Uniqueness and Representation Theorems for the Inhomogeneous Heat Equation

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1. INTRODUCTION

We consider solutions of the equation \( \theta w = f \), where

\[
\theta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - \frac{\partial}{\partial t}
\]

is the heat operator, on strips and half-spaces of the form \( \mathbb{R}^n \times [0, a[ \), where \( 0 < a \leq \infty \). Here \( x = (x_1, \ldots, x_n) \) is a typical point of \( n \)-dimensional euclidean space \( \mathbb{R}^n \), and \( t \in \mathbb{R} \). Since we require the difference \( u \) of two solutions of \( \theta w = f \) to satisfy \( \theta u = 0 \) in the classical sense, we must assume that \( \theta w \) is continuous, and hence that \( f \) is continuous, on \( \mathbb{R}^n \times [0, a[ \).

We first consider uniqueness of solution to the Cauchy problem, that is, whether there can exist two continuous functions on \( \mathbb{R}^n \times [0, a[ \) which satisfy \( \theta w = f \) on \( \mathbb{R}^n \times [0, a[ \) and which coincide on \( \mathbb{R}^n \times \{0\} \). As is well known, if we do not impose some condition on the size of the possible solutions \( w \), we do not obtain uniqueness even if \( f = 0 \) [3, p. 31]. It is typical of earlier work in this direction that conditions are imposed both on \( |f| \) and \( |w| \), so that the problem immediately reduces to finding whether, under the same condition as that imposed on \( |w| \), there is a continuous function \( u \) on \( \mathbb{R}^n \times [0, a[ \) which satisfies \( \theta u = 0 \) on \( \mathbb{R}^n \times [0, a[ \), \( u(\cdot, 0) = 0 \) and \( u \neq 0 \) [3, p. 29]. The only exception, in the case \( f \neq 0 \), is in the paper [1] of Besala and Krzyżański, where conditions are imposed on \( |f| \) and \( w^+ = \max\{w, 0\} \) (or \( w^- = \max\{-w, 0\} \)), so that it is not obvious what growth condition the difference of two solutions would satisfy. Here we show that their condition on \( w^- \) can be weakened, and that simultaneously their condition on \( |f| \) can be replaced by one on \( f^- \) (see Theorem 3). However, we work only with \( \theta \), whereas they considered more general parabolic operators.
We then turn our attention to representation of solutions of $\theta w = f$ in the form

$$w(x, t) = \int_{\mathbb{R}^n} W(x - y, t) d\mu(y) - \int_0^t \int_{\mathbb{R}^n} W(x - y, t - s) f(y, s) dy,$$

where $\mu$ is a signed measure and

$$W(x, t) = (4\pi t)^{-n/2} \exp(-\|x\|^2/(4t))$$

for $(x, t) \in \mathbb{R}^n \times [0, \infty[$ (see Theorem 4). Here we again have one-sided conditions on $w$ and $f$, and in addition a condition on $f$ at the boundary $\mathbb{R}^n \times \{0\}$ which still allows $f(x, t)$ to be unbounded as $t \to 0$. This contrasts with [3, p. 251], where $f$ is assumed continuous up to $\mathbb{R}^n \times \{0\}$.

Finally, for $n = 1$, we use a result of Gehring [4] to show that, under conditions which guarantee the representation (1.1), there is at most one solution of the problem

$$\theta w = f \quad \text{on} \quad \mathbb{R} \times [0, \infty[,$$

$$-\infty < \liminf_{t \to 0^+} w(x, t) \leq \limsup_{t \to 0^+} w(x, t) < +\infty \quad \text{for all} \ x \in \mathbb{R},$$

$$\lim_{t \to 0^+} w(x, t) = \phi(x) \quad \text{for almost all} \ x \in \mathbb{R},$$

where $\phi$ is any given real-valued function on $\mathbb{R}$.

The use of the one-sided condition on $f$ is possible because we can, by addition of a given function, reduce the condition to $f \geq 0$, so that solutions of $\theta w = f$ are then subtemperatures [7, 8, 9]. We can then use theorems in [8, 9] to show that the given conditions on $w^+$ imply that similar conditions on $w^-$ are satisfied, and also (implicitly) that $f(x, t)$ cannot grow too rapidly as $\|x\| \to \infty$, so that conditions on $|w|$ and $|f|$ follow. These results are analogues of theorems about subharmonic functions on strips and half-planes given in [2, 5].

