Available online at www.sciencedirect.com

ScienceDirect

Journal of Approximation Theory 162 (2010) 2149–2167

Journal of Approximation Theory

www.elsevier.com/locate/jat

Painlevé V and a Pollaczek–Jacobi type orthogonal polynomials

Yang Chen[a,*], Dan Dai[b,c]

[a] Department of Mathematics, Imperial College London, 180 Queens Gates, London SW7 2BZ, UK
[b] Department of Mathematics, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong
[c] Department of Mathematics, Katholieke Universiteit Leuven, Celestijnenlaan 200 B, 3001 Leuven, Belgium

Received 7 July 2009; received in revised form 23 March 2010; accepted 8 July 2010
Available online 15 July 2010

Communicated by Andrei Martinez-Finkelshtein

Abstract

We study a sequence of polynomials orthogonal with respect to a one-parameter family of weights

$$w(x) := w(x,t) = \mathrm{e}^{-t/x} x^{\alpha} (1-x)^{\beta}, \quad t \geq 0,$$

defined for $x \in [0,1]$. If $t = 0$, this reduces to a shifted Jacobi weight. Our ladder operator formalism and the associated compatibility conditions give an easy determination of the recurrence coefficients.

For $t > 0$, the factor $\mathrm{e}^{-t/x}$ induces an infinitely strong zero at $x = 0$. With the aid of the compatibility conditions, the recurrence coefficients are expressed in terms of a set of auxiliary quantities that satisfy a system of difference equations. These, when suitably combined with a pair of Toda-like equations derived from the orthogonality principle, show that the auxiliary quantities are particular Painlevé V and/or allied functions.

It is also shown that the logarithmic derivative of the Hankel determinant,

$$D_n(t) := \det\left(\int_0^1 x^{i+j} \mathrm{e}^{-t/x} x^{\alpha} (1-x)^{\beta} \mathrm{d}x\right)_{i,j=0}^{n-1},$$

satisfies the Jimbo–Miwa–Okamoto σ-form of the Painlevé V equation and that the same quantity satisfies a second-order non-linear difference equation which we believe to be new.

* Corresponding author.
E-mail addresses: ychen@ic.ac.uk, yangbrookchen@yahoo.co.uk (Y. Chen), dandai@cityu.edu.hk (D. Dai).

0021-9045/$ - see front matter
doi:10.1016/j.jat.2010.07.005

1. Introduction

For polynomials orthogonal with respect to weights w absolutely continuous on $[-1, 1]$ and satisfying the Szegő condition

$$\int_{-1}^{1} \frac{|\ln w(x)|}{\sqrt{1-x^2}} \mathrm{d}x < \infty,$$

a general theory of Szegő [26, pp 296–312] gives a comprehensive description of the large n behavior of the polynomials both for $x \in (-1, 1)$ and for $x \notin [-1, 1]$ and the recurrence coefficients; see also [16]. For a recent account and extension of Szegő's theory, see [21,13,22].

With the introduction of a deformation parameter $t \geq 0$, we have a Pollaczek–Jacobi type weight defined as

$$w(x) := w(x; t) = \mathrm{e}^{-t/x} x^{\alpha}(1-x)^{\beta}, \quad x \in [0, 1],\ \alpha > 0,\ \beta > 0, \tag{1.1}$$

which violates the Szegő condition. For convenience we have taken the interval of orthogonality to be $[0, 1]$. The weight function of the Pollaczek polynomial and a generalization due to Szegő behave like

$$\exp\left(-\frac{c}{\sqrt{1-x^2}}\right), \quad \text{as } x \to \pm 1,$$

where c is a positive constant; see [26, pp 393–400] for a detailed description. We should like to mention, as a brief guide to the reader, some recent literature on the Pollaczek polynomials; see [27] for the asymptotic behavior of the polynomials for $x \notin [-1, 1]$, [2,29] for the asymptotic behavior of their zeros and [19,20,28] for applications to physical problems. Regarding weighted polynomial approximation in L_p with respect to a class of exponential weights on $[-1, 1]$ that violate the Szegő conditions, see [9,18].

Note that our weight is in some sense more "singular" since the Szegő condition is strongly violated.

The purpose of this paper is to give a complete description of the recurrence coefficients of the associated orthogonal polynomials. As can be seen later these are expressed in terms of a set of auxiliary quantities which are ultimately particular Painlevé V and allied functions.

In Section 2, with the aid of certain supplementary conditions (S_1), (S_2) and (S_2') derived from a pair of operators and the recurrence relations, we obtain a system of difference equations satisfied by certain auxiliary quantities (R_n, R_n^*, r_n, r_n^*) and the recurrence coefficients (α_n, β_n). More importantly, the equation (S_2') is in some sense the "first integral" of (S_1) and (S_2), and automatically performs a sum of R_j^* from $j = 0$ to $j = n-1$ (see (2.35)). This turns out to be the logarithmic derivative of the Hankel determinant (generated by our Pollaczek–Jacobi type weight) with respect to t; see (4.4).

We should mention here that a similar approach was adopted in [1,8] where (S_1) and (S_2') were sufficient for the purpose. However, for the problem at hand, taking into account the difference equations, there are ultimately three auxiliary quantities R_n, r_n^* and r_n. The equation (S_2) turns out to be crucial for later development.

In Section 3, we make use of the results of Section 2 to express α_n and β_n in terms of the auxiliary quantities and show that these reduce to recurrence coefficients of the "shifted" Jacobi polynomials when $t = 0$ through an easy computation.

The t dependences of the recurrence coefficients and the auxiliary quantities are derived in Section 4, resulting in a pair of Toda-like equations.

In Section 5, combining the results from previous sections we express r_n^*, r_n and R_n in terms of

$$H_n(t) := t\frac{\mathrm{d}}{\mathrm{d}t}\ln D_n(t) \tag{1.2}$$

and $H_n'(t)$. We show that a functional equation $f(H_n, H_n', H_n'') = 0$ resulting from eliminating the auxiliary quantities in favor of H_n and H_n' is the Jimbo–Miwa–Okamoto σ-form of a Painlevé V equation.

In Section 6, from the difference equations found in Section 2 and the expressions for α_n and β_n found in Section 3, we express R_n, β_n in terms of $\mathsf{p}_1(n)$, the coefficients of z^{n-1} of our monic polynomials $P_n(z)$. And since $\mathsf{p}_1(n)$ is easily related to H_n, the resulting functional equation $g(H_n, H_{n+1}, H_{n-1}) = 0$ is the discrete analog of the σ-form mentioned in the abstract; see (6.13). We believe that this equation is new.

In Section 7 we derive a second-order o.d.e. satisfied by R_n which is also a Painlevé V form since H_n is shown to satisfy the σ-form.

The large n behavior of the recurrence coefficients will be described in a future publication.

