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Oscillation criteria for first-order impulsive differential equations with positive and negative coefficients☆

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Abstract

Some sufficient conditions are obtained for oscillation of all solutions of the first-order impulsive differential equation with positive and negative coefficients

$$[x(t) - R(t)x(t-r)]' + P(t)x(t-\tau) - Q(t)x(t-\sigma) = 0, \quad \tau \geqslant \sigma > 0, \ t \geqslant t_0,$$

$$x(t_k^+) = I_k(x(t_k)), \quad k = 1, 2, \ldots .$$

Our results improve the known results in the literature.

MSC: 34K15; 34C10

Keywords: Positive and negative coefficients; Differential equations; Oscillation solution; Impulsive

1. Introduction

It is well known that the theory of impulsive differential equations is being recognized to be not only richer than the corresponding theory of differential equations without impulses but also provides a more adequate mathematical model for numerous processes and phenomena studied in physics, biology, engineering, etc. [4,8]. However, the theory of impulsive functional differential equations is developing comparatively slowly due to numerous theoretical and technical difficulties caused by their peculiarities. In particular, to the best of our knowledge, there is little in the way of results for the oscillation of impulsive delay differential equations of neutral type despite the extensive development of the oscillatory and nonoscillatory properties of neutral differential equations without impulses (for example, see [5,6,2,3,7,9–12]). In this paper, we consider the oscillation of all solutions of the following impulsive neutral delay

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differential equations with positive and negative coefficients,

$$[x(t)-R(t)x(t-r)]'+P(t)x(t-\tau)-Q(t)x(t-\sigma)=0, \quad \tau\geqslant\sigma>0,\ t\geqslant t_0, \tag{1.1}$$

$$x(t_k^+)=I_k(x(t_k)), \quad k=1,2,\ldots, \tag{1.2}$$

where

(A_1) $r>0$, $\tau\geqslant\sigma>0$, $0<t_0<t_1<\cdots<t_k\to\infty$ as $k\to\infty$;

(A_2) $R\in PC([t_0,\infty),R^+)$, $P,Q\in C([t_0,\infty),R^+)$, $H(t)=P(t)-Q(t-\tau+\sigma)\geqslant 0$ and $H(t)\not\equiv 0$ on $(t_{k-1},t_k](k\geqslant 1)$, where $R^+=[0,\infty)$, $PC(I,X)=\{g\colon I\to X\colon g(t)$ is continuous for $t\in I$ and $t\neq t_k$, $g(t_k^+)$ and $g(t_k^-)=\lim_{t\to t_k^-}g(t)$ exist with $g(t_k^-)=g(t_k)$ $(k=1,2,\ldots)\}$;

(A_3) $I_k(x)$ is continuous and there exist positive number b_k such that $b_k\leqslant I_k(x)/x\leqslant 1$ for $k=1,2,\ldots$.

When $I_k(x)=x$ for $k=1,2,\ldots$, (1.1) and (1.2) reduce to the first-order neutral delay differential equations with positive and negative coefficients

$$[x(t)-R(t)x(t-r)]'+P(t)x(t-\tau)-Q(t)x(t-\sigma)=0. \tag{1.3}$$

There are many good results on the oscillation of (1.3), see for example [2,3,7,9,10], but all of them consider the three cases when $W(t)\equiv 1$, $W(t)\leqslant 1$, $W(t)\geqslant 1$, where

$$W(t)=R(t)+\int_{t-\tau+\sigma}^{t}Q(s)\,\mathrm{d}s. \tag{1.4}$$

In this paper, we introduce the function

$$W_s(t)=R(t)+\int_{t-s}^{t}Q(u)\,\mathrm{d}u+\int_{t}^{t-s+\tau-\sigma}P(u)\,\mathrm{d}u, \tag{1.5}$$

where $s\in[0,\tau-\sigma]$. Note that when $s=\tau-\sigma$, (1.5) becomes

$$W_{\tau-\sigma}(t)=W(t).$$

We establish oscillation criteria for (1.1) and (1.2). Our results improve the known results in the literature. With Eqs. (1.1) and (1.2), one associates an initial condition of the form

$$x_{t_0}=\phi(s), \quad s\in[-\rho,0], \tag{1.6}$$

where $x_{t_0}=x(t_0+s)$ for $-\rho\leqslant s\leqslant 0$ and $\phi(\cdot)\in C([-\rho,0],R)$.

