné Theorem guarantees the existence of a further compact disk $C_1$ in $Y$ such that $C$ is compact in $(Y_{C_1}, C_1)$. $C_1$, in turn, is compact in $(Y_{C_2}, C_2)$ for a third compact disk $C_2$ in $Y$. According to [13, Lemma 1], there exists a linear subspace $Y_r$ of $(Y_{C_2}, C_2)$ together with a norm $r$ on $Y_r$ such that $(Y_r, r)$ is a reflexive Banach space, and such that the embedding $(Y_{C_1}, C_1) \hookrightarrow (Y_{C_2}, C_2)$ factors continuously through $(Y_r, r)$. Thus, we arrive at the following diagram:

\[ \xymatrix{ H(K') \subset C \ar[r]^\text{compact} & (Y_{C_1}, C_1) \ar[r]^\text{compact} & (Y_{C_2}, C_2) \ar[r]^\text{compact} & Y. } \]

In particular, $H(K')$ is relatively compact in $(Y_r, r)$, and thus, again by Theorem 1.10(f), a relatively compact subset of $K_b(X, (Y_r, r))$. This completes the proof of part (a). The proof for part (b) can easily be read from diagram (*) (note that on $C$ the topology of $(Y_r, r)$ and the original topology of $Y$ coincide).

Related techniques will be used in a subsequent publication [19] to establish factorizations and series representations for compact sets of compact range vector measures.
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