2. Preliminaries

Let $P_n(x)$ be the monic polynomials of degree n in x and orthogonal with respect to the weight function $w(x;t)$ defined in (1.1), that is

$$\int_0^1 P_m(x)P_n(x)w(x;t)\mathrm{d}x = h_n\delta_{m,n}. \tag{2.1}$$

(The polynomials $P_n(x)$ and the constant h_n all depend on t, but we suppress the dependence for brevity.) An immediate consequence of the orthogonality condition is the recurrence relation

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x), \quad n = 0, 1, \ldots. \tag{2.2}$$

Here we take the "initial" conditions to be $P_0(z) := 1$ and $\beta_0 P_{-1}(z) := 0$. Note that $P_n(z)$ has the following form:

$$P_n(z) = z^n + \mathsf{p}_1(n)z^{n-1} + \cdots. \tag{2.3}$$

Substituting (2.3) into (2.2) we see that

$$\alpha_n = \mathsf{p}_1(n) - \mathsf{p}_1(n+1). \tag{2.4}$$

Taking a telescopic sum of above equation and noting that $\mathsf{p}_1(0) := 0$, we have

$$-\sum_{j=0}^{n-1}\alpha_j = \mathsf{p}_1(n). \tag{2.5}$$

The Hankel determinant generated by our weight is

$$\begin{aligned} D_n(t) &:= \det\left(\mu_{j+k}(t)\right)_{j,k=0}^{n-1} \\ &= \prod_{j=0}^{n-1} h_j, \end{aligned} \tag{2.6}$$

where

$$\mu_k(t) := \int_0^1 x^k e^{-t/x} x^\alpha (1-x)^\beta dx \\ = e^{-t}\Gamma(1+\beta)U(1+\beta,-\alpha-k,t)$$

and $U(a,b,z)$ is the Kummer function of the second kind; see [25].

The Hankel determinant will turn out to play an important role in our determination of α_n and β_n for the weight given by (1.1). Furthermore, through several auxiliary variables (which naturally appear in the theory) we obtain expressions for the α_n and β_n terms of $H_n(t)$ given in (1.2) and its derivatives with respect to t. Eqs. (2.1)–(2.6) can be found in Szegő's treatise [26] on orthogonal polynomials.

If $w(x)$, the weight function, is Lipschitz continuous then the following ladder operator relations hold:

$$\left(\frac{d}{dz} + B_n(z)\right) P_n(z) = \beta_n A_n(z) P_{n-1}(z), \tag{2.7}$$

$$\left(\frac{d}{dz} - B_n(z) - \mathsf{v}'(z)\right) P_{n-1}(z) = -A_{n-1}(z) P_n(z) \tag{2.8}$$

with

$$A_n(z) := \frac{1}{h_n}\int_0^1 \frac{\mathsf{v}'(z)-\mathsf{v}'(y)}{z-y}[P_n(y)]^2 w(y)dy, \tag{2.9}$$

$$B_n(z) := \frac{1}{h_{n-1}}\int_0^1 \frac{\mathsf{v}'(z)-\mathsf{v}'(y)}{z-y} P_{n-1}(y)P_n(y)w(y)dy \tag{2.10}$$

and $\mathsf{v}(z) := -\ln w(z)$.

Note that, as a consequence of the recurrence relation and the Christoffel–Darboux formula, $A_n(z)$ and $B_n(z)$ are not independent but must satisfy the following supplementary conditions valid for $z \in \mathbb{C}\cup\{\infty\}$.

Theorem 2.1. *The functions $A_n(z)$ and $B_n(z)$ defined by (2.9) and (2.10) satisfy the identities*

$$B_{n+1}(z) + B_n(z) = (z-\alpha_n)A_n(z) - \mathsf{v}'(z), \tag{S_1}$$

$$1 + (z-\alpha_n)[B_{n+1}(z) - B_n(z)] = \beta_{n+1}A_{n+1}(z) - \beta_n A_{n-1}(z). \tag{S_2}$$

Proof. The ladder operator relations (2.7) and (2.8) and the supplementary conditions (S_1) and (S_2) have been derived by many authors in different forms over the years [3–5,24]. Also, see [6,7] for a recent proof. □

It turns out that there is another identity involving $\sum_{j=0}^{n-1} A_j$ which will provide further insight into the determination of α_n and β_n. We state the result in the following theorem.

Theorem 2.2.

$$B_n^2(z) + \mathsf{v}'(z)B_n(z) + \sum_{j=0}^{n-1} A_j(z) = \beta_n A_n(z)A_{n-1}(z). \tag{S_2'}$$

Proof. This can be obtained as follows: First, we multiply (S_2) by $A_n(z)$ and replace $(z-\alpha_n)A_n(z)$ in the resulting equation by $B_{n+1}(z)+B_n(z)+\mathsf{v}'(z)$ with (S_1) to get

$$B_{n+1}^2(z)-B_n^2(z)+\mathsf{v}'(z)(B_{n+1}(z)-B_n(z))+A_n(z)$$
$$=\beta_{n+1}A_{n+1}(z)A_n(z)-\beta_nA_n(z)A_{n-1}(z).$$

Taking the sum of the above equation from 0 to $n-1$, we obtain our theorem with the initial conditions $B_0(z)=0$ and $\beta_0A_{-1}(z)=0$.

Let us explain a little more about the way in which we get the above initial conditions. In (2.10) we rewrite

$$\frac{P_{n-1}(y)}{h_{n-1}},$$

as

$$\frac{\beta_n}{h_n}P_{n-1}(y).$$

Consequently $B_0(z)=0$, since $\beta_0P_{-1}(y)=0$ according to the initial condition associated with the recurrence relations (2.2). $\beta_0A_{-1}(z)=0$ for the same reason. □

For the problem at hand,

$$\mathsf{v}(z):=-\ln w(z)=\frac{t}{z}-\alpha\ln z-\beta\ln(1-z). \tag{2.11}$$

Then we immediately have

$$\mathsf{v}'(z)=-\frac{t}{z^2}-\frac{\alpha}{z}-\frac{\beta}{z-1} \tag{2.12}$$

and

$$\frac{\mathsf{v}'(z)-\mathsf{v}'(y)}{z-y}=\frac{t}{z^2y}+\frac{\alpha y+t}{zy^2}+\frac{\beta}{(z-1)(y-1)}. \tag{2.13}$$

Substituting the above formula into the definitions of $A_n(z)$ and $B_n(z)$ in (2.9) and (2.10), we have the following proposition.

Proposition 2.3. *We have*

$$A_n(z)=\frac{R_n^*}{z^2}+\frac{R_n}{z}-\frac{R_n}{z-1}, \tag{2.14}$$

$$B_n(z)=\frac{r_n^*}{z^2}-\frac{n-r_n}{z}-\frac{r_n}{z-1}, \tag{2.15}$$

where

$$R_n^*:=\frac{t}{h_n}\int_0^1[P_n(y)]^2w(y)\frac{\mathrm{d}y}{y}, \tag{2.16}$$

$$R_n:=\frac{\beta}{h_n}\int_0^1[P_n(y)]^2w(y)\frac{\mathrm{d}y}{1-y}, \tag{2.17}$$

$$r_n^*:=\frac{t}{h_{n-1}}\int_0^1 P_{n-1}(y)P_n(y)w(y)\frac{\mathrm{d}y}{y}, \tag{2.18}$$

$$r_n := \frac{\beta}{h_{n-1}} \int_0^1 P_{n-1}(y)P_n(y)w(y)\frac{\mathrm{d}y}{1-y}. \tag{2.19}$$

