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Fixed block configuration group divisible designs with block size six

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ABSTRACT

We present constructions and results about GDDs with two groups and block size six. We study those GDDs in which each block has configuration (s, t), that is in which each block has exactly s points from one of the two groups and t points from the other. We show the necessary conditions are sufficient for the existence of GDD$(n, 2, 6; \lambda_1, \lambda_2)$s with fixed block configuration $(3, 3)$. For configuration $(1, 5)$, we give minimal or near-minimal index examples for all group sizes $n \geq 5$ except $n = 10, 15, 160$, or 190. For configuration $(2, 4)$, we provide constructions for several families of GDD$(n, 2, 6; \lambda_1, \lambda_2)$s.

1. Introduction

A group divisible design GDD$(n, m, k; \lambda_1, \lambda_2)$ is a collection of k element subsets of a v-set $\mathbf{X}$ called blocks which satisfies the following properties: each point of $\mathbf{X}$ appears in the same number, r, of the b blocks; the $v = nm$ elements of $\mathbf{X}$ are partitioned into m subsets (called groups) of size n each; pairs of points within the same group are called first associates of each other and appear in λ_1 blocks; pairs of points not in the same group are second associates and appear in λ_2 blocks together. If we require that $m = 2$ and each block intersects one group in s points and $t = k - s$ points in the other, we say the design has a fixed block configuration (s, t).

In [3] the authors settled the existence for group divisible designs with block size three and first and second associates, m groups of size n where $m, n \geq 3$. The problem of finding necessary and sufficient conditions for $m = 2$ or $v = 2n$ and block size four was established in [8]. In [9], the necessary conditions are shown to be sufficient for $3 \leq n \leq 8$. New conditions and results were presented in [5] with three groups and block size four, in particular, constructions were given to show that the necessary conditions are sufficient for all GDDs with three groups and group sizes two, three, and five with two exceptions. In [6], Hurd, Mishra and Sarvate gave new results for general fixed block configuration GDD$(n, 2, k; \lambda_1, \lambda_2)$, as well as new necessary and sufficient conditions for $k = 5$ and configuration $(2, 3)$. Hurd and Sarvate in [7] gave similar results for $k = 5$ and configuration $(1, 4)$. Unless otherwise stated, $m = 2$ is assumed from now on.

The purpose of this article is to establish similar results for GDDs with block size six and two groups. In this paper, we consider each possible configuration type: $(3, 3)$, $(2, 4)$ and $(1, 5)$.

1.1. Necessary conditions

For GDDs with block size six and two groups there are two necessary conditions on b, the number of blocks, and r, the number of blocks a point appears in. These conditions are easy to prove.

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Table 1
Possible values of n with respect to λ_1, λ_2.

(mod 15)	$\lambda_1 \equiv 0 \pmod 5$	$\lambda_1 \equiv 1 \pmod 5$	$\lambda_1 \equiv 2 \pmod 5$	$\lambda_1 \equiv 3 \pmod 5$	$\lambda_1 \equiv 4 \pmod 5$
$\lambda_2 \equiv 0$	Any n	$n \equiv 1 \pmod 5$	$n \equiv 1 \pmod 5$	$n \equiv 1 \pmod 5$	$n \equiv 1 \pmod 5$
$\lambda_2 \equiv 1$	Impossible	$n \equiv 3, 8 \pmod{15}$	$n \equiv 9, 14 \pmod{15}$	$n \equiv 2, 12 \pmod{15}$	Impossible
$\lambda_2 \equiv 2$	Impossible	$n \equiv 12 \pmod{15}$	$n \equiv 3, 8 \pmod{15}$	Impossible	$n \equiv 9 \pmod{15}$
$\lambda_2 \equiv 3$	$n \equiv 0 \pmod 5$	$n \equiv 4 \pmod 5$	Impossible	$n \equiv 3 \pmod 5$	$n \equiv 2 \pmod 5$
$\lambda_2 \equiv 4$	$n \equiv 0 \pmod{15}$	Impossible	$n \equiv 2, 12 \pmod{15}$	$n \equiv 9, 14 \pmod{15}$	$n \equiv 3, 8 \pmod{15}$
$\lambda_2 \equiv 5$	$n \equiv 0 \pmod 3$	$n \equiv 6 \pmod{15}$	$n \equiv 6, 11 \pmod{15}$	$n \equiv 6, 11 \pmod{15}$	$n \equiv 6, 11 \pmod{15}$
$\lambda_2 \equiv 6$	$n \equiv 0 \pmod 5$	$n \equiv 3 \pmod 5$	$n \equiv 4 \pmod 5$	$n \equiv 2 \pmod 5$	Impossible
$\lambda_2 \equiv 7$	Impossible	$n \equiv 2, 12 \pmod{15}$	$n \equiv 3, 8 \pmod{15}$	Impossible	$n \equiv 9, 14 \pmod{15}$
$\lambda_2 \equiv 8$	Impossible	$n \equiv 4, 9 \pmod{15}$	Impossible	$n \equiv 3, 8 \pmod{15}$	$n \equiv 2, 12 \pmod{15}$
$\lambda_2 \equiv 9$	$n \equiv 0 \pmod 5$	Impossible	$n \equiv 2 \pmod 5$	$n \equiv 4 \pmod 5$	$n \equiv 3 \pmod 5$
$\lambda_2 \equiv 10$	$n \equiv 0 \pmod 3$	$n \equiv 6, 11 \pmod{15}$	$n \equiv 6, 11 \pmod{15}$	$n \equiv 6 \pmod{15}$	$n \equiv 6, 11 \pmod{15}$
$\lambda_2 \equiv 11$	Impossible	$n \equiv 3 \pmod{15}$	$n \equiv 9, 14 \pmod{15}$	$n \equiv 2, 12 \pmod{15}$	Impossible
$\lambda_2 \equiv 12$	$n \equiv 0 \pmod 5$	$n \equiv 2 \pmod 5$	$n \equiv 3, 13 \pmod{15}$	Impossible	$n \equiv 4, 9 \pmod{15}$
$\lambda_2 \equiv 13$	Impossible	$n \equiv 9, 14 \pmod{15}$	Impossible	$n \equiv 3, 8 \pmod{15}$	$n \equiv 2, 12 \pmod{15}$
$\lambda_2 \equiv 14$	Impossible	Impossible	$n \equiv 2, 12 \pmod{15}$	$n \equiv 9, 14 \pmod{15}$	$n \equiv 3 \pmod 5$

Theorem 1.1. *The following conditions are necessary for the existence of a* GDD$(n, 2, 6; \lambda_1, \lambda_2)$.

(1) *The number of blocks is* $b = \frac{\lambda_1(n)(n-1)+\lambda_2 n^2}{15}$.
(2) *The number of blocks a point appears in is* $r = \frac{\lambda_1(n-1)+\lambda_2 n}{5}$.

These two necessary conditions on b and r determine possibilities for the parameter n and the indices λ_1 and λ_2. Table 1 summarizes this relationship.

There are at least two other necessary conditions:

Theorem 1.2. *Suppose a* GDD$(n, 2, 6; \lambda_1, \lambda_2)$ *exists. Then:*

(1) $b \geq \max(2r - \lambda_1, 2r - \lambda_2)$
(2) $\lambda_2 \leq 19\lambda_1(n-1)/(11n)$.

Proof. For condition (1), consider the set of blocks containing the points x and y. There are r blocks containing x and $r - \lambda_i$ blocks which contain y but do not contain x. So there are at least $2r - \lambda_i$ blocks. For condition (2) let b_6 be the number of blocks with all 6 points from one group, b_5 be the number of blocks with 5 points from 1 group, and the remaining point from the other group, b_4 be the number of blocks with 4 points from 1 group, and the remaining 2 points from the other group, and b_3 be the number of blocks with 3 points from each group. Counting the contribution of these blocks towards the number of pairs of points from the same group in the blocks together gives: $15b_6+10b_5+7b_4+6b_3 = 2\lambda_1\binom{n}{2} = n(n-1)\lambda_1$. Counting the pairs of points from different groups gives $5b_5 + 8b_4 + 9b_3 = n^2\lambda_2$. Thus we have:

$$
\begin{aligned}
&-15b_6 - 5b_5 + b_4 + 3b_3 = n^2\lambda_2 - n^2\lambda_1 + n\lambda_1 \leq b_4 + 3b_3 \leq 4b = 4n[\lambda_1(n-1) + \lambda_2 n]/15\\
&\Rightarrow 15(n^2\lambda_2 - n^2\lambda_1 + n\lambda_1) \leq 4(n^2\lambda_1 - n\lambda_1 + n^2\lambda_2)\\
&\Rightarrow 11n^2\lambda_2 - 19n^2\lambda_1 + 19n\lambda_1 \leq 0\\
&\Rightarrow \lambda_2 \leq \frac{19(n-1)\lambda_1}{11n}. \quad \square
\end{aligned}
$$

Condition (2) shows that while $\lambda_2 \geq \lambda_1$ is possible, we always have $\lambda_2 < 2\lambda_1$. We can apply the theorem to assert the following:

Corollary 1.3. *The family* GDD$(n, 2, 6; s, 2st)$ *does not exist for any integers* $s, t > 0$.

