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Asymptotics of the number of threshold functions on a two-dimensional rectangular grid

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ABSTRACT

Let $m, n \geq 2$, $m \leq n$. It is well-known that the number of (two-dimensional) threshold functions on an $m \times n$ rectangular grid is

$$\begin{aligned} t(m, n) &= \frac{6}{\pi^2} (mn)^2 + O(m^2 n \log n) + O(mn^2 \log \log n) \\ &= \frac{6}{\pi^2} (mn)^2 + O(mn^2 \log m). \end{aligned}$$

We improve the error term by showing that

$$t(m, n) = \frac{6}{\pi^2} (mn)^2 + O(mn^2).$$

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1. Introduction

Consider a rectangular grid $G = G(m, n) = \{0, \dots, m-1\} \times \{0, \dots, n-1\}$, where $m, n \geq 2$. A function $\tau : G \rightarrow \{0, 1\}$ is a (two-dimensional) *threshold function* if there is a line separating the sets $\tau^{-1}(\{0\})$ and $\tau^{-1}(\{1\})$. In other words, there are real numbers a, b and c such that $\tau(x, y) = 0$ if and only if $ax + by + c \leq 0$. Acketa and Žunić [1, Theorem 3] proved (with somewhat different formulation) that the number of these functions is

$$t(m, n) = f(m, n) + 2, \tag{1}$$

where

$$f(m, n) = \sum_{\substack{-m < i < m \\ -n < j < n \\ (i,j)=1}} (m - |i|)(n - |j|)$$

and (i, j) denotes the greatest common divisor of i and j . See also [9, pp. 9–10]. In particular,

$$t(n) = t(n, n) = f(n) + 2, \tag{2}$$

where

$$f(n) = \sum_{\substack{-n < i < n \\ (i,j)=1}} (n - |i|)(n - |j|).$$

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We are interested in asymptotic formulas. By Koplowitz et al. [6, Theorem 2],

$$t(n) = \frac{6}{\pi^2} n^4 + O(n^3 \log n). \quad (3)$$

Let $m \leq n$. By Acketa and Žunić [1, p. 168],

$$t(m, n) = \frac{6}{\pi^2} (mn)^2 + O(m^2 n \log n) + O(mn^2 \log \log n). \quad (4)$$

By Alekseyev [2, Theorem 25],

$$t(m, n) = \frac{6}{\pi^2} (mn)^2 + O(mn^2 \log m), \quad (5)$$

but see also [11]. The error terms of (4) and (5) are incomparable. For example, (4) is better than (5) if $m = \sqrt{n}$, while (5) is better if m remains constant.

We will in Section 2 improve the error terms of these formulas. The improvements of (3) follow directly from well-known results, but the improvement of (4) and (5) must be proved; we will do so in Section 4. Before it, we will introduce the basic facts and notations needed in the proof in Section 3. We will complete our paper with conclusions and remarks in Section 5.

2. Improved formulas

We can drop out the logarithm from (3):

$$t(n) = \frac{6}{\pi^2} n^4 + O(n^3). \quad (6)$$

We have also a sharper but more complicated formula

$$t(n) = \frac{6}{\pi^2} n^4 + O\left(n^3 \exp\left(-A(\log n)^{\frac{3}{5}} (\log \log n)^{-\frac{1}{5}}\right)\right) \quad (7)$$

for some $A > 0$. Assuming the Riemann hypothesis (RH), this still improves into

$$t(n) = \frac{6}{\pi^2} n^4 + O\left(n^{\frac{5}{2} + \varepsilon}\right) \quad (8)$$

for all $\varepsilon > 0$. To prove (7) and (8), we simply note that $t(n)$ and $f(n)$ have by (2) the same asymptotic behavior and refer to [3, Lemma 6].

We can also drop out the logarithms from (4) and (5). Generalizing (6), we will in Section 4 prove that

$$t(m, n) = \frac{6}{\pi^2} (mn)^2 + O(mn^2) \quad (9)$$

if $m \leq n$. We conjecture [10] that the error term can be improved by substituting $n \mapsto \sqrt{mn}$ in (7) and (8). That is,

$$t(m, n) = \frac{6}{\pi^2} (mn)^2 + O\left((mn)^{\frac{3}{2}} \exp\left(-A(\log mn)^{\frac{3}{5}} (\log \log mn)^{-\frac{1}{5}}\right)\right) \quad (10)$$

for some $A > 0$ and, under RH,

$$t(m, n) = \frac{6}{\pi^2} (mn)^2 + O\left((mn)^{\frac{5}{4} + \varepsilon}\right) \quad (11)$$

for all $\varepsilon > 0$.

