On the Semiring of Cone Preserving Maps*

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ABSTRACT

If \( K \) is a proper cone in \( \mathbb{R}^n \), then the cone of all linear operators that preserve \( K \), denoted by \( \pi(K) \), forms a semiring under usual operator addition and multiplication. Recently J. C. Horne examined the ideals of this semiring. He proved that if \( K_1, K_2 \) are polyhedral cones such that \( \pi(K_1) \) and \( \pi(K_2) \) are isomorphic as semirings, then \( K_1 \) and \( K_2 \) are linearly isomorphic. The study of this semiring is continued in this paper.

In Sec. 3 ideals of \( \pi(K) \) which are also faces are characterized. In Sec. 4 it is shown that \( \pi(K) \) has a unique minimal two-sided ideal, namely, the dual cone of \( \pi(K^*) \), where \( K^* \) is the dual cone of \( K \). Extending Horne's result, it is also proved that the cone \( K \) is characterized by this unique minimal two-sided ideal of \( \pi(K) \). The set of all faces of \( \pi(K) \) inherits a quotient semiring structure from \( \pi(K) \). Properties of this face-semiring are given in Sec. 5. In particular, it is proved that this face-semiring admits no nontrivial congruence relation iff the duality operator of \( \pi(K) \) is injective. In Sec. 6 the maximal one-sided and two-sided ideals of \( \pi(K) \) are identified. In Sec. 8 it is shown that \( \pi(K) \) never satisfies the ascending-chain condition on principal one-sided ideals. Some partial results on the question of topological closedness of principal one-sided ideals of \( \pi(K) \) are also given.

0. INTRODUCTION

Let \( K \) be a cone (convex, pointed, closed, and full) in the euclidean space \( \mathbb{R}^n \). Denote by \( \pi(K) \) the set of all linear operators that preserve \( K \). It is well known that \( \pi(K) \) is a cone in \( \text{Hom}(\mathbb{R}^n) \), the vector space of all linear operators of \( \mathbb{R}^n \). There is an immense literature devoted to the spectral

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properties of the individual operators in \( \pi(K) \), generalizing the classical
Perron-Frobenius theorems on nonnegative matrices (see Barker and Schneider [6]
and its references). In the past few years, many research
workers in the field have devoted their attention to the geometric properties
of the cone \( \pi(K) \) itself. Several results on the decomposability of this cone
and its extreme operators have been found (see Barker and Loewy [5],
Fiedler, Haynsworth, and Pták [9], Fiedler and Pták [11], Lowey and
Schneider [15, 16], and O'Brien [17]). There are also some characterizations
of special types of cones \( K \) in terms of their corresponding cones \( \pi(K) \) (see
Barker and Loewy [5], Tam [20, 21]). Recently Horne [12] looked at the
algebraic properties of \( \pi(K) \) as a semiring (under the usual operations of
operator addition and composition). He examined the ideal structure of this
semiring and proved that for polyhedral cones \( K \) this semiring characterizes
\( K \). Barker [4] also studied certain left and right ideals in this semiring for
perfect cones \( K \). In this paper we continue to study the algebraic properties
of the semiring \( \pi(K) \). We shall show that this semiring possesses many
peculiar properties.

A facial ideal of \( \pi(K) \) is an ideal which is also a face. In Sec. 3 we give a
characterization of facial ideals. For instance, a right facial ideal consists of
all \( A \in \pi(K) \) such that \( AF \subseteq F \) for some fixed face \( F \). We also show that
\( A \in \pi(K) \) is (say) a right zero divisor iff \( A \) belongs to a proper right facial
ideal.

In Sec. 4 we show that every semiring \( \pi(K) \) has a unique minimal
two-sided ideal. This ideal is in fact the dual cone of \( \pi(K^*) \), where \( K^* \) is the
dual cone of \( K \). (The duality is defined with respect to usual inner products;
see Sec. 1.) With an exceptional case, this minimal two-sided ideal \( \mathcal{M} \) has the
following property: If \( A \) is a noninvertible element of \( \pi(K) \) and \( A \not\in \mathcal{M} \), then
there exists \( B \in \mathcal{M} \) such that \( AB \in \mathcal{M} \). Extending Horne’s result, we also
prove that this minimal two-sided ideal characterizes the cone \( K \) [and hence
the semiring \( \pi(K) \)].

The set of all faces of \( \pi(K) \) inherits a semiring structure from \( \pi(K) \). In
Section 5 we study the properties of this face-semiring. If \( K = R^n_+ \) (the
nonnegative orthant), then the face-semiring of \( \pi(K) \) can be identified with
the semiring of all \( n \times n \) matrices over the lattice \( \{0, 1\} \) (with operations
\( 1 + 1 = 1 \), etc.). We show that the face-semiring of \( \pi(K) \) is simple (i.e., it has
no nontrivial congruence relations) iff the duality operator of \( \pi(K) \) is
injective (see Sec. 1 for the definition of the duality operator). In particular,
the semiring of \( (0, 1) \)-matrices mentioned above is simple. The semiring
\( \pi(R^n_+) \) and this matrix semiring were studied by Horne [12]; however, he did
not mention that the faces of \( \pi(R^n_+) \) form a semiring. We show that many of
his results remain valid for arbitrary semirings \( \pi(K) \).

In Sec. 6 we describe maximal left (or right) ideals of \( \pi(K) \). They are in
1-1 correspondence with the indecomposable subcones in a direct
decomposition of $K$. We also identify all maximal two-sided ideals (their number does not exceed the number of indecomposable subcones just mentioned).

In our characterization of maximal two-sided ideals of $\pi(K)$, we encounter an interesting relation between indecomposable subcones which appear in a direct decomposition of $K$. Two examples are given in Sec. 7 to illustrate the relation.

In Sec. 8 we answer a question which was raised by Horne [12]: we show that the semiring $\pi(K)$ never satisfies the ascending-chain condition on principal one-sided ideals. Then we consider the problem of topological closedness of principal one-sided ideals in this semiring. Some partial results are given.

Finally, in Sec. 9 we mention some open questions that may be of interest.

1. **Preliminaries**

A nonempty subset $K$ in a finite-dimensional real vector space $V$ is called a cone if $K + K \subseteq K$ and $\alpha K \subseteq K$ for all $\alpha \geq 0$; $K$ is pointed if $K \cap (-K) = \{0\}$; $K$ is reproducing if $K - K = V$. If $K$ is closed (in the usual topology of $V$) and satisfies all the above properties, $K$ is called a proper cone.

The interior, closure, and boundary of a convex set $S$ in $V$ will be denoted respectively by $\text{int } S$, $\text{cl } S$, and $\partial S$. The relative interior of $S$ in its affine hull will be denoted by $S^\Delta$. For a cone $K$ in $V$, $K$ is reproducing iff $\text{int } K = \emptyset$.

We assume familiarity with the elementary properties of cones. For convenience and to fix notation, below we collect some of the definitions and cite relevant references.

Let $K$ be a pointed, closed cone. A nonempty subset $F$ of $K$ is called a face of $K$, denoted by $F \sqsubseteq K$, if $F$ is itself a cone and in addition satisfies the following: if $x, y \in K$ such that $x + y \in F$, then $x, y \in F$. If $S \subseteq K$, then the smallest face containing $S$ is called the face generated by $S$ and is denoted by $F(S)$. If $S = \{x\}$, we write $F(x)$ for simplicity. If $x \neq 0$ and if $F(x) = \{\alpha x : \alpha > 0\}$, then $F(x)$ is called an extreme ray and $x$ an extreme vector of $K$. The set of all extreme vectors of $K$ is denoted by $\text{Ext } K$. It is known that $\mathcal{F}(K)$, the collection of all faces of $K$, becomes a complete lattice of finite length under the operations meet and join given by $F \wedge G = F \cap G$ and $F \vee G = \Phi(F \cup G)$. We shall often tacitly make use of the basic properties of faces (see Barker [2], Barker and Schneider [6, Sec. 2], and Fiedler and Pták [10, Sec. 1]). In particular, we need the following result (Barker and Schneider [6, Lemma 2.20]): Let $F \sqsubseteq K$ and let $x \in K$. Then $x \in F^\Delta$ iff $F = F(x)$. 
Duality plays an important role in the theory of cones. Although the concept can be defined in a more general setting in terms of sets in a vector space and its dual space (see, for instance, Fiedler, Haynsworth, and Pták [9]), for the sake of simplicity and brevity we shall restrict ourselves to euclidean spaces. By making obvious changes, most of the results we shall obtain can be extended to more general situations.

Let $S$ be a subset of a euclidean space $V$ with inner product $(\cdot, \cdot)$. The set $S^* = \{z \in V : (z, y) > 0 \text{ for all } y \in S\}$ is called the dual of $S$. For properties of $S$, see Berman [7, Chapter 1]. The dual $S^*$ of a proper cone $S$ is also a proper cone, known as the dual cone of $S$; furthermore $S^{**} = S$, and we have the following useful characterization of the interior of $S$ (see Schneider and Vidyasagar [19]):

$$y \in \text{int } S \iff (z, y) > 0 \text{ for all nonzero vectors } z \in S^*.$$

In the sequel we shall use $K$ to denote a proper cone in $\mathbb{R}^n$, the vector space of all $n$-dimensional real column vectors ($n \geq 1$). The inner product in $\mathbb{R}^n$ is given by $(z, y) = z^T y$, where $z^T$ is the transpose of $z$. We shall identify an $n \times n$ real matrix with the linear operator (of $\mathbb{R}^n$) which it represents. Thus if $y, z \in \mathbb{R}^n$, $zy^T$ is the linear operator given by $zy^T(x) = (y^T x) z$. In $\text{Hom}(\mathbb{R}^n)$, the space of all linear operators of $\mathbb{R}^n$, we introduce the usual inner product: $\langle B, A \rangle = \text{trace } B^T A$. We have a useful relation between the inner product of $\mathbb{R}^n$ and that of $\text{Hom}(\mathbb{R}^n)$:

For any linear operator $A \in \text{Hom}(\mathbb{R}^n)$ and vectors $y, z \in \mathbb{R}^n$,

$$\langle z, Ay \rangle = \langle zy^T, A \rangle.$$

By the duality operator of $K$, we mean the mapping $d_K : \mathcal{T}(K) \to \mathcal{T}(K^*)$ given by $d_K(F) = (\text{span } F)^\perp \cap K^*$, where $(\text{span } F)^\perp$ is the orthogonal complement of the linear span of $F$. We call $d_K(F)$ the dual face of $F$ and write simply $d(F)$ when there is no ambiguity. The concept has been introduced and studied independently by Barker [3] and Tam [21]. A face $F$ of $K$ is said to be exposed if $F = d_K(G)$ for some $G \in \mathcal{T}(K^*)$. Geometrically, a proper face of $K$ is exposed iff it is the intersection of $K$ with a hyperplane. Barker [3] has shown that the mapping $d_{K^*} \circ d_K$ is a closure operation on $\mathcal{T}(K)$. We shall denote it by $\text{cl}_K$, or simply $\text{cl}$ when there is no danger of confusion. (It should not be mixed up with the topological closure, as every face is topologically closed.) Barker [3] has also proved that $F \subseteq K$ is exposed iff $\text{cl } F = F$. We omit the simple proof of
PROPOSITION 1.1.  The following are equivalent:

(i) $d_K$ is injective,
(ii) every face of $K$ is exposed,
(iii) for any $F \subseteq K$, $\text{cl } F = F$,
(iv) $d_K$ is surjective.

2. THE CONE $\pi(K)$

Following the notation of Schneider and Vidyasagar [19], we write

$$\pi(K) = \{ A \in \text{Hom}(R^n) : AK \subseteq K \}$$
$$\pi^+(K) = \{ A \in \pi(K) : A(K \setminus \{0\}) \subseteq \text{int } K \}$$

Matrices in $\pi(K)$ are called positive operators on $K$. It is known that $\pi(K)$ is a proper cone of Hom($R^n$) and that $\pi^+(K) = \text{int } \pi(K)$. With respect to the usual inner product of Hom($R^n$), the dual cone $\pi(K)^*$ is in fact the positive hull of the set of all matrices of the form $zy^T$ with $y \in K$ and $z \in K^*$ (Tam [20, Theorem 1]).

