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Uniqueness of Solutions of Parabolic Semilinear Nonlocal-Boundary Problems

LUDWIK BYSZEWSKI*

Department of Applied Mathematics, Florida Institute of Technology, 150 West University Boulevard, Melbourne, Florida 32901

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The purpose of the paper is to give two theorems about the uniqueness of solutions of parabolic semilinear nonlocal-boundary problems. The paper is a continuation of previous papers by Byszewski and the generalization of some results from [R. Rabczuk, "Introduction to Differential Inequalities," PWN, Warsaw, 1976 [Polish]; J. Chabrowski, On nonlocal problems for parabolic equations, *Nagoya Math. J.* **93** (1984), 109–131]. The theorems obtained in this paper can be applied in the theories of diffusion and heat conduction with better effects than the analogous theorems about parabolic initial-boundary problems and than the analogous theorems about parabolic periodic-boundary problems.

1. INTRODUCTION

In papers [1–5] the author studied parabolic and hyperbolic nonlinear problems together with nonlocal conditions. The coefficients in these conditions had the values belonging to the intervals $[-1, 0]$ and $(-1, 1)$. In this paper we give two theorems about the uniqueness of solutions of parabolic semilinear boundary problems together with nonlocal conditions. In these theorems the coefficients in the nonlocal conditions have the values belonging to the interval $[-1, 1]$. Therefore, the problems considered in the paper are more general than the analogous parabolic initial-boundary problems and than the analogous parabolic periodic-boundary problems. To prove the results of the paper, a method is used other than in the author's earlier papers about nonlocal problems. The proofs of the theorems from the paper are based on the Green formula about the integration by parts.

The paper is a continuation of papers [1–7] and the generalizations of some results from [10] (see [10, Sect. 45]) and [8] (see [8, Theorem 4]).

Analogously, as in [1–3] the results obtained in this paper can be applied for some problems in the theories of diffusion and heat conduction with better effects than the analogous known classical parabolic problems.

* Permanent address: Cracow University of Technology, Institute of Mathematics, Warszawska 24, 31-155 Cracow, Poland.

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2. PRELIMINARIES

The notation, definitions, and assumptions from this section are valid throughout this paper.

Let t_0 be a real finite number, $0 < T < \infty$ and $x = (x_1, ..., x_n) \in R^n$. Define the domain

$$D := D_0 \times (t_0, t_0 + T),$$

where D_0 is an open and bounded domain in R^n such that the boundary ∂D_0 satisfies the following conditions:

(i) If $n \geq 2$ then ∂D_0 is an union of a finite number of surface patches of class C^1 which have no common interior points but have common boundary points.

(ii) If $n \geq 3$ then all the edges of ∂D_0 are sums of a finite numbers of $(n-2)$-dimensional surface patches of class C^1.

Conditions (i) and (ii) are understood in the sense that for $n = 3$ the edges of ∂D_0 are the arcs and for $n = 2$ the surface patches of ∂D_0 are also the arcs. Since for $n = 2$ the edges of ∂D_0 are the points and since for $n = 1$ the surface patches of ∂D_0 are also the points then conditions (i) and (ii) are only formulated for $n \geq 2$ and $n \geq 3$, respectively.

By n_x, where $x \in \partial D_0$, we denote the interior normal to ∂D_0 at x. If it does not lead to misunderstanding the interior normal n_x will be denoted by n.

The symbols L and P are reserved for two operators given by the formulae

$$L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial}{\partial x_j} \right) \tag{2.1}$$

and

$$P = L + c - \frac{\partial}{\partial t}, \tag{2.2}$$

where $a_{ij} = a_{ij}(x, t)$ $(i, j = 1, ..., n)$ and $c = c(x, t)$ are given functions defined for $(x, t) \in \bar{D}$. Moreover, we assume that $a_{ij}(x, t) = a_{ji}(x, t)$ $(i, j = 1, ..., n)$ for $(x, t) \in \bar{D}$.

By $Z(D)$ we denote the set of the functions $u(x, t)$ continuous in $\bar{D}$, possessing continuous derivatives $(\partial u/\partial x_i)$ $(i = 1, ..., n)$ in $\bar{D}$ and possessing continuous and bounded derivatives $(\partial^2 u/\partial x_i\, \partial x_j)$ $(i, j = 1, ..., n)$, $\partial u/\partial t$ in D.