In [6], Hurd et al. proved the following two results for GDDs with fixed block configuration. We repeat their results here.

Theorem 1.4 ([7]). *Suppose a* GDD$(n, 2, k; \lambda_1, \lambda_2)$ *has configuration* (s, t). *Then the number of blocks with s points (respectively t) from the first group is equal to the number of blocks with s points (respectively t) from the second group. Consequently, for any s and t, the number of blocks b is necessarily even.*

Theorem 1.5 ([7]). *For any* GDD$(n, 2, k; \lambda_1, \lambda_2)$ *with configuration* (s, t), *the second index is given by* $\lambda_2 = \left(\frac{\lambda_1(n-1)}{n}\right)\left(\frac{k(k-1)-2\beta}{2\beta}\right)$ *where* $\beta = \binom{s}{2} + \binom{t}{2}$.

For the remainder of this paper, we will refer to the results in this section as the "necessary" conditions.

2. GDDs with configuration (3, 3)

In this section, we introduce a basic construction for configuration (3, 3) GDDs with specific indices and present the minimal indices for any configuration (3, 3) GDD$(n, 2, 6; \lambda_1, \lambda_2)$. We begin by providing an example of a configuration (3, 3) GDD where $\lambda_1 = 4$ and $\lambda_2 = 5$.

Example 1. GDD(6, 2, 6; 4, 5). Let $A = \{0, 1, 2, 3, 4, 5\}$ and $B = \{a, b, c, d, e, f\}$. Then the $b = 20$ blocks are:

$$\{0, 1, 2, a, b, c\}, \{0, 1, 2, d, e, f\}, \{0, 1, 3, a, b, d\}, \{0, 1, 3, c, e, f\}, \{0, 2, 4, a, c, e\},$$
$$\{0, 2, 4, b, d, f\}, \{0, 3, 5, a, d, f\}, \{0, 3, 5, b, c, e\}, \{0, 4, 5, a, e, f\}, \{0, 4, 5, b, c, d\},$$
$$\{1, 2, 5, b, c, f\}, \{1, 2, 5, a, e, d\}, \{1, 3, 4, b, d, e\}, \{1, 3, 4, a, c, e\}, \{1, 4, 5, b, e, f\},$$
$$\{1, 4, 5, a, c, d\}, \{2, 3, 4, c, d, e\}, \{2, 3, 4, a, b, f\}, \{2, 3, 5, c, d, f\}, \{2, 3, 5, a, b, e\}.$$

By applying Theorem 1.5 to configuration (3, 3) GDDs, we get the following result.

Corollary 2.1. *For any configuration* (3, 3) GDD$(n, 2, 6; \lambda_1, \lambda_2)$, *we have* $\lambda_2 = \frac{3\lambda_1(n-1)}{2n}$.

2.1. *A basic construction for configuration* (3, 3)

A balanced incomplete block design BIBD(v, k, λ) is a pair $(V, \mathcal{B})$ where V is a set of points with cardinality v and $\mathcal{B}$ is a collection of bk-subsets of V called blocks such that each element of V is contained in exactly r blocks and any 2-subset of V is contained in exactly λ blocks. If $k = 3$, the design is also called a triple system, and is abbreviated by TS(v, λ). We use triple systems in the following construction.

Theorem 2.2. *If there exists a* TS(n, λ) *with b blocks and replication number r, then there exists a configuration* (3, 3) GDD$(n, 2, 6; \lambda b, r^2)$. *Further if such a GDD exists, then there exists a* TS$(n, \lambda b)$.

Proof. Suppose there exists a TS(n, λ). Consider two copies of this triple system, TS$_1(n, \lambda)$ and TS$_2(n, \lambda)$. Form the complete bipartite graph G with bipartition $(V(G_1), V(G_2))$ where $V(G_1)$ is the set of blocks of TS$_1(n, \lambda)$ and $V(G_2)$ is the set of blocks of TS$_2(n, \lambda)$. For each edge $\{u, v\} \in G$, take the six element subset $u \cup v$ to be a block. Thus we may think of the collection of blocks in the desired GDD as the edge set of G. Consider a pair of first associates. It appears λ times in TS$_i(n, \lambda)$, $i = 1, 2$. Therefore, in the given construction it appears in exactly λb blocks of size six, where b is the number of blocks in a TS(n, λ). Now consider a pair of second associates $\{v_1, v_2\}$ where $v_i \in TS_i(n, \lambda)$. Any point appears exactly r times in a TS(n, λ), thus the pair $\{v_1, v_2\}$ is contained in exactly r^2 blocks of the resulting GDD.

Suppose such a GDD exists with groups G_1 and G_2, with the collection of blocks $\mathcal{B}$. Let $\mathcal{B}' = \{B \cap G_2 : B \in \mathcal{B}\}$. Then $(G_2, \mathcal{B}')$ is a TS$(n, \lambda b)$ on the points of G_2. □

The construction given in Theorem 2.2 can easily be generalized to any configuration (k, k) GDD. Thus we have the following corollary.

Corollary 2.3. *If there exists a* BIBD(n, k, λ) *with b blocks and replication number r, then there exists a configuration* (k, k) GDD$(n, 2, 2k; \lambda b, r^2)$.

2.2. *Minimal indices*

There exists a TS(7, 1), and thus by Theorem 2.2 there exists a GDD(7, 2, 6; 7, 9). From Corollary 2.1, $\lambda_2 = \frac{3\lambda_1(6)}{14} = \frac{9\lambda_1}{7}$, so the construction given in Theorem 2.2 gives a design with the minimum possible indices. However, there also exists a TS(9, 1) which means that there exists a GDD(9, 2, 6; 12, 16) by Theorem 2.2. In this case we have that $\lambda_2 = \frac{3\lambda_1(8)}{18} = \frac{4\lambda_1}{3}$. Here the minimum values for (λ_1, λ_2) are (3, 4). So the construction given in Theorem 2.2 does not give a design with the minimum possible indices. In general, Corollary 2.1 says that for any configuration (3, 3) GDD, if for some value of n, the minimum possible indices are (λ_1, λ_2), then any other GDD with that configuration will have the indices $(w\lambda_1, w\lambda_2)$ for some positive integer w. We can find the minimal indices by using Corollary 2.1 and by the equations given in Theorem 1.1. Any configuration (3, 3) GDD with indices $(w\lambda_1, w\lambda_2)$ can be obtained by taking w copies of the blocks in the minimal design. Therefore, we focus on constructing configuration (3, 3) GDDs with indices (λ_1, λ_2). We may then say that the necessary conditions are sufficient for the existence of any configuration (3, 3) GDD with that n.