Fiedler, Haynsworth, and Pták have considered positive operators in a more general setting. They studied the proper cone $\pi(K_1, K_2)$ which consists of all linear operators $A$ such that $AK_1 \subseteq K_2$, where $K_1$ and $K_2$ are proper cones in finite-dimensional real vector spaces (especially when $K_1$ and $K_2$ are polyhedral).

The author has studied the face lattice of the cone $\pi(K)$. In [21], he introduced a simple kind of faces, namely, those of the form $\pi_{F, G}$ where

$$\pi_{F, G} = \{ A \subseteq \pi(K) : AF \subseteq G \}, \quad F, G \subseteq K.$$ 

In particular, it was proved that maximal faces of $\pi(K)$ are of such form.

Barker [4] has also considered the faces $\pi_{F, 0}$ and $\pi_{K, F}$ for perfect cones $K$, though in different notation. By modifying Barker's proof of $\text{int } \pi(K) = \pi^+(K)$ in [1], we obtain

PROPOSITION 2.1.  For any $F \subseteq K$, $\pi_{K, F}^A = \{ A \in \pi(K) : A(K \setminus \{0\}) \subseteq F^A \}$.

We also need the following simple fact, whose proof we omit:

For any $F, G \subseteq K$, $\pi_{F, G} = \pi(K)$ iff $F = 0$ or $G = K$. 

It is easy to check that the linear transformation of $\text{Hom}(R^n)$ given by $A \rightarrow A^T$ is an isometry which carries $\pi(K)$ onto $\pi(K^*)$. We shall often make use of this correspondence between the cones $\pi(K)$ and $\pi(K^*)$. For any subset $S$ of $\pi(K)$, we denote by $S^T$ the set $\{A^T : A \in S\}$. Then $\Phi(A)^T = \Phi(A^T)$ and $A \in \text{Ext} \pi(K)$ iff $A^T \in \text{Ext} \pi(K^*)$. We shall need the corollary of the following proposition.

**Proposition 2.2.** If $F$, $G$ are faces of $K$, then $\pi^T_{F,G} \subseteq \pi_{d(G),d(F)}$; the equality holds if, in addition, the face $G$ is exposed.

**Proof.** Let $A$ be a linear operator in $\pi_{F,G}$. By definition $AF \subseteq G$. Hence, for any vectors $y \in F$ and $z \in d(G)$, we have $0 = (z, Ay) = (A^Tz, y)$. This shows that $A^T(d(G)) \subseteq d(F)$. Therefore $\pi^T_{F,G} \subseteq \pi_{d(G),d(F)}$. Now assume further that $G$ is exposed. Taking transpose of both sides of the above inclusion, we obtain $\pi_{F,G} \subseteq \pi_{d(G),d(F)}$. On the other hand, using the first part of our proposition, $\pi_{d(G),d(F)} \subseteq \pi_{d(F),d(G)}$. Hence $\pi_{F,G} \subseteq \pi_{d(G),d(F)} \subseteq \pi_{d(F),d(G)}$. Notice that $\pi_{F,G} = \pi_{F,d(G)}$, as $G$ is exposed. The inclusion $\pi_{F,d(G)} \subseteq \pi_{d(F),d(G)}$ is obvious. It follows that $\pi_{F,G} = \pi_{d(G),d(F)}$ and therefore $\pi^T_{F,G} = \pi_{d(G),d(F)}$.

We readily obtain

**Corollary 2.3.** If the duality operator $d_K$ is injective, then $\pi^T_{F,G} = \pi_{d(G),d(F)}$ for any $F \subseteq G \subseteq K$.

In the remainder of this section we collect some material concerning the direct decomposition of $\pi(K)$ which will be needed in the sequel.

Following Loewy and Schneider [15], we say that a cone $K$ is a direct sum of $K_1$ and $K_2$ and we write $K = K_1 \oplus K_2$ if (a) span $K_1 \cap$ span $K_2 = \{0\}$ and (b) $K = K_1 + K_2$. (Then $K_1, K_2 \subseteq K$.) The cone $K$ is called decomposable if there exist nonzero subsets $K_1$ and $K_2$ such that $K = K_1 \oplus K_2$. Otherwise $K$ is indecomposable. The following result is fundamental (Fiedler and Pták [10, (2,6)]):

For any cone $K$, there exist indecomposable cones $K_1, \ldots, K_r$ such that $K = K_1 \oplus \cdots \oplus K_r \ (r \geq 1)$. This decomposition of $K$ is unique (except for a possible renumbering).

Barker and Loewy [5] proved that $K$ is indecomposable iff $\pi(K)$ is indecomposable. Fiedler, Haynsworth, and Pták [9] proved that $\pi(K_1, K_2)$ is decomposable if either $K_1$ or $K_2$ is decomposable; in fact, the converse is also true, as can be shown by a modification of Barker and Loewy's proof.

Now let $K$ be a decomposable cone, $K = K_1 \oplus \cdots \oplus K_r$, its decomposition as a sum of indecomposable subcones. Denote by $P_i$ the corresponding
projections (which are also positive on $K$). Note that if $A'$ is a linear operator in $\pi(K, i_j), 1 \leq i, j \leq r$, then there is a unique linear operator $A$ in $\pi(K)$ such that $A|_{K_j} = A'$ and $A|_{K_h} = 0$ for $h \neq i$. Then $A$ is said to be induced by $A'$. Clearly if $A$ is a linear operator in $\pi(K)$, then $A = \sum_{1 \leq i, j \leq r} P_i A P_j$; furthermore each of the operators $P_i A P_j$ is induced by some operator in $\pi(K, i_j)$. From the foregoing discussions we can deduce

**Proposition 2.4.** Let $K = K_1 \oplus \cdots \oplus K_r$ denote the unique representation of $K$ as a direct sum of indecomposable subcones. Then $\pi(K) = \bigoplus_{1 \leq i, j \leq r} S_{i,j}$, where each subcone $S_{i,j}$ can be identified with the indecomposable cone $\pi(K_i, K_j)$.

**Corollary 2.5.** Under the same assumption, let $P_i$ be the corresponding projections. Then $\Phi(I)$, the face generated by the identity operator $I$, has exactly $r$ extreme rays, namely, $\Phi(P_1), \ldots, \Phi(P_r)$.

**Proof.** This follows readily from the proposition and the following two facts: (1) $\text{Ext} \pi(K) = \bigcup_{1 \leq i, j \leq r} \text{Ext} S_{i,j}$, (2) $K$ is indecomposable iff the identity operator $I \in \text{Ext} \pi(K)$ (Loewy and Schneider [15, Theorem 3.3]).

We shall also need

**Proposition 2.6.** Let $K = K_1 \oplus K_2$ and let $A' \in \pi(K_1)$. Let $A$ be the linear operator in $\pi(K)$ such that $A|_{K_1} = A'$ and $A|_{K_2} = 0$. Denote by $K_1^V$, $\pi(K_1^V)^V$ the duals of $K_1$ and $\pi(K_1^V)$ in their respective linear spans. Then $A \in \pi(K^*)^* \iff A' \in \pi(K_1^V)^V$.

**Proof.** "Only if" part: Suppose $A$ is a nonzero linear operator in $\pi(K^*)^*$. Then $A$ can be expressed as a finite sum $\sum y_a z_a^T$ where $y_a$ are nonzero vectors of $K$ and $z_a$ nonzero vectors of $K^*$. Since $A$ is induced by some linear operator of $\pi(K_1)$, necessarily $y_a \in K_1$ and $z_a \in d(K_2)$. For each $\alpha$, write $z_\alpha = z_\alpha'' + z_\alpha'$ with $z_\alpha'' \in \text{span} K_1$ and $z_\alpha' \in (\text{span} K_1) \perp$. Then $A|_{K_1} = \sum y_a z_\alpha'^T$, and so $A' = \sum y_a z_\alpha'^T$. Furthermore $z_\alpha' \in K_1^V$. Hence $A' \in \pi(K_1^V)^V$.

"If" part: Suppose $A'$ is nonzero and is in $\pi(K_1^V)^V$. Then $A'$ can be written as $\sum y_a z_\alpha'^T$ where $y_a$ are nonzero vectors of $K_1$ and $z_\alpha'$ nonzero vectors of $K_1^V$. For each $\alpha$, choose $z_\alpha'' \in (\text{span} K_1) \perp$ such that $z_\alpha = z_\alpha' + z_\alpha'' \in (\text{span} K_2) \perp$. This is possible because

$$\dim [(\text{span} \{ z_\alpha' \} + (\text{span} K_1) \perp) + \dim (\text{span} K_2) \perp = 1 + (n - \dim K_1) + (n - \dim K_2) = n + 1,$$
so that \[ \text{span}\{z_{\alpha}\} + (\text{span} K_1)^{\perp} \cap (\text{span} K_2)^{\perp} \neq 0 \] whereas \((\text{span} K_1)^{\perp} \cap (\text{span} K_2)^{\perp} = 0\). Hence \(z_{\alpha} \in (\text{span} K_2)^{\perp} \cap K_1^* = d(K_2)\). (Recall that \(K_1^*\) is the dual of \(K_1\) in \(R^n\).) Now it is easy to see that \(A = \sum y_{\alpha} z_{\alpha}^T\) and hence belongs to \(\pi(K^*)^*\).

3. FACIAL IDEALS

A semiring is a set with two binary operations (addition and multiplication) such that:

1. Addition is associative and commutative.
2. Multiplication is associative and distributive over addition.

We do not assume that a semiring possesses a zero element or an identity element.

Clearly \(\pi(K)\) forms a semiring (with a zero element and an identity element) under the usual operations of operator addition and composition. As in Horne [12], we call a nonempty subset \(\mathcal{J}\) of the semiring \(\pi(K)\) a right ideal if it is closed under addition and has the property that if \(A \in \mathcal{J}\) and \(B \in \pi(K)\) then \(AB \in \mathcal{J}\). Left ideals and two-sided ideals of \(\pi(K)\) are defined similarly. By the term "ideal" alone we refer to one-sided ideals or two-sided ideals. An ideal of \(\pi(K)\) which is also a face will be called a facial ideal. A linear operator of the semiring \(\pi(K)\) is an invertible element iff \(A^{-1}\) exists and belongs to \(\pi(K)\). It is easy to show that \(A\) is an invertible element of \(\pi(K)\) iff \(AK = K\). Loewy and Schneider [15] have proved that \(K\) is indecomposable iff each invertible element of \(\pi(K)\) is extreme. Horne [13] has also studied the group of invertible elements of \(\pi(K)\). Straightforward verifications show that the correspondence \(A \mapsto A^T\) between \(\pi(K)\) and \(\pi(K^*)\) is a semiring antisomorphism. So if we obtain a statement about the right ideals of \(\pi(K)\), we shall have a corresponding statement for its left ideals, and vice versa.

The structure of the ring \(\text{Hom}(R^n)\) is well known (for reference, see Jacobson [14, Chapter 8]). \(\text{Hom}(R^n)\) is a simple ring; it has no nontrivial two-sided ideals. A right ideal of \(\text{Hom}(R^n)\) consists of all linear operators of \(R^n\) which map \(R^n\) into a fixed subspace. A left ideal of \(\text{Hom}(R^n)\) consists of all linear operators which annihilate a fixed subspace. Clearly every ideal of \(\text{Hom}(R^n)\) is a subspace of \(\text{Hom}(R^n)\). Furthermore the intersection of a right ideal of \(\text{Hom}(R^n)\) with \(\pi(K)\) is a right ideal of \(\pi(K)\); and by the above characterization of the right ideals of \(\text{Hom}(R^n)\), it in fact consists of all linear operators in \(\pi(K)\) which map \(K\) into \(K \cap S\) for some fixed subspace \(S\) of \(R^n\). Similarly the set of all linear operators in \(\pi(K)\) which annihilate some
fixed subspace of $R^n$ is a left ideal. Nevertheless, we shall see that unless $K$ is simplicial, $\pi(K)$ always has nontrivial two-sided ideals. Also there are one-sided ideals of $\pi(K)$ which are not of the type described above.