3. Basic facts and notations

The “delta function” δ , defined by

$$\delta(k) = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{if } k \geq 2, \end{cases}$$

and the Möbius function μ , defined by

$$\mu(k) = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{if } k \text{ has one or more repeated prime factors,} \\ (-1)^r & \text{if } k \text{ is a product of } r \text{ distinct primes,} \end{cases}$$

satisfy

$$\delta(k) = \sum_{d|k} \mu(d). \tag{12}$$

We give some asymptotic formulas involving $\mu(k)$. Let $0 \leq c_1, \dots, c_m \leq 1$. Since

$$c_k |\mu(k)| \leq 1 \quad (k = 1, \dots, m), \tag{13}$$

we have

$$\sum_{k=1}^m c_k \mu(k) = O(m). \tag{14}$$

We also have

$$\sum_{k=1}^m \frac{\mu(k)}{k} = O(1), \tag{15}$$

which in fact can be sharpened by replacing O with o [7, p. 194]. Furthermore, (13) implies easily that

$$\sum_{k=1}^m c_k \frac{\mu(k)}{k} = O(\log m). \tag{16}$$

Finally, we give

$$\sum_{k=1}^m \frac{\mu(k)}{k^2} = \frac{6}{\pi^2} + O\left(\frac{1}{m}\right), \tag{17}$$

which in fact can be sharpened by substituting on the right-hand side $m \mapsto m(\log m)^a$ for any $a > 0$ [7, p. 194].

Let us define

$$\{x\} = x - \lfloor x \rfloor, \quad \alpha(m, n) = \sum_{d=1}^m \frac{\mu(d)}{d} \left\{ \frac{n}{d} \right\}, \quad \alpha(m) = \alpha(m, m).$$

Because $0 \leq \{x\} < 1$, we see by (16) that

$$\alpha(m, n) = O(\log m). \tag{18}$$

4. Proof of (9)

By (1), an equivalent task is to prove that

$$f(m, n) = \frac{6}{\pi^2} (mn)^2 + O(mn^2) \tag{19}$$

if $m \leq n$. We divide the proof into six parts.

1. Evaluating $f(m + 1, n + 1)$. We have

$$\begin{aligned} f(m + 1, n + 1) &= \sum_{\substack{-m \leq i \leq m \\ -n \leq j \leq n \\ (i,j)=1}} (m + 1 - |i|)(n + 1 - |j|) \\ &= 4 \sum_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n \\ (i,j)=1}} (m + 1 - i)(n + 1 - j) + 2(m + 1)n + 2m(n + 1) \\ &= 4 \sum_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n \\ (i,j)=1}} [(m + 1)(n + 1) - (n + 1)i - (m + 1)j + ij] + 4mn + 2m + 2n \\ &= 4 \left[(m + 1)(n + 1) \sum_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n \\ (i,j)=1}} 1 - (n + 1) \sum_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n \\ (i,j)=1}} i - (m + 1) \sum_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n \\ (i,j)=1}} j + \sum_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n \\ (i,j)=1}} ij \right] + O(mn) \\ &= 4[(m + 1)(n + 1)s_1(m, n) - (n + 1)s_2(m, n) - (m + 1)s_3(m, n) + s_4(m, n)] + O(mn), \tag{20} \end{aligned}$$

where

$$s_1(m, n) = \sum_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n \\ (i,j)=1}} 1, \quad s_2(m, n) = \sum_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n \\ (i,j)=1}} i,$$

$$s_3(m, n) = \sum_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n \\ (i,j)=1}} j, \quad s_4(m, n) = \sum_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n \\ (i,j)=1}} ij.$$

2. Evaluating $s_1(m, n)$. Remember that $m \leq n$. Denote $i = ad, j = bd$ and apply (12). Then, by (14), (15), (17) and (18),