Theorem 2.4. *The minimal indices* (λ_1, λ_2) *for any configuration* (3, 3) GDD$(n, 2, 6; \lambda_1, \lambda_2)$ *are summarized in Table 2.*

Proof. We know that $\lambda_2 = \frac{3\lambda_1(n-1)}{2n}$ from Corollary 2.1. If $n \equiv 0 \pmod 3$ and $n \equiv 1 \pmod 2$, then $n \equiv 3 \pmod 6$. Thus λ_1 is a multiple of $n/3$ and λ_2 is a multiple of $(n-1)/2$. If $n \equiv 0 \pmod 3$ and $n \equiv 0 \pmod 2$, then $n \equiv 0 \pmod 6$, so λ_1 is a

Table 2
Summary of minimal indices for configuration (3, 3).

n	λ_1	λ_2
$n \equiv 0 \pmod 6$	$2n/3$	$(n-1)$
$n \equiv 1 \pmod 6$	n	$3(n-1)/2$
$n \equiv 2 \pmod 6$	$6n$	$9(n-1)$
$n \equiv 3 \pmod 6$	$n/3$	$(n-1)/2$
$n \equiv 4 \pmod 6$	$2n$	$3(n-1)$
$n \equiv 5 \pmod 6$	$3n$	$9(n-1)/2$

multiple of $2n/3$ and λ_2 is a multiple of $(n-1)$. If $n \equiv 1 \pmod 3$ and $n \equiv 1 \pmod 2$, then $n \equiv 1 \pmod 6$, implying λ_1 is a multiple of n and λ_2 is a multiple of $3(n-1)/2$. If $n \equiv 1 \pmod 3$ and $n \equiv 0 \pmod 2$, then $n \equiv 4 \pmod 6$, and λ_1 is a multiple of $2n$ and λ_2 is a multiple of $3(n-1)$. If $n \equiv 2 \pmod 3$ and $n \equiv 1 \pmod 2$, then $n \equiv 5 \pmod 6$. This implies that λ_1 is a multiple of n and λ_2 is a multiple of $3(n-1)/2$. However, if we take these values to be the minimal indices, then the number of blocks given by Theorem 1.1 would not be integer valued. The smallest values for (λ_1, λ_2) that give integer values for b are $(\lambda_1, \lambda_2) = (3n, \frac{9}{2}(n-1))$. Finally consider the case when $n \equiv 2 \pmod 3$ and $n \equiv 0 \pmod 2$. Here $n \equiv 2 \pmod 6$, which means that λ_1 is a multiple of $2n$ and λ_2 is a multiple of $3(n-1)$. If we take these values to be the minimal indices, then the number of blocks given by Theorem 1.1 would not be integer valued. Thus the smallest values for (λ_1, λ_2) that give integer values for b are $(\lambda_1, \lambda_2) = (6n, 9(n-1))$. $\square$

3. Constructing configuration (3, 3) GDDs

In this section, we give a similar construction to the one given in Theorem 2.2 based on α-resolvable triple systems. We then show that this construction produces designs with minimal indices for all configuration (3, 3) GDDs with block size six and two groups.

A set of blocks in a design is called a parallel class if it partitions the point set. A partition of the blocks of a design into parallel classes is a resolution, and such a design is called resolvable. An α-parallel class in a design is a set of blocks which contain every point of the design exactly α times. A design that can be resolved into α-parallel classes is called α-resolvable. We abbreviate an α-resolvable design as an α-RBIBD(n, k, λ). If $\alpha = 1$, then we refer to the design as an RBIBD(n, k, λ). It is an easy exercise to work out the necessary conditions for the existence of an α-RBIBD(n, k, λ) which appear in [1]. We record these in Theorem 3.1.

Theorem 3.1. *The necessary conditions for the existence of an α-resolvable* BIBD(n, k, λ) *are,*

(1) $\lambda(n-1) \equiv 0 \pmod{(k-1)\alpha}$
(2) $\lambda n(n-1) \equiv 0 \pmod{k(k-1)}$
(3) $\alpha n \equiv 0 \pmod k$.

Jungnickle et al. [11] showed that these conditions are sufficient when $k = 3$.

Lemma 3.2 *([11]). The necessary conditions for the existence of an α-resolvable* BIBD$(n, 3, \lambda)$ *are sufficient, except for $n = 6$, $\alpha = 1$ and $\lambda \equiv 2 \pmod 4$.*

Vasiga et al. [12] showed that the necessary conditions are sufficient for $k = 4$.

Lemma 3.3 *([12]). The necessary conditions for the existence of an α-resolvable* BIBD$(n, 4, \lambda)$ *are sufficient, with the exception of $(\alpha, n, \lambda) = (2, 10, 2)$.*

We use α-resolvable designs to obtain the following result.

Lemma 3.4. *Suppose there exists an α-resolvable* TS(n, λ) *with s α-parallel classes, where each parallel class contains t blocks. Then there exists a configuration* (3, 3) GDD$(n, 2, 6; \lambda t, \alpha^2 s)$.

Proof. For $i = 1, 2$, let D_i be an α-resolvable TS(n, λ). Resolve the blocks of D_i into α-parallel classes $C_1^i, C_2^i, \ldots, C_s^i$. Construct a graph G in the following manner. For $j = 1, 2, \ldots, s$, create the complete bipartite graph G_j with bipartition $(V(G_j^1), V(G_j^2))$ where $V(G_j^1)$ is the set of blocks in C_j^1 and $V(G_j^2)$ is the set of blocks in C_j^2. Let $E(G) = \bigcup_{j=1}^{s} E(G_j)$. For each edge, $\{u, v\} \in G$, form a block of size six by taking $u \cup v$. Thus we may think of the collection of blocks in the desired GDD as the edge set of G.

Consider a pair of first associates. It will appear in exactly λ blocks of D_i. Therefore, in the given construction, it will appear in λt blocks of size six. Now consider a pair of second associates $\{v_1, v_2\}$ where $v_1 \in D_1$ and $v_2 \in D_2$. This pair appears in exactly α^2 blocks of size six per α-parallel class, thus $\lambda_2 = \alpha^2 s$. $\square$

We now consider values of n (mod 6) and apply Lemma 3.4 in each case to obtain the desired configuration (3, 3) GDD with minimal indices (λ_1, λ_2).

Theorem 3.5. *The necessary conditions are sufficient for the existence of a configuration* $(3, 3)$ $\mathrm{GDD}(n, 2, 6; \frac{n}{3}, \frac{n-1}{2})$ *when* $n \equiv 3 \pmod 6$.

Proof. Let $n \equiv 3 \pmod 6$. Then by Lemma 3.2 there exists a 1-resolvable $\mathrm{TS}(n, 1)$ with $\frac{n-1}{2}$ parallel classes, each containing $\frac{n}{3}$ blocks. By applying the construction in Lemma 3.4 we obtain a GDD with indices $(\lambda_1, \lambda_2) = \left(\frac{n}{3}, \frac{n-1}{2}\right)$, which are the minimal indices given in Theorem 2.4. □

Theorem 3.6. *The necessary conditions are sufficient for the existence of* $\mathrm{GDD}(n, 2, 6; n, \frac{3}{2}(n-1))$ *when* $n \equiv 1 \pmod 6$ *with configuration* $(3, 3)$.

Proof. Let $n \equiv 1 \pmod 6$. By Lemma 3.2 there exists a 3-resolvable $\mathrm{TS}(n, 1)$ with $\frac{n-1}{6}$ 3-parallel classes, each containing n blocks. If we apply the construction in Lemma 3.4, we obtain a GDD with minimal indices $(\lambda_1, \lambda_2) = (n, \frac{3(n-1)}{2})$. □

Theorem 3.7. *The necessary conditions are sufficient for the existence of* $\mathrm{GDD}(n, 2, 6; 6n, 9(n-1))$ *when* $n \equiv 2 \pmod 6$ *with configuration* $(3, 3)$.

Proof. Let $n \equiv 2 \pmod 6$. Then by Lemma 3.2 there exists a 3-resolvable $\mathrm{TS}(n, 6)$ with $(n-1)$ 3-parallel classes, each containing n blocks. Applying Lemma 3.4 yields a GDD with minimal indices $(\lambda_1, \lambda_2) = (6n, 9(n-1))$. □

Theorem 3.8. *The necessary conditions are sufficient for the existence of* $\mathrm{GDD}(n, 2, 6; 2n, 3(n-1))$ *when* $n \equiv 4 \pmod 6$ *with configuration* $(3, 3)$.

Proof. Let $n \equiv 4 \pmod 6$. By Lemma 3.2, there exists a 3-resolvable $\mathrm{TS}(n, 2)$ with $\frac{n-1}{3}$ 3-parallel classes each containing n blocks. We may apply Lemma 3.4 to obtain a GDD with minimal indices $(\lambda_1, \lambda_2) = (2n, 3(n-1))$. □

Theorem 3.9. *The necessary conditions are sufficient for the existence of* $\mathrm{GDD}(n, 2, 6; 3n, \frac{9}{2}(n-1))$ *when* $n \equiv 5 \pmod 6$ *with configuration* $(3, 3)$.

Proof. Let $n \equiv 5 \pmod 6$. Then by Lemma 3.2 there exists a 3-resolvable $\mathrm{TS}(n, 3)$ with $\frac{n-1}{2}$ 3-parallel classes, each containing n blocks. We may apply Lemma 3.4 to obtain a GDD with minimal indices $(\lambda_1, \lambda_2) = (3n, \frac{9(n-1)}{2})$. □

Theorem 3.10. *The necessary conditions are sufficient for the existence of* $\mathrm{GDD}(n, 2, 6; \frac{2}{3}n, n-1)$ *for* $n \equiv 0 \pmod 6$ *with configuration* $(3, 3)$.