2. Notation and Terminology

We recall the definition of a subtemperature given in [7], for which we require some preliminary notation.

Given $(x_0, t_0) \in \mathbb{R}^{n+1}$ and $c > 0$, we denote by $\Omega(x_0, t_0; c)$ the domain defined by $W(x_0 - x, t_0 - t) > (4\pi c)^{-n/2}$, and we denote by $Q$ the function defined for $(x, t) \in \mathbb{R}^n \times [0, \infty[$ by

$$Q(x, t) = |x|^2 \{4| |x|^2 t^2 + (|x|^2 - 2nt)^2\}^{-1/2}.$$  

Let $w$ be a function defined on a domain $D$. We say that $w$ is a subtemperature on $D$ if it satisfies the following conditions:
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(i) \(-\infty \leq w(x, t) < +\infty\) for all \((x, t) \in D;\)

(ii) \(w\) is upper semicontinuous on \(D;\)

(iii) for each \((x, t) \in D\) there exists a point \((x_0, t_0) \in D, t_0 > t,\) such that \((x, t)\) can be joined to \((x_0, t_0)\) by a polygonal line in \(D\) along which the \(t\)-coordinate is strictly monotonic, and \(w(x_0, t_0) > -\infty;\)

(iv) for each \((x_0, t_0) \in D,\) the inequality

\[
w(x_0, t_0) \leq (4\pi c)^{-n/2} \int_{\partial D(x_0, t_0)} Q(x_0 - \xi, t_0 - \tau) w(\xi, \tau) \, ds
\]

holds whenever \(Q(x_0, t_0; c) \subseteq D\) (here \(s\) denotes surface measure on \(\partial D)).\)

A function \(w\) is a supertemperature if \(-w\) is a subtemperature. Any solution \(v\) of \(\theta v \geq 0\) is a subtemperature [7; Theorem 14], and a function \(u\) is a temperature, a solution of \(\theta u = 0,\) if and only if both \(u\) and \(-u\) are subtemperatures [7; p. 388].

The only supertemperatures we shall use here are defined on some set of the form \(\mathbb{R}^n \times ]\alpha, \beta[\), where \(-\infty \leq \alpha < \beta \leq +\infty.\) The following type of super-temperature is particularly important. Let \(\mu\) be a positive Borel measure on \(\mathbb{R}^n \times ]\alpha, \beta[\) such that \(\mu(K) < \infty\) whenever \(K\) is a compact subset of \(\mathbb{R}^n \times ]\alpha, \beta[\), and consider the function \(W_\mu\) given by

\[
W_\mu(x, t) = \int_{\mathbb{R}^n \times t} W(x - y, t - s) \, d\mu(y, s) \quad (2.1)
\]

for \((x, t) \in \mathbb{R}^n \times ]\alpha, \beta[\). This is a Green potential in the terminology of [9]. If \(W_\mu(x_0, t_0) < +\infty,\) then \(W_\mu\) is a supertemperature on \(\mathbb{R}^n \times ]\alpha, t_0[\) by [9; Lemma 16]. If \(W_\mu\) is a supertemperature, it is called a potential.

If \(w\) is a subtemperature on \(D,\) and \(u\) is a temperature such that \(w \leq u\) on \(D,\) then \(u\) is called a thermic majorant of \(w\) on \(D.\) If, in addition, \(u\) has the property that \(u \leq v\) for every thermic majorant \(v\) of \(w\) on \(D,\) then \(u\) is called the least thermic majorant of \(w\) on \(D [7; Section 11].\) Similar statements, with the inequalities reversed, serve to define a thermic minorant and the greatest thermic minorant of a supertemperature on \(D [9, Section 7].\)

We shall employ the integral means \(M_\beta (b > 0)\) of [8]. Let \(0 < t < b < \infty,\) and let \(\nu\) be a measurable function on \(\mathbb{R}^n \times \{t\}.\) The integral mean \(M_\beta(\nu; t)\) is defined by the formula

\[
M_\beta(\nu; t) = \int_{\mathbb{R}^n} W(x, b - t) \nu(x, t) \, dx
\]

whenever the integral exists.
We require also the following well-known formulae

\[ \int_{\mathbb{R}^n} W(x - y, t - s) \, dy = 1, \]
\[ \int_{\mathbb{R}^n} W(x - y, t - s) W(y - z, s - r) \, dy = W(x - z, t - r), \]

which hold whenever \( x, z \in \mathbb{R}^n \) and \( r < s < t \). They can be deduced from [8; Lemma 3].