$$\begin{aligned} s_1(m, n) &= \sum_{i=1}^m \sum_{j=1}^n \delta((i, j)) = \sum_{i=1}^m \sum_{j=1}^n \sum_{d|(i,j)} \mu(d) = \sum_{d=1}^m \mu(d) \sum_{a=1}^{\lfloor \frac{m}{d} \rfloor} \sum_{b=1}^{\lfloor \frac{n}{d} \rfloor} 1 \\ &= \sum_{d=1}^m \mu(d) \lfloor \frac{m}{d} \rfloor \lfloor \frac{n}{d} \rfloor = \sum_{d=1}^m \mu(d) \left(\frac{m}{d} - \left\{ \frac{m}{d} \right\} \right) \left(\frac{n}{d} - \left\{ \frac{n}{d} \right\} \right) \\ &= mn \sum_{d=1}^m \frac{\mu(d)}{d^2} - m \sum_{d=1}^m \frac{\mu(d)}{d} \left\{ \frac{n}{d} \right\} - n \sum_{d=1}^m \frac{\mu(d)}{d} \left\{ \frac{m}{d} \right\} + \sum_{d=1}^m \mu(d) \left\{ \frac{m}{d} \right\} \left\{ \frac{n}{d} \right\} \\ &= mn \left(\frac{6}{\pi^2} + O\left(\frac{1}{m}\right) \right) - m\alpha(m, n) - n\alpha(m) + O(m) \\ &= \frac{6}{\pi^2} mn - m\alpha(m, n) - n\alpha(m) + O(n). \end{aligned} \tag{21}$$

3. Evaluating $s_2(m, n)$. Similarly,

$$\begin{aligned} 2s_2(m, n) &= 2 \sum_{i=1}^m \sum_{j=1}^n i \delta((i, j)) = 2 \sum_{i=1}^m \sum_{j=1}^n i \sum_{d|(i,j)} \mu(d) \\ &= 2 \sum_{d=1}^m \mu(d) d \sum_{a=1}^{\lfloor \frac{m}{d} \rfloor} \sum_{b=1}^{\lfloor \frac{n}{d} \rfloor} a = 2 \sum_{d=1}^m \mu(d) d \frac{1}{2} \lfloor \frac{m}{d} \rfloor \left(\lfloor \frac{m}{d} \rfloor + 1 \right) \lfloor \frac{n}{d} \rfloor \\ &= \sum_{d=1}^m \mu(d) d \left(\frac{m}{d} - \left\{ \frac{m}{d} \right\} \right) \left(\frac{m}{d} - \left\{ \frac{m}{d} \right\} + 1 \right) \left(\frac{n}{d} - \left\{ \frac{n}{d} \right\} \right) \\ &= m^2 n \sum_{d=1}^m \frac{\mu(d)}{d^2} + mn \sum_{d=1}^m \frac{\mu(d)}{d} - m^2 \sum_{d=1}^m \frac{\mu(d)}{d} \left\{ \frac{n}{d} \right\} \\ &\quad - 2mn \sum_{d=1}^m \frac{\mu(d)}{d} \left\{ \frac{m}{d} \right\} - m \sum_{d=1}^m \mu(d) \left\{ \frac{n}{d} \right\} - n \sum_{d=1}^m \mu(d) \left\{ \frac{m}{d} \right\} + n \sum_{d=1}^m \mu(d) \left\{ \frac{m}{d} \right\}^2 \\ &\quad + 2m \sum_{d=1}^m \mu(d) \left\{ \frac{m}{d} \right\} \left\{ \frac{n}{d} \right\} + \sum_{d=1}^m \mu(d) d \left\{ \frac{m}{d} \right\} \left\{ \frac{n}{d} \right\} - \sum_{d=1}^m \mu(d) d \left\{ \frac{m}{d} \right\}^2 \left\{ \frac{n}{d} \right\} \\ &= m^2 n \left(\frac{6}{\pi^2} + O\left(\frac{1}{m}\right) \right) + mnO(1) - m^2 \alpha(m, n) - 2mn\alpha(m) + mO(m) \\ &\quad + nO(m) + nO(m) + mO(m) + O(m^2) + O(m^2) \\ &= \frac{6}{\pi^2} m^2 n - m^2 \alpha(m, n) - 2mn\alpha(m) + O(mn), \end{aligned}$$

and so

$$s_2(m, n) = \frac{3}{\pi^2} m^2 n - \frac{1}{2} m^2 \alpha(m, n) - mn\alpha(m) + O(mn). \tag{22}$$

4. Evaluating $s_3(m, n)$. A simple modification of Part 3 gives

$$s_3(m, n) = \frac{3}{\pi^2} mn^2 - mn\alpha(m, n) - \frac{1}{2} n^2 \alpha(m) + O(mn). \tag{23}$$