If $R$ is a face of $\pi(K)$ whose linear span is a right ideal of $\text{Hom}(R^n)$, then in view of $R=\pi(K)\cap \text{span} R$ and the above discussion, $R$ is a right facial ideal. Conversely, suppose $R$ is a right facial ideal of $\pi(K)$. Let $A\in \text{span} R$ and $B\in \text{Hom}(R^n)$. Then there exist linear operators $A_1, A_2\subseteq R$ and $B_1, B_2\subseteq \pi(K)$ such that $A=A_1-A_2$ and $B=B_1-B_2$. Hence $AB=(A_1B_1+A_2B_2)-(A_1B_2+A_2B_1)\in R-R=\text{span} R$. Similarly, we check that $\text{span} R$ is closed under addition. So $\text{span} R$ is a right ideal of $\text{Hom}(R^n)$. We have in fact shown that the right facial ideals of $\pi(K)$ are exactly those faces whose linear spans are right ideals of $\text{Hom}(R^n)$. Similar statements hold for the left and the two-sided facial ideals. As $\text{Hom}(R^n)$ has no nontrivial two-sided ideals, $\pi(K)$ has no nontrivial two-sided facial ideals. As to the one-sided facial ideals, we have

**Proposition 3.1.** A subset of $\pi(K)$ is a right facial ideal iff it is of the form $\pi_{K,F}$ for some $F\subseteq K$.

**Proof.** "If" part: Straightforward verification.

"Only if" part: Let $\Phi(A)$ be a right facial ideal of $\pi(K)$. We claim that for any $y\in \text{int} K$, $\Phi(A)=\pi_{K,F_A}$ for some $F_A\subseteq K$. Choose a vector $z\in \text{int} K^*$. Since $\Phi(A)$ is a right ideal, $Ayz\in \Phi(A)$. Notice that $Ayz^T(K\setminus \{0\})\subseteq \Phi(A)^0$. So by Proposition 2.1, $Ayz^T\in \pi_{K^*,F_A}$ and hence $\pi_{K,F_A}=\Phi(Ayz^T)^0$. Therefore $\Phi(A)=\pi_{K,F_A}$ as required.

**Proposition 3.2.** A subset of $\pi(K)$ is a left facial ideal iff it can be expressed in the form $\Phi(yz^T)$ with $y\in \text{int} K$, $z\in K^*$. If, in addition, $d_K$ is surjective, the left facial ideals of $\pi(K)$ are of the form $\pi_{F,0}$ with $F\subseteq K$.

**Proof.** "If" part: Straightforward verification.

"Only if" part: Let $\Phi(A)$ be a left facial ideal of $\pi(K)$. Then $\Phi(A^T)=\Phi(A)^T$ is a right facial ideal of $\pi(K^*)$. Let $y\in \text{int} K$, $z\in \text{int} K^*$. From the proof of Proposition 3.1 (replace $K$ by $K^*$, $A$ by $A^T$, etc.), $\Phi(A^T)=\Phi(A^Tz^Ty^T)$. Hence $\Phi(A)=\Phi(yz^TA^Tz^Ty^T)\subseteq \Phi(yz^T)$, where $z=A^Tz\in K^*$ and $y\in \text{int} K$.

Suppose, in addition, $d_K$ is surjective. Then by Proposition 1.1, $d_K$ is injective. Again from the proof of Proposition 3.1, $\Phi(A^T)=\pi_{K^*,F_A}(z\in \text{int} K^*)$. Hence $\Phi(A)=\pi_{K^*,F_A}(\Phi(z_A^T))=\pi_{F,0}$, where $F=d_K(\Phi(z_A^T))\subseteq K$. The last equality follows from Corollary 2.3.

We have the following simple description of the zero divisors of $\pi(K)$.
PROPOSITION 3.3. A linear operator \( A \in \pi(K) \) is a right (or left) zero divisor iff \( A \) belongs to a proper right (or left) facial ideal.

Proof. Suppose that \( A \) belongs to a proper right facial ideal \( R \) of \( \pi(K) \). By Proposition 3.2, \( R = \pi_{K,F} \) for some proper face \( F \) of \( K \). Choose nonzero vectors \( y \in K \) and \( z \in d_F(K) \). Then \( yz^T \neq 0 \in \pi(K) \), and for any \( x \in K \), \( (yz^T)Ax = y(z^TAx) = 0 \) (because \( Ax \in F \) and \( z \perp F \)). As \( K - K = R^n \), this implies \( (yz^T)A = 0 \). In other words, \( A \) is a right zero divisor.

Next, assume the existence of a nonzero linear operator \( B \in \pi(K) \) such that \( BA = 0 \). Let \( y \in \text{int} \, K \). Observe that \( Ay \notin \text{int} \, K; \) otherwise \( B = 0 \). Hence \( \Phi(Ay) \) is a proper face of \( K \) and \( \pi_{K,\Phi(Ay)} \) is a proper right facial ideal containing \( A \).

The second half of the proposition follows easily from the first half and the following two facts: (1) \( A \) is a left zero divisor of \( \pi(K) \) iff \( A^T \) is a right zero divisor of \( \pi(K^*) \), and (2) \( L \) is a left facial ideal of \( \pi(K) \) iff \( L^T \) is a right facial ideal of \( \pi(K^*) \).

4. THE IDEAL \( \pi(K^*)^* \)

Denote by \( \text{pos} \, S \) the positive hull of the set \( S \). As mentioned in Sec. 2, the dual cone of \( \pi(K) \) is the set \( \text{pos}\{zy^T: y \in K \text{ and } z \in K^*\} \). Thus \( \pi(K^*)^* \) is the set \( \text{pos}\{yz^T: y \in K \text{ and } z \in K^*\} \), and is in fact a subcone of \( \pi(K) \). There are several known results which suggest that the properties of \( K \) are greatly determined by this subcone of \( \pi(K) \). For instance:

\( K \) is simplicial \( \iff \pi(K) = \pi(K^*)^* \iff \text{the identity operator } I \in \pi(K^*)^* \) (see Tam [20]).

\( K \) is simplicial, self-dual \( \iff \pi(K) = \pi(K^*) \) (see Barker and Loewy [5]).

These results are natural because \( \pi(K^*)^* \) is the dual cone of \( \pi(K^*) \), \( \pi(K^*) \) is isometric to \( \pi(K) \) and clearly \( \pi(K) \) determines the properties of \( K \). When \( \pi(K) \) is given the structure of a semiring, we can say more.

PROPOSITION 4.1. \( \pi(K) \) has a unique minimal (nonzero) two-sided ideal, namely \( \pi(K^*)^* \). Furthermore, \( \pi(K^*)^* \) is a principal two-sided ideal generated by each of its nonzero elements. Also \( K \) is simplicial iff \( \pi(K) \) has no nontrivial two-sided ideals.

Proof. It is easy to verify that \( \pi(K^*)^* \) is a two-sided ideal. Let \( \mathcal{I} \) be a nonzero two-sided ideal of \( \pi(K) \). Choose a nonzero element \( A \in \mathcal{I} \). As \( A \) is
nonzero, it is possible to find $y' \in \text{int } K$, $z' \in \text{int } K^*$ such that $z'^T A y' = 1$. For any $y \in K$ and $z \in K^*$, we have, $y z^T = (y z^T) A (y' z'^T) \in \mathcal{G}$. It follows that $\mathcal{G}$ includes $\pi(K^*)^*$, and the first statement in the proposition is proved. The last two statements now follow easily from the first.

Horne [12] has proved that if $K_1$, $K_2$ are polyhedral cones such that $\pi(K_1)$ and $\pi(K_2)$ are isomorphic as semirings, then $K_1$ and $K_2$ are linearly isomorphic. We shall see that the assumption that $K_1$ and $K_2$ are polyhedral can be dropped. In fact we can prove

**Theorem 4.4.** If the unique minimal two-sided ideals of $\pi(K_1)$ and $\pi(K_2)$ are isomorphic as semirings, then $K_1$ and $K_2$ are linearly isomorphic.

We sketch the ideas of our proof below.

Let $K_1$ and $K_2$ be proper cones in $\mathbb{R}^n_+$ and $\mathbb{R}^n_+$ respectively. Suppose that there is an isomorphism $T$ between the semirings $\pi(K_1)$ and $\pi(K_2)$. Here by the term "semiring isomorphism" we do not assume (as Horne implicitly did) that $T$ preserves multiplication by nonnegative scalars. However, we shall show in the next lemma that this property of $T$ follows from its additivity.

**Lemma 4.2.** If $T: \pi(K_1) \rightarrow \pi(K_2)$ is additive, then for any $A \in \pi(K_1)$ and $\lambda > 0$, $T(\lambda A) = \lambda T(A)$.

**Proof.** Let $A \in \pi(K_1)$. It is easy to deduce from the additivity of $T$ that for any nonnegative rational number $r$, $T(rA) = rT(A)$. Suppose $\lambda$ is a nonnegative irrational number. Then for any rational number $r$ less than $\lambda$, $T(\lambda A) = T((\lambda - r)A + rA) = T((\lambda - r)A) + rT(A)$ and hence $T(\lambda A) - rT(A) \in \pi(K_2)$. By the closedness of $\pi(K_2)$ this implies that $T(\lambda A) - \lambda T(A) \in \pi(K_2)$. Similarly, by considering rational numbers greater than $\lambda$, we deduce that $\lambda T(A) - T(\lambda A) \in \pi(K_2)$. Hence $T(\lambda A) = \lambda T(A)$.

It is easy to check that Lemma 1.2, Theorem 1.3, and Lemma 1.4 in Horne [12] can be proved without the assumption that the cones under consideration are polyhedral. Horne's key lemma, Lemma 1.5, is also true for general cones. However, his proof does not hold in the general case, as it depends on the following fact, which is untrue for a general cone $K$:

If $y_1, y_2 \in \text{Ext } K$ such that $d_K(\Phi(y_1)) = d_K(\Phi(y_2))$, then $\Phi(y_1) = \Phi(y_2)$.

We can in fact state Horne's key lemma without assuming that the vectors to be considered are extreme.

**Lemma 4.3.** Let $T: \pi(K_1) \rightarrow \pi(K_2)$ be a semiring isomorphism. Let $z$ be a nonzero vector in $K_1^*$, and let $y_1, y_2$ be nonzero vectors in $K_1$. Let
\( T(y_1 z^T) = y'_1 z'_1^T \), and let \( T(y_2 z^T) = y'_2 z'_2^T \), where \( y_1, y_2 \in K_2 \) and \( z'_1, z'_2 \in K_2^* \). (We may assume these because \( T \) sends rank-1 operators to rank-1 operators.) Then the vectors \( z'_1 \) and \( z'_2 \) are equivalent (that is, they are positive multiples of each other).

**Proof.** Assume the contrary: that the vectors \( z'_1 \) and \( z'_2 \) are not equivalent. Notice that under \( T \) the rank-1 operator \( y_1 z^T + y_2 z^T \) is mapped to the operator \( y'_1 z'_1^T + y'_2 z'_2^T \), which is also rank-1. As \( z'_1 \) and \( z'_2 \) are not dependent, this implies that \( y'_1 \) and \( y'_2 \) are equivalent.

We claim that \( d_{K_1}(\Phi(y_1)) = d_{K_1}(\Phi(y_2)) \). Let \( w \) be a nonzero vector in \( d_{K_1}(\Phi(y_1)) \). Choose a nonzero vector \( x \) in \( K_1 \). Then \( xw^T \in \pi(K_1) \) and \( (xw^T)(y_1 z^T) = 0 \). Hence \( T(xw^T)T(y_1 z^T) = [T(xw^T)y'_1] z'_1^T = 0 \), so that \( T(xw^T)y'_1 = 0 \), as \( z'_1 \neq 0 \). Since \( y'_2 \) is equivalent to \( y'_1 \), \( T(xw^T)y'_2 = 0 \). As \( T \) is bijective, we can retrace our steps and obtain \( 0 = (xw^T)(y_2 z^T) = (w^T y_2)(xz^T). \) Thus \( w \in d_{K_1}(\Phi(y_2)) \), and so \( d_{K_1}(\Phi(y_1)) \subseteq d_{K_1}(\Phi(y_2)) \). Similarly we prove that \( d_{K_1}(\Phi(y_2)) \subseteq d_{K_1}(\Phi(y_1)) \) and hence our claim.

