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Discrete Mathematics 132 (1994) 115–126
DISCRETE MATHEMATICS

Compatible systems of representatives☆

Martin Kochol

Institute for Informatics, Slovak Academy of Sciences, Dúbravská cesta 9, 842 35 Bratislava, Slovak Republic

Received 11 February 1992; revised 31 August 1992

Abstract

The main result of this paper can be quickly described as follows. Let G be a bipartite graph and assume that for any vertex v of G a strongly base orderable matroid is given on the set of edges adjacent with v. Call a subgraph of G a *system of representatives* of G if the edge neighborhood of each vertex of this subgraph is independent in the corresponding matroid. Two systems of representatives we call *compatible* if they have no common edge. We give a necessary and sufficient condition for G to have k pairwise compatible systems of representatives with at least d edges. Unfortunately, this condition is not sufficient if we deal with arbitrary matroids. Furthermore, we establish a listing variant of the Edmonds' covering theorem for strongly base orderable matroids.

1. Introduction

Transversal theory has its basis in the classical theorem of Hall [10] on distinct representatives that gives a necessary and sufficient condition for a family $\mathscr{A}=(A_t\colon t\in T)$ of subsets of S to possess a system of distinct representatives (i.e., a family $(x_t\colon t\in T)$ of elements of S such that $x_t\in A_t$ for any $t\in T$ and $x_t\neq x_{t'}$ if $t\neq t'$).

There are many variations and generalizations of this theorem. The results regarding transversals and matroids are very interesting, especially the two classical theorems established by Rado [18] and Edmonds and Fulkerson [9] (sell also [17]). The first result describes a necessary and sufficient condition for a finite family of sets to possess a transversal which is independent in a given matroid and the second result says that the partial transversals of a finite family of sets form matroid.

Another generalization is the study of k-transversals (see [16, 19, 20]) where each $s\in S$ can be used as a representative of several but at most k sets of $\mathscr{A}$. In [11] we have

☆This research was partially supported by Grant of Slovak Academy of Sciences No. 88 and by EC Cooperation Action IC 1000 "Algorithms for Future Technologies".

SSDI 0012-365X(92)00567-R

introduced M-transversals which are based on the idea that for any $s \in S$, the set of the indices t for which s represents A_t is independent in a given matroid M. Using this idea we have generalized theorems of Hall and Welsh (see [11]) and the theorem of Edmonds and Fulkerson (see [12–14]).

Welsh [19] introduced p-transversals where each set of $\mathscr{A}$ is represented by its subset of a given cardinality.

Asratian [2, 3] called two systems of representatives $(x_t\colon t \in T)$ and $(x_t'\colon t \in T)$ of $\mathscr{A}$ *compatible* if $x_t \neq x_t'$ for any $t \in T$ and presented a necessary and sufficient condition for $\mathscr{A}$ to possess k pairwise compatible systems of distinct representatives.

Joining the ideas of M-transversals and p-transversals we deal with so called "$(\mathscr{M}_S, \mathscr{M}_T)$-systems of representatives" in this paper and generalize the results of Asratian. In the third section we present a necessary and sufficient condition for a finite family of sets to possess k pairwise "compatible" $(\mathscr{M}_S, \mathscr{M}_T)$-systems of representatives if we deal with strongly base orderable matroids only. Our condition is also necessary for arbitrary matroids, but not sufficient as will be shown in the fourth section. In the last section we present a listing variant of the covering theorem of Edmonds [7]. The second section is of preliminary character.

We suppose that the reader is familiar with the theory of matroids. For a survey of matroid theory we refer to Welsh [20]. A comprehensive survey of transversal theory is in Mirsky [16] and also in Welsh [20].

2. Basic notations and definitions

If M is a matroid on S with the rank function ρ then $M^{(k)}$ denotes the *union* of M with itself k times and $\rho^{(k)}$ denotes the rank of $M^{(k)}$,

$$\rho^{(k)}(A) = \min_{X \subseteq A} (k\rho X + |A \setminus X|) \quad (A \subseteq S). \tag{1}$$

If $Y \subseteq S$ then $M|Y$ denotes the *restriction* of M to Y. The rank function of $M|Y$ is just the restriction of ρ to Y (see [20]). *Uniform matroids* of rank k on S are denoted by $U_{k,S}$. Note that $U_{r,S}^{(k)} = U_{kr,S}$.