Proof. Using (2.13), (2.9) can be rewritten as

$$A_n(z) = \frac{1}{h_n}\left[\frac{1}{z^2}\int_0^1 [P_n(y)]^2 w(y)\frac{t}{y}\mathrm{d}y + \frac{1}{z}\int_0^1 [P_n(y)]^2 w(y)\frac{\alpha y + t}{y^2}\mathrm{d}y + \frac{1}{z-1}\int_0^1 [P_n(y)]^2 w(y)\frac{\beta}{y-1}\mathrm{d}y\right]. \tag{2.20}$$

Applying integration by parts, we have

$$\int_0^1 [P_n(y)]^2 w(y)\mathsf{v}'(y)\mathrm{d}y = -\int_0^1 [P_n(y)]^2 \mathrm{d}w(y) = \int_0^1 2P_n'(y)P_n(y)w(y)\mathrm{d}y = 0. \tag{2.21}$$

Then it follows from (2.12) and the above formula that

$$\int_0^1 [P_n(y)]^2 w(y)\frac{\alpha y + t}{y^2}\mathrm{d}y = -\int_0^1 [P_n(y)]^2 w(y)\frac{\beta}{y-1}\mathrm{d}y. \tag{2.22}$$

Combining (2.20) and (2.22) gives us (2.14).

In a very similar way, we get (2.15) from (2.10). One just needs to take into account the following equality:

$$\int_0^1 P_{n-1}(y)P_n(y)w(y)\frac{\alpha y + t}{y^2}\mathrm{d}y = -nh_{n-1} - \int_0^1 P_{n-1}(y)P_n(y)w(y)\frac{\beta}{y-1}\mathrm{d}y. \quad \square \tag{2.23}$$

Now we have four more auxiliary quantities R_n, R_n^*, r_n, r_n^*, in addition to the two unknowns α_n and β_n. However, from (S_1), (S_2) and (S_2'), we obtain relations among these quantities.

Proposition 2.4. *From (S_1), we obtain the following equations:*

$$r_{n+1}^* + r_n^* = t - \alpha_n R_n^*, \tag{2.24}$$

$$R_n^* - R_n = -2n - 1 - \alpha - \beta, \tag{2.25}$$

$$r_{n+1} + r_n = (1-\alpha_n)R_n - \beta, \tag{2.26}$$

where the constants R_n, R_n^*, r_n and r_n^* are defined in (2.16)–(2.19), respectively.

Proof. Substituting (2.14) and (2.15) into (S_1), we get

$$B_{n+1}(z) + B_n(z) = \frac{r_{n+1}^* + r_n^*}{z^2} - \frac{-r_{n+1} - r_n + 2n + 1}{z} - \frac{r_{n+1} + r_n}{z-1} \tag{2.27}$$

and

$$(z-\alpha_n)A_n(z) - \mathsf{v}'(z) = (z-\alpha_n)\left[\frac{R_n^*}{z^2} + \frac{R_n}{z} - \frac{R_n}{z-1}\right] - \left[-\frac{t}{z^2} - \frac{\alpha}{z} + \frac{\beta}{1-z}\right]$$
$$= \frac{t - \alpha_n R_n^*}{z^2} + \frac{\alpha + R_n^* - \alpha_n R_n}{z} + \frac{\beta - (1-\alpha_n)R_n}{z-1}. \tag{2.28}$$

Comparing the coefficients in the above two formulas, it follows that

$$r_{n+1}^* + r_n^* = t - \alpha_n R_n^*, \tag{2.29}$$

$$-r_{n+1} - r_n + 2n + 1 = -\alpha - R_n^* + \alpha_n R_n, \tag{2.30}$$

$$-r_{n+1} - r_n = \beta - (1 - \alpha_n) R_n. \tag{2.31}$$

Combining the above three formulas immediately proves our proposition. □

Proposition 2.5. *From (S_2'), we obtain the following equations:*

$$(r_n^*)^2 - t r_n^* = \beta_n R_n^* R_{n-1}^*, \tag{2.32}$$

$$r_n^2 + \beta r_n = \beta_n R_n R_{n-1}, \tag{2.33}$$

$$(t - 2r_n^*)(n - r_n) - \alpha r_n^* = \beta_n (R_n^* R_{n-1} + R_{n-1}^* R_n) \tag{2.34}$$

and

$$\sum_{j=0}^{n-1} R_j^* = n(t - \alpha - n) - (2n + \alpha + \beta)(r_n^* - r_n), \tag{2.35}$$

where the constants R_n, R_n^*, r_n and r_n^* are defined in (2.16)–(2.19), respectively.

Proof. From (2.14) and (2.15), we know that

$$\begin{aligned} B_n^2(z) &+ \mathsf{v}'(z) B_n(z) + \sum_{j=0}^{n-1} A_j(z) \\ &= \frac{(r_n^*)^2 - t r_n^*}{z^4} + \frac{(t - 2r_n^*)(n - r_n) - \alpha r_n^*}{z^3} + \frac{(n - r_n)^2 + \alpha(n - r_n)}{z^2} \\ &\quad + \frac{r_n^2 + \beta r_n}{(z-1)^2} + \frac{-2r_n r_n^* - \beta r_n^* + t r_n}{z^2(z-1)} + \frac{(\beta + 2r_n)(n - r_n) + \alpha r_n}{z(z-1)} \\ &\quad + \sum_{j=0}^{n-1} \left[\frac{R_j^*}{z^2} + \frac{R_j}{z} - \frac{R_j}{z-1} \right]. \end{aligned} \tag{2.36}$$

Using (2.14) again, we have

$$\begin{aligned} \beta_n A_n(z) A_{n-1}(z) = \frac{\beta_n R_n^* R_{n-1}^*}{z^4} &+ \beta_n R_n R_{n-1} \left[\frac{1}{z^2} - \frac{2}{z(z-1)} + \frac{1}{(z-1)^2} \right] \\ &+ \beta_n (R_n^* R_{n-1} + R_{n-1}^* R_n) \left[\frac{1}{z^3} - \frac{1}{z^2(z-1)} \right]. \end{aligned} \tag{2.37}$$

Note that (2.36) equals (2.37) due to (S_2'). Then let us compare their coefficients. At $O(z^{-4})$, $O(z-1)^{-2}$ and $O(z^{-3})$, equating the coefficients we have (2.32)–(2.34) in our proposition, respectively. At $O(z^{-2})$, using the fact that

$$\frac{1}{z-1} = -1 - z - z^2 - \cdots, \quad \text{as } z \to 0,$$

we obtain

$$(n - r_n)^2 + \alpha(n - r_n) - (-2r_n r_n^* - \beta r_n^* + t r_n)$$

$$= \beta_n R_n R_{n-1} + \beta_n (R_n^* R_{n-1} + R_{n-1}^* R_n) - \sum_{j=0}^{n-1} R_j^*. \tag{2.38}$$

Combining (2.34) and the above formula yields

$$\sum_{j=0}^{n-1} R_j^* = \beta_n R_n R_{n-1} + (t - 2r_n^* - \alpha)(n - r_n) - (n - r_n)^2$$
$$+ (t - 2r_n^*) r_n - (\alpha + \beta) r_n^*. \tag{2.39}$$

Substituting (2.33) into (2.39) gives us (2.35). □

Remark 2.6. From (S_2), using calculations similar to those in the above proposition, we get one more equation as follows:

$$-r_{n+1} + r_n + r_{n+1}^* - r_n^* + \alpha_n = 0. \tag{2.40}$$

To continue, we rewrite (2.40) as

$$-\alpha_n = r_{n+1}^* - r_n^* - (r_{n+1} - r_n).$$

Performing a telescopic sum and recalling (2.5), we find the very handy relation

$$\mathsf{p}_1(n) = r_n^* - r_n, \tag{2.41}$$

where we have used the initial conditions $r_0(t) = r_0^*(t) := 0$. As we shall see later, (2.41) will play a crucial role in the derivation of the Painlevé equation.