Proof. Let $n \equiv 0 \pmod 6$ with $n \geq 12$. Then by Lemma 3.2 there exists a 1-resolvable $\mathrm{TS}(n, 2)$ with $n-1$ parallel classes, each containing $\frac{n}{3}$ blocks. If we apply the construction given in Lemma 3.4 we obtain a GDD with minimal indices $(\lambda_1, \lambda_2) = (\frac{2n}{3}, n-1)$. If $n = 6$, we may not use the construction described in Lemma 3.2. However if $n = 6$, the minimal indices $(\lambda_1, \lambda_2) = (4, 5)$ and Example 1 gives a $\mathrm{GDD}(6, 2, 6; 4, 5)$. □

Since we have given a construction for all possible values of $n \pmod 6$, we may give the following result.

Theorem 3.11. *The necessary conditions are sufficient for the existence of all configuration* $(3, 3)$ $\mathrm{GDD}(n, 2, 6; \lambda_1, \lambda_2)$ *with minimal indices.*

4. GDDs with configuration (2, 4)

In this section we present the minimal indices for any configuration $(2, 4)$ $\mathrm{GDD}(n, 2, 6; \lambda_1, \lambda_2)$. By Theorem 1.5 we have the following relation between λ_1 and λ_2 for any configuration $(2, 4)$ GDD.

Theorem 4.1. *For any configuration* $(2, 4)$ $\mathrm{GDD}(n, 2, 6; \lambda_1, \lambda_2)$ *we have* $\lambda_2 = \frac{8\lambda_1(n-1)}{7n}$.

For any configuration $(2, 4)$ GDD if for some value of n, the minimum possible indices are (λ_1, λ_2), then any other GDD with that configuration will have the indices $(w\lambda_1, w\lambda_2)$ for some positive integer w. We may find the minimum indices by using the equation in Theorem 4.1, the equations in Theorem 1.1, and the condition in Theorem 1.4. As in the case with configuration $(3, 3)$, we focus on constructing GDDs with minimal indices since we may then say the necessary conditions are sufficient for the existence of any configuration $(2, 4)$ GDD with that n.

Theorem 4.2. *The minimal indices* (λ_1, λ_2) *for any configuration* $(2, 4)$ $\mathrm{GDD}(n, 2, 6; \lambda_1, \lambda_2)$ *are summarized in Table 3.*

Proof. By Theorem 4.1, we know that $\lambda_2 = \frac{8\lambda_1(n-1)}{7n}$. If $n \not\equiv 1 \pmod 7$ and n is odd, then this implies that $n \equiv 3, 5, 7, 9, 11, 13 \pmod{14}$. Thus λ_1 is a multiple of $7n$ and λ_2 is a multiple of $8(n-1)$. If $n \equiv 1 \pmod 7$ and n is odd, then $n \equiv 1 \pmod{14}$. In this case, λ_1 must be a multiple of n and λ_2 a multiple of $(8/7)(n-1)$. If $n \not\equiv 1 \pmod 7$ and $n \equiv 0 \pmod 8$, then we have that $n \equiv 0, 16, 24, 32, 40, 48 \pmod{56}$, so λ_1 is a multiple of $7n/8$ and λ_2 is a multiple of $n-1$. If $n \not\equiv 1 \pmod 7$ and $n \equiv 2 \pmod 8$, then $n \equiv 2, 10, 18, 26, 34, 42 \pmod{56}$ implying λ_1 is a multiple of $7n/2$ and λ_2

Table 3
Summary of minimal indices for configuration (2, 4).

n	λ_1	λ_2
$n \equiv 0, 16, 24, 32, 40, 48 \pmod{56}$	$7n/8$	$n-1$
$n \equiv 2, 6, 10, 14, 18, 26, 30, 34, 38, 42, 46, 54 \pmod{56}$	$7n/2$	$4(n-1)$
$n \equiv 4, 12, 20, 28, 44, 52 \pmod{56}$	$7n/4$	$2(n-1)$
$n \equiv 8 \pmod{56}$	$n/8$	$(n-1)/7$
$n \equiv 22, 50 \pmod{56}$	$n/2$	$4(n-1)/7$
$n \equiv 36 \pmod{56}$	$n/4$	$2(n-1)/7$
$n \equiv 3, 5, 7, 9, 11, 13, 17, 19, 21, 23, 25, 27, 31, 33, 35, 37, 39, 41, 45, 47, 49, 51, 53, 55 \pmod{56}$	$7n$	$8(n-1)$
$n \equiv 1, 15, 29, 43 \pmod{56}$	n	$8(n-1)/7$

is a multiple of $4(n-1)$. If $n \not\equiv 1 \pmod 7$ and $n \equiv 4 \pmod 8$, then $n \equiv 4, 12, 20, 28, 44, 52 \pmod{56}$. Thus λ_1 is a multiple of $7n/4$ and λ_2 is a multiple of $2(n-1)$. If $n \not\equiv 1 \pmod 7$ and $n \equiv 6 \pmod 8$, then $n \equiv 6, 14, 30, 38, 46, 54 \pmod{56}$; here λ_1 is a multiple of $7n/2$ and λ_2 is a multiple of $4(n-1)$. If $n \equiv 1 \pmod 7$ and $n \equiv 0 \pmod 8$, then we have that $n \equiv 8 \pmod{56}$. Here, it follows that λ_1 is a multiple of $n/8$ and λ_2 is a multiple of $(n-1)/7$. If $n \equiv 1 \pmod 7$ and $n \equiv 2 \pmod 8$, then we have that $n \equiv 50 \pmod{56}$. Here, it follows that λ_1 is a multiple of $n/2$ and λ_2 is a multiple of $4(n-1)/7$. If $n \equiv 1 \pmod 7$ and $n \equiv 4 \pmod 8$, then we have that $n \equiv 36 \pmod{56}$. Here, it follows that λ_1 is a multiple of $n/4$ and λ_2 is a multiple of $2(n-1)/7$. If $n \equiv 1 \pmod 7$ and $n \equiv 6 \pmod 8$, then $n \equiv 22 \pmod{56}$; and it follows λ_1 is a multiple of $n/2$ and λ_2 is a multiple of $4(n-1)/7$. □

5. Constructing (2, 4) GDD(n, 2, 6; λ_1, λ_2)

We use Theorem 4.2 and Lemma 3.3 to construct configuration (2, 4) GDDs with minimal indices, when possible. We begin with a general construction.

Lemma 5.1. *If there exists an α-resolvable* BIBD$(n, 4, \lambda)$ *with n even and $\lambda = 3\alpha$, then there exists a configuration* $(2, 4)$ GDD$(n, 2, 6; \frac{n}{2}(\lambda + \frac{\alpha}{2}), 2\alpha(n-1))$.

Proof. Let the two groups be $X = \{1, 2, \dots, n\}$, and $X' = \{1', 2', \dots, n'\}$. First, let $D = (X, \mathcal{B})$ be an α-resolvable BIBD$(n, 4, \lambda)$. Let F be a 1-factorization of K_n on the point set X'. Resolve the blocks of D into α parallel classes. There will be $\lambda(n-1)/3\alpha = (n-1)$ classes with $(n\alpha)/4$ blocks in each class. Construct a graph G with vertex set $\mathcal{B} \cup \{\{x', y'\} : \{x', y'\} \in X'\}$ in the following manner. For $j = 1, 2, \dots, (n-1)$, create the complete bipartite graph G_j with bipartition $(V(G_j^1), V(G_j^2))$ where $V(G_j^1)$ is the set of blocks in an α parallel class, C_j, and $V(G_j^2)$ is a 1-factor of K_n. Let $E(G) = \bigcup_{j=1}^{s} E(G_j)$. For each edge, $\{u, v\}$ in G, take $u \cup v$ to be a block of size six. This collection of blocks is exactly half of the blocks in the desired GDD. The second step of the construction is to let D be an α-resolvable BIBD$(n, 4, \lambda)$ on the point set X'. Let F be a 1-factorization of K_n on the point set X. Repeat the given construction, forming blocks of size six by taking the union of the subsets u and v for each edge $\{u, v\} \in G$. This gives us the remaining desired blocks.