A function $x(t)$ is said to be a solution of Eqs. (1.1) and (1.2) satisfying the initial value condition (1.6) if

(i) $x(t)=\phi(t-t_0)$ for $t_0-\rho\leqslant t\leqslant t_0$, $x(t)$ is continuous for $t\geqslant t_0$ and $t\neq t_k$ $(k=1,2,\ldots)$;

(ii) $x(t)-R(t)x(t-r)$ is continuously differentiable for $t>t_0$, $t\neq t_k$, $t\neq t_k+\tau$, $t\neq t_k+\sigma$, $t\neq t_k+r$, and satisfies (1.1);

(iii) $x(t_k^+)$ and $x(t_k^-)$ exist with $x(t_k^-)=x(t_k)$ and satisfy (1.2).

As is customary, a solution of Eqs. (1.1) and (1.2) is said to be nonoscillatory if it is eventually positive or eventually negative, otherwise, it will be called oscillatory.

Throughout all of our paper, we always assume that (A_1)–(A_3) hold and let $\rho=\max\{r,\tau\}$, $\delta=\min\{r,\sigma\}$.

2. Main results

Lemma 2.1. *Assume that* $h_1\in C([a,b],R^+)$, $h_2\in PC([a,b],R^+)$, *then*

$$h_2(\bar{\xi})\int_a^b h_1(t)\,\mathrm{d}t\leqslant\int_a^b h_1(t)h_2(t)\,\mathrm{d}t\leqslant h_2(\xi)\int_a^b h_1(t)\,\mathrm{d}t,$$

where $a\leqslant\xi,\bar{\xi}\leqslant b$.

Proof. Suppose $c_i \in [a, b]$ and $h_2(t)$ is not continuous at the points c_i, $i = 1, 2, \ldots, k$, then

$$\int_a^b h_1(t)h_2(t)\,\mathrm{d}t = \int_a^{c_1} h_1(t)h_2(t)\,\mathrm{d}t + \int_{c_1}^{c_2} h_1(t)h_2(t)\,\mathrm{d}t + \cdots + \int_{c_k}^{b} h_1(t)h_2(t)\,\mathrm{d}t$$
$$= h_2(\xi_1)\int_a^{c_1} h_1(t)\,\mathrm{d}t + \cdots + h_2(\xi_{k+1})\int_{c_k}^{b} h_1(t)\,\mathrm{d}t.$$

Let $h_2(\xi) = \max_{1\leqslant i\leqslant k+1}\{h_2(\xi_i)\}$ and $h_2(\bar{\xi}) = \min_{1\leqslant i\leqslant k+1}\{h_2(\xi_i)\}$, clearly $a \leqslant \xi, \bar{\xi} \leqslant b$ and

$$h_2(\bar{\xi})\int_a^b h_1(t)\,\mathrm{d}t \leqslant \int_a^b h_1(t)h_2(t)\,\mathrm{d}t \leqslant h_2(\xi)\int_a^b h_1(t)\,\mathrm{d}t.$$

The proof is complete. □

Lemma 2.2. *Assume that the following two conditions hold*:

(l_1) *there exists a real number* $s \in [0, \tau - \sigma]$ *such that*

$$W_s(t) \leqslant 1 \quad \text{for } t \geqslant t_0; \tag{2.1}$$

(l_2) $b_0 = 1$, $0 < b_k \leqslant 1$ *for* $k = 1, 2, \ldots$ *and*

$$\begin{cases} R(t_k^+) \geqslant R(t_k) & \text{for } t_k - r \neq t_{k_i},\ \ k_i < k, \\ b_k R(t_k^+) \geqslant R(t_k) & \text{for } t_k - r = t_{k_i},\ \ k_i < k, \end{cases}$$

where $b_k = b_{k_i}$ *when* $t_k - r = t_{k_i}\,(k_i < k)$.