Remark 2.7. We may expect that (S_1) and (S_2) should "contain" all that is necessary. However, these are non-linear equations and their combination (S_2') carries extra information. It transpires that all three are needed to provide a complete description of the recurrence coefficients.

3. The recurrence coefficients

In this section we shall express the recurrence coefficients α_n and β_n in terms of the auxiliary quantities R_n, r_n and r_n^*. Note that we do not require R_n^* since it is R_n up to a linear form in n; see (2.25).

Lemma 3.1. *The diagonal recurrence coefficients α_n are expressed in terms of R_n, r_n and r_n^* as follows:*

$$(2n + 2 + \alpha + \beta)\alpha_n = 2(r_n^* - r_n) + R_n - \beta - t. \tag{3.1}$$

Proof. We eliminate R_n^* from (2.24) with the aid of (2.25) and find

$$r_{n+1}^* + r_n^* = t + \alpha_n (2n + 1 + \alpha + \beta - R_n). \tag{3.2}$$

Subtracting the above formula from (2.26), we get

$$r_{n+1} - r_{n+1}^* + r_n - r_n^* = R_n - \beta - t - (2n + 1 + \alpha + \beta)\alpha_n. \tag{3.3}$$

Recalling (2.40), we see that the left hand side of the above formula is $\alpha_n + 2(r_n - r_n^*)$. Then (3.1) immediately follows. □

Remark 3.2. For $n = 0$, we find, from the definition of $\alpha_0(t)$ and $R_0(t)$, that

$$\alpha_0(t) = \frac{U(1+\beta, -\alpha-1, t)}{U(1+\beta, -\alpha, t)}, \tag{3.4}$$

$$R_0(t) = \frac{U(\beta, -\alpha, t)}{U(1+\beta, -\alpha, t)}, \tag{3.5}$$

where U is the second solution of Kummer's equation; see [25]. We verify the validity of (3.1) at $n = 0$ by substituting the above two formulas.

Remark 3.3. For $t \to \infty$,

$$R_0(t) = t\left(1 + \frac{\alpha + 2(1+\beta)}{t} + \mathrm{O}\left(1/t^2\right)\right). \tag{3.6}$$

The next lemma gives an expression for β_n.

Lemma 3.4. *The off-diagonal recurrence coefficients β_n are expressed in terms of r_n and r_n^* as follows:*

$$\{1 - (2n+\alpha+\beta)^2\}\beta_n = -(r_n^* - r_n)^2 - (\beta + t)r_n + (t - \alpha - 2n)r_n^* + nt. \tag{3.7}$$

Proof. We eliminate R_n^* in favor of R_n using (2.32) and replace $\beta_n R_n R_{n-1}$ by $r_n^2 + \beta r_n$ with (2.33) to find

$$\begin{aligned}(r_n^*)^2 - t r_n^* = r_n^2 + \beta r_n + \beta_n[(2n+\alpha+\beta)^2 - 1]\\ - \beta_n[(2n+1+\alpha+\beta)R_{n-1} + (2n-1+\alpha+\beta)R_n].\end{aligned}$$

The same substitutions as that made in (2.34) produce

$$\begin{aligned}&(t - 2r_n^*)(n - r_n) - \alpha r_n^*\\ &\quad = 2(r_n^2 + \beta r_n) - \beta_n[(2n+1+\alpha+\beta)R_{n-1} + R_n(2n-1+\alpha+\beta)].\end{aligned} \tag{3.8}$$

Subtracting the above two formulas gives us (3.7). □

Remark 3.5. We consider the case when $t = 0$. In this situation $R_n^*(0) = r_n^*(0) = 0$. Therefore, from (2.25) and (2.35) we find

$$R_n(0) = 2n + 1 + \alpha + \beta$$

and

$$r_n(0) = \frac{n(n+\alpha)}{2n+\alpha+\beta},$$

respectively. Finally, from (3.1) and (3.7), we have

$$\alpha_n(0) = \frac{2n^2 + 2n(\alpha+\beta+1) + (1+\alpha)(\alpha+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}, \tag{3.9}$$

$$\beta_n(0) = \frac{n(n+\alpha)[n^2 + (\alpha+2\beta)n + \beta(\alpha+\beta)]}{(2n+\alpha+\beta)^2[(2n+\alpha+\beta)^2 - 1]}. \tag{3.10}$$

Note that

$$\lim_{n\to\infty} \alpha_n(0) = \frac{1}{2},$$
$$\lim_{n\to\infty} \beta_n(0) = \frac{1}{16}.$$

They are in agreement with the classical theory in [21].

4. The t dependence

Note that our weight function depends on t. As a consequence, the coefficients of our polynomials, the recurrence coefficients and the auxiliary quantities defined in (2.16)–(2.19) all depend on t. In this section, we are going to study the evolution of auxiliary quantities in t. First of all, we state a lemma which concerns the derivative of $\mathsf{p}_1(n)$ with respect to t.

Lemma 4.1. *We have*

$$t\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{p}_1(n) = r_n^*. \tag{4.1}$$

Proof. By the orthogonal property (2.1), we know that

$$\int_0^1 P_n(x)P_{n-1}(x)w(x;t)\mathrm{d}x = 0.$$

Differentiating the above formula with respect to t gives us

$$\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}P_n(x)P_{n-1}(x)w(x;t)\mathrm{d}x + \int_0^1 P_n(x)P_{n-1}(x)\frac{\mathrm{d}}{\mathrm{d}t}w(x;t)\mathrm{d}x = 0.$$

Using (1.1), (2.1) and (2.3), we get

$$h_{n-1}\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{p}_1(n) - \int_0^1 P_n(x)P_{n-1}(x)w(x)\frac{\mathrm{d}x}{x} = 0.$$

Taking into account (2.18), (4.1) follows immediately. □

From (2.41) and the above lemma, it is easily seen that

$$t\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{p}_1(n) = r_n^* = t\frac{\mathrm{d}}{\mathrm{d}t}r_n^* - t\frac{\mathrm{d}}{\mathrm{d}t}r_n \tag{4.2}$$

or

$$t\frac{\mathrm{d}}{\mathrm{d}t}r_n^* = r_n^* + t\frac{\mathrm{d}}{\mathrm{d}t}r_n. \tag{4.3}$$

Next, we have the following property concerning the Hankel determinant D_n.

Lemma 4.2. *We have*

$$t\frac{\mathrm{d}}{\mathrm{d}t}\ln D_n(t) = -\sum_{j=0}^{n-1} R_j^*, \tag{4.4}$$

where R_j^* is defined in (2.16).