Consider a pair of first associates, $\{x, y\} \in X$. It will appear exactly λ times in D. Therefore in the given construction, it will appear $n\lambda/2$ times in the blocks constructed in the first step, and then the same pair will appear in $n\alpha/4$ blocks constructed in the second step. Thus $\lambda_1 = \frac{n}{2}(\lambda + \frac{\alpha}{2})$. Now consider a pair of second associates $\{x, y'\}$, where $x \in X$ and $y' \in X'$. Here x will appear with y' exactly $\alpha(n-1)$ times in the blocks constructed at each step, so $\lambda_2 = 2\alpha(n-1)$. □

We use the above construction to obtain the following results:

Corollary 5.2. *Let* $n \equiv 2, 6, 10, 14, 18, 26, 30, 34, 38, 42, 46, 54 \pmod{56}$. *Then the necessary conditions are sufficient for the existence of a configuration* $(2, 4)$ GDD$(n, 2, 6; \frac{7n}{2}, 4(n-1))$.

Proof. Let n be assumed as above. By Lemma 3.3, there exists a 2-resolvable BIBD$(n, 4, 6)$. Apply Lemma 5.1 to obtain a GDD with minimal indices $(\lambda_1, \lambda_2) = (\frac{7n}{2}, 4(n-1))$. □

Corollary 5.3. *Let* $n \equiv 4, 12, 20, 28, 44, 52 \pmod{56}$. *Then the necessary conditions are sufficient for the existence of a configuration* $(2, 4)$ GDD$(n, 2, 6; \frac{7n}{4}, 2(n-1))$.

Proof. Let n be assumed as above. By Lemma 3.3, there exists a resolvable BIBD$(n, 4, 3)$. So we may apply Lemma 5.1 to obtain a GDD with minimal indices $(\lambda_1, \lambda_2) = (\frac{7n}{4}, 2(n-1))$. □

We define a near-minimal GDD as a GDD which has indices exactly twice the minimal size.

Corollary 5.4. *If* $n \equiv 0, 8 \pmod{24}$, *then there exists a near minimal configuration* $(2, 4)$ GDD$(n, 2, 6; \frac{7n}{4}, 2(n-1))$.

Proof. Let n be assumed as above. By Lemma 3.3, there exists a resolvable BIBD$(n, 4, 3)$. Apply Lemma 5.1 to obtain a near-minimal GDD with indices $(\frac{7n}{4}, 2(n-1))$. □

The above construction gives near-minimal GDDs for $n \equiv 0, 8 \pmod{24}$. If $n = 8$, note that a GDD(8, 2, 6; 1, 1) is a BIBD(16, 6, 1). By Fisher's necessary condition [2], $b \geq v$; however in this case $b = 8$ and $v = 16$. Thus this design cannot exist with any configuration, so we conclude that the minimal indices cannot be obtained, and we have the following theorem.

Theorem 5.5. *There does not a exist a configuration* (2, 4) GDD(8, 2, 6; 1, 1).

We use a slightly different construction for $n \equiv 16 \pmod{24}$.

Theorem 5.6. *If* $n \equiv 16 \pmod{24}$ *then the necessary conditions are sufficient for the existence of a configuration* (2, 4) GDD$(n, 2, 6; 7n/8, (n-1))$.

Proof. Let $n \equiv 16 \pmod{24}$, and let $X = \{1, 2, \dots, n\}$ and $X' = \{1', 2', \dots, n'\}$ be the point sets for the two groups in the desired design. By Lemma 3.3, there exists an RBIBD$(n, 4, 1)$, $(X, \mathcal{B})$. For the first step of the construction, resolve the blocks of $(X, \mathcal{B})$ into parallel classes, $C_1, \dots, C_{(n-1)/3}$. There will be $n/4$ blocks in each parallel class. For $j = 1, 2, \dots, (n-1)/3$, partition C_j into equal parts A_j and B_j. We construct a 1-factorization of K_n on the point set X'. On each parallel class C_j, decompose the blocks of C_j into three 1-factors $F_{j,1}$, $F_{j,2}$, and $F_{j,3}$ as follows. For each block $g \in C_j$ we let g_1, g_2, and g_3 be a 1-factorization of g. For $i = 1, 2, 3$, let

$$F^A_{j,i} = \{\{x', y'\} : \{x, y\} \in g_i \in A_j\}$$

and

$$F^B_{j,i} = \{\{x', y'\} : \{x, y\} \in g_i \in B_j\}.$$

Then define

$$F_{j,i} = F^A_{j,i} \cup F^B_{j,i}.$$

Now construct a graph G with vertex set $\mathcal{B} \cup \{\{x', y'\} : \{x, y\} \subseteq B \in \mathcal{B}\}$ in the following manner. For $j = 1, 2, \dots, (n-1)/3$, form each of these bipartite graphs.

- $G_{j,1}$ is the complete $(A_j, F_{j,1})$-bipartite graph,
- $G_{j,2}$ is the complete $(B_j, F_{j,2})$-bipartite graph,
- $G_{j,3}$ is the complete $(A_j, F^A_{j,3})$-bipartite graph,
- $G_{j,4}$ is the complete $(B_j, F^B_{j,3})$-bipartite graph.

Then let

$$E(G) = \left(\bigcup_{j=1}^{(n-1)/3} E(G_{j,1})\right) \cup \left(\bigcup_{j=1}^{(n-1)/3} E(G_{j,2})\right) \cup \left(\bigcup_{j=1}^{(n-1)/3} E(G_{j,3})\right) \cup \left(\bigcup_{j=1}^{(n-1)/3} E(G_{j,4})\right).$$

For each edge $\{u, v\}$ in G, take the six element subset $u \cup v$ to be a block. This collection of blocks is exactly half of the blocks of size six in the desired GDD.

To obtain the other half, we take the second step of the construction. Here we switch the roles of X and X' in the RBIBD and the 1-factorization. In other words, we let $(X', \mathcal{B}')$ be an RBIBD$(n, 4, 1)$, and we construct a 1-factorization of K_n on the point set X. We construct a graph H with vertex set $\mathcal{B}' \cup \{\{x, y\} : \{x', y'\} \subseteq B \in \mathcal{B}'\}$ in a similar manner to G. For $j = 1, 2, \dots, (n-1)/3$, partition C_j into parts A_j and B_j. Construct a 1-factorization of K_n on the point set X in the following manner. On each parallel class C_j, decompose the blocks of C_j into three 1-factors $F_{j,1}$, $F_{j,2}$, and $F_{j,3}$ as follows. For each block $g \in C_j$ we let g_1, g_2, and g_3 be a 1-factorization of g. For $i = 1, 2, 3$, let

$$F^A_{j,i} = \{\{x, y\} : \{x', y'\} \in g_i \in A_j\}$$

and

$$F^B_{j,i} = \{\{x, y\} : \{x', y'\} \in g_i \in B_j\}.$$

Then define

$$F_{j,i} = F^A_{j,i} \cup F^B_{j,i}.$$

For $j = 1, 2, \dots, (n-1)/3$, form each of these bipartite graphs.

- $H_{j,1}$ is the complete $(B_j, F_{j,1})$-bipartite graph,
- $H_{j,2}$ is the complete $(A_j, F_{j,2})$-bipartite graph,
- $H_{j,3}$ is the complete $(A_j, F^B_{j,3})$-bipartite graph,
- $H_{j,4}$ is the complete $(B_j, F^A_{j,3})$-bipartite graph.

Then let

$$E(H) = \left(\bigcup_{j=1}^{(n-1)/3} E(H_{j,1})\right) \cup \left(\bigcup_{j=1}^{(n-1)/3} E(H_{j,2})\right) \cup \left(\bigcup_{j=1}^{(n-1)/3} E(H_{j,3})\right) \cup \left(\bigcup_{j=1}^{(n-1)/3} E(H_{j,4})\right).$$

For each edge $\{u, v\}$ in H, take the six element subset $u \cup v$ to be a block.