Let

$$w(t) = x(t) - R(t)x(t-r) - \int_{t-s}^{t} Q(u)x(u-\sigma)\,\mathrm{d}u - \int_{t}^{t-s+\tau-\sigma} P(u)x(u-\tau)\,\mathrm{d}u, \tag{2.2}$$

then

(i) *if* $x(t)$ *is a solution of* (1.1) *and* (1.2) *such that* $x(t) > 0$ *for* $t \geqslant t_0$, *then*

$$w(t) > 0 \quad \text{for large } t;$$

(ii) *if* $x(t)$ *is a solution of* (1.1) *and* (1.2) *such that* $x(t) < 0$ *for* $t \geqslant t_0$, *then*

$$w(t) < 0 \quad \text{for large } t.$$

Proof. (i) Let $l = \min\{k \geqslant 0\colon t_k > t_0 + \rho\}$, from (1.1) and (2.2) we have

$$w'(t) = -H(t - s + \tau - \sigma)x(t - s - \sigma) \leqslant 0, \quad t_k < t \leqslant t_{k+1},\ k = l, l+1, \ldots. \tag{2.3}$$

From (2.2), we have

$$w(t_k^+) = x(t_k^+) - R(t_k^+)x((t_k - r)^+) - \int_{t_k-s}^{t_k} Q(u)x(u-\sigma)\,\mathrm{d}u - \int_{t_k}^{t_k-s+\tau-\sigma} P(u)x(u-\tau)\,\mathrm{d}u. \tag{2.4}$$

In view of $0 < b_k \leqslant 1$ and condition (l_2), when $t_k - r = t_{k_i}\,(k_i < k)$, then

$$R(t_k^+)x((t_k - r)^+) \geqslant R(t_k^+)b_{k_i}x(t_k - r) = b_k R(t_k^+)x(t_k - r) \geqslant R(t_k)x(t_k - r), \tag{2.5}$$

when $t_k - r \neq t_{k_i}$ $(k_i < k)$, then

$$R(t_k^+)x((t_k - r)^+) \geqslant R(t_k)x(t_k - r). \tag{2.6}$$

So from (2.4)–(2.6) we have

$$\begin{aligned} w(t_k^+) &= I_k(x(t_k)) - R(t_k^+)x((t_k-r)^+) - \int_{t_k-s}^{t_k} Q(u)x(u-\sigma)\,\mathrm{d}u - \int_{t_k}^{t_k-s+\tau-\sigma} P(u)x(u-\tau)\,\mathrm{d}u \\ &\leqslant x(t_k) - R(t_k)x(t_k-r) - \int_{t_k-s}^{t_k} Q(u)x(u-\sigma)\,\mathrm{d}u - \int_{t_k}^{t_k-s+\tau-\sigma} P(u)x(u-\tau)\,\mathrm{d}u = w(t_k). \end{aligned} \tag{2.7}$$

Eqs. (2.3) and (2.7) imply $w(t)$ is nonincreasing on $[t_l, \infty)$. We firstly claim $w(t_k)\geqslant 0$ for $k=l, l+1, \ldots$. Otherwise, suppose that there exists some $k\geqslant l$ such that $w(t_k)=-\mu<0$. From (2.3) and (2.7) $w(t)\leqslant -\mu<0$ for $t\geqslant t_k$. We claim that $x(t)$ is bounded. Otherwise, there exists a sequence of points $\{s_n\}_{n=1}^{\infty}$ such that $s_n\to\infty$, $x(s_n^+)\to\infty$ as $n\to\infty$ and

$$x(s_n^+) = \max\{x(t)\colon t_k\leqslant t\leqslant s_n\}, \quad n=1,2,\ldots,$$

where $x(s_n^+)=x(s_n)$ if s_n is not an impulsive point. From (2.1) and (2.2), we have

$$\begin{aligned} x(s_n^+) &= -\mu + R(s_n^+)x(s_n^+-r) + \int_{s_n-s}^{s_n} Q(u)x(u-\sigma)\,\mathrm{d}u + \int_{s_n}^{s_n-s+\tau-\sigma} P(u)x(u-\tau)\,\mathrm{d}u \\ &\leqslant -\mu + \left(R(s_n^+)x(s_n^+-r) + \int_{s_n-s}^{s_n} Q(u)\,\mathrm{d}u + \int_{s_n}^{s_n-s+\tau-\sigma} P(u)\,\mathrm{d}u\right)x(s_n^+) \\ &\leqslant -\mu + x(s_n^+). \end{aligned}$$

It means $\mu\leqslant 0$, which is a contradiction. Thus, $x(t)$ is bounded. From (2.2) we have

$$x(t)\leqslant -\mu + R(t)x(t-r) + \int_{t-s}^{t} Q(u)x(u-\sigma)\,\mathrm{d}u + \int_{t}^{t-s+\tau-\sigma} P(u)x(u-\tau)\,\mathrm{d}u.$$