The term *decreasing* is used here in the wide sense.

3. A Characterization of Potentials

We now give a characterization of potentials of the form (2.1) on a strip \( \mathbb{R}^n \times ]0, a[ \) in terms of the means \( M_b \). The characterization is analogous to [2; Theorem 3] for superharmonic potentials on a strip, and to [6; Theorem 5, Corollary 3] for such functions on a half-space.

**Theorem 1.** If \( W_\mu \) is a potential on \( \mathbb{R}^n \times ]0, a[ \), \( 0 < a \leq \infty \), and \( b < a \), then \( M_b(W_\mu; \cdot) \) is real-valued and increasing on \( ]0, b[, \) and \( M_b(W_\mu; t) \to 0 \) as \( t \to 0 \).

Conversely, if \( w \) is a non-negative supertemperature on \( \mathbb{R}^n \times ]0, a[ \) and, for some \( b < a \), \( \liminf_{t \to a} M_b(w; t) = 0 \), then \( w \) is a potential.

**Proof.** Let \( W_\mu \) be a potential on \( \mathbb{R}^n \times ]0, a[ \), and let \( 0 < b < a \). Then, if \( 0 < t < b \),

\[ M_b(W_\mu; t) = \int_{\mathbb{R}^n} W(x, b - t) \, dx \int_{\mathbb{R}^n \times ]0, t[} W(x - y, t - s) \, d\mu(y, s) \]
\[ = \int_{\mathbb{R}^n \times ]0, t[} d\mu(y, s) \int_{\mathbb{R}^n} W(x, b - t) \, W(y - x, t - s) \, dx \]
\[ = \int_{\mathbb{R}^n \times ]0, t[} W(y, b - s) \, d\mu(y, s) \]
\[ = W_{\mu_t}(0, b), \]

where \( \mu_t \) is the restriction of \( \mu \) to \( \mathbb{R}^n \times ]0, t[ \). Since the function \( t \mapsto W_{\mu_t} \) is increasing on \( ]0, a[ \), and \( W_{\mu_t} = W_\mu \) is finite a.e. [7; Theorem 1], \( W_{\mu_t} \) is finite a.e. for each \( t \). Further, the support of \( \mu_t \) is contained in \( \mathbb{R}^n \times [0, t] \), so that \( W_{\mu_t} \) is a temperature on \( \mathbb{R}^n \times ]t, a[ \), by [9; Theorem 18], and hence \( W_{\mu_t}(0, b) \) is finite whenever \( t < b < a \). Hence \( M_b(W_\mu; \cdot) \) is increasing and finite-valued.
Next, as $t \to 0$, $W_{\mu t}$ decreases and is bounded below by zero. Since $W_{\mu t}$ is a temperature on $\mathbb{R}^n \times ]t, a[$, it follows that $u = \lim_{t \to 0} W_{\mu t}$ is a temperature on $\mathbb{R}^n \times ]0, a[$ [7; Lemma 9], and $0 \leq u \leq W_{\mu}$. Since $W_{\mu}$ is a potential its greatest thermic minorant is zero [9; Theorem 17], so that $u = 0$ and

$$\lim_{t \to 0} M_b(W_{\mu}; t) = \lim_{t \to 0} W_{\mu t}(0, b) = u(0, b) = 0.$$  

This proves the first part.

Now let $w$ be a non-negative supertemperature on $\mathbb{R}^n \times ]0, a[$ such that for some $b \in ]0, a[$, $\lim \inf M_b(w; t) = 0$ as $t \to 0$. Then there is a measure $\nu$ on $\mathbb{R}^n \times ]0, a[$ such that $w = W_{\nu} + \nu$, where $\nu$ is the greatest thermic minorant of $w$ on $\mathbb{R}^n \times ]0, a[$, by [9; Theorem 22]. It suffices to prove that $\nu = 0$, in view of [9; Theorem 22, Corollary]. Since $w \geq 0$ we have $\nu \geq 0$, so that $M_b(\nu; t) = \nu(0, b)$, by [8; Theorem 1, Corollary 2]. Hence, if $0 < t < b$,

$$M_b(w; t) = M_b(W_{\nu}; t) + \nu(0, b)$$

so that, by our hypothesis and the first part of the theorem,

$$\nu(0, b) - \lim \inf_{t \to 0} M_b(w; t) - \lim_{t \to 0} M_b(W_{\nu}; t) = 0.$$  