Consider a pair of first associates, $\{x, y\}$. In the first step of the construction, when $\{x, y\} \in X$ appears in the RBIBD, it appears exactly once. Thus it will be in a block of size six exactly $n/2 + n/4 = 3n/4$ times. In the second step of the construction when $\{x, y\}$ is in the role of a 1-factor, it will appear in a block of size six $n/8$ times. Thus $\lambda_1 = 7n/8$. Now consider a pair of second associates, $\{x, y'\}$ where $x \in X$ and $y' \in X'$. Without loss of generality, we may assume $\{x, y\} \in C_j$ for some j. In the first step of the construction, there are four cases to consider. Each point is either in A_j or B_j. Suppose x and y are both in Aj. Then in the construction, $\{x, y'\}$ appears twice. If $x \in A_j$ and $y \in B_j$, then $\{x, y'\}$ appears once. If $x \in B_j$ and $y \in A_j$, then $\{x, y'\}$ appears once and if x and y are both in B_j, then $\{x, y'\}$ appears twice. In the second step of the construction when we reverse the roles, if x and y are both in A_j, then $\{x, y'\}$ appears once. If $x \in A_j$ and $y \in B_j$, then $\{x, y'\}$ appears twice. If $x \in B_j$ and $y \in A_j$, then $\{x, y'\}$ appears twice, and if x and y are both in B_j, then $\{x, y'\}$ appears once. Thus for each parallel class, each pair $\{x, y'\}$ appears a total of 3 times. Therefore, each pair of second associates will appear in a total of $3(n - 1/3) = n - 1$ blocks of size six in the construction. □

Theorem 5.7. *Let* $n \equiv 3, 5, 7, 9, 11, 13 \pmod{14}$. *Then the necessary conditions are sufficient for the existence of a configuration* $(2, 4)$ GDD$(n, 2, 6; 7n, 8(n - 1))$.

Proof. Let the two groups be $X = \{1, 2, \ldots, n\}$ and $X' = \{1', 2', \ldots, n'\}$. By Lemma 3.3, there exists a 4-resolvable BIBD$(n, 4, 6)$. First, let D be such a design with point set X. Resolve the blocks of D into 4-parallel classes. There will be $(n-1)/2$ classes with n blocks in each class. Construct a graph G in the following manner. For $j = 1, 2, \ldots, (n-1)/2$ create the complete bipartite graph G_j with bipartition $(V(G_j^1), V(G_j^2))$ where $V(G_j^1)$ are the blocks of a 4-parallel class and $V(G_j^2)$ are the pairs obtained by developing $\{0', j'\} \pmod n$. Let $E(G) = \bigcup_{j=1}^{(n-1)/2} E(G_j)$. For each edge $\{u, v\} \in G$, form a block of size six by taking $u \cup v$. This collection of blocks is half of the blocks in the desired GDD. Secondly, by reversing the roles of X and X' and repeating the same construction we obtain all desired blocks.

Consider a pair of first associates, $\{x, y\} \in X$. It will appear exactly six times in D. Therefore, in the given construction, it will appear $6n$ times in the blocks constructed in the first step, and this pair will appear in an additional n blocks constructed in the second step. Thus $\lambda_1 = 7n$. Now consider a pair of second associates $\{x, y'\}$ where $x \in X$ and $y' \in X'$. Here x will appear with y' exactly $4(n - 1)$ times in blocks constructed during each step of the construction, and thus $\lambda_2 = 8(n - 1)$. □

If $n \equiv 1, 15, 29, 43 \pmod{56}$, then the above construction gives a GDD with 7 times the minimal indices. However, the following construction gives a configuration $(2, 4)$ GDD$(15, 2, 6; 15, 16)$ with minimum possible indices.

Theorem 5.8. *The necessary conditions are sufficient for the existence of a configuration* $(2, 4)$ GDD$(15, 2, 6; 15, 16)$.

Proof. By Lemma 3.3, there exists a RBIBD$(16, 4, 1)$. It has five parallel classes with four blocks in each class. Let D be such an RBIBD on the point set $Y = \{\infty, 0, 1, 2, \ldots, 14\}$. For each of the five blocks containing ∞, namely the blocks of the form $\{\infty, x, y, z\}$, construct the pairs $\{x', y'\}$, $\{x', z'\}$, and $\{y', z'\}$. Clearly, from these five blocks, we obtain 15 pairs. The first step of the construction is to let the two groups be $X = \{0, 1, \ldots, 14\}$ and $X' = \{0', 1', \ldots, 14'\}$. Let $\hat{D}$ be the set of 15 blocks from D which do not contain ∞. Create the complete bipartite graph G with bipartition $(V(G_1), V(G_2))$ where $V(G_1)$ is the collection of blocks $\hat{D}$, and $V(G_2)$ is the set of 15 pairs obtained from the blocks of D which contain ∞. For each edge $\{u, v\} \in G$, take the six element subset $u \cup v$ to be a block. This collection of blocks is half of the blocks in the desired GDD. In the second step of the construction, we obtain the rest of the blocks. Let D be an RBIBD$(16, 4, 1)$ on the point set $Y' = \{\infty, 0', 1', 2', \ldots, 14'\}$. Similar to the first step, for each of the blocks containing ∞, (blocks of the form $\{\infty, x', y', z'\}$), construct the pairs $\{x, y\}$, $\{x, z\}$, $\{y, z\}$. Again, let $\hat{D}$ be the set of blocks from D which do not contain ∞. Create the complete bipartite graph H with bipartition $(V(H_1), V(H_2))$ where $V(H_1)$ is the set of 15 pairs obtained from the blocks of D which contain ∞, and $V(H_2)$ is the set of blocks $\hat{D}$. For each edge $\{u, v\} \in H$, take the six element subset $u \cup v$ to be a block.

Consider a pair of first associates, $\{x, y\} \in X$. If $\{x, y\}$ was in a block with ∞ in D, then it appears exactly 0 times in the blocks of size six constructed in the first step, and this pair appears in 15 blocks constructed in the second step. If $\{x, y\}$ was not in a block with ∞ in the D, then it appears in exactly 15 blocks of size six constructed in the first step, and it appears in 0 blocks constructed in the second step. Therefore, each pair of first associates appears $\lambda_1 = 15$ times. Now consider a pair of second associates $\{x, y'\}$ where $x \in X$ and $y' \in X'$. In the first step of the construction, x is in four of the blocks and y' is in two of the pairs, so $\{x, y'\}$ is in eight blocks of size six. In the second step, x is in two pairs and y' is in four blocks, so $\{x, y'\}$ is again in eight blocks of size six. Thus, $\lambda_2 = 16$. □

5.1. Summary of minimality

Table 4 summarizes the results given in this section. It shows when the necessary conditions are sufficient for (2, 4) GDDs with minimal indices. Further, the table indicates when the results show the necessary conditions are sufficient for

Table 4
Summary of constructions and minimality for configuration (2, 4).

n	λ_1	λ_2	
$n \equiv 0, 16, 24, 32, 40, 48 \pmod{56}$ and $n \equiv 16 \pmod{24}$	$7n/8$	$n-1$	Minimal
$n \equiv 0, 16, 24, 32, 40, 48 \pmod{56}$ and $n \equiv 0, 8 \pmod{24}$	$7n/8$	$n-1$	Near minimal
$n \equiv 2, 10, 18, 26, 34, 42, 6, 14, 30, 38, 46, 54 \pmod{56}$	$7n/2$	$4(n-1)$	Minimal
$n \equiv 4, 12, 20, 28, 44, 52 \pmod{56}$	$7n/4$	$2(n-1)$	Minimal
$n \equiv 8 \pmod{56}$ and $n \equiv 16 \pmod{24}$	$n/8$	$(n-1)/7$	7 times the minimal
$n \equiv 8 \pmod{56}$ and $n \equiv 0, 8 \pmod{24}$	$n/8$	$(n-1)/7$	14 times the minimal
$n \equiv 22, 50 \pmod{56}$	$n/2$	$4(n-1)/7$	7 times the minimal
$n \equiv 36 \pmod{56}$	$n/4$	$2(n-1)/7$	7 times the minimal
$n \equiv 3, 5, 7, 9, 11, 13 \pmod{14}$	$7n$	$8(n-1)$	Minimal
$n \equiv 1 \pmod{14}$, $n \neq 15$	n	$8(n-1)/7$	7 times the minimal
$n = 15$	15	16	Minimal

Table 5
Existence of BIBD(n, 5, λ) and resulting configuration (1, 5) GDDs.