From Lemma 2.1 we have

$$\begin{aligned} x(t_k+n\rho) &\leqslant -\mu + R(t_k+n\rho)x(t_k+n\rho-r) + \int_{t_k+n\rho-s}^{t_k+n\rho} Q(u)x(u-\sigma)\,\mathrm{d}u \\ &\quad + \int_{t_k+n\rho}^{t_k+n\rho-s+\tau-\sigma} P(u)x(u-\tau)\,\mathrm{d}u \\ &\leqslant -\mu + R(t_k+n\rho)x(t_k+n\rho-r) + x(t_k+n\rho-\sigma-\xi_1)\int_{t_k+n\rho-s}^{t_k+n\rho} Q(u)\,\mathrm{d}u \\ &\quad + x(t_k+n\rho-\tau+\xi_2)\int_{t_k+n\rho}^{t_k+n\rho-s+\tau-\sigma} P(u)\,\mathrm{d}u, \end{aligned}$$

where $0\leqslant\xi_1\leqslant s$, $0\leqslant\xi_2\leqslant\tau-\sigma-s$. Set

$$x(L_1) = \max\{x(t_k+n\rho-r), x(t_k+n\rho-\sigma-\xi_1), x(t_k+n\rho-\tau+\xi_2)\},$$

clearly, $t_k+(n-1)\rho\leqslant L_1\leqslant t_k+n\rho-\delta$, then

$$\begin{aligned} x(t_k+n\rho) &\leqslant -\mu + \left(R(t_k+n\rho) + \int_{t_k+n\rho-s}^{t_k+n\rho} Q(u)\,\mathrm{d}u + \int_{t_k+n\rho}^{t_k+n\rho-s+\tau-\sigma} P(u)\,\mathrm{d}u\right)x(L_1) \\ &\leqslant -\mu + x(L_1). \end{aligned}$$

Similarly,

$$x(L_1)\leqslant -\mu + x(L_2),$$

where $t_k+(n-2)\rho\leqslant L_1-\rho\leqslant L_2\leqslant L_1-\rho\leqslant t_k+n\rho-2\delta$, which means

$$x(t_k+n\rho)\leqslant -2\mu + x(L_2).$$

In general, one can easily prove that

$$x(t_k + n\rho) \leqslant -n\mu + x(L_n), \tag{2.8}$$

where, $t_k \leqslant L_n \leqslant t_k + n(\rho - \delta)$ for $n = 1, 2, \ldots$.

Because $x(t)$ is bounded, from (2.8), we have

$$x(t_k + n\rho) \leqslant -n\mu + x(L_n) \to -\infty \quad (n \to \infty),$$

which is a contradiction for $x(t) > 0$, $t \geqslant t_0$. Then $w(t_k) \geqslant 0$ for $k = l, l+1, \ldots$.

To prove $w(t) > 0$ for $t > t_l$, we firstly prove that $w(t_k) > 0$. If it is not true, then there exists some $\bar{k} \geqslant l$ such that $w(t_{\bar{k}}) = 0$, thus from (2.3) and (2.7), we obtain

$$\begin{aligned} w(t_{\bar{k}+1}) &\leqslant w(t_{\bar{k}}^{+}) - \int_{t_{\bar{k}}}^{t_{\bar{k}+1}} H(t - s + \tau - \sigma)x(t - s - \sigma)\,\mathrm{d}t \\ &\leqslant w(t_{\bar{k}}) - \int_{t_{\bar{k}}}^{t_{\bar{k}+1}} H(t - s + \tau - \sigma)x(t - s - \sigma)\,\mathrm{d}t < 0. \end{aligned}$$

This contradiction shows that $w(t_k) > 0$ for $k = l, l+1, \ldots$. Therefore, from (2.3) we have

$$w(t) \geqslant w(t_{k+1}) > 0, \quad t \in (t_k, t_{k+1}].$$

So,

$$w(t) > 0 \quad \text{for } t \geqslant t_l.$$

The proof of (ii) is similar and thus is omitted.