Proof. Note that the constant h_n defined in (2.1) depends on the parameter t. Then, from (1.1) and (2.1), we have

$$h_n' = -\int_0^1 [P_n(x)]^2 w(x) \frac{dx}{x}. \tag{4.5}$$

Recalling (2.16), we get from the above formula

$$h_n' = -\frac{R_n^* h_n}{t}, \tag{4.6}$$

which gives us

$$t\frac{d}{dt} \ln h_n = -R_n^*. \tag{4.7}$$

Then, our lemma immediately follows from the above formula and (2.6). □

From the above lemmas, we also derive differential relations for the recurrence coefficients α_n and β_n. These are the non-standard Toda equations.

Lemma 4.3. *The recurrence coefficients α_n and β_n satisfy the following differential equations:*

$$t\frac{d}{dt}\alpha_n = r_n^* - r_{n+1}^*, \tag{T_1}$$

$$t\frac{d}{dt}\beta_n = (R_{n-1}^* - R_n^*)\beta_n, \tag{T_2}$$

where R_n^* and r_n^* are defined in (2.16) *and* (2.18)*, respectively.*

Proof. (T_1) follows from (2.4) and (4.1). And (T_2) follows from (4.6) and the fact that $\beta_n = h_n/h_{n-1}$. □

5. A non-linear differential equation satisfied by H_n

In this section we express R_n, r_n^* and r_n in terms of H_n and its derivative with respect to t, and obtain a functional equation involving H_n, H_n' and H_n''. For this purpose, we first express r_n and r_n^* in terms of H_n and H_n' in the next lemma.

Lemma 5.1.

$$r_n^* = \frac{nt + tH_n'}{2n+\alpha+\beta}, \tag{5.1}$$

$$r_n = \frac{n(n+\alpha) + tH_n' - H_n}{2n+\alpha+\beta}. \tag{5.2}$$

Proof. From (2.35) and (4.4) we have

$$\begin{aligned} -H_n &= nt - n(n+\alpha) - (2n+\alpha+\beta)(r_n^* - r_n) \\ &= nt - n(n+\alpha) - (2n+\alpha+\beta)\mathsf{p}_1(n). \end{aligned} \tag{5.3}$$

Taking the derivative of the above formula with respect to t and using (4.1), we find

$$-H_n' = n - (2n+\alpha+\beta)\frac{r_n^*}{t}. \tag{5.4}$$

Eq. (5.1) then follows from the above one. And Eq. (5.2) follows from eliminating r_n^* from (5.3) and (5.4). □

Then we try to get a similar lemma for R_n. To achieve this, we first derive the relations among R_n, r_n, and r_n^*.

Proposition 5.2. *The auxiliary quantity $R_n(t)$ satisfies the following quadratic equations:*

$$\begin{aligned}&\frac{2n+1+\alpha+\beta}{R_n}(r_n^2+\beta r_n)+\frac{R_n}{2n+1+\alpha+\beta}\\&\quad\times\left[(r_n^*-r_n)^2+(2n+\alpha-t)r_n^*+(\beta+t)r_n-nt\right]\\&=2r_n^2+(t+2\beta-2r_n^*)r_n+(2n+\alpha)r_n^*-nt\end{aligned}\tag{5.5}$$

and

$$\begin{aligned}&\frac{1-(2n+\alpha+\beta)^2}{R_n}(r_n^2+\beta r_n)\\&\quad+\left[(r_n^*-r_n)^2-t(r_n^*-r_n)+\beta r_n+(2n+\alpha)r_n^*-nt\right]R_n\\&=2r_n^2+(t+2\beta-2r_n^*)r_n+(2n+\alpha)r_n^*-nt-(2n+\alpha+\beta)t\frac{\mathrm{d}}{\mathrm{d}t}r_n.\end{aligned}\tag{5.6}$$

Proof. In (3.8), we replace $\beta_n R_{n-1}$ by $(r_n^2+\beta r_n)/R_n$ with (2.33) and get

$$\begin{aligned}&(2n+1+\alpha+\beta)\frac{r_n^2+\beta r_n}{R_n}+(2n-1+\alpha+\beta)\beta_n R_n\\&=2(r_n^2+\beta r_n)+\alpha r_n^*+(t-2r_n^*)(r_n-n).\end{aligned}$$

Replacing β_n in the above formula with the aid of (3.7), we have (5.5).

Then, we look back to (T_2) by using (2.25) and (2.33):

$$\begin{aligned}t\frac{\mathrm{d}}{\mathrm{d}t}\beta_n&=(R_{n-1}^*-R_n^*)\beta_n\\&=(R_{n-1}-R_n+2)\beta_n\\&=(2-R_n)\beta_n+\frac{r_n^2+\beta r_n}{R_n}.\end{aligned}\tag{5.7}$$

Applying $t\frac{\mathrm{d}}{\mathrm{d}t}$ to (3.7) gives us

$$\begin{aligned}&[1-(2n+\alpha+\beta)^2]t\frac{\mathrm{d}}{\mathrm{d}t}\beta_n\\&=-2(r_n^*-r_n)r_n^*+t(r_n^*-r_n)+tr_n^*-\beta t\frac{\mathrm{d}}{\mathrm{d}t}r_n-(2n+\alpha)t\frac{\mathrm{d}}{\mathrm{d}t}r_n^*+nt,\end{aligned}$$

where we have made use of (4.2) to arrive at the last step. Substituting (5.7) into the above formula gives us (5.6). □

Directly from the above proposition, we express R_n and $1/R_n$ in terms of r_n, r_n^* and $tr_n'(t)$.

Proposition 5.3. *The auxiliary quantity R_n has the following representations:*

$$R_n(t) = \frac{(2n+1+\alpha+\beta)[2r_n^2 + (t+2\beta-2r_n^*)r_n + (2n+\alpha)r_n^* - nt - tr_n'(t)]}{2[(r_n^*-r_n)^2 + (2n+\alpha-t)r_n^* + (\beta+t)r_n - nt]}, \tag{5.8}$$

$$\frac{1}{R_n(t)} = \frac{2r_n^2 + (t+2\beta-2r_n^*)r_n + (2n+\alpha)r_n^* - nt + tr_n'(t)}{2(2n+1+\alpha+\beta)(\beta+r_n)r_n}. \tag{5.9}$$

Proof. These are found by solving for R_n and $1/R_n$ from (5.5) and (5.6). □

Finally we arrive at the following theorem.

Theorem 5.4. *The logarithmic derivative of the Hankel determinant with respect to t,*

$$H_n(t) := t\frac{\mathrm{d}}{\mathrm{d}t}\ln D_n(t),$$

satisfies the following non-linear second-order ordinary differential equation:

$$(tH_n'')^2 = [n(n+\alpha+\beta) - H_n + (\alpha+t)H_n']^2 + 4H_n'(tH_n' - H_n)(\beta - H_n'). \tag{5.10}$$

Proof. Multiplying (5.8) and (5.9) gives us

$$\begin{aligned} t^2[r_n'(t)]^2 = {} & t^2r_n^2 + 2r_n[-2(2n+\alpha+\beta)(r_n^*)^2 \\ & + (4n+\alpha+2\beta)tr_n^* - nt^2] + [(2n+\alpha)r_n^* - nt]^2. \end{aligned} \tag{5.11}$$

Substituting (5.1) and (5.2) into (5.11) gives us (5.10). □

Remark 5.5. It turns out that H_n satisfies the Jimbo–Miwa–Okamoto σ-form of P_V for a special choice for parameters.