Let $u \in Z$, $x_0 \in \partial D_0$, and $t \in [t_0, t_0 + T]$. The expression

$$\frac{du(x, t)}{dv(x_0, t)} := \sum_{i=1}^{n} \frac{\partial u(x_0, t)}{\partial x_i} \sum_{j=1}^{n} a_{ij}(x_0, t) \cos(n_{x0}, x_j) \tag{2.3}$$

is called the transversal derivative of the function u at the point (x_0, t).

If it does not lead to misunderstanding the transversal derivative $du(x, t)/dv(x_0, t)$ will be denoted by $(d/dv)\, u(x_0, t)$ or du/dv.

For the given functions f, ϕ, ψ, h defined on $D \times R$, D_0, $\partial D_0 \times [t_0, t_0 + T]$, D_0, respectively, the first Fourier's nonlocal problem in D consists in finding a function $u \in Z(D)$, satisfying the equation

$$(Pu)(x, t) = f(x, t, u(x, t)) \qquad \text{for} \quad (x, t) \in D, \tag{2.4}$$

the nonlocal condition

$$u(x, t_0) + h(x)\, u(x, t_0 + T) = \phi(x) \qquad \text{for} \quad x \in D_0, \tag{2.5}$$

and the boundary condition

$$u(x, t) = \psi(x, t) \qquad \text{for} \quad (x, t) \in \partial D_0 \times [t_0, t_0 + T]. \tag{2.6}$$

A function u possessing the above properties is called the solution of the first Fourier's nonlocal problem (2.4)–(2.6) in D.

If condition (2.6) from the first Fourier's nonlocal problem (2.4)–(2.6) is replaced by the condition

$$\frac{d}{dv} u(x, t) + k(x, t)\, u(x, t) = \psi(x, t) \qquad \text{for} \quad (x, t) \in \partial D_0 \times [t_0, t_0 + T], \tag{2.7}$$

where k is the given function defined on $\partial D_0 \times [t_0, t_0 + T]$, then problem (2.4), (2.5), and (2.7) is said to be the mixed nonlocal problem in D. A function $u \in Z(D)$, satisfying Eq. (2.4) and conditions (2.5), (2.7) is called the solution of the mixed nonlocal problem (2.4), (2.5), and (2.7) in D.

We shall use the following:

LEMMA 2.1 (See [9, Sect. 17.11]). *If $\xi = \xi(x)$ and $\eta = \eta(x)$ are continuous functions for $x \in \bar{D}_0$ and $\partial \xi(x)/\partial x_i$, $(\partial \eta(x)/\partial x_i)$ $(i = 1, ..., n)$ are continuous and bounded functions for $x \in D_0$, then*

$$\int_{D_0} \xi(x) \frac{\partial \eta(x)}{\partial x_i}\, dx = -\int_{\partial D_0} \xi(x)\, \eta(x) \cos(n, x_i)\, d\sigma - \int_{D_0} \eta(x) \frac{\partial \xi(x)}{\partial x_i}\, dx \ (i = 1, ..., n),$$

where $d\sigma$ is a surface element in R^n.

3. Theorems about Uniqueness

In this section we shall prove two theorems about the uniqueness of solutions of nonlocal semilinear parabolic boundary problems.

THEOREM 3.1. *Assume that*

1. a_{ij} $(i, j=1, ..., n)$ *are continuous in* $\bar{D}$, $(\partial a_{ij}/\partial x_k)$ $(i, j, k=1, ..., n)$ *are continuous and bounded in* D, *and* c *is continuous in* $\bar{D}$.

2. $\sum_{i,j=1}^{n} a_{ij}(x, t)\, \lambda_i \lambda_j \geqslant 0$ *for arbitrary* $(x, t) \in D$ *and* $(\lambda_1, ..., \lambda_n) \in R^n$.

3. $c(x, t) \leqslant 0$ *for* $(x, t) \in D$.

4. h *is a continuous function in* $\bar{D}_0$ *such that* $|h(x)| \leqslant 1$ *for* $x \in D_0$.