Let $S_1, \ldots, S_k$ be mutually disjoint sets and $S = \bigcup_{i=1}^{k} S_i$. If $M_1, \ldots, M_k$ are matroids on $S_1, \ldots, S_k$ with the rank functions $\rho_1, \ldots, \rho_k$ and the collections of independent sets $\mathscr{I}_1, \ldots, \mathscr{I}_k$ respectively, then

$$\mathscr{I} = \{X;\, X = X_1 \cup \cdots \cup X_k,\, X_i \in \mathscr{I}_i \ (1 \leqslant i \leqslant k)\}$$

is the collection of independent sets of matroid M on S whose rank function ρ is given by

$$\rho(A) = \sum_{i=1}^{k} \rho_i(A \cap S_i) \quad (A \subseteq S). \tag{2}$$

We call this matroid the *product* (or *direct sum*) of $M_1, \ldots, M_k$ and denote it by $\prod_{i=1}^{k} M_i$ (see e.g. [1]).

One of the basic results of matroid theory is the Edmonds' intersection theorem [8].

Theorem 2.1. *Let M_1, M_2 be matroids on S with rank functions ρ_1, ρ_2, respectively. Then M_1 and M_2 have a common independent set of cardinality k if and only if for all $A \subseteq S$*

$$\rho_1 A + \rho_2(S \setminus A) \geqslant k.$$

The focus of our attention will be the following class of matroids.

Definition 2.2. We say a matroid M is *strongly base orderable* if for any two bases B_1, B_2 there exists a bijection $\pi: B_1 \to B_2$ such that for all subsets $A \subseteq B_1$ $(B_1 \setminus A) \cup \pi A$ is a base of M. (It is easy to show that $(B_2 \setminus \pi A) \cup A$ is also a base of M.)

Strongly base orderable matroids were introduced in [4, 5, 15]. It is known that the cycle matroid of a graph is strongly base orderable if and only if it does not contain a subgraph homeomorphic from K_4. Gammoids are strongly base orderable, therefore transversal matroids and uniform matroids are strongly base orderable (see [20, Ch. 14] for more details).

Davies and McDiarmid [6, Theorems 3, 4] have proved the following generalization of the Edmonds' intersection theorem for strongly base orderable matroids.

Theorem 2.3. *Let M_1, M_2 be strongly base orderable matroids on a finite set S with rank functions ρ_1, ρ_2, respectively and let d be a positive integer. Then the following conditions are equivalent.*

(a) *M_1 and M_2 have k disjoint common independent sets of cardinality d.*

(b) *$M_1^{(k)}$ and $M_2^{(k)}$ have a common independent set of cardinality kd.*

(c) *For all subsets X and Y of S*

$$k(\rho_1 X + \rho_2 Y) + |S \setminus (X \cup Y)| \geqslant kd.$$

Note that the conditions (b) and (c) are equivalent for arbitrary matroids. In [20] is proved.

Lemma 2.4. *If M is a strongly base orderable matroid any minor of M is strongly base orderable. The union and product of strongly base orderable matroids are strongly base orderable.*

If it is clear from the context that we are referring to a set rather than an element we abbreviate $\{x\}$ to x. For example $X \cup x$ means $X \cup \{x\}$, $X \times x$ means $X \times \{x\}$.

Throughout this paper T denotes a finite index set and $\mathscr{A}$ denotes the finite family $(A_t\colon t \in T)$ of subsets of a finite set S. Suppose that $S \cap T = \emptyset$.

A family $\mathscr{X}=(X_t\colon t\in T)$ of subsets of S is called a *subsystem* of $\mathscr{A}$ if $X_t\subseteq A_t$ for any $t\in T$. Then the sum $\sum_{t\in T}|X_t|$ is called the *length* of $\mathscr{X}$.

If each X_t is a singleton then we write $\mathscr{X}=(x_t\colon t\in T)$ and $\mathscr{X}$ is called a *system of representatives* (*SR*) of $\mathscr{A}$. If each X_t is a singleton or empty set we get a *partial SR* of $\mathscr{A}$. Clearly a partial SR of $\mathscr{A}$ of length $|T|$ is an SR of $\mathscr{A}$. An SR $(x_t\colon t\in T)$ of $\mathscr{A}$ is called a *system of distinct representatives* of $\mathscr{A}$ if $x_t\neq x_{t'}$ for any $t\neq t'$.