The proof is complete. □

Lemma 2.3. *Assume that* (l_2) *holds and* $w(t)$ *is defined by* (2.2). *Suppose there exists a real number* $s \in [0, \tau - \sigma]$ *such that*

$$W_s(t) = R(t) + \int_{t-s}^{t} Q(u)\,\mathrm{d}u + \int_{t}^{t-s+\tau-\sigma} P(u)\,\mathrm{d}u \geqslant 1 \quad \text{for } t \geqslant t_0. \tag{2.9}$$

Further assume that the second-order impulsive differential inequality

$$\begin{cases} y''(t) + \dfrac{1}{\rho} H(t - s + \tau - \sigma)y(t) \leqslant 0, & t \geqslant t_0, \ \ t \neq t_k, \\ y(t_k^{+}) = y(t_k), & k = 1, 2, \ldots, \\ y'(t_k^{+}) \leqslant y'(t_k), & k = 1, 2, \ldots \end{cases} \tag{2.10}$$

has no eventually positive solution. Then

(i) *if* $x(t)$ *is a solution of* (1.1) *and* (1.2) *such that* $x(t) > 0$ *for* $t \geqslant t_0$, *then*

$$w(t) < 0 \quad \textit{for large } t;$$

(ii) *if* $x(t)$ *is a solution of* (1.1) *and* (1.2) *such that* $x(t) < 0$ *for* $t \geqslant t_0$, *then*

$$w(t) > 0 \quad \textit{for large } t.$$

Proof. (i) From the proof of Lemma 2.2, $w(t)$ is nonincreasing for $t \geqslant t_l$, where $l = \min\{k > 0, t_k \geqslant t_0 + \rho\}$. Suppose that (i) is not true, without loss of generality we assume $w(t) \geqslant 0$ for $t \geqslant t_l$. Set $M = 2^{-1} \min\{x(t) : t_l - \rho \leqslant t \leqslant t_l\}$, then $M > 0$ and $x(t) > M$ for $t_l - \rho \leqslant t \leqslant t_l$. We claim that

$$x(t) > M, \quad t \in (t_l, t_{l+1}]. \tag{2.11}$$

If (2.11) does not hold, then there exists a $t^* \in (t_l, t_{l+1}]$ such that $x(t^*) = M$ and $x(t) > M$ for $t_l - \rho \leqslant t < t^*$. From (2.2) we have

$$M = x(t^*) = w(t^*) + R(t^*)x(t^* - r) + \int_{t^*-s}^{t*} Q(u)x(u-\sigma)\,\mathrm{d}u + \int_{t^*}^{t^*-s+\tau-\sigma} P(u)x(u-\tau)\,\mathrm{d}u$$
$$> \left(R(t^*) + \int_{t^*-s}^{t*} Q(u)\,\mathrm{d}u + \int_{t^*}^{t^*-s+\tau-\sigma} P(u)\,\mathrm{d}u\right) M \geqslant M,$$

which is a contradiction. So (2.11) holds. Noting that $w(t_{l+1}^+) \geqslant 0$ and (2.5), (2.6), we have

$$x(t_{l+1}^+) = w(t_{l+1}^+) + R(t_{l+1}^+)x((t_{l+1} - r)^+) + \int_{t_{l+1}-s}^{t_{l+1}} Q(u)x(u-\sigma)\,\mathrm{d}u + \int_{t_{l+1}}^{t_{l+1}-s+\tau-\sigma} P(u)x(u-\tau)\,\mathrm{d}u$$
$$\geqslant R(t_{l+1})x(t_{l+1} - r) + \int_{t_{l+1}-s}^{t_{l+1}} Q(u)x(u-\sigma)\,\mathrm{d}u + \int_{t_{l+1}}^{t_{l+1}-s+\tau-\sigma} P(u)x(u-\tau)\,\mathrm{d}u$$
$$> \left(R(t_{l+1}) + \int_{t_{l+1}-s}^{t_{l+1}} Q(u)\,\mathrm{d}u + \int_{t_{l+1}}^{t_{l+1}-s+\tau-\sigma} P(u)\,\mathrm{d}u\right) M \geqslant M.$$

Repeating the above argument, by induction, we obtain

$$x(t) > M, \quad t \geqslant t_l - \rho. \tag{2.12}$$

Because $w(t) \geqslant 0$ and $w(t)$ is nonincreasing, $\lim_{t\to\infty} w(t)$ exists. Let $\lim_{t\to\infty} w(t) = a$. There are two possible cases.