5. *The functions* $f(x, t, z)$ *and* $\partial f(x, t, z)/\partial z$ *are continuous for* $(x, t) \in \bar{D}$, $z \in R$. *Moreover,* $\partial f(x, t, z)/\partial z > 0$ *for* $(x, t) \in \bar{D}$, $z \in R$.

Then the first Fourier's nonlocal problem (2.4)–(2.6) *admits at most one solution in* D.

Proof. Suppose that u_1 and u_2 are two solutions of problem (2.4)–(2.6) in D and let

$$w := u_1 - u_2. \tag{3.1}$$

Then the following formulae hold:

$$(Pw)(x, t) = f(x, t, u_1(x, t)) - f(x, t, u_2(x, t)) \quad \text{for} \quad (x, t) \in D, \tag{3.2}$$

$$w(x, t_0) + h(x)\, w(x, t_0 + T) = 0 \quad \text{for} \quad x \in D_0, \tag{3.3}$$

$$w(x, t) = 0 \quad \text{for} \quad (x, t) \in \partial D_0 \times [t_0, t_0 + T]. \tag{3.4}$$

From the assumption that u_1, $u_2 \in Z(D)$, from assumption 5 and from the mean value theorem, there exists $\theta \in (0, 1)$ such that

$$f(x, t, u_1(x, t)) - f(x, t, u_2(x, t))$$
$$= w(x, t) \frac{\partial f(x, t, u_2(x, t) + \theta w(x, t))}{\partial z} \quad \text{for} \quad (x, t) \in \bar{D}. \tag{3.5}$$

By (3.5), (3.2), by assumption 1, by (2.2), (2.1) and by Lemma 2.1,

$$\int_{t_0}^{t_0+T} \left[\int_{D_0} w^2 \frac{\partial f(x, t, u_2 + \theta w)}{\partial z}\, dx \right] dt$$
$$= \int_{t_0}^{t_0+T} \left[\int_{D_0} wPw\, dx \right] dt$$
$$= \int_{t_0}^{t_0+T} \left[\int_{D_0} wLw\, dx \right] dt + \int_{t_0}^{t_0+T} \left[\int_{D_0} cw^2\, dx \right] dt$$
$$- \int_{t_0}^{t_0+T} \left[\int_{D_0} \frac{\partial w}{\partial t} w\, dx \right] dt$$

$$= -\int_{t_0}^{t_0+T}\left[\int_{\partial D_0} w \sum_{i=1}^{n} \cos(n, x_i) \sum_{j=1}^{n} a_{ij} \frac{\partial w}{\partial x_j}\, d\sigma\right] dt$$

$$-\int_{t_0}^{t_0+T}\left[\int_{D_0} \sum_{i,j=1}^{n} a_{ij} \frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j}\, dx\right] dt$$

$$+\int_{t_0}^{t_0+T}\left[\int_{D_0} c w^2\, dx\right] dt - \int_{t_0}^{t_0+T}\left[\int_{D_0} \frac{\partial w}{\partial t} w\, dx\right] dt. \tag{3.6}$$

From (3.6), (3.4) and from assumptions 2, 3,

$$\int_{t_0}^{t_0+T}\left[\int_{D_0} w^2 \frac{\partial f(x, t, u_2+\theta w)}{\partial z}\, dx\right] dt \leqslant -\int_{t_0}^{t_0+T}\left[\int_{D_0} \frac{\partial w}{\partial t} w\, dx\right] dt. \tag{3.7}$$

Using integration by parts, it is easy to see that

$$\int_{t_0}^{t_0+T}\left[\int_{D_0} \frac{\partial w}{\partial t} w\, dx\right] dt = \frac{1}{2}\int_{D_0} w^2(x, t_0+T)\, dx - \frac{1}{2}\int_{D_0} w^2(x, t_0)\, dx. \tag{3.8}$$

Formulae (3.7) and (3.8) imply the inequality

$$\int_{t_0}^{t_0+T}\left[\int_{D_0} w^2 \frac{\partial f(x, t, u_2+\theta w)}{\partial z}\, dx\right] dt$$

$$\leqslant -\frac{1}{2}\int_{D_0} [w^2(x, t_0+T) - w^2(x, t_0)]\, dx. \tag{3.9}$$