We shall call two subsystems $\mathscr{X}=(X_t\colon t\in T)$ and $\mathscr{X}'=(X'_t\colon t\in T)$ of $\mathscr{A}$ *compatible* if $X_t\cap X'_t=\emptyset$ for any $t\in T$.

We shall also use the following notation. If $\mathscr{A}$ is a family $(A_t\colon t\in T)$ of subsets of S, then, for any $s\in S$, denote

$$A_s=\{t\in T;\ s\in A_t\}\ \ (\subseteq T).$$

Set $S'=S\times T$ and for any $t\in T$ and $s\in S$, let

$$A'_t=A_t\times t,\qquad A'_s=s\times A_s,$$

i.e., $A'_t,\ A'_s\subseteq S\times T=S'$. Then denote by

$$A'=\bigcup_{t\in T}A'_t=\bigcup_{s\in S}A'_s\ \ (\subseteq S').$$

Similar notation we shall use also for any subsystem $\mathscr{X}$ of $\mathscr{A}$.

Let $\mathscr{M}_S=(M_s\colon s\in S)$ be a family of matroids on T and let $\mathscr{M}_T=(M_t\colon t\in T)$ be a family of matroids on S. Then a subsystem $\mathscr{X}=(X_t\colon t\in T)$ of $\mathscr{A}$ we shall call *$(\mathscr{M}_S, \mathscr{M}_T)$-system of representatives* (in abbreviation *$(\mathscr{M}_S, \mathscr{M}_T)$-SR*) of $\mathscr{A}$ if X_t is independent in M_t and X_s is independent in M_s for any $t\in T$, $s\in S$.

If, for any $s\in S$, M_s is equal to the same matroid M we write $(M, \mathscr{M}_T)$-SR instead of $(\mathscr{M}_S, \mathscr{M}_T)$-SR. Furthermore, if $M=U_{k,T}$ then we write $(k, \mathscr{M}_T)$-SR instead of $(U_{k,T}, \mathscr{M}_T)$-SR. $(\mathscr{M}_S, M)$-SR, $(\mathscr{M}_S, k)$-SR, (M, k)-SR, (k, M)-SR and (k, k')-SR can be defined in a similar way.

$(\mathscr{M}_S, 1)$-SR of $\mathscr{A}$ are in fact partial systems of representatives of $\mathscr{A}$. In this case we omit 1 and $(\mathscr{M}_S, 1)$-SR are called *partial* $\mathscr{M}_S$-SR. Analogously can be introduced *partial* M-SR (if $M_s=M$ for any $s\in S$) and *partial* k-SR (if $M_s=U_{k,T}$ for any $s\in S$). Partial $\mathscr{M}_S$-SR (partial M-SR, partial k-SR) of $\mathscr{A}$ of length $|T|$ are called $\mathscr{M}_S$-SR (M-SR, k-SR) of $\mathscr{A}$ (i.e., we delete the word "partial"). This notation is in accordance with the notation from [20] and [11]. According to our notation a system of distinct representatives is just a 1-SR or a (1, 1)-SR of $\mathscr{A}$ of length $|T|$.

As before, let $\mathscr{M}_S=(M_s\colon s\in S)$ be a family of matroids on T and $\mathscr{M}_T=(M_t\colon t\in T)$ be a family of matroids on S. Then denote by M'_s the matroid on $s\times T$ induced from M_s by the bijection $t\mapsto(s, t)$ for any $s\in S$ and by M'_t the matroid on $S\times t$ induced from M_t by the bijection $s\mapsto(s, t)$ for any $t\in T$. Finally, denote $M'_S=\prod_{s\in S}M'_s$, $M'_T=\prod_{t\in T}M'_t$. M'_S and M'_T are matroids on S'.

Then $\mathscr{X}=(X_t\colon t\in T)$ is an $(\mathscr{M}_S, \mathscr{M}_T)$-SR of $\mathscr{A}$ if and only if X' is independent in $M'_S|A'$ and $M'_T|A'$. Thus from the properties of $M'_S|A'$, $M'_T|A'$ and Theorem 2.1 follows the next theorem, which was proved also in [13]. (Let us stress the fact that it is true for families of arbitrary matroids.)