Hence, by the strong minimum principle [7; Theorem 7] $v = 0$ on $\mathbb{R}^n \times ]0, b[$. Since $v \geq 0$, it follows from the representation theorem for non-negative temperatures [8; Lemma 3] that $v = 0$ on $\mathbb{R}^n \times ]0, a[$, and the proof is complete.

4. MEAN VALUES OF SUBTEMPERATURES

In this section we show how growth conditions on the positive part of a subtemperature lead to similar conditions on the negative part. Although Theorem 1 is employed, the essential step is an application of the decomposition theorem [9; Theorem 22].

We recall the definitions of the classes $\Sigma_b$ and $\Phi_b$ of [8]. For $0 < b < \infty$, we say that $w \in \Sigma_b$ if $w$ is a subtemperature on $\mathbb{R}^n \times ]0, b[$ and $M_b(w^+; \cdot)$ is locally integrable on $]0, b[$. Then $M_b(w^+; \cdot)$ is, in fact, decreasing on $]0, b[$, by [8; Theorem 7]. We say that $w \in \Phi_b$ if $w \in \Sigma_b$ and $\lim \inf_{t \to 0} M_b(w^+; t) < \infty$; then $M_b(w^+; \cdot)$ is bounded.

**Theorem 2.** Let $0 < a \leq \infty$, and let $w \in \Sigma_b$ for every $b < a$. Then $M_b(w; \cdot)$ is decreasing, real-valued, and upper semicontinuous on $]0, b[$ for each $b < a$.

**Proof.** We suppose first that $w \in \Phi_b$ for every $b \in ]0, a[$. Then there is a non-negative temperature $u$ on $\mathbb{R}^n \times ]0, a[$ such that $w \leq u$ [8; Theorem 19]. Then $u - w$ is a non-negative supertemperature, so that there is a measure $\mu$
such that $u - w = W\mu + v$ on $\mathbb{R}^n \times [0, a]$, where $v$ is a non-negative temperature on $\mathbb{R}^n \times [0, a]$, by [9; Theorem 22]. Hence $-w = W\mu + (v - u)$ is the sum of a potential and a temperature that can be expressed as the difference of two non-negative temperatures. Hence, if $0 < t < b < a$,

$$M_b(w; t) = -M_b(W\mu; t) - v(0, b) + u(0, b)$$

by [8; Theorem 1, Corollary 2]. Theorem 1 now implies that $M_b(w; t) > -\infty$.

Now suppose that $w \in \Sigma_b$ for every $b < a$. Fix $b$, take $\epsilon \in ]0, a - b[$, and put $w_\epsilon(x, t) = w(x, t + \epsilon)$. Then $0 < b < a - \epsilon$ and $w_\epsilon$ is a subtemperature on $\mathbb{R}^n \times [0, a - \epsilon[$. Further, whenever $0 < t < c < a - \epsilon$,

$$M_c(w_\epsilon; x + \epsilon - t - \epsilon) w_\epsilon(x, t + \epsilon) \ dx = M_c(w_\epsilon; x + \epsilon)$$

and since $0 < c + \epsilon < a$, our hypothesis and [8; Theorem 7] imply that $M_c(w_\epsilon; x + \epsilon) w_\epsilon(x, t + \epsilon) \ dx$ is bounded on $[c, c + \epsilon[$, and $w_\epsilon \in \Phi_\epsilon$ whenever $0 < c < a - \epsilon$. Therefore the first part of this proof implies that $M_c(w_\epsilon; x + \epsilon) w_\epsilon(x, t + \epsilon) \ dx$ is finite-valued whenever $0 < c < a - \epsilon$. Since $0 < b - \epsilon < a - \epsilon$, and $M_b(w_\epsilon; t) = M_b(w; t + \epsilon)$ whenever $0 < t < b - \epsilon$, $M_b(w; t)$ is finite whenever $\epsilon < t < b$. Since $\epsilon$ is an arbitrary member of $]0, a - b[$, the finiteness of $M_b$ follows. The other properties now follow from [8; Theorem 7].