Case I: $a = 0$. Let $T_1 > t_l$ be such that $w(t) \leqslant M/2$ for $t \geqslant T_1$. Then for any $\bar{t} > T_1$, we have

$$\rho^{-1} \int_{\bar{t}}^{t+\rho} w(v)\,\mathrm{d}v \leqslant M < x(t), \quad t \in [\bar{t}, \bar{t} + \rho].$$

Case II: $a > 0$. Then $w(t) \geqslant a$ for $t \geqslant t_l$. From (2.2) and (2.11), we get

$$x(t) \geqslant a + R(t)x(t-r) + \int_{t-s}^{t} Q(u)x(u-\sigma)\,\mathrm{d}u + \int_{t}^{t-s+\tau-\sigma} P(u)x(u-\tau)\,\mathrm{d}u$$
$$> a + \left(R(t) + \int_{t-s}^{t} Q(u)\,\mathrm{d}u + \int_{t}^{t-s+\tau-\sigma} P(u)\,\mathrm{d}u\right) M \geqslant a + M, \quad t \geqslant t_l.$$

By induction, it is easy to see that $x(t) \geqslant na + M$, for $t \geqslant t_l + (n-1)\rho$, and so $\lim_{t\to\infty} x(t) = \infty$, which implies that there exists a $T > T_1$ such that

$$\rho^{-1} \int_{T}^{t+\rho} w(v)\,\mathrm{d}v \leqslant 2w(T) < x(t), \quad t \in [T, T + \rho].$$

Combining Cases I and II we see that

$$x(t) > \rho^{-1} \int_{T}^{t+\rho} w(v)\,\mathrm{d}v, \quad t \in [T, T + \rho].$$

Let $l^* = \min\{k \geqslant l : t_k > T + \rho\}$, we claim that

$$x(t) > \rho^{-1} \int_{T}^{t+\rho} w(v)\,\mathrm{d}v, \quad t \in [T + \rho, t_{l^*}]. \tag{2.13}$$

Otherwise, there exists a $t^* \in (T + \rho, t_{l^*}]$ such that

$$x(t^*) = \rho^{-1} \int_{T}^{t^*+\rho} w(v)\,\mathrm{d}v \quad \text{and} \quad x(t) > \rho^{-1} \int_{T}^{t+\rho} w(v)\,\mathrm{d}v, \quad t \in (T + \rho, t^*).$$

Then, from (2.2), we have

$$\begin{aligned}
\rho^{-1}\int_T^{t^*+\rho} w(v)\,\mathrm{d}v = x(t^*) &= w(t^*) + R(t^*)x(t^*-r) + \int_{t^*-s}^{t^*} Q(u)x(u-\sigma)\,\mathrm{d}u \\
&\quad + \int_{t^*}^{t^*-s+\tau-\sigma} P(u)x(u-\tau)\,\mathrm{d}u \\
&> \rho^{-1}\int_{t^*}^{t^*+\rho} w(v)\,\mathrm{d}v + \rho^{-1}R(t^*)\int_T^{t^*-r+\rho} w(v)\,\mathrm{d}v + \rho^{-1}\int_{t^*-s}^{t^*} Q(u)\int_T^{u-\sigma+\rho} w(v)\,\mathrm{d}v\,\mathrm{d}u \\
&\quad + \rho^{-1}\int_{t^*}^{t^*-s+\tau-\sigma} P(u)\int_T^{u-\tau+\rho} w(v)\,\mathrm{d}v\,\mathrm{d}u \\
&\geqslant \rho^{-1}\int_{t^*}^{t^*+\rho} w(v)\,\mathrm{d}v + \rho^{-1}R(t^*)\int_T^{t^*} w(v)\,\mathrm{d}v + \rho^{-1}\int_{t^*-s}^{t^*} Q(u)\int_T^{t^*-s-\sigma+\rho} w(v)\,\mathrm{d}v\,\mathrm{d}u \\
&\quad + \rho^{-1}\int_{t^*}^{t^*-s+\tau-\sigma} P(u)\int_T^{t^*-\tau+\rho} w(v)\,\mathrm{d}v\,\mathrm{d}u \\
&\geqslant \rho^{-1}\int_{t^*}^{t^*+\rho} w(v)\,\mathrm{d}v + \rho^{-1}R(t^*)\int_T^{t^*} w(v)\,\mathrm{d}v + \rho^{-1}\int_{t^*-s}^{t^*} Q(u)\int_T^{t^*} w(v)\,\mathrm{d}v\,\mathrm{d}u \\
&\quad + \rho^{-1}\int_{t^*}^{t^*-s+\tau-\sigma} P(u)\int_T^{t^*} w(v)\,\mathrm{d}v\,\mathrm{d}u \\
&= \rho^{-1}\int_{t^*}^{t^*+\rho} w(v)\,\mathrm{d}v + \rho^{-1}\int_T^{t^*} w(v)\,\mathrm{d}v \\
&\quad \times \left(R(t^*) + \int_{t^*-s}^{t^*} Q(u)\,\mathrm{d}u + \int_{t^*}^{t^*-s+\tau-\sigma} P(u)\,\mathrm{d}u\right) \\
&\geqslant \rho^{-1}\int_{t^*}^{t^*+\rho} w(v)\,\mathrm{d}v + \rho^{-1}\int_T^{t^*} w(v)\,\mathrm{d}v = \rho^{-1}\int_T^{t^*+\rho} w(v)\,\mathrm{d}v.
\end{aligned}$$