Theorem 2.5. *Let $\mathscr{A}=(A_t\colon t\in T)$ be a family of subsets of a finite set S and let $\mathscr{M}_S=(M_s\colon s\in S)$ ($\mathscr{M}_T=(M_t\colon t\in T)$) be a family of matroids on T(S). Let ρ_s (ρ_t) denote the rank function of M_s (M_t) for any $s\in S$ ($t\in T$).* Then *$\mathscr{A}$ has an ($\mathscr{M}_S$, $\mathscr{M}_T$)*-SR *of length d if and only if for any subsystem $\mathscr{X}=(X_t\colon t\in T)$ of $\mathscr{A}$ holds*

$$\sum_{s\in S}\rho_s(X_s)+\sum_{t\in T}\rho_t(A_t\setminus X_t)\geqslant d.$$

Finally, let $\mathscr{M}_S^{(k)}$ ($\mathscr{M}_T^{(k)}$) denote the family of matroids ($\mathscr{M}_s^{(k)}\colon s\in S$) (($M_t^{(k)}\colon t\in T$)) on $T(S)$, where $M_s^{(k)}$ ($M_t^{(k)}$) is the union of M_s (M_t) with itself k times.

3. Characterization of compatible ($\mathscr{M}_S$, $\mathscr{M}_T$)-SR

Theorem 3.1. *Let $\mathscr{A}=(A_t\colon t\in T)$ be a family of subsets of a finite set S and let $\mathscr{M}_S=(M_s\colon s\in S)$ ($\mathscr{M}_T=(M_t\colon t\in T)$) be a family of strongly base orderable matroids on T(S). Let ρ_s (ρ_t) denote the rank function of M_s (M_t) for any $s\in S$ ($t\in T$) and d be a positive integer. Then the following conditions are equivalent.*

(a) *$\mathscr{A}$ has k pairwise compatible ($\mathscr{M}_S$, $\mathscr{M}_T$)-SR of length d.*
(b) *$\mathscr{A}$ has an ($\mathscr{M}_S^{(k)}$, $\mathscr{M}_T^{(k)}$)-SR of length kd.*
(c) *For all subsystems $\mathscr{X}=(X_t\colon t\in T)$ and $\mathscr{Y}=(Y_t\colon t\in T)$ of $\mathscr{A}$*

$$k\left(\sum_{s\in S}\rho_s(X_s)+\sum_{t\in T}\rho_t(Y_t)\right)+\sum_{t\in T}|A_t\setminus(X_t\cup Y_t)|\geqslant kd.$$

Proof. $\mathscr{A}$ has k pairwise compatible ($\mathscr{M}_S$, $\mathscr{M}_T$)-SR of length d if and only if $M_S'|A'$ and $M_T'|A'$ have k pairwise disjoint common independent sets of cardinality d. $\mathscr{A}$ has an ($\mathscr{M}_S^{(k)}$, $\mathscr{M}_T^{(k)}$)-SR of length kd if and only if $(M_S')^{(k)}|A'$ and $(M_T')^{(k)}|A'$ have a common independent set of cardinality kd. By Lemma 2.4, $M_S'|A'$ and $M_T'|A'$ are strongly base orderable. Thus Theorem 3.1 follows from Theorem 2.3 and the properties of $M_S'|A'$, $M_T'|A'$. □

Note that in general (i.e., if $\mathscr{M}_S$ and $\mathscr{M}_T$ are families of arbitrary matroids) the conditions (b) and (c) are equivalent and (a) implies both (b) and (c).

Now we deal with consequences of this theorem to $\mathscr{M}_S$-SR.

Corollary 3.2. *Let $\mathscr{A}=(A_t\colon t\in T)$ be a finite family of subsets of a finite set S and let $\mathscr{M}_S=(M_s\colon s\in S)$ be a family of strongly base orderable matroids on T. Let ρ_s denote the rank function of M_s for any $s\in S$. Then the following conditions are equivalent.*

(a) *$\mathscr{A}$ has k pairwise compatible $\mathscr{M}_S$-SR.*

(b) *$\mathscr{A}$ has an $(\mathscr{M}_S^{(k)}, k)$-SR of length $k|T|$.*

(c) *For any $J \subseteq T$ and any subsystem $\mathscr{X}=(X_t\colon t\in T)$ of $\mathscr{A}$*

$$\sum_{s\in S}(k\rho_s(X_s\cap J)+|(A_s\cap J)\setminus X_s|)\geqslant k|J|.$$

Proof. Replacing, in Theorem 3.1, M_t by $U_{1,S}$ we get the conditions (a) and (b). Now ρ_t is the rank function of the matroid $U_{1,S}$ (i.e., $\rho_t X=\mathrm{sg}|X|$). Furthermore, $d=|T|$.