As a corollary, we give a new characterization of the class $\cap_{b<\alpha} \Sigma_b$.

**COROLLARY.** Let $0 < a \leq \infty$, and let $w$ be a subtemperature on $\mathbb{R}^n \times [0, a]$. Then $w \in \cap_{b<\alpha} \Sigma_b$ if and only if the inequality

$$w(x, t) \leq \int_{\mathbb{R}^n} W(x - y, t - s) w(y, s) \ dy$$

(4.1)

holds for $0 < s < t < a$ and the integral is finite.

**Proof.** If $w \in \cap_{b<\alpha} \Sigma_b$, then (4.1) follows from Theorem 2 and [8; Theorem 3]. Conversely, if (4.1) holds then

$$w^+(x, t) \leq \int_{\mathbb{R}^n} W(x - y, t - s) w^+(y, s) \ dy$$

whenever $0 < s < t < a$, and [8; Theorem 4] shows that $w \in \cap_{b<\alpha} \Sigma_b$.

It follows from the above corollary and [8; Theorem 6] that, if $w$ is a temperature, (4.1) holds if and only if it holds with equality.
5. The Cauchy Problem for \( \partial u = f \)

If \( a \in ]0, \infty[ \), we define the temperature \( V_a \) on \( \mathbb{R}^n \times ]0, a[ \) by

\[
V_a(x, t) = (a - t)^{-n/2} \exp(\|x\|/4(a - t)).
\]

The condition on \( f \) which we shall use in the sequel is

\[
f(x, t) \geq -kV_a(x, t)
\]

on \( \mathbb{R}^n \times ]0, a[ \), for some positive constant \( k \). First we give a lemma which enables us to convert this condition to \( f \geq 0 \).

**Lemma 1.** Suppose that \( f \) is defined on \( \mathbb{R}^n \times ]0, a[, \) where \( 0 < a < \infty \), and that \( w \) is continuous on \( \mathbb{R}^n \times [0, a[ \). Then \( w \) satisfies \( \partial w = f \) on \( \mathbb{R}^n \times ]0, a[, \) if and only if the function \( v \), given by

\[
v(x, t) = w(x, t) - ktV_a(x, t)
\]

for \( (x, t) \in \mathbb{R}^n \times ]0, a[ \), satisfies \( \partial v = f + kV_a \). Further, whenever \( 0 < t < b < a \), we have

\[
M_b(v^+; t) \leq M_b(w^+; t) \leq M_b(v^+; t) + kt(a - b)^{-n/2}.
\] (5.1)

Finally, \( tV_a(x, t) \to 0 \) as \( t \to 0^+ \), uniformly for \( x \) in any compact subset of \( \mathbb{R}^n \).

**Proof.** The only non-trivial part of the lemma is (5.1), and that follows from [8; Theorem 1, Corollary 2].

**Theorem 3.** Suppose that \( f \) is continuous on \( \mathbb{R}^n \times ]0, a[, \) where \( 0 < a < \infty \), that \( \phi \) is continuous on \( \mathbb{R}^n \), and that there is a positive constant \( k \) such that

\[
f(x, t) \geq -kV_a(x, t)
\] (5.2)

for all \( (x, t) \in \mathbb{R}^n \times ]0, a[, \) Then there is at most one function \( w \), continuous on \( \mathbb{R}^n \times [0, a[, \) which satisfies

\[
\partial w = f \quad \text{on} \quad \mathbb{R}^n \times ]0, a[, \quad w(\cdot, 0) = \phi,
\] (5.3)

subject to the condition that for each \( b < a \), \( M_b(w^+; \cdot) \) is locally integrable on \( [0, b[ \).

**Proof.** We can assume that \( f \geq 0 \). For if \( w \) satisfies (5.3) and the given condition on \( M_b(w^+; \cdot) \), then \( \tilde{w} \), given by

\[
\tilde{w}(x, t) = w(x, t) - ktV_a(x, t) \quad ((x, t) \in \mathbb{R}^n \times [0, a[)
\]

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satisfies $\theta \tilde{w} = f + kV_\alpha \geq 0$, $\tilde{w}(\cdot, 0) = \phi$, and $M_b(\tilde{w}^+; \cdot)$ is locally integrable on $[0, b[, \text{ for each } b < a$, in view of Lemma 1. Hence we can suppose that the solutions sought are subtemperatures, and the condition on their means $M_b$ is precisely that which defined the class $\cap_{b < a} \mathcal{W}_b$ in [8].