From (3.9) and (3.3), we have

$$\int_{t_0}^{t_0+T}\left[\int_{D_0} w^2 \frac{\partial f(x, t, u_2+\theta w)}{\partial z}\, dx\right] dt$$

$$\leqslant -\frac{1}{2}\int_{D_0} w^2(x, t_0+T)[1 - h^2(x)]\, dx. \tag{3.10}$$

By (3.10) and by assumption 4, we obtain

$$\int_{t_0}^{t_0+T}\left[\int_{D_0} w^2 \frac{\partial f(x, t, u_2+\theta w)}{\partial z}\, dx\right] dt \leqslant 0.$$

From the above inequality and from assumption 5,

$$w^2(x, t) \leqslant 0 \qquad \text{for} \quad (x, t) \in D$$

and therefore

$$w(x, t) = 0 \qquad \text{for} \quad (x, t) \in D.$$

The proof of Theorem 3.1 is thereby complete.

THEOREM 3.2. *Suppose that assumptions* 1–5 *from Theorem* 3.1 *are satisfied. Assume, additionally, that*

6. *The function* k *is continuous in* $\partial D_0 \times [t_0, t_0 + T]$ *and*

$$k(x, t) \leqslant 0 \qquad \text{for} \quad (x, t) \in \partial D_0 \times [t_0, t_0 + T]. \tag{3.11}$$

Then the mixed nonlocal problem (2.4), (2.5), *and* (2.7) *admits at most one solution in* D.

Proof. Suppose that u_1 and u_2 are two solutions of problem (2.4), (2.5), and (2.7) in D, and let

$$w := u_1 - u_2. \tag{3.1}$$

Then the following formulae hold:

$$(Pw)(x, t) = f(x, t, u_1(x, t)) - f(x, t, u_2(x, t)) \qquad \text{for} \quad (x, t) \in D, \tag{3.2}$$

$$w(x, t_0) + h(x)\, w(x, t_0 + T) = 0 \qquad \text{for} \quad x \in D_0, \tag{3.11}$$

$$\frac{d}{dv} w(x, t) + k(x, t)\, w(x, t) = 0 \qquad \text{for} \quad (x, t) \in \partial D_0 \times [t_0, t_0 + T]. \tag{3.12}$$

Using the same argument as in the proof of Theorem 3.1 and using the definition of du/dv (see (2.3)) we have

$$\begin{aligned} \int_{t_0}^{t_0+T} \left[\int_{D_0} w^2 \frac{\partial f(x, t, u_2 + \theta w)}{\partial z} dx \right] dt &= -\int_{t_0}^{t_0+T} \left[\int_{\partial D_0} w \frac{dw}{dv} d\sigma \right] dt \\ &\quad - \int_{t_0}^{t_0+T} \left[\int_{D_0} \sum_{i,j=1}^{n} a_{ij} \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} dx \right] dt \\ &\quad + \int_{t_0}^{t_0+T} \left[\int_{D_0} c w^2 \, dx \right] dt - \int_{t_0}^{t_0+T} \left[\int_{D_0} \frac{\partial w}{\partial t} w \, dx \right] dt. \end{aligned} \tag{3.13}$$

From (3.13), (3.12), from assumptions 2, 3, and from (3.8), (3.3), we obtain

$$\begin{aligned} \int_{t_0}^{t_0+T} \left[\int_{D_0} w^2 \frac{\partial f(x, t, u_2 + \theta w)}{\partial z} dx \right] dt \\ \leqslant \int_{t_0}^{t_0+T} \left[\int_{\partial D_0} k w^2 \, d\sigma \right] dt - \frac{1}{2} \int_{D_0} w^2(x, t_0 + T)[1 - h^2(x)] \, dx. \end{aligned} \tag{3.14}$$

By (3.14), (3.11) and by assumption 4, we obtain the inequality

$$\int_{t_0}^{t_0+T}\left[\int_{D_0} w^2 \frac{\partial f(x, t, u_2+\theta w)}{\partial z}\, dx\right] dt \leqslant 0.$$

From the above inequality and from assumption 5,

$$w^2(x, t) \leqslant 0 \qquad \text{for} \quad (x, t) \in D$$

and, therefore, the proof of Theorem 3.2 is complete.

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