5. Evaluating $s_4(m, n)$. Now we need quite a heavy calculation. We obtain

$$\begin{aligned}
 4s_4(m, n) &= 4 \sum_{i=1}^m \sum_{j=1}^n ij\delta((i, j)) = 4 \sum_{i=1}^m \sum_{j=1}^n ij \sum_{d|(i,j)} \mu(d) \\
 &= 4 \sum_{d=1}^m \mu(d)d^2 \sum_{a=1}^{\lfloor \frac{m}{d} \rfloor} \sum_{b=1}^{\lfloor \frac{n}{d} \rfloor} ab = 4 \sum_{d=1}^m \mu(d)d^2 \frac{1}{2} \lfloor \frac{m}{d} \rfloor \left(\lfloor \frac{m}{d} \rfloor + 1 \right) \frac{1}{2} \lfloor \frac{n}{d} \rfloor \left(\lfloor \frac{n}{d} \rfloor + 1 \right) \\
 &= \sum_{d=1}^m \mu(d)d^2 \left(\frac{m}{d} - \left\lfloor \frac{m}{d} \right\rfloor \right) \left(\frac{m}{d} - \left\lfloor \frac{m}{d} \right\rfloor + 1 \right) \left(\frac{n}{d} - \left\lfloor \frac{n}{d} \right\rfloor \right) \left(\frac{n}{d} - \left\lfloor \frac{n}{d} \right\rfloor + 1 \right) \\
 &= m^2 n^2 \sum_{d=1}^m \frac{\mu(d)}{d^2} + m^2 n \sum_{d=1}^m \frac{\mu(d)}{d} + mn^2 \sum_{d=1}^m \frac{\mu(d)}{d} - 2m^2 n \sum_{d=1}^m \frac{\mu(d)}{d} \left\lfloor \frac{n}{d} \right\rfloor \\
 &\quad - 2mn^2 \sum_{d=1}^m \frac{\mu(d)}{d} \left\lfloor \frac{m}{d} \right\rfloor - m^2 \sum_{d=1}^m \mu(d) \left\lfloor \frac{n}{d} \right\rfloor - n^2 \sum_{d=1}^m \mu(d) \left\lfloor \frac{m}{d} \right\rfloor \\
 &\quad + mn \sum_{d=1}^m \mu(d) + m^2 \sum_{d=1}^m \mu(d) \left\lfloor \frac{n}{d} \right\rfloor^2 + n^2 \sum_{d=1}^m \mu(d) \left\lfloor \frac{m}{d} \right\rfloor^2 \\
 &\quad - 2mn \sum_{d=1}^m \mu(d) \left\lfloor \frac{m}{d} \right\rfloor - 2mn \sum_{d=1}^m \mu(d) \left\lfloor \frac{n}{d} \right\rfloor + 4mn \sum_{d=1}^m \mu(d) \left\lfloor \frac{m}{d} \right\rfloor \left\lfloor \frac{n}{d} \right\rfloor - m \sum_{d=1}^m \mu(d)d \left\lfloor \frac{n}{d} \right\rfloor \\
 &\quad - n \sum_{d=1}^m \mu(d)d \left\lfloor \frac{m}{d} \right\rfloor + 2m \sum_{d=1}^m \mu(d)d \left\lfloor \frac{m}{d} \right\rfloor \left\lfloor \frac{n}{d} \right\rfloor + 2n \sum_{d=1}^m \mu(d)d \left\lfloor \frac{m}{d} \right\rfloor \left\lfloor \frac{n}{d} \right\rfloor \\
 &\quad + m \sum_{d=1}^m \mu(d)d \left\lfloor \frac{n}{d} \right\rfloor^2 + n \sum_{d=1}^m \mu(d)d \left\lfloor \frac{m}{d} \right\rfloor^2 - 2m \sum_{d=1}^m \mu(d)d \left\lfloor \frac{m}{d} \right\rfloor \left\lfloor \frac{n}{d} \right\rfloor^2 \\
 &\quad - 2n \sum_{d=1}^m \mu(d)d \left\lfloor \frac{m}{d} \right\rfloor \left\lfloor \frac{n}{d} \right\rfloor^2 + \sum_{d=1}^m \mu(d)d^2 \left\lfloor \frac{m}{d} \right\rfloor \left\lfloor \frac{n}{d} \right\rfloor - \sum_{d=1}^m \mu(d)d^2 \left\lfloor \frac{m}{d} \right\rfloor^2 \left\lfloor \frac{n}{d} \right\rfloor \\
 &\quad - \sum_{d=1}^m \mu(d)d^2 \left\lfloor \frac{m}{d} \right\rfloor \left\lfloor \frac{n}{d} \right\rfloor^2 + \sum_{d=1}^m \mu(d)d^2 \left\lfloor \frac{m}{d} \right\rfloor^2 \left\lfloor \frac{n}{d} \right\rfloor^2 \\
 &= m^2 n^2 \left(\frac{6}{\pi^2} + O\left(\frac{1}{m}\right) \right) + m^2 n O(1) + mn^2 O(1) - 2m^2 n \alpha(m, n) - 2mn^2 \alpha(m) \\
 &\quad + m^2 O(m) + n^2 O(m) + mn O(m) + m^2 O(m) + n^2 O(m) + mn O(m) \\
 &\quad + mn O(m) + mn O(m) + m O(m^2) + n O(m^2) + m O(m^2) + n O(m^2) \\
 &\quad + m O(m^2) + n O(m^2) + m O(m^2) + n O(m^2) + O(m^3) + O(m^3) + O(m^3) \\
 &\quad + O(m^3) = \frac{6}{\pi^2} m^2 n^2 - 2m^2 n \alpha(m, n) - 2mn^2 \alpha(m) + O(mn^2).
 \end{aligned}$$