Now choose a nonzero vector \( y_3 \) such that \( d_{K_1}(\Phi(y_3)) \neq d_{K_1}(\Phi(y_1)) \). Suppose that \( T(y_3 z^T) = y'_3 z'_3^T \) where \( y'_3 \in K_2 \) and \( z'_3 \in K_2^* \). Since \( d_{K_1}(\Phi(y_3)) \neq d_{K_1}(\Phi(y_1)) \), by what we have proved above, necessarily \( z'_1 \) and \( z'_2 \) are equivalent. Similarly \( z'_2 \) and \( z'_3 \) are also equivalent; hence so are \( z'_1 \) and \( z'_3 \). This is a contradiction.

Horne's argument after Lemma 1.5 in his paper can now be used to deduce his Theorem 1.8, without the assumption that the cones are polyhedral. In passing, we point out that there is an interesting alternative way to complete the proof. Roughly it goes like this:

Since \( \pi(K_1) \) is a reproducing cone in \( \text{Hom}(R^{n_1}) \), \( T \) can be extended in a natural way to a linear transformation from \( \text{Hom}(R^{n_1}) \) to \( \text{Hom}(R^{n_2}). \) Using Lemma 4.3, its analogue, and the fact that \( T \) sends rank-1 operators in \( \pi(K_1) \) to operators of the same rank in \( \pi(K_2) \), we can show that \( T \) is a rank-1 preserver. Then by Theorem 1 in Djoković [8] (identify \( yz^T \) with \( y \otimes z \) etc.), we can deduce that \( T = A \otimes (A^T)^{-1} \) for some isomorphism \( A \) such that \( AK_1 = K_2 \). We omit the details.

Finally we observe that in the above proof of Horne's main theorem, it is elements in the minimal two-sided ideals of \( \pi(K_1) \) and \( \pi(K_2) \) that are really involved. We need not use elements outside the minimal two-sided ideals. We have, in fact, proved Theorem 4.4.

\( \pi(K^*)^* \) not only is the unique minimal two-sided ideal of the semiring \( \pi(K) \) and determines it up to isomorphism, but also possesses a rather peculiar property. With an exceptional case, \( \pi(K^*)^* \) satisfies the following: if \( A \) is a noninvertible element of \( \pi(K) \) and \( A \notin \pi(K^*)^* \), then there exists \( B \in \pi(K) \setminus \pi(K^*)^* \) such that \( AB \in \pi(K^*)^* \).
Lemma 4.5. The cone $K$ can be written as $K' \ominus \Phi(y)$ for some nonzero extreme vector $y \in K$ and $K' \subseteq K$ iff there exist nonzero vectors $y \in \partial K$ and $z \in \partial K^*$ such that for every $y' \in \text{Ext} K$ and $z' \in \text{Ext} K^*$ satisfying $(z', y') = 0$, we have either $(z, y') = 0$ or $(z', y) = 0$.

Proof. "Only if" part: Assume that $K = K' \ominus \Phi(y)$ (where $y \neq 0$). Choose a nonzero vector $z \in d(K')$. Then the pair of vectors $y, z$ has the required properties: if $y' \in K'$, there is no problem; if $y' \notin K'$, then $y' = ay$ for some positive scalar $a$, and so $(z', y) = 0$.

"If" part: Suppose the pair of vectors $y$ and $z$ satisfies the assumptions of the lemma. It is sufficient to show that there exists exactly one extreme ray of $K$ outside the proper face $d(\Phi(z))$. Assume the contrary. Then outside $d(\Phi(z))$ there will be more than one extreme rays of $K$ which are also exposed, because by Straszewicz's theorem (see Rockafellar [10, Theorem 18.6]), every nonexposed extreme vector is the limit of a sequence of exposed extreme vectors. But then there exists an exposed extreme vector $y'$ outside $d(\Phi(z))$ such that $y \notin \Phi(y') = \text{cl}_K \Phi(y')$. We can then choose an extreme vector $z'$ of $K^*$ such that $(z', y') = 0$ and $(z', y) \neq 0$. This together with $(z, y') \neq 0$, which follows from our choice of $y'$, contradicts the assumption on the pair of vectors $y$ and $z$. [In fact we can show further that $\Phi(y)$ is exactly the unique extreme ray of $K$ outside $d(\Phi(z))$.]

Lemma 4.6. If $K$ does not satisfy either of the equivalent conditions stated in Lemma 4.5, then $d_{\pi(K)}^*(\Phi(I)) \subseteq \pi^*(K^*)$.

Proof. Suppose there exists $A \in d_{\pi(K)}^*(\Phi(I))$ such that $A \in \partial \pi(K^*)$. Then for some nonzero vectors $y \in \partial K$ and $z \in \partial K^*$, $(y, Ax) = 0$. If $y'$ and $z'$ are respectively extreme vectors of $K$ and $K^*$ such that $(z', y') = 0$, then $z'y^T \in d_{\pi(K)}^*(\Phi(I))$. Hence, since $A \in d_{\pi(K)}^*(\Phi(I))$, $A = \alpha z'y^T + \Sigma z_i y_i^T$ for some vectors $y_i \in K, z_i \in K^*$ and a positive scalar $\alpha$. It follows that $(y, (z'y^T)z) = 0$ and so either $(z', y) = 0$ or $(z, y') = 0$. Thus $K$ satisfies the equivalent conditions stated in Lemma 4.5.

Lemma 4.7. For any $A, B \in \text{Hom}(R^n), AB \in \pi(K^*)$ iff $B \in [\text{cl} A^T \pi (K^*)]^*$.

Proof. First observe that for any $A, B, C \in \text{Hom}(R^n), \langle AB, C \rangle = \text{trace}(AB)^TC = \text{trace} B^T(A^TC) = \langle B, A^TC \rangle$. Hence,

$AB \in \pi(K^*)$ iff $\langle AB, C \rangle > 0$ for all $C \in \pi(K^*)$

iff $\langle B, A^TC \rangle > 0$ for all $C \in \pi(K^*)$

iff $B \in [\text{cl} A^T \pi (K^*)]^*$.
**Theorem 4.8.** Suppose that when $K$ is expressed as a direct sum of indecomposable cones, none of the indecomposable subcones is a single ray. Then for any noninvertible element $A$ of $\pi(K)$ outside $\pi(K^*)^*$ there exists an element $B \subset \pi^+(K) \setminus \pi(K^*)^*$ such that $AB \in \pi(K^*)^*$.

**Proof.** Since $A$ is a noninvertible element of $\pi(K)$, $A^T$ is a noninvertible element of $\pi(K^*)$. By Proposition 8.2 (see Sec. 8) the closed principal right ideal of $\pi(K^*)$ generated by $A^T$ is proper, i.e. $\text{cl } A^T \pi(K^*) \subseteq \pi(K^*)$. Hence $[\text{cl } A^T \pi(K^*)]^* \supseteq \pi(K^*)^*$; here the inclusion is proper because

$$\text{cl } A^T \pi(K^*) = [\text{cl } A^T \pi(K^*)]^*$$

and $\pi(K^*) = \pi(K^*)^*$

(Berman [7, Theorem 2.2]). Certainly the identity matrix $I$ does not belong to the proper ideal $\text{cl } A^T \pi(K^*)$. So there exists a matrix $C \in [\text{cl } A^T \pi(K^*)]^*$ such that $\langle I, C \rangle < 0$. Choose a matrix $D \in d_{\pi(K^*)}^*(\Phi(I))$. By assumption, $K$ cannot be expressed as $K' \oplus \Phi(y)$ for some $K' \leq K$ and $y \in \text{Ext } K$; neither can $K^*$ possess this property. Hence by Lemma 4.6, $D \in \pi^+(K)$. Since

$$d_{\pi(K^*)}^*(\Phi(I)) \subseteq \pi(K^*)^* \subseteq [\text{cl } A^T \pi(K^*)]^*,$$

certainly $D$ also belongs to $[\text{cl } A^T \pi(K^*)]^*$. Choose $\varepsilon > 0$ sufficiently small so that $D + \varepsilon C \in \pi^+(K)$, and write $B = D + \varepsilon C$. Then $B \in \pi^+(K) \cap [\text{cl } A^T \pi(K^*)]^*$. By Lemma 4.7, $AB \in \pi(K^*)^*$. Also $B \notin \pi(K^*)^*$, because

$$\langle I, B \rangle = \langle I, D \rangle + \langle I, \varepsilon C \rangle = \varepsilon \langle I, C \rangle < 0.$$

We have found a matrix $B$ with the required properties.

**Corollary 4.9.** $K$ is simplicial iff $\pi(K^*)^*$ is a prime two-sided ideal of $\pi(K)$.

**Proof.** "Only if" part: If $K$ is simplicial, then $\pi(K^*)^* = \pi(K)$. So certainly the two-sided ideal $\pi(K^*)^*$ is prime. "If" part: Suppose $K$ is not simplicial, and let $K = K_1 \oplus \cdots \oplus K_r$ denote the unique representation of $K$ as a direct sum of indecomposable cones $(r > 1)$. Clearly at least one of the $K_i$ is not a single ray, say $K_1$. Write $K' = K_2 \oplus \cdots \oplus K_r$ ($K' = 0$ if $K$ is indecomposable). Then $K = K_1 \oplus K'$. As $K_1$ is not simplicial, there exists a linear operator $A' \in \pi(K_1) \setminus \pi(K_1')$ [where $K_1'$, $\pi(K_1')^*$ denote the duals of $K_1$ and $\pi(K_1')$ in their respective linear spans]. By Theorem 4.8 there exists $B' \in \pi(K_1) \setminus \pi(K_1')^*$ such that
\( A'B' \in \pi(K)^V \). Let \( A, B \) be the linear operators in \( \pi(K) \) induced respectively by \( A' \) and \( B' \). Then by Proposition 2.6, \( A, B \in \pi(K^*)^* \), but \( AB \in \pi(K^*)^* \). So \( \pi(K^*)^* \) is not prime.

**Remark 4.10.** In Theorem 4.8 we cannot replace the assumptions on \( K \) by the weaker one "\( K \) is not simplicial." To show this, suppose \( K = K_1 \oplus \Phi(y) \), where \( K_1 \) is indecomposable and is not a single ray. Clearly there exists a vector \( z \in d(K_1) \) such that \( (z, y) = 1 \). It is easy to see that \( yz^T \) and \( P = I - yz^T \) \( \in \pi(K) \) are the corresponding projections. Here \( P \in \pi(K^*)^* \); for otherwise, \( I \in \pi(K^*)^* \) and hence \( K \) is simplicial, which is a contradiction. However, there does not exist \( B \in \pi(K) \setminus \pi(K^*)^* \) such that \( PB \in \pi(K^*)^* \). For if \( B \) is such a linear operator, then \( B = PB + y(B'z)^T \in \pi(K^*)^* \).

5. **The Face-Semiring of \( \pi(K) \)**

Denote by \( \preceq \) the partial ordering in \( \text{Hom}(R^n) \) induced by the cone \( \pi(K) \). It is easy to see that for any operators \( A, B \in \pi(K), \Phi(A) = \Phi(B) \) iff \( \alpha A \preceq B \preceq \beta A \) for some positive scalars \( \alpha \) and \( \beta \).