BIBD	Existence	Resulting GDD
$(n, 5, 1)$	$n \equiv 1, 5 \pmod{20}$	GDD$(n, 2, 6; n, (n-1)/2)$
$(n, 5, 2)$	$n \equiv 1, 5 \pmod{10}$, $n \neq 15$	GDD$(n, 2, 6; 2n, n-1)$
$(n, 5, 4)$	$n \equiv 0, 1 \pmod{10}$, $n \neq 10, 160, 190$	GDD$(n, 2, 6; 4n, 2(n-1))$
$(n, 5, 5)$	$n \equiv 1 \pmod{4}$	GDD$(n, 2, 6; 5n, 5/(2(n-1)))$
$(n, 5, 10)$	$n \equiv 1 \pmod{2}$	GDD$(n, 2, 6; 10n, 5(n-1))$
$(n, 5, 20)$	All n	GDD$(n, 2, 6; 20n, 10(n-1))$

configuration (2, 4) GDDs with near minimal, seven times the minimal possible or fourteen times the minimal possible indices.

6. GDDs with configuration (1, 5)

In this section we focus on the minimal indices for configuration (1, 5) GDD$(n, 2, 6; \lambda_1, \lambda_2)$. Hurd and Sarvate gave a construction for configuration $(1, k)$ GDD$(n, 2, k+1; \lambda_1, \lambda_2)$ using a BIBD(n, k, Λ)s [7]. We repeat their result here:

Theorem 6.1 *([7]). The existence of a* BIBD(n, k, Λ) *implies the existence of a configuration* $(1, k)$ GDD$(n, 2, k+1; \lambda_1, \lambda_2)$ *with* $\lambda_1 = \Lambda n$ *and* $\lambda_2 = 2\Lambda(n-1)/(k-1)$.

Further, in [4] Hanani showed the existence of some classes of BIBD$(n, 5, \lambda)$. Using his result and Theorem 6.1 we obtain the following (1, 5) configuration GDD$(n, 2, 6; \lambda_1, \lambda_2)$s summarized in Table 5.

However, this construction does not always give optimal values of λ_1 and λ_2. By Theorem 1.5, we have the following relation between λ_1 and λ_2.

Corollary 6.2. *For any configuration* (1, 5) GDD$(n, 2, 6; \lambda_1, \lambda_2)$ *we have* $\lambda_2 = \frac{\lambda_1(n-1)}{2n}$.

From Corollary 6.2 we see that for some value of n the minimum possible indices are (λ_1, λ_2). As in the other two configurations, we may find the minimal indices by Corollary 6.2 and Theorem 1.1. Further, any other GDD with configuration (1, 5) will have indices $(w\lambda_1, w\lambda_2)$ for some positive integer w. The minimal indices are summarized in the next theorem.

Theorem 6.3. *The minimal indices* (λ_1, λ_2) *for any configuration* (1, 5) GDD$(n, 2, 6; \lambda_1, \lambda_2)$ *are summarized in Table 6.*

Proof. By Corollary 6.2, we have that $\lambda_2 = \frac{\lambda_1(n-1)}{2n}$. This implies that if $n \equiv 1 \pmod{2}$ then λ_1 must be a multiple of n and λ_2 must be a a multiple of $(n-1)/2$. However, if $n \equiv 11, 15 \pmod{20}$ then the indices given do not give an even number of blocks which is required by Theorem 1.4. So for $n \equiv 11, 15 \pmod{20}$, if we take two times the minimum possible indices, the number of blocks will be integer valued implying $(\lambda_1, \lambda_2) = (2n, (n-1))$. Also, using the given indices for $n \equiv 3, 7, 9 \pmod{10}$ results in a non-integer value for the number of blocks given by Theorem 1.1. Thus we must take 5 times these, so the minimal indices are $(\lambda_1, \lambda_2) = (5n, 5(n-1)/2)$. Finally, if $n \equiv 1, 5 \pmod{20}$, then the necessary conditions in Theorem 1.1 are met.

If $n \equiv 0 \pmod{2}$, then Corollary 6.2 tells us that λ_1 must be a multiple of $2n$ and λ_2 must be a multiple of $n-1$. However if $n \equiv 2, 4, 8 \pmod{10}$, then these values give a non-integer value for the number of blocks. If we take 5 times these indices then the necessary condition in Theorem 1.1 is satisfied, and so the minimal indices are $(\lambda_1, \lambda_2) = (10n, 5(n-1))$. Notice that for $n \equiv 0, 6 \pmod{10}$, the given indices are $(\lambda_1, \lambda_2) = (2n, n-1)$ which are the minimum possible. □

Table 6
Summary of minimal indices for configuration (1, 5).

n	λ_1	λ_2
$n \equiv 0, 6, 10, 11, 15, 16 \pmod{20}$	$2n$	$(n-1)$
$n \equiv 1, 5 \pmod{20}$	n	$(n-1)/2$
$n \equiv 2, 4, 8, 12, 14, 18 \pmod{20}$	$10n$	$5(n-1)$
$n \equiv 3, 7, 9, 13, 17, 19 \pmod{20}$	$5n$	$5(n-1)/2$

7. Constructing configuration (1, 5) GDDs

In this section we focus on constructing (1, 5) GDDs with minimal indices. Theorem 6.1 gives us the following results.

Corollary 7.1. *The necessary conditions are sufficient for the existence of a configuration* (1, 5) GDD$(n, 2, 6; n, (n-1)/2)$ *for* $n \equiv 1, 5 \pmod{20}$.

Corollary 7.2. *The necessary conditions are sufficient for the existence of a configuration* (1, 5) GDD$(n, 2, 6; 2n, n-1)$ *for* $n \equiv 11, 15 \pmod{20}$, $n \neq 15$.

Notice that in the previous two constructions, the design is minimal. We use a resolvable BIBD$(n, 5, 4)$ in the following construction. In [10], it is given that a resolvable BIBD$(n, 5, 4)$ exists for $n \equiv 0 \pmod{10}$ except for $n = 10, 160, 190$.

Theorem 7.3. *Let* $n \equiv 0 \pmod{10}, n \neq 10, 160, 190$. *Then the necessary conditions are sufficient for the existence of a configuration* (1, 5) GDD$(n, 2, 6; 2n, n-1)$.

Proof. Let $n \equiv 0 \pmod{10}, n \neq 10, 160, 190$. Assume the two groups are $X = \{1, 2, \dots, n\}$ and $X' = \{1', 2', \dots, n'\}$. There exists a RBIBD$(n, 5, 4)$ with $b = n(n-1)/5$ blocks, and each point appearing $r = (n-1)$ times. First, let $D = (X, \mathcal{B})$ be such a design with parallel classes $C_1, C_2, \dots, C_{n-1}$. Construct a graph G with vertex set $\mathcal{B} \cup X'$ in the following manner. For $j = 1, 2, \dots, n-1$, create the bipartite graph G_j with bipartition $(V(G_j^1), V(G_j^2))$ where $V(G_j^1)$ are the $n/5$ blocks of C_j and $V(G_j^2)$ are the points in X' as follows. Partition C_j into equal parts A_j and B_j. Let

$$F_j^A = \{x : x \in A_j\}$$

and

$$F_j^B = \{x : x \in B_j\}.$$

Form each of these bipartite graphs.

- $G_{j,1}$ is the complete (A_j, F_j^A)-bipartite graph,
- $G_{j,2}$ is the complete (B_j, F_j^B)-bipartite graph.

Then let $E(G_j) = E(G_{j,1}) \cup E(G_{j,2})$. Thus each vertex in G_j^1 has degree $n/2$ and each vertex in G_j^2 has degree $n/10$. Let $E(G) = \bigcup_{j=1}^{n-1} E(G_j)$. For each edge $\{u, v\} \in G$, form a block of size six by taking $u \cup v$. This collection of blocks is exactly half of the blocks in the desired GDD. Secondly, let $D = (X', \mathcal{B}')$ be an RBIBD$(n, 5, 4)$ on X' with parallel classes $C_1, C_2, \dots, C_{n-1}$. Construct a graph H on the vertex set $\mathcal{B}' \cup X$ as follows. For $j = 1, 2, \dots, n-1$, create the bipartite graph H_j with bipartition $(V(H_j^1), V(H_j^2))$ where $V(H_j^1)$ is the set of $n/5$ blocks from C_j and $V(H_j^2)$ are the points in X. As in the first step, partition C_j into equal parts A_j and B_j, and let F_j^A and F_j^B be defined in the same way. Form the bipartite graphs $H_{j,1}$ and $H_{j,2}$ as follows.

- $H_{j,1}$ is the complete (A_j, F_j^B)-bipartite graph,
- $H_{j,2}$ is the complete (B_j, F_j^A)-bipartite graph.