This contradiction shows (2.13) holding. Similarly, from condition (l_2), (2.2) and (2.13) we have

$$\begin{aligned}
x(t_{l^*}^+) &= w(t_{l^*}^+) + R(t_{l^*}^+)x((t_{l^*}-r)^+) + \int_{t_{l^*}-s}^{t_{l^*}} Q(u)x(u-\sigma)\,\mathrm{d}u + \int_{t_{l^*}}^{t_{l^*}-s+\tau-\sigma} P(u)x(u-\tau)\,\mathrm{d}u \\
&\geqslant w(t_{l^*}^+) + R(t_{l^*})x(t_{l^*}-r) + \int_{t_{l^*}-s}^{t_{l^*}} Q(u)x(u-\sigma)\,\mathrm{d}u + \int_{t_{l^*}}^{t_{l^*}-s+\tau-\sigma} P(u)x(u-\tau)\,\mathrm{d}u \\
&> \rho^{-1}\int_{t_{l^*}}^{t_{l^*}+\rho} w(v)\,\mathrm{d}v + \rho^{-1}\int_T^{t_{l^*}} w(v)\,\mathrm{d}v \\
&= \rho^{-1}\int_T^{t_{l^*}+\rho} w(v)\,\mathrm{d}v.
\end{aligned}$$

Repeating the above procedure, by induction, we can see that

$$x(t) > \rho^{-1}\int_T^{t+\rho} w(v)\,\mathrm{d}v, \quad t \geqslant T. \tag{2.14}$$

Thus, by (2.2), (2.3), we have

$$\begin{aligned}
w'(t) &\leqslant -H(t-s+\tau-\sigma)x(t-s-\sigma) \\
&\leqslant \frac{-H(t-s+\tau-\sigma)}{\rho}\int_T^{t-s-\sigma+\rho} w(v)\,\mathrm{d}v \\
&\leqslant \frac{-H(t-s+\tau-\sigma)}{\rho}\int_T^{t} w(v)\,\mathrm{d}v,
\end{aligned}$$

where $t \geqslant T + \rho$ and $t \neq t_k$. Set

$$y(t) = \rho^{-1} \int_T^t w(v)\,\mathrm{d}v.$$

Then $y(t_k^+) = y(t_k)$, $y'(t_k^+) = \rho^{-1} w(t_k^+) \leqslant \rho^{-1} w(t_k) = y'(t_k)$ for $k = l, l+1, \ldots$. Thus $y(t) > 0$, $y'(t_k^+) > 0$ for $t > T + \rho$ and $y(t)$ satisfies (2.10), which contradicts the assumption that (2.10) has no eventually positive solution. So $w(t)$ is eventually negative. The proof of (i) is complete.

(ii) The proof of (ii) is similar and thus is omitted.

The proof of Lemma 2.3 is complete. □

The following Lemma 2.4 follows from the similar arguments to that in [1, Theorem 1] by letting $\varphi(x) = x$. We omit the details.

Lemma 2.4. *Consider the impulsive differential inequality*

$$\begin{aligned} &y''(t) + G(t)y(t) \leqslant 0, \quad t \geqslant t_0, \quad t \neq t_k, \\ &y(t_k^+) \geqslant y(t_k), \quad k = 1, 2, \ldots, \\ &y'(t_k^+) \leqslant C_k y'(t_k), \quad k = 1, 2, \ldots, \end{aligned} \tag{2.15}$$

where $0 \leqslant t_0 < t_1 < \cdots < t_k \to \infty$ *as* $k \to \infty$, $G(t) \in PC([t_0, \infty), R^+)$ *and* $C_k > 0$. *If*

$$\sum_{i=0}^{\infty} \int_{t_i}^{t_{i+1}} \frac{1}{C_0 C_1 \cdots C_i} G(t)\,\mathrm{d}t = \infty,$$

where $C_0 = 1$. *Then inequality* (2.15) *has no solution* $y(t)$ *such that* $y(t) > 0$ *for* $t \geqslant t_0$.