If

$$\widetilde{H}_n := H_n - n(n+\alpha+\beta), \tag{5.12}$$

then (5.10) becomes

$$\begin{aligned} (t\widetilde{H}_n'')^2 = {} & -4t(\widetilde{H}_n')^3 + (\widetilde{H}_n')^2\left[4\widetilde{H}_n + (\alpha+2\beta+t)^2 + 4n(n+\alpha+\beta) - 4\beta(\alpha+\beta)\right] \\ & + 2\widetilde{H}_n'\left[-(\alpha+2\beta+t)\widetilde{H}_n - 2n\beta(n+\alpha+\beta)\right] + \widetilde{H}_n^2. \end{aligned}$$

Comparing the above formula with the Jimbo–Miwa–Okamoto [17,23] σ-form of P_V, we can choose a possible identification of the parameters of [17] as

$$\nu_0 = 0,\ \nu_1 = -(n+\alpha+\beta),\ \nu_2 = n,\ \nu_3 = -\beta. \tag{5.13}$$

When a suitable limit is taken, (5.10) can be reduced to the σ-form of a Painlevé III equation given in [8]. To see this, we replace x by y/β, and t by s/β, in

$$\int_0^1 \mathrm{e}^{-t/x}x^\alpha(1-x)^\beta P_n^2(x)\mathrm{d}x = h_n(t),$$

resulting in

$$\int_0^\beta \mathrm{e}^{-s/y}y^\alpha\left(1-\frac{y}{\beta}\right)^\beta \widetilde{P}_n^2(y)\mathrm{d}y = \beta^{2n+\alpha+1}h_n(s/\beta), \tag{5.14}$$

where $\widetilde{P}_n(y) = \beta^n P_n(y/\beta)$. Now since the left hand side of (5.14) tends to

$$\int_0^\infty e^{-s/y-y} y^\alpha \widetilde{P}_n^2(y)dy,$$

as $\beta \to \infty$, we see that

$$\lim_{\beta\to\infty} \beta^{2n+1+\alpha} h_n(s/\beta)$$

becomes the square of the L^2 norm of the orthogonal polynomial studies in [8]. Consequently,

$$\lim_{\beta\to\infty} \beta^{n(n+\alpha)} D_n(s/\beta)$$

becomes the Hankel determinant

$$\det\left(\int_0^\infty y^{i+j} e^{-s/y-y} y^\alpha dy\right)_{i,j=0}^{n-1}.$$

Indeed on replacing t by s/β and letting $\beta \to \infty$, (5.10) becomes, keeping only the highest order term in β,

$$(sH_n'')^2 = (n + \alpha H_n')^2 + 4(sH_n' - H_n)H_n'(1 - H_n'), \tag{5.15}$$

which is (3.24) of [8] in a slightly different form.

Note that we have abused the notation: retaining H_n after the limit to avoid introducing extra symbols.

6. A non-linear difference equation satisfied by H_n

On the basis of the recent papers [1,8], we expect to find a second-order non-linear difference equation satisfied by H_n. To arrive at the difference equation, we want to express the recurrence coefficients α_n and β_n in terms of H_n and $H_{n\pm1}$. First, let us find a useful relation between H_n and $\mathsf{p}_1(n)$.

Lemma 6.1. *We have*

$$H_n = (2n + \alpha + \beta)\mathsf{p}_1(n) + n(n + \alpha - t) \tag{6.1}$$

or

$$\mathsf{p}_1(n) = \frac{H_n - n(n + \alpha - t)}{2n + \alpha + \beta}. \tag{6.2}$$

Proof. From (2.35) and (4.4) and the definition of H_n, we have

$$H_n = t\frac{d}{dt}\ln D_n(t) = -\sum_{j=0}^{n-1} R_j^* \tag{6.3}$$

$$= (2n + \alpha + \beta)(r_n^* - r_n) + n(n + \alpha - t). \tag{6.4}$$

Using (2.41) and rewriting the above formula gives our proposition. □

Note that since $\alpha_n = \mathsf{p}_1(n) - \mathsf{p}_1(n+1)$, we have a very simple expression for α_n in terms of H_n and H_{n+1}. Furthermore we can also obtain R_n in terms of H_n or $\mathsf{p}_1(n)$:

Lemma 6.2.

$$R_n = H_n - H_{n+1} + 2n + 1 + \alpha + \beta \tag{6.5}$$

$$= (2n + \alpha + \beta)\mathsf{p}_1(n) - (2n + 2 + \alpha + \beta)\mathsf{p}_1(n+1) + t + \beta. \tag{6.6}$$

Proof. These forms are found by combining (2.25), (6.3) and (6.1) together. □

For the formulas involving r_n and β_n, we have results as follows.

Lemma 6.3. *We have*

$$r_n = \frac{-[\mathsf{p}_1(n)]^2 - (2n + \alpha - t)\mathsf{p}_1(n) + nt - [1 - (2n + \alpha + \beta)^2]\beta_n}{2n + \alpha + \beta} \tag{6.7}$$

with

$$\beta_n = \frac{X_n}{Y_n}, \tag{6.8}$$

where

$$X_n := \frac{1}{2n + \alpha + \beta}\big[-2[\mathsf{p}_1(n)]^3 + (3t - \alpha + 2\beta - 2n)[\mathsf{p}_1(n)]^2 - (t^2 - 2(n - \beta)t - (2n + \alpha)\beta)\mathsf{p}_1(n) - (t + \beta)nt\big] \tag{6.9}$$

and

$$Y_n := (2n - 1 + \alpha + \beta)(2n + 1 + \alpha + \beta) + \frac{2}{2n + \alpha + \beta}\mathsf{p}_1(n) + (2n - 1 + \alpha + \beta)(2n + 2 + \alpha + \beta)\mathsf{p}_1(n+1) - (2n + 1 + \alpha + \beta) \times (2n - 2 + \alpha + \beta)\mathsf{p}_1(n-1) - (t + \beta)\left(\frac{1}{2n + \alpha + \beta} + 2n + \alpha + \beta\right). \tag{6.10}$$

Proof. To obtain (6.7) and (6.8), first we use (2.32) and (2.41) to get

$$r_n^2 + (2\mathsf{p}_1(n) - t)r_n + [\mathsf{p}_1(n)]^2 - t\mathsf{p}_1(n) = \beta_n R_n^* R_{n-1}^*. \tag{6.11}$$

Subtracting the above formula from (2.33), we have

$$(2\mathsf{p}_1(n) - t - \beta)r_n + [\mathsf{p}_1(n)]^2 - t\mathsf{p}_1(n) = \beta_n(R_n^* R_{n-1}^* - R_n R_{n-1}), \tag{6.12}$$

which is a linear equation with respect to r_n and β_n. Recall that (3.7) is also linear in r_n and β_n. Solving this linear system and taking into account (2.25) and (6.6), we prove our lemma. □

Finally, we obtain the following theorem.