Suppose \( \Phi(A) = \Phi(A') \) and \( \Phi(B) = \Phi(B') \). Then by the above remark, there exist positive scalars \( \alpha \) and \( \beta \) such that \( \alpha A \preceq A' \preceq \beta A \). Hence \( \alpha AB \preceq A'B \preceq \beta AB \), and so \( \Phi(AB) = \Phi(A'B) \). Similarly \( \Phi(A'B) = \Phi(A'B') \). Therefore \( \Phi(AB) = \Phi(A'B') \). The equality \( \Phi(A + B) = \Phi(A' + B') \) can also be established. Hence in \( \mathcal{F}(\pi(K)) \), the face lattice of \( \pi(K) \), we introduce addition and multiplication as follows:

\[
\Phi(A) + \Phi(B) = \Phi(A + B),
\]
\[
\Phi(A) \Phi(B) = \Phi(AB).
\]

Note that \( \mathcal{F}(\pi(K)) \) can be identified with the quotient set \( \pi(K)/\sim \), where \( \sim \) is the equivalence relation in \( \pi(K) \) defined by \( A \sim B \) iff \( \Phi(A) = \Phi(B) \). Furthermore, the addition and multiplication in \( \mathcal{F}(\pi(K)) \) are compatible with this equivalence relation. Hence \( \mathcal{F}(\pi(K)) \) inherits a quotient semiring structure from \( \pi(K) \), the mapping \( \Phi: \pi(K) \rightarrow \mathcal{F}(\pi(K)) \) given by \( A \mapsto \Phi(A) \) being the canonical semiring epimorphism. Together with the lattice operations meet \( \wedge \) and join \( \vee \), \( \mathcal{F}(\pi(K)) \) becomes a lattice ordered semiring.

If \( K = R^*_+ \), then \( \mathcal{F}(\pi(K)) \) can be identified with the semiring of \( n \times n \) matrices over the lattice \( \{0, 1\} \) (with operations \( 1 + 1 = 1 \), etc.). Horne [12] has found out all the maximal right ideals of this matrix semiring. He then made use of the natural homomorphism which sends a nonnegative matrix to
its zero pattern to identify the maximal right ideals of \( \pi(R^*_+) \) (the semiring of \( n \times n \) nonnegative matrices). Lemma 2.1, Corollary 2.2, and Theorem 2.3 in his paper can be extended as below. However, as we shall see in the next two sections, his characterization of the maximal right ideals cannot be generalized to arbitrary semirings \( \pi(K) \).

**Proposition 5.1.** An element \( A \in \pi(K) \) is invertible iff \( \Phi(A) \subseteq \pi(K) \).

**Proof.** The "only if" part is obvious. To prove the "if" part, suppose \( \Phi(A) \) is invertible in \( \pi(K) \). Then there exists an operator \( B \in \pi(K) \) such that \( \Phi(A)\Phi(B) = \Phi(AB) = \Phi(I) \). Hence for some positive scalars \( \alpha \) and \( \beta \), \( \alpha I \leq AB \leq \beta I \). Thus for each extreme vector \( x \), \( ABx = \alpha x \) for some positive scalar \( \alpha \). This shows that \( AB(\text{Ext } K) = \text{Ext } K \). Hence \( AB(K) = K \) and so \( AK = K \). Therefore, \( A \) is an invertible element of \( \pi(K) \).

The proofs of Corollary 5.2 and Proposition 5.3 below are similar to those for Corollary 2.2 and Theorem 2.3 in Horne's paper.

**Corollary 5.2.** If \( \mathcal{I} \) is a proper left, right or two-sided ideal in \( \pi(K) \), then \( \Phi(\mathcal{I}) \) is a proper ideal of the same type in \( \pi(K) \).

**Proposition 5.3.** Let \( \mathcal{I} \) be a maximal left (or right) ideal in the semiring \( \pi(K) \). Then \( \Phi(\mathcal{I}) \) is a maximal ideal of the same type in \( \pi(K) \), and \( \mathcal{I} = \Phi^{-1}(\Phi(\mathcal{I})) \). In particular \( \Phi \) induces a bijection between the maximal left (right) ideals of \( \pi(K) \) and those of \( \pi(K) \).

An ideal of \( \pi(K) \) of the form \( \Phi^{-1}(\mathcal{I}) \) for some ideal \( \mathcal{I} \) of \( \pi(K) \) is called an \( \Phi \)-ideal. The \( \Phi \)-ideals \( \mathcal{I} \) are ideals characterized by the property that if \( A \in \mathcal{I} \), then \( B \in \mathcal{I} \) for every \( B \) in the relative interior of \( \Phi(A) \). Certainly, facial ideals of \( \pi(K) \) are \( \Phi \)-ideals. In fact, as can be easily verified, facial ideals are characterized as the inverse images under \( \Phi \) of intervals which are also ideals of \( \pi(K) \) (as a semiring).

Is the semiring \( \pi(K) \) simple? We have the following answer.

**Proposition 5.4.** In the face-semiring \( \pi(K) \) the relation \( \sim \) defined by \( \Phi(A) \sim \Phi(B) \) iff \( d(\Phi(A)) = d(\Phi(B)) \) is a congruence relation. Furthermore, the quotient semiring \( \pi(K)/\sim \) is simple, i.e., it admits no nontrivial congruence relation.

**Proof.** It is obvious that \( \sim \) is an equivalence relation in \( \pi(K) \). Denote the equivalence class which contains \( \Phi(A) \) by \( \Phi(A) \). It is required to show that the operations addition and multiplication in \( \pi(K)/\sim \) given
in the following way are well defined:

\[ (1) \quad \Phi (A) + \Phi (B) = \Phi (A + B), \]
\[ (2) \quad \Phi (A) \Phi (B) = \Phi (AB). \]

To prove that the addition operation is well defined, it is sufficient to show that if \( \Phi (A) \sim \Phi (A') \), then \( \Phi (A) + \Phi (C) \sim \Phi (A') + \Phi (C) \), where \( A, A', C \in \pi (K) \). Now, for any vectors \( y \in K \) and \( z \in K^* \),

\[ zy^T \perp \Phi (A) + \Phi (C) \quad \text{iff} \quad zy^T \perp A + C \quad \text{[because} \quad \Phi (A) + \Phi (C) = \Phi (A + C)\text{]} \]

iff \( zy^T \perp A \) and \( zy^T \perp C \)

iff \( zy^T \perp A' \) and \( zy^T \perp C \)

iff \( zy^T \perp \Phi (A') + \Phi (C) \).

Hence, since \( \pi (K)^* \) consists of nonnegative linear combinations of matrices of the form \( zy^T \) with \( y \in K \) and \( z \in K^* \), we have \( d(\Phi (A) + \Phi (C)) = d(\Phi (A') + \Phi (C)) \) and so \( \Phi (A) + \Phi (C) \sim \Phi (A') + \Phi (C) \).

To prove that the multiplication operation is well defined, it is sufficient to show that if \( A, A' \in \pi (K) \) such that \( \Phi (A) \sim \Phi (A') \) then \( \Phi (A) \Phi (C) \sim \Phi (A') \Phi (C) \) and \( \Phi (C) \Phi (A) \sim \Phi (C) \Phi (A') \) for any \( C \in \pi (K) \). Now for any vectors \( y \in K \) and \( z \in K^* \),

\[ zy^T \perp \Phi (A) \Phi (C) \quad \text{iff} \quad \langle zy^T, AC \rangle = 0 \]
\[ \quad \text{iff} \quad (z, A(Cy)) = 0 \]
\[ \quad \text{iff} \quad \langle z(Cy)^T, A \rangle = 0 \]
\[ \quad \text{iff} \quad \langle z(Cy)^T, A' \rangle = 0 \]
\[ \quad \text{iff} \quad zy^T \perp \Phi (A') \Phi (C). \]

Hence \( \Phi (A) \Phi (C) \sim \Phi (A') \Phi (C) \). Similarly,

\[ zy^T \perp \Phi (C) \Phi (A) = 0 \quad \text{iff} \quad (z, CAy) = 0 \]
\[ \quad \text{iff} \quad (CTz, Ay) = 0 \]
\[ \quad \text{iff} \quad \langle (CTz)y^T, A \rangle = 0 \]
\[ \quad \text{iff} \quad \langle (CTz)y^T, A' \rangle = 0 \]
\[ \quad \text{iff} \quad (CTz, A'y) = 0 \]
\[ \quad \text{iff} \quad zy^T \perp \Phi (C) \Phi (A'). \]

Hence \( \Phi (C) \Phi (A) \sim \Phi (C) \Phi (A') \).
Consider a congruence relation in the semiring $\mathcal{S}(\pi(K))/\sim$, which is not the identity relation. Denote the equivalence class containing the element $\Phi(A)$ of $\mathcal{S}(\pi(K))/\sim$ by $[\Phi(A)]$. As the congruence relation is not the identity relation, there exist $A, B \in \pi(K)$ such that $\Phi(A) \neq \Phi(B)$ but $[\Phi(A)] = [\Phi(B)]$. Without loss of generality, assume $d(\Phi(A)) \leq d(\Phi(B))$. Then there exist nonzero vectors

$$y \in K \quad \text{and} \quad z \in K^*$$

such that

$$zy^T \perp A \quad \text{but} \quad zy^T \not\perp B.$$  

Choose vectors $y_0 \in \text{int} K$ and $z_0 \in \text{int} K^*$. Then since $[\Phi(A)] = [\Phi(B)]$ and our relation in $\mathcal{S}(\pi(K))/\sim$ is compatible with its addition and multiplication, we have

$$\left[ \Phi(y_0 z^T) \right] \left[ \Phi(A) \right] \left[ \Phi(yz_0^T) \right] = \left[ \Phi(y_0 z^T) \right] \left[ \Phi(B) \right] \left[ \Phi(yz_0^T) \right].$$

Hence $[\Phi((z^T Ay_0 y_0 z_0^T))] = [\Phi((z^T By_0 y_0 z_0^T))]$. Since (by our choice of the vectors $y$ and $z$) $z^T Ay = 0$ and $z^T By \neq 0$, this gives

$$[\Phi(0)] = [\pi(K)].$$

Observe that for any $C \in \pi(K)$, we have

$$\left[ \Phi(0) \right] + \left[ \Phi(C) \right] = \left[ \pi(K) \right] + \left[ \Phi(C) \right].$$

Hence, $[\Phi(C)] = [\Phi(C) + \Phi(0)] = [\pi(K) + \Phi(C)] = [\pi(K)]$.

Thus the quotient semiring $\mathcal{S}(\pi(K))/\sim$ has just one equivalence class, and our congruence relation is the trivial zero relation.

The following corollary is obvious.

**Corollary 5.5.** The face-semiring $\mathcal{S}(\pi(K))$ is simple iff the duality operator $d_{\pi(K)}$ is injective.

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6. **MAXIMAL IDEALS OF $\pi(K)$**

Horne [12] proved that $K$ is indecomposable iff $\mathcal{M}$, the set of all noninvertible elements of $\pi(K)$, forms a two-sided ideal. [Then certainly $\mathcal{M}$
is the unique maximal two-sided, as well as one-sided, ideal of \( \pi(K) \). He also succeeded in identifying all the maximal right ideals of \( \pi(R^n) \). It is our purpose to characterize the maximal one-sided and two-sided ideals of \( \pi(K) \) for a general cone \( K \).

When \( K \) is indecomposable, there is no problem. So hereafter in this section (unless stated otherwise) we assume that \( K \) is decomposable, and let \( K = K_1 \oplus \cdots \oplus K_r \) denote the unique representation of \( K \) as a direct sum of indecomposable subcones. Denote by \( P_i \) the corresponding projection onto span \( K_i \) such that \( P_1 + \cdots + P_r = I \). The principal right ideal generated by an element \( A \) of \( \pi(K) \) is denoted by \( R(A) \), i.e., \( R(A) = A \pi(K) \). Similarly the principal left ideal generated by \( A \) is denoted by \( L(A) \), and the principal two-sided ideal by \( I(A) \).