Suppose, therefore, that $u$ and $v$ are two solutions of (5.3) in the class of subtemperatures $\cap_{b < a} \mathcal{W}_b$, and for the present let $w$ denote either function. Then whenever $b < a$, $M_b(w^+; \cdot)$ and $M_b(w^-; \cdot)$ are both locally integrable on $[0, b[$, the latter since, whenever $0 < t \leq c < b$,

$$-\infty < M_b(w^+; c) \leq M_b(w; t) < M_b(w^+; t)$$

by Theorem 2. Hence $M_b(w^+; \cdot)$ and $M_b(w^-; \cdot) = M_b(w^+; \cdot) - M_b(w; \cdot)$ are both locally integrable on $[0, b[$. Therefore

$$M_b((u - v)^+; \cdot) \leq M_b(u^+; \cdot) + M_b(v^-; \cdot)$$

is locally integrable on $[0, b[$, so that $u - v$ is a temperature which belongs to $\cap_{b < a} \mathcal{W}_b$, is continuous on $\mathbb{R}^n \times [0, a[$, and satisfies $u(\cdot, 0) - v(\cdot, 0) = 0$. Therefore $u = v$, by [8; Theorem 15].

**Corollary 1.** Suppose that $f$ is continuous on $\mathbb{R}^n \times [0, a[\, 0 < a \leq \infty$, that $\phi$ is continuous on $\mathbb{R}^n$, and that there are positive constants $k, h$ such that

$$f(x, t) \geq -k \exp(h \| x \|^2)$$

for all $(x, t) \in \mathbb{R}^n \times [0, a[$. Then there is at most one function $w$, continuous on $\mathbb{R}^n \times [0, a[$, which satisfies (5.3) subject to the condition that for some positive constant $m$

$$\int_0^a dt \int_{\mathbb{R}^n} w^+(x, t) \exp(-m \| x \|^2) dx < \infty. \quad (5.5)$$

**Proof.** By using simple inequalities like those used in the proof of [8; Theorem 15, Corollary], we can show that (5.4) and (5.5) imply that (5.2) and the condition on the $M_b$ in Theorem 3 are satisfied on $\mathbb{R}^n \times [0, a[\, \rho$ for some $\rho \leq a$. Hence there is uniqueness on this strip. If $\rho < a$, by taking $w(\cdot, \rho)$ as the new initial data we can repeat the argument on $\mathbb{R}^n \times [\rho, \min\{2\rho, a\}[$, thus obtaining uniqueness on $\mathbb{R}^n \times [0, \min\{2\rho, a\}[$. Further repetition eventually gives the desired result.

As a second corollary, we give the special case where the differential operator is $\theta$ of Besala and Krzyżaniski’s result [1; Theorem 4].

**Corollary 2.** Suppose that the continuity conditions in Corollary 1 are satisfied, and that there are positive constants $k, h$ such that

$$|f(x, t)| \leq k \exp(h \| x \|^2)$$
for all \((x, t) \in \mathbb{R}^n \times ]0, a[\). Then there is at most one solution \(w\) of (5.3) which satisfies

\[ w(x, t) \leq q \exp(m \| x \|^2) \]

on \(\mathbb{R}^n \times ]0, a[\), for some positive constants \(q, m\).

6. A REPRESENTATION THEOREM

Theorem 1 shows that the means \(M_t(W, t)\) of a potential tend to 0 as \(t \to 0\), but in order to prove our representation theorem, we need a condition on \(f\) which ensures that \(Wf\) itself tends to zero as \(\mathbb{R}^n \times ]0, a[\) is approached.