If $\mathscr{D}=(D_t\colon t\in T)$ is a family of subsets of S, then for any $K\subseteq T$, denote by $\mathscr{D}^{(K)}=(D_t^{(K)}\colon t\in T)$ the subsystem of $\mathscr{D}$ such that $D_t^{(K)}=\emptyset$ if $t\notin K$ and $D_t^{(K)}=D_t$ if $t\in K$.

If $\mathscr{X}=(X_t\colon t\in T)$ and $\mathscr{Y}=(Y_t\colon t\in T)$ are two subsystems of $\mathscr{A}$ and $J=\{t\in T;\ Y_t=\emptyset\}$, then

$$\sum_{s\in S}\rho_s(X_s)+\sum_{t\in T}\rho_t(Y_t)\geqslant\sum_{s\in S}\rho_s(X_s^{(J)})+\sum_{t\in T}\rho_t(A_t^{(T\setminus J)}),$$

and, for any $t\in T$,

$$|A_t\setminus(X_t\cup Y_t)|\geqslant|A_t\setminus(X_t^{(J)}\cup A_t^{(T\setminus J)})|.$$

Therefore, in Theorem 3.1(c) we can restrict our attention to $\mathscr{X}=\mathscr{X}^{(J)}$ and $\mathscr{Y}=\mathscr{A}^{(T\setminus J)}$, in other words the condition (c) from Theorem 3.1 is equivalent with

(c′) For any $J\subseteq T$ and any subsystem $\mathscr{X}=(X_t\colon t\in T)$ of $\mathscr{A}$

$$k\left(\sum_{s\in S}\rho_s(X_s^{(J)})+\sum_{t\in T}\rho_t(A_t^{(T\setminus J)})\right)+\sum_{t\in T}|A_t\setminus(X_t^{(J)}\cup A_t^{(T\setminus J)})|\geqslant k|T|.$$

But

$$X_s^{(J)}=X_s\cap J,\qquad \sum_{t\in T}\rho_t(A_t^{(T\setminus J)})=|T\setminus J|$$

and

$$\sum_{t\in T}|A_t\setminus(X_t^{(J)}\cup A_t^{(T\setminus J)})|=\sum_{s\in S}|(A_s\cap J)\setminus X_s|.$$

Thus (c′) is equivalent with the condition (c) from Corollary 3.2, concluding the proof. □

The following result was proved in fact by Asratian [2].

Corollary 3.3. *Let $\mathscr{A}=(A_t\colon t\in T)$ be a finite family of subsets of a finite set S. Then the following conditions are equivalent.*

(a) *$\mathscr{A}$ has k pairwise compatible systems of distinct representatives.*

(b) *$\mathscr{A}$ has a (k, k)-SR of length $k|T|$.*

(c) *For any $J\subseteq T$*

$$\sum_{s\in S}(\min\{k,|A_s\cap J|\})\geqslant k|J|.$$

Proof. Replace ρ_s by the rank function of $U_{1,T}$ in Corollary 3.2. As pointed out earlier $U_{1,T}$ is strongly base orderable matroid.

Fix $J \subseteq T$. Then take the subsystem $\mathscr{J} = (J_t\colon t \in T)$ of $\mathscr{A}$ such that

$$J_s = A_s \cap J \quad \text{if } |A_s \cap J| \geqslant k,$$

$$J_s = \emptyset \quad \text{if } |A_s \cap J| < k.$$

Then for any subsystem $\mathscr{X} = (X_t\colon t \in T)$ of $\mathscr{A}$ and any $s \in S$

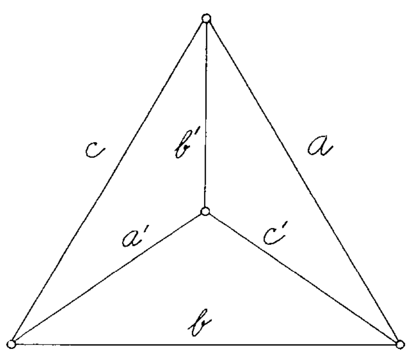
$$k\rho_s(X_s \cap J) + |(A_s \cap J) \setminus X_s| \geqslant k\rho_s(J_s \cap J) + |(A_s \cap J) \setminus J_s|$$
$$= \min\{k, |A_s \cap J|\}.$$

Thus the subsystem $\mathscr{X}$ from Corollary 3.2(c) can be replaced by the subsystem $\mathscr{J}$ and we get the condition (c) from Corollary 3.3. The details are left to the reader. □

4. Construction of contraexamples

Now we show that Theorem 3.1 does not hold in general. The first simple example is in fact a modification of an example presented in [6]. (By the way, the example over there shows that Theorem 2.3 is not true in general.)