Let us consider the problem for the right ideals first. Clearly an ideal of the semiring \( \pi(K) \) is proper iff it does not contain the identity matrix \( I \). Thus a maximal right ideal is a right ideal maximal with respect to the property of not containing \( I \). Now \( I = P_1 + \cdots + P_r \). So a maximal right ideal cannot contain all the \( P_i \). An intelligent guess is to consider a right ideal which contains all the \( P_i \) except one. This leads to the following:

**Theorem 6.1.** The semiring \( \pi(K) \) has exactly \( r \) maximal right ideals, namely,

\[
\mathcal{R}_i = \{ A \in \pi(K) : P_i \notin R(A) \}, \quad i = 1, 2, \ldots, r.
\]

**Proof.** We first show that each \( \mathcal{R}_i \) is a right ideal of \( \pi(K) \). Clearly the zero matrix 0 belongs to \( \mathcal{R}_i \), so \( \mathcal{R}_i \) is nonempty. For any matrices \( A \in \mathcal{R}_i \) and \( B \in \pi(K) \), necessarily \( AB \in \mathcal{R}_i \); otherwise, there exists \( C \in \pi(K) \) such that \( (AB)C = P_i \) and hence \( A(BC) = P_i \), which contradicts the assumption that \( A \in \mathcal{R}_i \). To show that \( \mathcal{R}_i \) is closed under addition, let \( A_1, A_2 \) be nonzero matrices in \( \mathcal{R}_i \) and suppose that \( A_1 + A_2 \notin \mathcal{R}_i \). Then \( P_i \in R(A_1 + A_2) \), and there exists \( B \in \pi(K) \) such that \( (A_1 + A_2)B = P_i \). Since \( P_i \in \text{Ext} \pi(K) \), this implies \( A_1B = \alpha_1 P_i \) and \( A_2B = \alpha_2 P_i \) for some nonnegative scalars \( \alpha_1 \) and \( \alpha_2 \). Clearly either \( \alpha_1 > 0 \) or \( \alpha_2 > 0 \), say \( \alpha_1 \). Then \( A_1(B/\alpha_1) = P_i \) and so \( A_1 \notin \mathcal{R}_i \). This is a contradiction. Therefore, \( A_1 + A_2 \in \mathcal{R}_i \).

Next we prove that each of the right ideals \( \mathcal{R}_i \) is maximal. Observe that \( P_1 + \cdots + \hat{P}_i + \cdots + P_r \in \mathcal{R}_i \) (where the term under the symbol \( \hat{} \) is to be deleted); otherwise there exists \( B \in \pi(K) \) such that \( (P_1 + \cdots + \hat{P}_i + \cdots + P_r)B = P_i \) and hence \( 0 = P_i(P_1 + \cdots + \hat{P}_i + \cdots + P_r)B = P_i^2 = P_i \), which is a contradiction. Let \( A \) be an element of \( \pi(K) \) outside \( \mathcal{R}_i \). Then for some \( B \in \pi(K) \), \( AB = P_i \). It follows that the right ideal generated by \( A \) and \( \mathcal{R}_i \) contains \( P_1 + P_2 + \cdots + P_r = I \) and hence is the whole semiring \( \pi(K) \). We have established the maximality of \( \mathcal{R}_i \).
Finally we show that each maximal right ideal of \( \pi(K) \) is one of the \( R_i \). Let \( R \) be a maximal right ideal of \( \pi(K) \). Certainly \( I \notin R \). Hence at least one of the projections \( P_1, P_2, \ldots, P_r \) does not belong to \( R \), say \( P_1 \). As \( P_1 \in \text{Ext} \pi(K) \), the right ideal \( R + R_1 \) does not contain \( P_1 \) and hence is proper. By the maximality of \( R \) and \( R_1 \), we have \( R = R + R_1 = R_1 \). The proof is complete. 

Similarly we have

**Theorem 6.2.** The semiring \( \pi(K) \) has exactly \( r \) maximal left ideals, namely,

\[
\mathcal{L}_i = \{ A \in \pi(K) : P_i \notin L(A) \}, \quad i = 1, 2, \ldots, r.
\]

The question of determining the maximal two-sided ideals of \( \pi(K) \) is more delicate. It can be verified that

\[
\mathcal{M}_i = \{ A \in \pi(K) : P_i \notin I(A) \}
\]

is a two-sided ideal. In general, \( \mathcal{M}_i \) is not maximal, but every maximal two-sided ideal is one of \( \mathcal{M}_1, \ldots, \mathcal{M}_r \). Before stating our theorem, we first prove

**Proposition 6.3.** Using the notation introduced above, the following are equivalent:

(i) \( \pi(K_i, K_j) \pi(K_j, K_i) = \pi(K_i) \),

(ii) either \( K_i \) and \( K_j \) are linearly isomorphic, or \( \dim K_i < \dim K_j \), in which case, for some subspace \( H \) of span \( K_i \) of dimension \( \dim K_j \), the cone \( H \cap K_i \) is linearly isomorphic to \( K_i \) and there exists a projection \( P \) of span \( K_i \) onto \( H \) such that \( P \in \pi(K_i) \),

(iii) there exist linear operators \( A' \in \pi(K_i, K_j) \) and \( B' \in \pi(K_j, K_i) \) such that \( A'B' = I_j \) (the identity operator on span \( K_i \)),

(iv) there exist linear operators \( A \) and \( B \in \pi(K) \) such that \( AP_i B = P_i \).

**Proof.** (i)\( \iff \) (iii): Obvious.

(iii)\( \implies \) (iv): Let \( A, B \) be the linear operators in \( \pi(K) \) induced respectively by \( A' \) and \( B' \) (i.e. \( A_{\text{span } K_i} = A', A_{\text{span } K_h} = 0 \) for \( h \neq i \), etc.). It is easy to check that \( AP_i B = P_i \).
(iv)⇒(iii): Let $A', B'$ be the linear operators in $\pi(K_i, K_j)$, $\pi(K_j, K_i)$ given respectively by $A' = P_i A|_{\text{span} K_i}$, and $B' = P_i B|_{\text{span} K_j}$. It is straightforward to show that $A'B' = I_i$.

(ii)⇒(iii): Clearly for linearly isomorphic cones $K_i$ and $K_j$, (iii) holds. Assume the second case when $\dim K_j > \dim K_i$. Denote the linear isomorphism which sends $K_j$ onto $H \cap K_j$ by $B'$. Obviously the linear operator $B'$ belongs to $\pi(K_j, K_i)$. Let $A' = B'^{-1} P$. It is easy to check that $A' \in \pi(K_i, K_j)$ and $A'B' = I_i$.

(iii)⇒(ii): From the identity $A'B' = I_i$, we see that $\dim K_i > \dim K_j$. If $\dim K_i = \dim K_j$, then clearly $B'$ is a linear isomorphism between $K_i$ and $K_j$. So suppose $\dim K_i < \dim K_j$. Denote the image space of $B'$ by $H$. Since $B'$ is injective, $\dim H = \dim K_i$. Hence $B'$ induces a linear isomorphism between $\text{span} K_i$ and $H$. In fact, this linear isomorphism sends $K_i$ onto $H \cap K_i$ (so that these cones are linearly isomorphic); for if $B'(K_j)$ is a proper subset of $H \cap K_i$, then since $A'|_H$ is injective and $A'B'K_j = K_j$, it follows that $A'(H \cap K_i)$ will properly include $K_j$, which is a contradiction. Finally it is easy to check that the linear operator $P \in \text{Hom}(\text{span} K_i)$ given by $P = B'A'$ belongs to $\pi(K_i)$ and is a projection onto $H$.

**Theorem 6.4.** Using the notation introduced above, the two-sided ideal $\mathcal{M}_i$ is maximal iff $\pi(K_i, K_j) \pi(K_j, K_i) \neq \pi(K_i)$ for all $i$ satisfying $\dim K_i > \dim K_j$. Furthermore every maximal two-sided ideal of $\pi(K)$ is one of $\mathcal{M}_1, \ldots, \mathcal{M}_r$.

**Proof.** "If" part: The two-sided ideal $\mathcal{M}_i$ is certainly proper, as $P_i \notin \mathcal{M}_i$. To show that it is maximal, let $A$ be an element of $\pi(K)$ outside $P_i$. We contend that the two-sided ideal $\mathfrak{g}$ generated by $A$ and $\mathcal{M}_i$ is $\pi(K)$. For that purpose, it is sufficient to show that $\mathfrak{g}$ contains $P_1, P_2, \ldots, P_r$. First of all, since $A \notin \mathcal{M}_i$ by the definition of $\mathcal{M}_i$, $P_i$ belongs to $I(A)$ and hence to $\mathfrak{g}$. Consider the projections $P_i$ for $i \neq j$. We separate into two cases.

Case 1. $\dim K_i < \dim K_j$ or $\dim K_j > \dim K_i$. By Proposition 6.3, there do not exist $B, C \in \pi(K)$ such that $BP_i C = P_i$. So $P_i$ is in $\mathcal{M}_i$ and hence in $\mathfrak{g}$.

Case 2. $\dim K_i = \dim K_j$. If $P_i \notin \mathcal{M}_i$, there is no problem. Suppose $P_i \in \mathcal{M}_i$. Then for some linear operators $B, C \in \pi(K)$, $BP_i C = P_i$. Since $\dim K_i = \dim K_j$, by Proposition 6.3, $K_i$ and $K_j$ are linearly isomorphic, which in turn, by Proposition 6.3, again implies that $B'P_i C' = P_i$ for some $B', C' \in \pi(K)$. But as mentioned above, $P_i \in I(A)$. Hence $P_i \in \mathfrak{g}$.

"Only if" part: Suppose that for some $i$ such that $\dim K_i > \dim K_j$ we have $\pi(K_i, K_j) \pi(K_j, K_i) = \pi(K_i)$. We claim that $\mathcal{M}_i$ is properly included in
and hence is not maximal. By Proposition 6.3, there exist linear operators $B, C \in \pi(K)$ such that $BP_i C = P_i$. Hence for any $A \in \pi(K)$,

$$A \notin \mathcal{M}_i \Rightarrow B'AC' = P_i \text{ for some } B', C' \in \pi(K)$$

$$\Rightarrow (BB')A(C'C) = P_i, \text{ where } BB', CC' \in \pi(K)$$

$$\Rightarrow A \notin \mathcal{M}_i.$$ 

This shows that $\mathcal{M}_j \subseteq \mathcal{M}_i$. Notice that $P_j \in \mathcal{M}_i$, because there do not exist linear operators $B, C \in \pi(K)$ such that $BP_i C = P_j$, as $\dim K_1 < \dim K_i$. On the other hand, $P_i \notin \mathcal{M}_i$. So $\mathcal{M}_i$ properly includes $\mathcal{M}_j$.

Last part: Let $\mathcal{M}$ be a maximal two-sided ideal of $\pi(K)$. Clearly at least one of the projections $P_1, \ldots, P_r$ does not belong to $\mathcal{M}$. Choose one such projection with the greatest possible rank, say $P_j$. Observe that if $\dim K_j > \dim K_i$, then necessarily we have $\pi(K_1, K_j) \pi(K_1, K_i) \neq \pi(K_j)$; otherwise, by Proposition 6.3, there exist linear operators $A, B \in \pi(K)$ such that $AP_j B = P_j$ and hence $P_j \notin \mathcal{M}$, which contradicts our choice of $P_j$. Thus by the first part of our theorem, $\mathcal{M}_j$ is maximal. Notice that $P_j \notin \mathcal{M} + \mathcal{M}_j$, as $P_j \in \text{Ext } \pi(K)$. Hence $\mathcal{M} + \mathcal{M}_j$ is a proper two-sided ideal of $\pi(K)$, and by the maximality of $\mathcal{M}$ and $\mathcal{M}_j$, $\mathcal{M} = \mathcal{M} + \mathcal{M}_j = \mathcal{M}_j$.

\textbf{Remark 6.5.} Let $K$ be a decomposable cone, and suppose that the indecomposable subcones $K_1, \ldots, K_r$ in the unique representation of $K$ are linearly isomorphic. Then $\pi(K)$ has just one maximal two-sided ideal. Furthermore, the principal two-sided ideal generated by each of the projections $P_1, \ldots, P_r$ is the whole semiring $\pi(K)$, though the projections are noninvertible.

It is of some interest to compare our characterization of the maximal right ideals with Horne's characterization in the case of $K = R^n_+$. Horne proved the following:

Denote by $e$ any extreme vector of $R^n_+$. Then the set of all linear operators $A \in \pi(R^n_+)$ such that $e \notin AR^nR^n_+$ is a maximal right ideal, and every maximal right ideal is of this form.