**Lemma 2.** Let \(f\) be a non-negative measurable function on \(\mathbb{R}^n \times ]0, a[\) such that

\[
 u(x, t) = \int_0^t ds \int_{\mathbb{R}^n} W(x - y, t - s) f(y, s) dy
\]

(6.1)
is finite a.e. on \(\mathbb{R}^n \times ]0, a[\), and let \(x_0 \in \mathbb{R}^n\). If there is a neighborhood \(\mathcal{N}\) of \((x_0, 0)\), and a function \(\psi\) such that \(f(y, s) \leq \psi(s)\) for all \((y, s) \in \mathcal{N}\) and

\[
 \int_0^t \psi(s) ds \to 0 \quad \text{as} \quad t \to 0,
\]

then \(u(x, t) \to 0\) as \((x, t) \to (x_0, 0)\).

**Proof.** Since the integral in (6.1) is finite, \(u\) is a potential on \(\mathbb{R}^n \times ]0, a[\), by [9; Lemma 16]. If we extend \(f\) to \(\mathbb{R}^n \times ]-\infty, a[\) by putting \(f(y, s) = 0\) whenever \(s < 0\), then \(u\) can be written in the form

\[
 u(x, t) = \int_{\mathbb{R}^n \times ]-\infty, t[} W(x - y, t - s) f(y, s) dy ds.
\]

(6.2)

Hence \(u\) can be extended, by (6.2), to a potential on \(\mathbb{R}^n \times ]-\infty, a[\) such that \(u(x, t) = 0\) whenever \(t \leq 0\). Putting \(\mathbb{R}^n \times ]-\infty, a[ = S\), for each \(r \in \mathbb{R}\), we write

\[
 u(x, t) = \left( \int_{S^r_0 \cap N} + \int_N \right) W(x - y, t - s) f(y, s) dy ds
\]

\[ = I_1(x, t) + I_2(x, t). \]

Then \(I_1\) is the potential of the restriction to \(S^r_0(S_0 \cup N)\) of \(f\), so that \(I_1\) is a
temperature on $S_0 \cup N$, by [9; Theorem 18], and $I_1 = 0$ on $S_0$. Hence, in particular, $I_1(x, t) \to 0$ as $(x, t) \to (x_0, 0)$. Further,

$$|I_2(x, t)| \leq \int_0^t \psi(s) \, ds \int_{\mathbb{R}^n} W(x - y, t - s) \, dy$$

$$= \int_0^t \psi(s) \, ds \to 0 \quad \text{as} \quad t \to 0.$$

The result follows.

**Theorem 4.** Suppose that $f$ is continuous on $\mathbb{R}^n \times ]0, a[$, $0 < a < \infty$, that there is a positive constant $k$ such that

$$f(x, t) \geq -k V_a(x, t)$$

for all $(x, t) \in \mathbb{R}^n \times ]0, a[$, and that for each $x_0 \in \mathbb{R}^n$ there is a neighborhood $N$ of $(x_0, 0)$ and a function $\psi$ such that $f(y, s) \leq \psi(s)$ for all $(y, s) \in N$ and

$$\int_0^t \psi(s) \, ds \to 0 \quad \text{as} \quad t \to 0.$$

If $w$ satisfies $\theta w = f$ on $\mathbb{R}^n \times ]0, a[$, then the following statements are equivalent.

(i) For each $b < a$, $M_b(w^+; \, ;)$ is bounded on $]0, b[$.

(ii) For each $b < a$, $M_b(w^+; \, ;)$ is locally integrable on $]0, b[$ and

$$\lim \inf_{a \to b} M_b(w^+; \, ;) \leq \infty.$$

(iii) For each $(x, t) \in \mathbb{R}^n \times ]0, a[$, $w(x, t)$ is given by

$$w(x, t) = \int_{\mathbb{R}^n} W(x - y, t) \, d\mu(y) - \int_0^t ds \int_{\mathbb{R}^n} W(x - y, t - s) f(y, s) \, dy, \quad (6.3)$$

where $\mu$ is a signed measure and the integrals are finite, and if $w$ is continuous on $\mathbb{R}^n \times [0, a[$ then $d\mu(y) = w(y, 0) \, dy$.