Example 4.1. Let $S = \{1, 2\}$, $T = \{a, a', b, b', c, c'\}$ and $A_t = S$ for any $t \in T$. Let M_1 be the cycle matroid of the graph K_4 depicted in Fig. 1, M_2 be the transversal matroid of the family $(\{a, a'\}, \{b, b'\}, \{c, c'\})$, $\mathscr{M}_S = (M_1, M_2)$ and $\mathscr{M}_T = (M_t = U_{1,S}\colon t \in T)$. Since T is independent in $M_1^{(2)}$, $M_2^{(2)}$ then $\mathscr{A}$ is an $(\mathscr{M}_S^{(2)}, 2)$-SR of $\mathscr{A}$. But $\mathscr{A}$ has no two compatible $\mathscr{M}_S$-SR since M_1 and M_2 do not have two disjoint common bases.



K_4

Fig. 1.

The matroids M_1 and M_2 from Example 4.1 are different. Now we present another example, where the matroids of $\mathcal{M}_S$ are equal to the same cycle matroid of a graph, i.e., we shall deal with M-SR.

Example 4.2. Let $S=\{0,1,\dots,12\}$ and $T=\{e_1,e_2,\dots,e_{36}\}$. Elements of T will be presented as edges of graphs.

If H is a graph with the edge set $E(H)=\{e_{i_1},e_{i_2},\dots,e_{i_r}\}$ $(\subseteq T)$ and k is a positive integer, then by $H+k$ we denote the graph with the edge set $E(H+k)=\{e_{i_1+k},e_{i_2+k},\dots,e_{i_r+k}\}$ and isomorphic with H such that there exists an isomorphism $\phi: V(H)\to V(H+k)$ satisfying: if $(u,v)=e_i$ then $(\phi u,\phi v)=e_{i+k}$. (Note that we shall use this operation such that $E(H)\cap E(H+k)=\emptyset$ and no confusions will occur.)

Let H_0 and G_1 be the graphs depicted in Fig. 2. Take $G_2=G_1+12$ and $G_3=G_1+24$. Let G be a graph with the edge set equal to T erasing from H_0, G_1, G_2, G_3 such that we glue the edges $e_1,e_2,e_{13},e_{14},e_{25},e_{26}$ from H_0 with the same edges of G_1, G_2, G_3. Note that from this construction we can erase several nonisomorphic graphs, but take G to be one of them. Let M be a cycle matroid of G. It is a matroid on T. Let H_1,H_2,H_3,H_4 be the subgraphs of G_1 depicted in Fig. 3 and

$$H_5=H_1+12,\qquad H_6=H_2+12,\qquad H_7=H_3+12,\qquad H_8=H_4+12,$$

$$H_9=H_1+24,\qquad H_{10}=H_2+24,\qquad H_{11}=H_3+24,\qquad H_{12}=H_4+24.$$

Take the family $\mathcal{A}=(A_t\colon t\in T)$ of subsets of S such that for any $s\in S$, A_s is the edge set of the graph H_s.