Horne's characterization suggests the following simple way of constructing right ideals of $\pi(K)$. (In Propositions 6.6, 6.7 and Corollaries 6.8, 6.9 below we do not assume that $K$ is decomposable.)
Proposition 6.6. Let $S$ be a subcone of $K$ (i.e., $S + S \subseteq S$ and $\alpha S \subseteq S$ for every $\alpha > 0$). Then the set of all linear operators in $\pi(K)$ which map $K$ into $S$ forms a right ideal.

Certainly the maximal right ideals of $\pi(R_+)$ and the right facial ideals of $\pi(K)$ are of this type. One may even guess that if $S$ is a maximal subcone of $K$ [so that $S$ is necessarily $(K \setminus \Psi(x)) \cup \{0\}$ for some extreme ray $\Psi(x)$ of $K$], then the set of all linear operators in $\pi(K)$ which map $K$ into $S$ is a maximal right ideal of $\pi(K)$. The guess, however, is wrong. As we know, if $K$ is indecomposable, then the set of all noninvertible elements of $\pi(K)$ forms the unique maximal right ideal of $\pi(K)$, but certainly it is not of the above type. When $K$ is decomposable, a maximal right ideal of $\pi(K)$ is again rarely of this type. This can be seen from Corollary 6.9 below. For brevity, we shall denote by $R_F$ the set of all linear operators in $\pi(K)$ which map $K$ into $(K \setminus F) \cup \{0\}$ when $F \lneq K$.

Proposition 6.7. For any $F \lneq K$, there exist $G \lneq K$ such that $F \oplus G = K$ iff $\pi(K)$ is a sum of the right ideals $\pi_K, F$ and $R_F$.

Proof. "Only if" part: Assume that $K = F \oplus G$. Denote by $p_F$ and $p_G$ the corresponding projections which are positive on $K$. Clearly $p_F \in \pi_{K, F}$ and $p_G \in R_F$. Hence, since $I = p_F + p_G$, $\pi_{K, F} + R_F = \pi(K)$.

"If" part: Assume that $\pi(K) = \pi_{K, F} + R_F$. Then for some linear operators $A \in \pi_{K, F}$ and $B \in R_F$, $I = A + B$. It suffices to show that $A$ is a projection onto $\text{span } F$. Let $x$ be a vector in $F$. We have $Ax + Bx = x$. Since $F$ is a face of $K$, $Rx \in F$. Hence $Rx = 0$ and $Ax = x$ as $R \in R_F$. So the restriction of $A$ to $\text{span } F$ is the identity mapping. Furthermore, $AR_F \subseteq \text{span } F$ as $A \in \pi_{K, F}$. Thus $A$ is a projection onto $\text{span } F$.

Corollary 6.8. $K$ is decomposable iff the semiring $\pi(K)$ is expressible as a sum of two proper right (or left) ideals.

Proof. The "only if" part follows from the proposition. The "if" part follows from the fact that if $K$ is indecomposable, then $K$ has a unique maximal right (or left) ideal, namely the set of all noninvertible elements.

Corollary 6.9. If $\Phi(x)$ is an extreme ray of $K$, then $R_{\Phi(x)}$ is a maximal right ideal of $\pi(K)$ iff $K = \Phi(x) \oplus K'$ for some $K' \lneq K$. 

Proof. "Only if" part: Observe that $\pi_{K, \Phi(x)} \cap R_{\Phi(x)} = 0$. Since $R_{\Phi(x)}$ is maximal, this implies that $\pi(K) = \pi_{K, \Phi(x)} + R_{\Phi(x)}$. Hence from the proposition $K = \Phi(x) \oplus K'$ for some $K' \leq K$.

"If" part: Suppose that $K = \Phi(x) \oplus K'$ for some $K' \leq K$. It suffices to show that if $A$ is an element of $\pi(K)$ outside $R_{\Phi(x)}$, then the identity operator $I$ belongs to the right ideal $R_{\Phi(x)} + R(A)$. Since $A \not\in R_{\Phi(x)}$, for some vector $y \in K$ we have $Ay = x$. Choose a nonzero vector $z \in d(K')$. Certainly $(z, x) \neq 0$. We may, in fact, assume that $z^TX = 1$. Notice also that $A(yz^T) \in R(A)$ and $(Ayz^T)x = x$. Let $B$ be the linear operator in $R^n$ defined by $Bx = 0$ and $B|_{\text{span} x} = \text{identity}$. It is easy to check that $B \in R_{\Phi(x)}$ and that $I = B + A(yz^T)$. The proof is complete.

Let $x, y$ be extreme vectors of $K$. It is not difficult to show that if $x, y$ do not belong to the same $K_i$, then $R_{\Phi(x)} + R_{\Phi(y)} = \pi(K)$; if $x, y$ both belong to $K_i$, then $R_{\Phi(x)} + R_{\Phi(y)}$ is contained in the maximal right ideal $\mathcal{R}_i$. One may venture to guess that the right ideal generated by the union of all the $R_{\Phi(x)}$ when $x$ runs through all the extreme vectors in $K_i$ is $\mathcal{R}_i$. The guess is true in the special case when $K$ is indecomposable. However, as we shall see in the examples in next section, the guess is in general wrong.

7. EXAMPLES

In our characterization of the maximal two-sided ideals of $\pi(K)$, we have encountered the relation $\pi(K_1, K_2) \pi(K_2, K_1) = \pi(K_2)$ between proper cones $K_1$ and $K_2$. We know, if the condition is satisfied, then there exist linear operators $A \in \pi(K_1, K_2)$ and $B \in \pi(K_2, K_1)$ such that $AB = I_2$ (the identity operator on span $K_2$). Certainly then $AK_1 = K_2$. One may ask the following question: if $AK_1 = K_2$, is it necessarily true that there exists $B \in \pi(K_2, K_1)$ such that $AB = I_2$? As can be seen, the question is also equivalent to the following interesting problem: if $AK_1 = K_2$, can we find a complement $H$ of Ker $A$ in span $K_1$ such that $A$ induces a linear isomorphism between the cones $H \cap K_1$ and $K_2$?

In Example 7.1 we shall give a negative answer to the above question. However, it may be of interest to note that the cones $K_1$ and $K_2$ given in that example also satisfy $\pi(K_1, K_2) \pi(K_2, K_1) = \pi(K_2)$. Our example also shows that the guess on the maximal right ideals of $\pi(K)$, for a decomposable cone $K$, mentioned at the end of last section is wrong.

In Example 7.2 we shall give a pair of cones $K_1, K_2$ which satisfies $\dim K_2 < \dim K_1$ and $\pi(K_1, K_2) \pi(K_2, K_1) \neq \pi(K_2)$.

Example 7.1. Let $K_1$ be the polyhedral cone in $\mathbb{R}^4$ generated by the extreme vectors $e_1 = (1, 0, 0, 0)^T$, $e_2 = (0, 1, 0, 0)^T$, $e_3 = (0, 0, 1, 0)^T$.
\( e_4 = (0,0,0,1)^T \), and \( e_5 = (1,1,1,-\frac{1}{2})^T \). Let \( K_2 \) be the polyhedral cone in \( R^3 \) generated by the extreme vectors \( f_1 = (1,0,0)^T, f_2 = (0,1,0)^T, f_3 = (0,0,1)^T \) and \( f_4 = (1,1,-1)^T \). It can be shown that the cones \( K_1 \) and \( K_2 \) are indecomposable. (In fact, they are what Fiedler and Pták [11] called minimal cones.)

Let \( A' : R^4 \to R^3 \) be the linear operator given by \( A'e_1 = f_1/2, A'e_2 = f_2/2, A'e_3 = f_3/2, \) and \( A'e_4 = f_4 \). By calculation \( A'e_5 = f_3 \). Hence \( A'K_1 = K_2 \). We are going to show that the identity operator \( I_2 \notin A'\pi(K_2, K_1) \).

Assume the contrary: that there exists a linear operator \( B \in \pi(K_2, K_1) \) such that \( A'B = I_2 \). Then necessarily \( A'(K_1 \cap BR^3) = K_2 \). Observe that under \( A' \) every extreme vector of \( K_1 \) is mapped to an extreme vector of \( K_2 \). Only one extreme vector is mapped to the extreme vector \( f_1 \), namely \( 2e_1 \). In fact this vector is the only preimage of \( f_1 \) under \( A' \) in \( K_1 \). The same remark is also true for the vectors \( f_2 \) and \( f_4 \). For the vector \( f_3 \), its preimages in \( K_1 \) lie in the face \( \Phi(e_3 + e_5) \). Hence the cone \( K_1 \cap BR^3 \) contains the vectors \( 2e_1, 2e_2, e_4 \) and a nonzero vector in \( \Phi(e_3 + e_5) \). It is easy to see that these vectors are linearly independent. So \( \dim BR^3 > 4 \), which is a contradiction.

Next we show that \( \dim \Phi(A') = 2 \). Relative to the canonical bases, the representative matrix \( M(A') \) of \( A' \) is given by

\[
M(A') = \begin{bmatrix}
\frac{1}{2} & 1 \\
\frac{1}{2} & 1 \\
\frac{1}{2} & -1
\end{bmatrix}.
\]

If \( B \in \Phi(A') \), then \( By \in \Phi(A'y) \) for all \( y \in K_1 \). It follows that \( M(B) \) is of the form

\[
\begin{bmatrix}
\alpha_1 & \beta \\
\alpha_2 & \beta \\
\alpha_3 & -\beta
\end{bmatrix},
\]

where \( \alpha_1, \alpha_2, \alpha_3 \) and \( \beta \) are nonnegative scalars. Since \( A'e_5 = f_3 \), we have \( Be_5 = \lambda f_3 \) for some nonnegative scalar \( \lambda \). Direct calculations yield

\[
\begin{bmatrix}
\alpha_1 - \frac{1}{2} \beta \\
\alpha_2 - \frac{1}{2} \beta \\
\alpha_3 + \frac{1}{2} \beta
\end{bmatrix} = -\lambda \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}.
\]

Thus \( \alpha_1 = \alpha_2 = \beta/2 \ (\ge 0) \) and \( \alpha_3 = \lambda - \beta/2 \ (\ge 0) \). Hence \( M(B) \) is of the
form

\[
\begin{pmatrix}
\beta/2 & \beta \\
\beta/2 & \beta \\
\alpha_3 & -\beta
\end{pmatrix},
\]

or

\[
\begin{pmatrix}
\frac{1}{2} & 1 \\
\frac{1}{2} & 1 \\
0 & -1
\end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
\]

where \( \alpha_3 \) and \( \beta \) are nonnegative scalars. We have thus shown that \( \dim \Phi(A') = 2 \).

Incidentally we have also shown that if \( C' : R^4 \to R^3 \) is the linear operator given by \( C'e_1 = f_1/2, C'e_2 = f_2/2, C'e_3 = 0, \) and \( C'e_4 = f_4, \) then \( C' \in \text{Ext } \pi(K_1, K_2) \). By calculation \( C'e_5 = f_5/2 \). So we also have \( C'K_1 = K_2 \). Observe that \( 2e_1 + 2e_3, 2e_2, 2e_5, \) and \( e_4 \) are linearly dependent vectors of \( K_1 \), their images under \( C' \) being respectively \( f_1, f_2, f_3, \) and \( f_4 \). Hence if \( B : R^3 \to R^4 \) is the linear operator given by \( Bf_1 = 2e_1 + 2e_3, Bf_2 = 2e_2, \) and \( Bf_3 = 2e_5, \) then \( B \in \pi(K_2, K_1) \) and \( C'B = I_2 \). So we have \( \pi(K_1, K_2) = \pi(K_2, K_1) \).

Now let \( K \) be the proper cone in \( R^7 \) given by \( K = \{(x, y) \in R^7 : x \in K_1 \) and \( y \in K_2 \} \). Then \( K_1, K_2 \) can be identified with subcones of \( K \) and we may write \( K = K_1 \oplus K_2 \). Let \( A \) be the linear operator in \( \pi(K) \) which is induced by the linear operator \( A' \) of \( \pi(K_1, K_2) \). As shown above, \( I_2 \notin A' \pi(K_2, K_1) \).