Theorem 6.4. *$\widetilde{H}_n$ satisfies the following non-linear second-order ordinary difference equation:*

$$nt - \frac{(2n + \alpha)(\widetilde{H}_n + n(\beta + t))}{2n + \alpha + \beta} + \frac{\widetilde{Z}_n}{2n + \alpha + \beta}\left[-2n - \alpha - t + \frac{2(\widetilde{H}_n + n(\beta + t))}{2n + \alpha + \beta}\right]$$

$$
\begin{aligned}
&+\frac{2\widetilde{Z}_n^2}{(2n+\alpha+\beta)^2}=\frac{1}{Z_n}\Bigg(-2\widetilde{H}_n^2+2\widetilde{H}_n(1+\widetilde{H}_{n+1})+\widetilde{H}_{n+1}(2n-1+\alpha+\beta)\\
&-\widetilde{H}_{n-1}(2n+1+\alpha+\beta-2\widetilde{H}_n+2\widetilde{H}_{n+1})\Bigg)\\
&\times\Bigg(-nt(\beta+t)+\frac{(\alpha\beta+2\beta(n-t)+(2n-t)t)(\widetilde{H}_n+n(\beta+t))}{2n+\alpha+\beta}\\
&+\frac{(-\alpha+2\beta-2n+3t)(\widetilde{H}_n+n(\beta+t))^2}{(2n+\alpha+\beta)^2}-\frac{2(\widetilde{H}_n+n(\beta+t))^3}{(2n+\alpha+\beta)^3}\Bigg),
\end{aligned}
\tag{6.13}
$$

where

$$
\begin{aligned}
Z_n &:= (2n+\alpha+\beta)\Bigg((2n-1+\alpha+\beta)(2n+1+\alpha+\beta)\\
&\quad-\left(2n+\alpha+\beta+\frac{1}{2n+\alpha+\beta}\right)(\beta+t)\\
&\quad-(2n+1+\alpha+\beta)(\widetilde{H}_{n-1}+(n-1)(\beta+t))\\
&\quad+\frac{2(\widetilde{H}_n+n(\beta+t))}{(2n+\alpha+\beta)^2}+(2n-1+\alpha+\beta)(\widetilde{H}_{n+1}+(n+1)(\beta+t))\Bigg),\\
\widetilde{Z}_n &:= nt+\frac{(-2n-\alpha+t)(\widetilde{H}_n+n(\beta+t))}{2n+\alpha+\beta}-\frac{(\widetilde{H}_n+n(\beta+t))^2}{(2n+\alpha+\beta)^2}-\frac{1-(2n+\alpha+\beta)^2}{Z_n}\\
&\quad\times\Bigg(-nt(\beta+t)+\frac{(\alpha\beta+2\beta(n-t)+(2n-t)t)(\widetilde{H}_n+n(\beta+t))}{2n+\alpha+\beta}\\
&\quad+\frac{(-\alpha+2\beta-2n+3t)(\widetilde{H}_n+n(\beta+t))^2}{(2n+\alpha+\beta)^2}-\frac{2(\widetilde{H}_n+n(\beta+t))^3}{(2n+\alpha+\beta)^3}\Bigg).
\end{aligned}
$$

Proof. The non-linear difference equation for $\mathsf{p}_1(n)$ is obtained by substituting (2.25), (2.41) and (6.6)–(6.8) into (2.34). Due to the relations among $\mathsf{p}_1(n)$, H_n and $\widetilde{H}_n$ in (5.12) and (6.1), the non-linear difference equation for $\widetilde{H}_n$ follows. □

7. $P_V((2n+1+\alpha+\beta)^2/2, -\beta^2/2, \alpha, -1/2)$

We end this paper with the derivation of a second-order ordinary differential equation for R_n which is expected for P_V since we have seen that H_n satisfies the Jimbo–Miwa–Okamoto σ-form.

For this purpose we state in the next lemma a Riccati equation satisfied by R_n.

Lemma 7.1. *The auxiliary quantity $R_n(t)$ satisfies the following Riccati equation:*

$$
tR_n' = 2R_n(r_n^* - r_n) + (2n+1+\alpha+\beta)(2r_n - R_n + \beta) + (R_n - \beta - t)R_n. \tag{7.1}
$$

Proof. First we apply $t\frac{d}{dt}$ to Eq. (3.1) and make use of (T_1) to replace $t\frac{d}{dt}\alpha_n$ by $r_n^* - r_{n+1}^*$.

In the next step we replace r_{n+1}^* by $t - \alpha_n R_n^* - r_n$ using (2.24). Finally, noting (3.1) and $R_n^* = R_n - (2n+1+\alpha+\beta)$, we arrive at (7.1). □

From (7.1) we see that

$$r_n^* = \frac{1}{2R_n}\left[tR_n' - (2n+1+\alpha+\beta)(2r_n - R_n + \beta)\right] + r_n - \frac{R_n - \beta - t}{2}. \tag{7.2}$$

Substituting the above formula into (5.8) and (5.9), we find a pair of linear equations in r_n and r_n'. Solving this system we have

$$r_n = F(R_n, R_n'), \tag{7.3}$$

$$r_n' = G(R_n, R_n'), \tag{7.4}$$

where $F(\cdot,\cdot)$ and $G(\cdot,\cdot)$ are functions that are explicitly known. Because the expressions are unwieldy, we have decided not to write them down.

By equating the derivative of (7.3) with respect to t and (7.4), we find the following:

$$\begin{aligned}
&[(2n+\alpha+\beta)(2n+1+\alpha+\beta) - (4n+2\alpha+2\beta+1)R_n + R_n^2 - tR_n'] \\
&\quad\times\Bigg[2t^2(2n+1+\alpha+\beta-R_n)R_nR_n'' + t^2(3R_n - 2n - 1 - \alpha - \beta)(R_n')^2 \\
&\quad + 2t(2n+1+\alpha+\beta-R_n)R_nR_n' + R_n^5 - 3(2n+1+\alpha+\beta)R_n^4 \\
&\quad + C_1(t)R_n^3 + C_2(t)R_n^2 - 3\beta^2(2n+1+\alpha+\beta)^2R_n + \beta^2(2n+1+\alpha+\beta)^3\Bigg] = 0,
\end{aligned} \tag{7.5}$$

where

$$C_1(t) := -t^2 + 2\alpha t + 3(2n+1+\alpha+\beta)^2 - \beta^2$$

$$C_2(t) := -(2n+1+\alpha+\beta)\left[t^2 + 2\alpha t + (2n+1+\alpha+\beta)^2 - 3\beta^2\right].$$

From (7.5) we have two equations, one of which is a Riccati equation whose solution is

$$R_n(t) = \frac{(2n+\alpha+\beta)\gamma_n t - (2n+1+\alpha+\beta)}{\gamma_n t - 1}, \tag{7.6}$$

where γ_n is an integration constant. Note that $R_n(t)$ tends to $2n+\alpha+\beta$ as $t \to \infty$. Because $R_0(t) \sim t$ from Remark 3.3, we discard this equation.

It turns out that the above differential equation for $R_n(t)$ is a particular Painlevé V one.