Hence

$$s_4(m, n) = \frac{3}{2\pi^2} (mn)^2 - \frac{1}{2} m^2 n \alpha(m, n) - \frac{1}{2} mn^2 \alpha(m) + O(mn^2). \tag{24}$$

6. Final computation. By (20), (21), (22), (23), (24) and (18),

$$\begin{aligned}
 a &= (m+1)(n+1)s_1(m, n) = (m+1)(n+1) \left(\frac{6}{\pi^2} mn - m\alpha(m, n) - n\alpha(m) + O(n) \right) \\
 &= \frac{6}{\pi^2} (mn)^2 + \frac{6}{\pi^2} m^2 n + \frac{6}{\pi^2} mn^2 + \frac{6}{\pi^2} mn - m^2 n \alpha(m, n) - mn^2 \alpha(m) - m^2 \alpha(m, n) \\
 &\quad - n^2 \alpha(m) - mn \alpha(m, n) - mn \alpha(m) - m \alpha(m, n) - n \alpha(m) + O(mn^2) \\
 &= \frac{6}{\pi^2} (mn)^2 - m^2 n \alpha(m, n) - mn^2 \alpha(m) + O(mn^2),
 \end{aligned}$$

$$\begin{aligned}
b &= (n+1)s_2(m, n) = (n+1) \left(\frac{3}{\pi^2} m^2 n - \frac{1}{2} m^2 \alpha(m, n) - mn\alpha(m) + O(mn) \right) \\
&= \frac{3}{\pi^2} (mn)^2 + \frac{3}{\pi^2} m^2 n - \frac{1}{2} m^2 n \alpha(m, n) - mn^2 \alpha(m) + O(mn^2), \\
c &= (m+1)s_3(m, n) = (m+1) \left(\frac{3}{\pi^2} mn^2 - mn\alpha(m, n) - \frac{1}{2} n^2 \alpha(m) + O(mn) \right) \\
&= \frac{3}{\pi^2} (mn)^2 + \frac{3}{\pi^2} mn^2 - m^2 n \alpha(m, n) - \frac{1}{2} mn^2 \alpha(m) + O(m^2 n), \\
d &= s_4(m, n) = \frac{3}{2\pi^2} (mn)^2 - \frac{1}{2} m^2 n \alpha(m, n) - \frac{1}{2} mn^2 \alpha(m) + O(mn^2), \\
f(m+1, n+1) &= 4(a-b-c+d) + O(mn) = \frac{6}{\pi^2} (mn)^2 + O(mn^2),
\end{aligned}$$

which implies also (19).

5. Conclusions and remarks

We improved (3) into (6) and further into (7). Assuming RH, we still improved (7) into (8). An interesting converse problem arises (cf. [3, p. 168]): Does (8) imply RH?

We also improved (4) and (5) into (9). Because of (7) and (8), we conjectured that (9) can be further improved into (10) and, assuming RH, still into (11).

More generally, given $q \geq 1$, define

$$f_q(m, n) = \sum_{\substack{-m < i < m \\ -n < j < n \\ (i,j)=q}} (m - |i|)(n - |j|).$$

The proof of (19) can be extended to show that

$$f_q(m, n) = \frac{6}{\pi^2 q^2} (mn)^2 + O(mn^2)$$

if $m \leq n$. Also the formulas (6)–(8) for $f(n)$ generalize. We have [4]

$$f_q(n) = \frac{6}{\pi^2 q^2} n^4 + r(n),$$

where $r(n)$ has the O -estimates given in the original formulas.

Besides the number of threshold functions, there are several other quantities whose asymptotics can be studied in a similar way. For example, asymptotic formulas for the number of gridlines [3] and q -gridlines [4] are well-known in $G(n)$. A simple modification of our procedure above gives such formulas in $G(m, n)$.

In the practical computation of all these quantities, recursive formulas [8,9,3,5] are useful. They have also been applied in computer experiments to find asymptotic formulas.

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