From this we can deduce that \( P_2 \notin \Phi(A) \). [Here \( P_2 \in \pi(K) \) is the projection corresponding to \( K_2 \).] Thus \( A \) belongs to the maximal right ideal \( \mathcal{R}_2 \). However, we shall show that \( A \) does not belong to \( \mathcal{R} \), the right ideal generated by all the \( R_{\Phi(x)} \) when \( \Phi(x) \) runs through the extreme rays of \( K_2 \).

Let \( C \) be the linear operator in \( \pi(K) \) which is induced by the linear operator \( C' \) of \( \pi(K_1, K_2) \). As shown above, \( \Phi(A') \) is a 2-dimensional face of \( \pi(K_1, K_2) \) with \( \Phi(C') \) as one of its extreme rays. Hence \( \Phi(A) \) is a 2-dimensional face of \( \pi(K) \) with \( \Phi(C) \) as one of its extreme rays. Observe that if \( A \) belongs to the right ideal \( \mathcal{R}_2 \), then in \( \text{pos}(C, A) \) there is a nonzero linear operator which also belongs to \( R_{\Phi(x)} \) for some extreme ray \( \Phi(x) \) of \( K_2 \).

However, it can be seen that every nonzero linear operator \( D \) in \( \text{pos}(C, A) \) satisfies \( DK = K_2 \) and hence is not of the required type. Therefore \( A \) does not belong to the right ideal \( \mathcal{R}_2 \).

Example 7.2. Let \( K_1 \) be a proper polyhedral cone in \( R^4 \) generated by the extreme vectors \( e_1, \ldots, e_6 \) which satisfy \( e_1 + e_6 = e_2 + e_5 = e_3 + e_4 \).
Let $e_1, e_2, e_3, e_4$ linearly independent. Let $K_2$ be a proper polyhedral cone in $R^3$ generated by the extreme vectors $f_1, \ldots, f_6$ which satisfy $f_1 + f_2 = f_3 + f_4 = f_5 + f_6$ ($f_1, f_2, f_3$ linearly independent). Then $K_1$ and $K_2$ are indecomposable. (It may be of interest to note that the cone $K_1$ is of the type described in Sec. 3 of Tam [22]. It has the property that each of its 4-dimensional subcones which is generated by a proper subset of $\{e_1, \ldots, e_6\}$ is decomposable.)

Let $A' \in \text{Hom}(R^4, R^3)$ be given by $A'e_i = f_i$, $1 \leq i \leq 4$. Then $A'e_6 = A'(e_3 + e_4 - e_1) = f_5 + f_4 - f_1 = f_6$. Similarly, $A'e_5 = f_5$. So $A'K_1 = K_2$ and $A' \in \pi(K_1, K_2)$.

We claim that $A' \in \text{Ext} \pi(K_1, K_2)$. Let $B \in \Phi(A')$. Observe that $A'$ induces a linear isomorphism between the polyhedral subcone $K'_1$ of $K_1$ generated by the extreme vectors $e_1, e_2, e_5, e_6$ and the polyhedral subcone $K'_2$ of $K_2$ generated by the extreme vectors $f_1, f_2, f_5, f_6$. Furthermore these cones are indecomposable. Using Theorem 3.3 in Loewy and Schneider [15], we can prove that $A'|_{\text{span} K'_1} \in \text{Ext} \pi(K'_1, K'_2)$. Also notice that $B|_{\text{span} K'_1}$ belongs to the face of $\pi(K'_1, K'_2)$ generated by $A'|_{\text{span} K'_1}$. This is because since $Ae_i = f_i$ and $f_i$ is extreme, there exist nonnegative scalars $\lambda_i$ such that $Be_i = \lambda_i f_i$. If $\lambda > \max(\lambda_1, \lambda_2, \lambda_5, \lambda_6)$, then $\lambda A - B \in \pi(K'_1, K'_2)$. Hence $B|_{\text{span} K'_1} = \alpha A'|_{\text{span} K'_1}$ for some scalar $\alpha > 0$. Similarly $B|_{\text{span} K'_1} = \beta A'|_{\text{span} K'_1}$ for some scalar $\beta > 0$, where $K'_1$ is the 3-dimensional indecomposable subcone of $K_1$ generated by the extreme vectors $e_1, e_3, e_4$, and $e_6$. But the cones $K'_1$ and $K'_2$ have the vector $e_1$ in common. Hence $\alpha = \beta$, and $B = \alpha A'$ ($\alpha > 0$), as $\text{span} K'_1 + \text{span} K'_2 = R^4$. In passing, we note that, using similar arguments, we can in fact show that for any $C \in \pi(K_1, K_2)$ such that $CK_1 = K_2$ we have $C \in \text{Ext} \pi(K_1, K_2)$.

As in previous example let $K = K_1 \oplus K_2$, and let $A \in \pi(K)$ be the linear operator which is induced by the operator $A'$ of $\pi(K_1, K_2)$. Since $A' \in \text{Ext} \pi(K_1, K_2)$, $A \in \text{Ext} \pi(K)$. As $AK_1 = K_2$, clearly $A$ is not in $R_{\Phi(x)}$ for some extreme ray $\Phi(x)$ of $K_2$. Neither does $A$ belong to the right ideal generated by these $R_{\Phi(x)}$, as $A$ is extreme. Nevertheless $A$ belongs to the maximal right ideal $R_2$. This follows from the fact that $\pi(K_1, K_2) \pi(K_2, K_1) \neq \pi(K_2)$, which we are going to show. Assume the contrary. Then there exist linear operators $B \in \pi(K_1, K_2)$ and $C \in \pi(K_2, K_1)$ such that $BC = I_2$. Certainly $BK_1 = K_2$. Since both $K_1$ and $K_2$ have six extreme rays, under $B$ each extreme vector in $K_2$ has a unique preimage in $K_1$. Thus by the same argument as in previous example we can show that $I_2 \notin B \pi(K_2, K_1)$, which is a contradiction.

REMARK 7.3. From a study of our examples, one may be tempted to conjecture the following: Let $K_1, K_2$ be indecomposable cones such that $\pi(K_1, K_2) \pi(K_2, K_1) \neq \pi(K_2)$. If $AK_1 = K_2$, then $A \in \text{Ext} \pi(K_1, K_2)$. However, we have found a counterexample to this conjecture.
8. PRINCIPAL IDEALS

Horne [12] has shown that $\pi(K)$ always contains a strictly infinite descending sequence of principal one-sided ideals (provided $\dim K > 1$). By modifying his method, we establish

**Proposition 8.1.** If $\dim K > 1$, then $\pi(K)$ contains an infinite strictly ascending sequence of principal right (or left) ideals.

**Proof.** Let $y_1$ and $y_2$ be vectors in $\partial K$ such that $y_1 + y_2 \in \text{int } K$. Choose vectors $z_1$ and $z_2$ in $\partial K^*$ such that $z_i^T y_j = \delta_{ij}$ (the Kronecker symbol). For each positive integer $n$, let $A_n = y_1 z_1^T + (y_2 + y_1/2^n) z_2^T$. Denote by $K'$ the set $K \cap \text{span}(y_1, y_2)$. Since $y_1 + y_2 \in \text{int } K$, $y_1$ and $y_2$ are necessarily extreme vectors of the 2-dimensional cone $K'$. For simplicity denote $A_n |_{\text{span}(y_1, y_2)}$ by $B_n$.

Observe that $B_n y_1 = y_1$ and $B_n y_2 = y_2 + y_1/2^n$ for each positive integer $n$. This implies that $B_1 K' \subseteq B_2 K' \subseteq \cdots \subseteq B_n K' \subseteq B_{n+1} K' \subseteq \cdots$.

By direct calculation, $A_{n+1}^2 = A_n$ for each positive integer $n$. So we have $R(A_1) \subseteq R(A_2) \subseteq \cdots \subseteq R(A_n) \subseteq R(A_{n+1}) \subseteq \cdots$. Notice that

$$A_n(K) = A_{n+1}(A_{n+1}(K)) \subseteq A_{n+1}(K') \subseteq B_{n+1}(K').$$

Hence $A_{n+2} \notin R(A_n)$; otherwise $A_{n+2} = A_n B$ for some $B \in \pi(K)$ and so $A_{n+2}(K) = A_n(BK) \subseteq A_n K$, which is a contradiction. It follows that $R(A_1), R(A_3), \ldots, R(A_{2n+1}), \ldots$ is an infinite strictly ascending sequence of principal right ideals.

On the question of topological closedness of principal one-sided ideals of $\pi(K)$ we have the following partial results.

**Proposition 8.2.** If $A \in \pi(K)$ is a noninvertible element, then the closed principal right (or left) ideal generated by $A$ is proper.

**Proof.** Observe that if $B \in R(A)$ then $BK \subseteq AK$. Hence if $B \in \text{cl } R(A)$ then $BK \subseteq \text{cl } AK$. As $A$ is noninvertible, $\text{cl } AK$ is a proper subset of $K$. Thus $I \notin \text{cl } R(A)$ and $\text{cl } R(A)$ is a proper ideal of $\pi(K)$.

**Proposition 8.3.** If $R(A)$ is closed, then so is $AK$.

**Proof.** Suppose that $AK$ is not closed. Then there exists a vector $y \in \text{cl } AK \setminus AK$. Choose any nonzero vector $z \in K^*$. Then $yz^T \in \text{cl } R(A) \setminus R(A)$.
Note that there exists a cone $K$ such that $AK$ is not closed for some $A \in \pi(K)$.

**Proposition 8.4.** If $A \in \pi(K)$ is not a left zero divisor, then $R(A)$ is closed.

**Proof.** Let $B_i \in \pi(K)$ such that $AB_i \to C$. We content that $C \in R(A)$. It is sufficient to show that the norms $\|B_i\|$ are bounded; for then from the sequence $(B_i)_{i \in \mathbb{N}}$ we can extract a convergent subsequence with a limit, say $B$, and so we have $C = AB \in R(A)$. Assume the contrary: that the norms $\|B_i\|$ are unbounded. Then we may assume $\|B_i\| \to \infty$ and $B_i/\|B_i\| \to B'$. But that implies $AB' = 0$. Clearly $B'$ belongs to $\pi(K)$ and is nonzero. So $A$ is a left zero divisor of $\pi(K)$. This is a contradiction. 

Similarly we have

**Proposition 8.5.** If $A \in \pi(K)$ is not a right zero divisor, then $L(A)$ is closed.

9. **Final Remarks**

(1) We have shown in Sec. 4 that if the semirings $\pi(K_1)$ and $\pi(K_2)$ are isomorphic, then the cones $K_1$ and $K_2$ are linearly isomorphic. We guess it is also true that:

If the cones $\pi(K_1)$ and $\pi(K_2)$ are linearly isomorphic, then so are the cones $K_1$ and $K_2$.

(2) The method we have used to construct the maximal ideals of $\pi(K)$ can be extended to produce more ideals. In fact, for any $G \subseteq \pi(K)$, if we let $R_G = \{A \in \pi(K) : R(A) \cap G = 0\}$, $L_G = \{A \in \pi(K) : L(A) \cap G = 0\}$, and $I_G = \{A \in \pi(K) : I(A) \cap G = 0\}$, then $R_G$, $L_G$, and $I_G$ are respectively right, left, and two-sided $\Phi$-ideals of $\pi(K)$. In particular, $R_{\Phi(\mathcal{P}_i)} = \mathcal{G}_i$, $L_{\Phi(\mathcal{P}_i)} = \mathcal{E}_i$, and $I_{\Phi(\mathcal{P}_i)} = \mathcal{M}_i$; and if $G = \Phi(yz^T)$, where $y \in \text{Ext } K$ and $z \in \text{Ext } K^*$, then $I_G = 0$ and $R_{\Phi(yz^T)} = R_{\Phi(y)}$. Further study of these ideals may be of interest.

**References**


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