Then let $E(H_j) = E(H_{j,1}) \cup E(H_{j,2})$, and $E(H) = \bigcup_{j=1}^{n-1} E(H_j)$. For each edge $\{u, v\} \in H$, form a block of size six by taking $u \cup v$.

In the RBIBD, each pair appears in four blocks; therefore, in the construction, each pair of first associates will be in exactly $4 \cdot (n/2) = 2n$ blocks of size six. Now consider a pair of second associates $\{x, y'\}$ where $x \in X$ and $y' \in X'$. The points x and y appear in every parallel class exactly once. So for each C_j, if x and y are both in A_j, then $\{x, y'\}$ appears once in the blocks constructed in the first step, and it appears zero times in the blocks constructed in the second step. If $x \in A_j$ and $y \in B_j$, or vice versa, then $\{x, y'\}$ appears once in the blocks constructed in the second step of the construction and zero times in the blocks constructed in the first step. Therefore $\{x, y'\}$ appears exactly once per C_j. Thus λ_2 is the number of parallel classes, which is $n-1$. □

Table 7
Summary of constructions and minimality for configuration (1, 5).

n	λ_1	λ_2	
$n \equiv 0, 10, 11, 15, 6, 16 \pmod{20}$, $n \neq 10, 15, 160, 190$	$2n$	$(n-1)$	Minimal
$n \equiv 1, 5 \pmod{20}$	n	$(n-1)/2$	Minimal
$n \equiv 2, 4, 8, 12, 14, 18 \pmod{20}$	$10n$	$5(n-1)$	Near-minimal
$n \equiv 3, 7, 9, 13, 17, 19 \pmod{20}$	$5n$	$5(n-1)/2$	Near-minimal

A near parallel class is a partial parallel class missing a single point. A near resolvable design NRB$(n, k, k-1)$ is a BIBD$(n, k, k-1)$ with the property that the blocks can be partitioned into near parallel classes. For such a design, every point is absent from exactly one class. The necessary condition for the existence of an NRB$(v, k, k-1)$ is $v \equiv 1 \pmod{k}$. It is known that the necessary condition is sufficient for the existence of a NRB$(v, k, k-1)$ if $k \leq 7$ (see [10]). We use near resolvable designs in the following construction.

Theorem 7.4. *Let* $n \equiv 6 \pmod{10}$. *Then the necessary conditions are sufficient for the existence of a configuration* (1, 5) GDD$(n, 2, 6; 2n, n-1)$.

Proof. Let $n \equiv 6 \pmod{10}$, and the two groups have point sets $X = \{1, 2, \ldots, n\}$ and $X' = \{1', 2', \ldots, n'\}$. Since $n \equiv 6 \pmod{10}$, there exists a NRB$(n, 5, 4)$. It has n near parallel classes with $(n-1)/5$ blocks in them each. Let $D = (X, \mathcal{B})$ be such a design, and resolve the blocks of D into near parallel classes $C_1, C_2, \ldots, C_n$ where C_i misses point i. Construct a graph G with vertex set $\mathcal{B} \cup X'$ in the following manner. For $j = 1, 2, \ldots, n/2$, create the complete bipartite graph G_j with bipartition $(V(G_j^1), V(G_j^2))$ where $V(G_j^1)$ are the blocks of C_j and $V(G_j^2)$ are the points $\{1', 2', \ldots, n/2'\}$. For $j = n/2+1, \ldots, n$, create the complete bipartite graph G_j with bipartition $(V(G_j^1), V(G_j^2))$ where $V(G_j^1)$ are the blocks of C_j and $V(G_j^2)$ are the points $\{(n/2+1)', (n/2+2)', \ldots, n'\}$. Let $E(G) = \bigcup_{j=1}^{n} E(G_j)$. For each edge $\{u, v\} \in G$, take the subset $u \cup v$ to be a block of size six. This collection of blocks is half of the blocks in the desired GDD. To get the other half, let $D = (X', \mathcal{B}')$ be the NRB$(n, 5, 4)$ on X' and repeat the construction with $V(G_j^1) = \mathcal{B}'$, $V(G_j^2)$ as the points $\{1, 2, \ldots, n/2\}$ for $j = n/2+1, \ldots, n$, and $V(G_j^2)$ as the points $\{(n/2)+1, \ldots, n\}$ for $j = 1, 2, \ldots, n/2$.

Consider a pair of first associates. It will appear $4(n/2) = 2n$ times in the blocks of size six. Now consider a pair of second associates where $x \in X$ and $y' \in X'$. If $x \in \{1, 2, \ldots, n/2\}$ and $y' \in \{1', 2', \ldots, (n/2)'\}$ then $\{x, y'\}$ will appear $(n/2) - 1$ times in the blocks constructed in the first step, and it will appear $n/2$ times in the blocks constructed in the second step. It is the same case if $x \in \{n/2+1, n/2+2, \ldots, n\}$ and $y' \in \{(n/2+1)', (n/2+2)', \ldots, n'\}$. If $x \in \{1, 2, \ldots, n/2\}$ and $y' \in \{(n/2+1)', (n/2+2)', \ldots, n'\}$, then $\{x, y'\}$ will appear $n/2$ times in the blocks constructed in the first step and $n/2 - 1$ times in the blocks constructed in the second step. It is the same case if $x \in \{n/2+1, n/2+2, \ldots, n\}$ and $y' \in \{1', 2', \ldots, n'\}$. Thus $\lambda_2 = n - 1$. □

Note that we have constructed minimal GDDs for $n \equiv 0, 1, 5, 6 \pmod{10}$ (for all but a few values). Recall that a near-minimal design is one that has exactly twice the minimal indices. By Theorem 6.1, the necessary conditions are sufficient for the existence of a near minimal GDD$(n, 2, 6; \lambda_1, \lambda_2)$ for $n \equiv 2, 3, 4, 7, 8, 9 \pmod{10}$. Though 5-resolvable designs have not yet been studied extensively, their existence will be useful, as we may then construct certain minimal configuration (1, 5) GDD$(n, 2, 6; \lambda_1, \lambda_2)$s as stated in the next result.

Theorem 7.5. *The existence of a* 5*-resolvable* BIBD$(n, 5, 10)$ *implies the existence of a configuration* (1, 5) GDD$(n, 2, 6; 5n, 5(n-1)/2)$ *for* $n \equiv 3, 7, 9 \pmod{10}$.

Proof. Let $n \equiv 3, 7, 9 \pmod{10}$ and assume there exists a 5-resolvable BIBD $(n, 5, 10)$. Assume the two groups are $X = \{1, 2, 3, \ldots, n\}$ and $X' = \{1', 2', 3', \ldots, n'\}$ and let $D = (X, \mathcal{B})$ be such a design. Resolve the blocks of D into 5-parallel classes $C_1, C_2, \ldots, C_{n-1/2}$, each having n blocks. Construct a graph G with vertex set $\mathcal{B} \cup X'$ in the following manner. For $j = 1, 2, \ldots, (n-1)/4$, create the complete bipartite graph G_j with bipartition $(V(G_j^1), V(G_j^2))$ where $V(G_j^1)$ are the blocks of C_j and $V(G_j^2)$ are the odd integers in X'. For $j = (n-1)/4+1, \ldots, (n-1)/2$, create the complete bipartite graph G_j with bipartition $(V(G_j^1), V(G_j^2))$ where $V(G_j^1)$ are the blocks of C_j and $V(G_j^2)$ are the even integers in X'. Let $E(G) = \bigcup_{j=1}^{(n-1)/2} E(G_j)$. For each edge $\{u, v\} \in G$, take the six element subset $u \cup v$ to be a block. This collection of blocks is exactly half of the blocks in the desired GDD. To get the other half, let D be a 5-RBIBD$(n, 5, 10)$ on X' and repeat the construction with $V(G_j^1) = \mathcal{B}'$, $V(G_j^2)$ as the even integers in X for $j = 1, 2, \ldots, (n-1)/4$, and $V(G_j^2)$ as the odd integers in X for $j = (n-1)/4+1, \ldots, (n-1)/2$.

Consider a pair of first associates. It will appear in ten blocks from D. Therefore, in the given construction it will appear $5n$ times in a block of size six. Now consider a pair of second associates $\{x, y'\}$. In each step of the construction, this pair appears in $5(n-1)/4$ blocks of size six, thus it appears in a total of $5(n-1)/2$ blocks of size six. □

7.1. Summary of minimality

We conclude this section with a summary of the GDDs we have constructed, and their minimality found in Table 7.

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