**Proof.** We first show that we can assume $f \geq 0$ throughout. By Lemma 1, if we put $v(x, t) = w(x, t) - kV_a(x, t)$, then $\theta v = f + kV_a \geq 0$, and $v$ satisfies (i) or (ii) if and only if $w$ satisfies the same condition. Next, if $w$ is given by (6.3) then it follows from the calculation

$$\int_0^t \int_{\mathbb{R}^n} W(x - y, t - s) V_a(y, s) \, dy = \int_0^t V_a(x, t) \, ds - tV_a(x, t)$$
(using [8; Lemma 3] and Fubini's theorem) that \( v \) is given by

\[
v(x, t) = \int_{\mathbb{R}^n} W(x - y, t) \, d\mu(y)
- \int_0^t ds \int_{\mathbb{R}^n} W(x - y, t - s) \{ f(y, s) + kV_a(y, s) \} \, dy,
\]

and conversely. Finally, since \( V_a \) is locally bounded on \( \mathbb{R}^n \times [0, a] \), the condition on \( f \) near \( \mathbb{R}^n \times \{0\} \) is equivalent to the same condition being attached to \( f + kV_a \). It follows that, by replacing \( w \) with \( v \) if necessary, we can assume that \( f \geq 0 \), and hence that all solutions of \( \partial w = f \) are subtemperatures.

The equivalence of (i) and (ii) follows from [8; Theorem 16].

Suppose that (ii) holds, so that \( w \in \bigcap_{b < a} \Phi_b \). By [8; Theorem 19], \( w \) has a positive thermic majorant \( h' \) on \( \mathbb{R}^n \times [0, a] \), so that \( w \) has a least thermic majorant \( h \) on that strip, by [7; Theorem 17] or [9; Lemma 8]. The greatest thermic minorant of the supertemperature \( h - w \) is therefore zero, so that \( h - w \) has the representation

\[
(h - w)(x, t) = \int_{\mathbb{R}^n \times [0, t]} W(x - y, t - s) f(y, s) \, dy \, ds \quad (6.4)
\]

in view of [9; Theorems 22 and 9]. Since \( h \leq h' \), and \( h' \) is a positive temperature, it follows from [8; Theorem 1, Corollary 2] that \( h \) is majorized by a member of \( \bigcap_{b < a} \Phi_b \), and hence that \( h \in \bigcap_{b < a} \Phi_b \). Therefore \( h \) has the representation

\[
h(x, t) = \int_{\mathbb{R}^n} W(x - y, t) \, d\mu(y) \quad (6.5)
\]
on \( \mathbb{R}^n \times [0, a] \), where \( d\mu(y) = h(y, 0) \, dy \) if \( h \) is continuous on \( \mathbb{R}^n \times [0, a] \), by [8; Theorem 21]. In view of (6.4) and Lemma 2,

\[
(h - w)(x, t) \to 0 \quad \text{as} \quad (x, t) \to (x_0, 0)
\]

for all \( x_0 \in \mathbb{R}^n \), so that continuity of \( h \) on \( \mathbb{R}^n \times [0, a] \) is equivalent to that of \( w \) and \( h(\cdot, 0) = w(\cdot, 0) \) whenever that continuity occurs. Therefore (iii) follows from (6.4) and (6.5).

Finally, if (iii) holds then (ii) follows easily from [8; Theorem 21] and our assumption that \( f \geq 0 \).

7. A Generalized Cauchy Problem

For the case \( n = 1 \), Theorem 4 can be used to prove that there is at most one solution of a generalized form of the Cauchy problem.
Theorem 5. Suppose that $f$ satisfies the same conditions as in Theorem 4, and that $\phi$ is any real-valued function on $\mathbb{R}$. Then there is at most one function $w$ on $\mathbb{R} \times ]0, a[$ which satisfies:

$$\theta w = f \quad \text{on} \quad \mathbb{R}^n \times ]0, a[,$$

$$-\infty < \liminf_{t \to 0} w(x, t) \leq \limsup_{t \to 0} w(x, t) < +\infty$$

for all $x \in \mathbb{R}$,

$$\lim_{t \to 0} w(x, t) = \phi(x)$$

for almost all $x \in \mathbb{R}$, and (ii) of Theorem 4.

Proof. If $u$ and $v$ are two such functions, then Theorem 4 implies that $u - v$ has the representation

$$(u - v)(x, t) = \int_{\mathbb{R}} W(x - y, t) \, dv(y)$$

for some signed measure $v$ on $\mathbb{R}$. Also, $\lim_{t \to 0}(u - v)(x, t)$ is finite whenever it exists and is zero for almost all $x \in \mathbb{R}$. Therefore, by [4; Theorem 10], $u = v$ on $\mathbb{R} \times ]0, a[.$

References