Theorem 7.2. *Let*

$$S_n(t) := \frac{R_n(t)}{2n+1+\alpha+\beta}. \tag{7.7}$$

Then $S_n(t)$ satisfies the following differential equation:

$$\begin{aligned}
S_n'' &= \frac{3S_n - 1}{2S_n(S_n-1)}(S_n')^2 - \frac{S_n'}{t} + \frac{(S_n-1)^2}{t^2}\left[\frac{(2n+1+\alpha+\beta)^2}{2}S_n - \frac{\beta^2}{2S_n}\right] \\
&\quad + \frac{\alpha S_n}{t} - \frac{S_n(S_n+1)}{2(S_n-1)},
\end{aligned} \tag{7.8}$$

which is $P_V((2n+1+\alpha+\beta)^2/2, -\frac{\beta^2}{2}, \alpha, -1/2)$.

Proof. Eq. (7.8) follows if we substitute

$$R_n(t) = (2n + 1 + \alpha + \beta) S_n(t)$$

into the second-order ODE implied by (7.5). □

Remark 7.3. The isomonodromy theory of Jimbo and Miwa [17], and the Riemann–Hilbert approach to orthogonal polynomials of Fokas et al. [14,15] and others may also be applied. We have made use of the ladder operator approach, with the associated compatibility conditions (S_1), (S_2) and (S_2'), since it is relatively straightforward to express the recurrence coefficients in terms of the auxiliary variables. This allows us to investigate the "time" evolution of such quantities and discover the Painlevé V of our problem.

Regarding the asymptotic analysis which arises for large n, the Riemann–Hilbert approach of Deift et al. [11,10,12] is particularly suited for this purpose. However, the appearance of an extra irregular singularity at the origin requires further study since the conventional paramatrix associated with milder singularities would no longer be appropriate.

Acknowledgments

Yang Chen is partially supported by EPSRC Grant #R27027. He would also like to thank the School of Mathematical Sciences, Fudan University, Shanghai, China, where this paper was completed, for its hospitality. Dan Dai is partially supported by K.U. Leuven research grants OT/04/21 and OT/08/33, and by the Belgian Interuniversity Attraction Pole P06/02. He would like to thank the ENIGMA programme for support of his visit to the Department of Mathematics, Imperial College London.

References

[1] E. Basor, Y. Chen, Painlevé V and the distribution function of a discontinuous linear statistics in the Laguerre unitary ensembles, J. Phys. A: Math. Theor. 42 (2009) 035203. (18pp).

[2] R. Bo, R. Wong, Asymptotic behavior of the Pollaczek polynomials and their zeros, Stud. Appl. Math. 96 (1996) 307–338.

[3] S. Bonan, D.S. Clark, Estimates of the orthogonal polynomials with weight $\exp(-x^m)$, m an even positive integer, J. Approx. Theory 46 (1986) 408–410.

[4] S. Bonan, D.S. Lubinsky, P. Nevai, Orthogonal polynomials and their derivatives, II, SIAM J. Math. Anal. 18 (1987) 1163–1176.

[5] S. Bonan, P. Nevai, Orthogonal polynomials and their derivatives, I, J. Approx. Theory 40 (1984) 134–147.

[6] Y. Chen, M. Ismail, Ladder operators and differential equations for orthogonal polynomials, J. Phys. A 30 (22) (1997) 7817–7829.

[7] Y. Chen, M. Ismail, Jacobi polynomials from compatibility conditions, Proc. Amer. Math. Soc. 133 (2) (2005) 465–472.

[8] Y. Chen, A. Its, Painlevé III and a singular linear statistics in Hermitian random matrix ensembles, I, J. Approx. Theory (2009) doi:10.1016/j.jat.2009.05.005.

[9] S.B. Damelin, Smoothness theorems for generalized symmetric Pollaczek weights on $(-1, 1)$, J. Comput. Appl. Math. 101 (1999) 87–103.

[10] P. Deift, T. Kriecherbauer, K.T.-R. McLaughlin, S. Venakides, X. Zhou, Strong asymptotics of orthogonal polynomials with respect to exponential weights, Comm. Pure Appl. Math. 52 (12) (1999) 1491–1552.

[11] P. Deift, T. Kriecherbauer, K.T.-R. McLaughlin, S. Venakides, X. Zhou, Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and application to universality questions in random matrix theory, Comm. Pure Appl. Math. 52 (11) (1999) 1335–1425.

[12] P. Deift, T. Kriecherbauer, K.T.-R. McLaughlin, S. Venakides, X. Zhou, A Riemann–Hilbert approach to asymptotic questions for orthogonal polynomials, 133 (1–2), 2001, 47–63.

[13] S.A. Denisov, On Rakhmanov's theorem for Jacobi matrices, Proc. Amer. Math. Soc. 132 (2004) 847–852.
[14] A.S. Fokas, A.R. Its, A.V. Kitaev, Discrete Painlevé equations and their appearance in quantum gravity, Comm. Math. Phys. 142 (2) (1991) 313–344.
[15] A.S. Fokas, A.R. Its, A.V. Kitaev, The isomonodromy approach to matrix models in 2D quantum gravity, Comm. Math. Phys. 147 (2) (1992) 395–430.
[16] Ya.L. Geronimus, Orthogonal polynomials: estimates, asymptotic formulas, and series of polynomials orthogonal on the unit circle and on an interval, Consultant Bureau, 1961.
[17] M. Jimbo, T. Miwa, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients, II, Physica D 2 (1981) 407–448.
[18] D.S. Lubinsky, Forward and converse theorems of polynomial approximation for exponential weights on $[-1, 1]$, I, II, J. Approx. Theory 91 (1–47) (1997) 48–83.
[19] A. Martínez-Finkelshtein, J.F. Sánchez-Lara, Shannon entropy of symmetric Pollaczek polynomials, J. Approx. Theory 145 (2007) 55–80.
[20] C.T. Munger, Ideal basis sets for the Dirac Coulomb problem: eigenvalue bounds and convergence proofs, J. Math. Phys. 48 (2007) 022301 (50pp).
[21] P. Nevai, Orthogonal polynomials, Mem. Amer. Math. Soc. 213 (1979).
[22] P. Nevai, V. Totik, Denisov's theorem on recurrence coefficients, J. Approx. Theory 127 (2004) 240–245.
[23] K. Okamoto, On the τ-function of the Painlevé equations, Physica D 2 (1981) 525–535.
[24] J. Shohat, A differential equation for orthogonal polynomials, Duke Math. J. 5 (1939) 401–417.
[25] L.J. Slater, Confluent Hypergeometric Functions, Cambridge University Press, 1960.
[26] G. Szegő, Orthogonal Polynomials, 4th ed., in: AMS Colloquium Publications, vol. 23, Amer. Math. Soc., Providence, RI, 1975.
[27] W. Van Assche, Pollaczek polynomials and summability methods, J. Math. Anal. Appl. 147 (1990) 498–505.
[28] G.I. Watson, Two-electron perturbation problems and Pollaczek polynomials, J. Phys. A 24 (1991) 4989–4998.
[29] J.-R. Zhou, Y.-Q. Zhao, An infinite asymptotic expansion for the extreme zeros of the Pollaczek polynomials, Stud. Appl. Math. 118 (2007) 255–279.