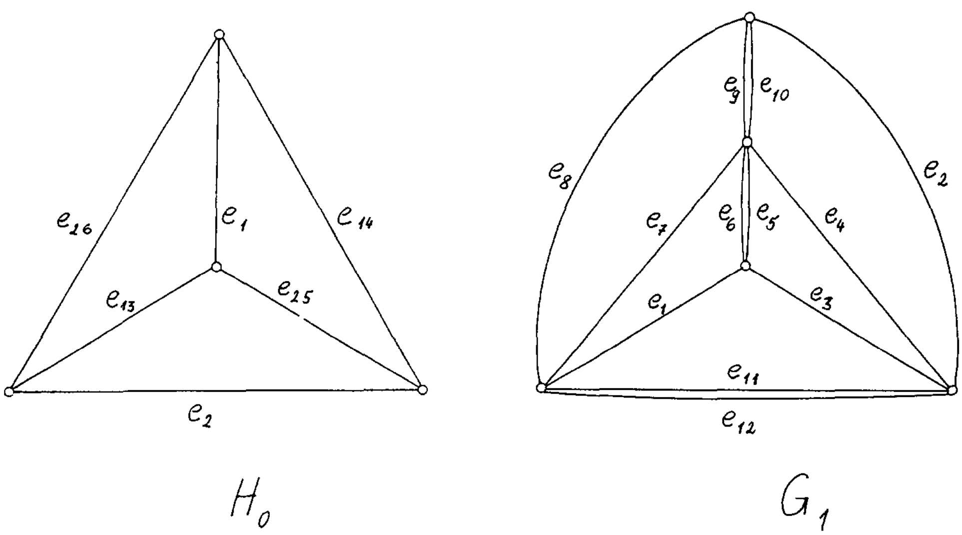


Fig. 2.

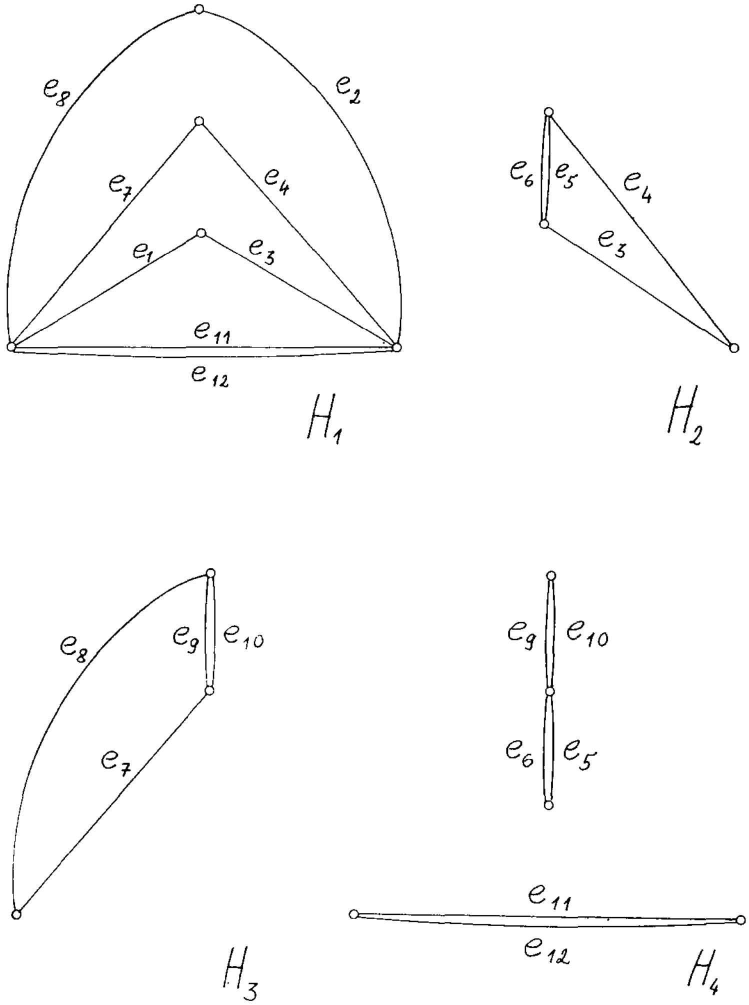


Fig. 3.

Clearly, any A_t ($t \in T$) has cardinality two and any H_s ($s \in S$) can be covered by two disjoint forests. Thus $\mathscr{A}$ is an $(M^{(2)}, 2)$-SR of $\mathscr{A}$. Let $\mathscr{A}$ has two compatible M-SR. Denote them by $\mathscr{X} = (X_t\colon t \in T)$ and $\mathscr{Y} = (Y_t\colon t \in T)$. Take the sets X' and Y'. Without the abuse of generality we can suppose that $(0, e_1) \in X'$. Then we can check that

$$(1, e_1) \in Y', \quad (1, e_3) \in X', \quad (2, e_3) \in Y', \quad (2, e_4) \in X',$$

$$(1, e_4) \in Y', \quad (1, e_7) \in X', \quad (3, e_7) \in Y', \quad (3, e_8) \in X',$$

$$(1, e_8) \in Y', \quad (1, e_2) \in X', \quad (0, e_2) \in Y'.$$

Thus X_0 (Y_0) contains just one element of the set $\{e_1, e_2\}$. Similarly it can be proved that X_0 (Y_0) contains just one element of the set $\{e_{13}, e_{14}\}$ and X_0 (Y_0) contains just one element of the set $\{e_{25}, e_{26}\}$. Then X_0 and Y_0 cannot be both independent in M – a contradiction. Thus $\mathscr{A}$ has no two compatible M-SR.