Theorem 2.1. *Assume that condition* (l_2) *holds and there exist two real numbers* $s_1, s_2 \in [0, \tau - \sigma]$ *such that*

$$W_{s_1}(t) = R(t) + \int_{t-s_1}^{t} Q(u)\,\mathrm{d}u + \int_{t}^{t-s_1+\tau-\sigma} P(u)\,\mathrm{d}u \leqslant 1 \tag{2.16}$$

and

$$W_{s_2}(t) = R(t) + \int_{t-s_2}^{t} Q(u)\,\mathrm{d}u + \int_{t}^{t-s_2+\tau-\sigma} P(u)\,\mathrm{d}u \geqslant 1 \tag{2.17}$$

for large t. Further assume that (2.10) *has no eventually positive solution, then every solution of* (1.1) *and* (1.2) *oscillates.*

Proof. In fact, suppose that Eqs. (1.1) and (1.2) have an eventually positive solution, then the conditions of Theorem 2.1 and Lemma 2.2 imply eventually $w(t) > 0$, while Lemma 2.3 implies eventually $w(t) < 0$. This contradiction shows that $x(t)$ cannot be an eventually positive solution of (1.1) and (1.2). On the other hand, if $x(t)$ is an eventually negative solution of (1.1) and (1.2), then Lemma 2.2 implies eventually $w(t) < 0$, while Lemma 2.3 implies eventually $w(t) > 0$. This contradiction shows that $x(t)$ cannot be an eventually negative solution of (1.1) and (1.2). Therefore, every solution of (1.1) and (1.2) oscillates. □

From Lemma 2.4 and Theorem 2.1, it is easy to see that the following Theorem 2.2 is true.

Theorem 2.2. *Assume* (2.16), (2.17) *and* (l_2) *hold, and that*

$$\frac{1}{\rho} \int_{t_0}^{\infty} H(t - s + \tau - \sigma)\,\mathrm{d}t = \infty. \tag{2.18}$$

Then every solution of (1.1) *and* (1.2) *oscillates.*

In fact, note that

$$G(t)=\frac{1}{\rho}H(t-s+\tau-\sigma),$$

thus, one has

$$\sum_{i=0}^{\infty}\int_{t_i}^{t_{i+1}}\frac{1}{C_0C_1\cdots C_i}G(t)\,\mathrm{d}t=\frac{1}{\rho}\int_{t_0}^{\infty}H(t-s+\tau-\sigma)\,\mathrm{d}t=\infty.$$

By Lemma 2.4 and Theorem 2.1 we see that Theorem 2.2 is true.

3. An example

Example 3.1. Consider the differential equation

$$[x(t)-\tfrac{1}{2}x(t-\tfrac{1}{2})]'+(\tfrac{1}{2}+t^{-1})x(t-2)-(\tfrac{1}{2}-t^{-1})x(t-1)=0,\quad t\geqslant 3, \tag{3.1}$$

$$x(t_k^+)=\frac{k}{k+1}x(t_k),\quad k=4,5,\ldots, \tag{3.2}$$

where $t_k=k$, $H(t)=t^{-1}+(t-1)^{-1}$.

Clearly, (A_1)–(A_4) and condition (l_2) hold,

$$W_1(t)=\frac{1}{2}+\int_{t-1}^{t}\left(\frac{1}{2}-s^{-1}\right)\mathrm{d}s=1-\int_{t-1}^{t}s^{-1}\,\mathrm{d}s\leqslant 1\quad\text{for } t\geqslant 3,$$

$$W_0(t)=\frac{1}{2}+\int_{t}^{t+1}\left(\frac{1}{2}+s^{-1}\right)\mathrm{d}s\geqslant 1\quad\text{for } t\geqslant 3,$$

$$\int_3^{\infty}H(t+\tau-\sigma)\,\mathrm{d}t=\int_3^{\infty}[(t+1)^{-1}+t^{-1}]\,\mathrm{d}t=\infty,$$

$$\int_3^{\infty}H(t)\,\mathrm{d}t=\int_3^{\infty}[t^{-1}+(t-1)^{-1}]\,\mathrm{d}t=\infty.$$

It follows that (2.16)–(2.18) hold. By Theorem 2.2, every solution of (3.1) and (3.2) oscillates.

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