Note that G_1 (thus also G) cannot be covered by two forests. Thus T is not independent in $M^{(2)}$. This fact will be interesting in connection with a problem presented in the next paragraph.

The matroids M_1 from Example 4.1 and M from Example 4.2 are not strongly base orderable because both are cycle matroids of graphs that contain K_4. But the class of graphic matroids is regarded as one of the simplest and has plenty of interesting properties. Thus Examples 4.1 and 4.2 in fact show that if we go behind the class of strongly base orderable matroids then Theorem 3.1 cannot be extended in relatively simple cases.

5. A generalization of the Edmonds' covering theorem

If M is a matroid on T with rank function ρ, then the covering theorem of Edmonds [7] says that M has k independent sets whose union is T if and only if for any $J \subseteq T$

$$k\rho J \geqslant |J|.$$

Now let us look at this theorem from the point of view of M-systems of representatives. We can prove the following theorem.

Theorem 5.1. *Let S and T be finite sets and M be a matroid on T with rank function ρ. Then the conditions* (a)–(d) *are equivalent*

(a) *M has k independent sets whose union is T,*

(b) *$k\rho J \geqslant |J|$ for any $J \subseteq T$,*

(c) *the family $\mathscr{B} = (B_t\colon t \in T)$ such that $B_t = \{1, \ldots, k\}$ for any $t \in T$ has an M-SR,*

(d) *any family $\mathscr{C} = (C_t\colon t \in T)$ of subsets of S such that $|C_t| = k$ has an M-SR.*

Moreover, if M is strongly base orderable then also the condition

(e) *any family $\mathscr{C} = (C_t\colon t \in T)$ of subsets of S such that $|C_t| = k$ has k pairwise compatible M-SR*

is equivalent with (a)–(d).

Proof. By Edmonds [7], (a) and (b) are equivalent. It is clear that also (a) and (c) are equivalent and that (d) implies (c). We prove that (b) implies (d).

Let (b) hold and $\mathscr{C} = (C_t\colon t \in T)$ be a family satisfying the conditions of (d). We have proved in [11] that $\mathscr{C}$ has an M-SR if and only if for any $J \subseteq T$

$$\sum_{s \in S} \rho(C_s \cap J) \geqslant |J|.$$

(This also follows from Corollary 3.2 or Theorem 2.5.) But, by (b), $\rho(C_s \cap J) \geqslant |C_s \cap J|/k$, thus

$$\sum_{s \in S} \rho(C_s \cap J) \geqslant \sum_{s \in S} \left(\frac{|C_s \cap J|}{k} \right) = \frac{k|J|}{k} = |J|$$

for any $J \subseteq T$ and $\mathscr{C}$ has an M-SR. Therefore (b) implies (d).

Clearly, (e) implies (d). We show that if M is strongly base orderable then (b) implies (e).

Let (b) hold. Then for any $J \subseteq T$ and any subsystem $\mathscr{X} = (X_t\colon t \in T)$ of $\mathscr{C}$

$$\sum_{s \in S} (k\rho(X_s \cap J) + |(C_s \cap J) \setminus X_s|) \geqslant \sum_{s \in S} (|X_s \cap J| + |(C_s \cap J) \setminus X_s|)$$
$$= \sum_{s \in S} |C_s \cap J| = k|J|.$$

Thus, by Theorem 3.1, if M is strongly base orderable then $\mathscr{C}$ has k pairwise compatible M-SR, concluding the proof. □

Items (d) and (e) describe something as "listing variants" of the Edmonds' covering theorem. Item (e) is of some interest also for another reason. As pointed out earlier, Theorem 3.1 cannot be extended in relatively simple cases. On the other hand there is a possibility that the conditions (a)–(e) from Theorem 5.1 are equivalent for arbitrary matroids or at least for a larger class of matroids than is the class of strongly base orderable matroids. Let us formulate our assumption.

Problem. Are the conditions (a)–(e) from Theorem 5.1 equivalent for any matroid?

This question is answered affirmatively for strongly base orderable matroids in Theorem 5.1. As pointed out at the end of the fourth section, Example 4.2 cannot be used as a negative solution of this problem, since the matroid M from this example cannot be covered by two of its independent sets.

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