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ScienceDirect
Journal of Functional Analysis 260 (2011) 1721–1733
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Level sets and composition operators on the Dirichlet space

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Received 24 March 2010; accepted 21 December 2010

Available online 3 January 2011

Communicated by Gilles Godefroy

Abstract

We consider composition operators in the Dirichlet space of the unit disc in the plane. Various criteria on boundedness, compactness and Hilbert–Schmidt class membership are established. Some of these criteria are shown to be optimal.

Keywords: Dirichlet space; Composition operators; Capacity

1. Introduction

In this note we consider composition operators in the Dirichlet space of the unit disc. A comprehensive study of composition operators in function spaces and their spectral behavior could be found in [3,11,16]. See also [6–8,12,13,17] for a treatment of some of the questions addressed in this paper.

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1 Research partially supported by a grant from AI PHC Volubilis MA 09209.
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0022-1236/$ – see front matter
doi:10.1016/j.jfa.2010.12.023

Let $\mathbb{D}$ be the unit disc in the complex plane and let $\mathbb{T} = \partial\mathbb{D}$ be its boundary. We denote by $\mathcal{D}$ the classical Dirichlet space. This is the space of all analytic functions f on $\mathbb{D}$ such that

$$\mathcal{D}(f) := \int_{\mathbb{D}} \left|f'(z)\right|^2 dA(z) < \infty,$$

where $dA(z) = dx\,dy/\pi$ stands for the normalized area measure in $\mathbb{D}$. We call $\mathcal{D}(f)$ the Dirichlet integral of f. The space $\mathcal{D}$ is endowed with the norm

$$\|f\|_{\mathcal{D}}^2 := \left|f(0)\right|^2 + \mathcal{D}(f).$$

It is standard that a function $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n$, holomorphic on $\mathbb{D}$, belongs to $\mathcal{D}$ if and only if

$$\sum_{n \geqslant 0} \left|\widehat{f}(n)\right|^2 (1+n) < \infty,$$

and that this series defines an equivalent norm on $\mathcal{D}$.

Since the Dirichlet space is contained in the Hardy space $\mathrm{H}^2(\mathbb{D})$, every function $f \in \mathcal{D}$ has non-tangential limits f^* almost everywhere on $\mathbb{T}$. In this case, however, more can be said. Indeed, Beurling [2] showed that if $f \in \mathcal{D}$ then $f^*(\zeta) = \lim_{r \to 1} f(r\zeta)$ exists for $\zeta \in \mathbb{T}$ outside of a set of logarithmic capacity zero.

Let φ be a holomorphic self-map of $\mathbb{D}$. The composition operator C_φ on $\mathcal{D}$ is defined by

$$C_\varphi(f) = f \circ \varphi, \quad f \in \mathcal{D}.$$

We are interested herein in describing the spectral properties of the composition operator C_φ, such as compactness and Hilbert–Schmidt class membership, in terms of the size of the level set of φ. For $s \in (0,1)$, the level set $E_\varphi(s)$ of φ is given by

$$E_\varphi(s) = \left\{\zeta \in \mathbb{T}\colon \left|\varphi(\zeta)\right| \geqslant s\right\}.$$

We give new characterizations of Hilbert–Schmidt class membership in the case of the Dirichlet space. We also establish the sharpness of these results.

2. A general criterion

For $\alpha > -1$, dA_α will denote the finite measure on $\mathbb{D}$ given by

$$dA_\alpha(z) := (1+\alpha)\left(1-|z|^2\right)^\alpha dA(z).$$

For $p \geqslant 1$ and $\alpha > -1$, the weighted Bergman space $\mathcal{A}_\alpha^p$ consists of the holomorphic functions f on $\mathbb{D}$ for which

$$\|f\|_{p,\alpha} := \left[\int_{\mathbb{D}} \left|f(z)\right|^p dA_\alpha(z)\right]^{1/p} < \infty.$$

We denote by $\mathcal{D}^p_\alpha$ the space consisting of analytic functions f on $\mathbb{D}$ such that

$$\|f\|^p_{\mathcal{D}^p_\alpha} := \big|f(0)\big|^p + \big\|f'\big\|^p_{p,\alpha} < \infty.$$

Appropriate choices of the parameter α give, with equivalent norm, all the standard holomorphic function spaces. Indeed, the Hardy space H^2 can be identified with $\mathcal{D}^2_1$. The classical Besov space is precisely $\mathcal{D}^p_{p-2}$, and if $p < \alpha + 1$, $\mathcal{D}^p_\alpha = \mathcal{A}^p_{\alpha-2}$. Finally, the classical Dirichlet space $\mathcal{D}$ is identical to $\mathcal{D}^2_0$.

We recall that, by the reproducing formula [16], for every $f \in \mathcal{A}^p_\alpha$,

$$f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-\overline{w}z)^{2+\alpha}}\, dA_\alpha(w), \quad z \in \mathbb{D}. \tag{1}$$

Lemma 2.1. *Let $p \geqslant 1$ and let $\sigma > -1$. Then, there exists a constant C depending only on p and σ such that for every $f \in \mathcal{A}^p_\sigma$,*

$$\big|f(z)\big|^p \leqslant C \int_{\mathbb{D}} \frac{|f(\lambda)|^p}{|1-\bar{\lambda}z|^{2+\sigma}}\, dA_\sigma(\lambda), \quad z \in \mathbb{D}.$$

Proof. By the above reproducing formula,

$$\frac{f(z)}{1-z\overline{w}} = \int_{\mathbb{D}} \frac{f(\lambda)}{1-\lambda\overline{w}} \frac{dA_\sigma(\lambda)}{(1-\bar{\lambda}z)^{2+\sigma}}, \quad z, w \in \mathbb{D},$$

for every $f \in \mathcal{A}^p_\sigma$. By Hölder's inequality, with $q = p/(p-1)$,

$$\frac{|f(z)|^p}{|1-z\overline{w}|^p} \leqslant \int_{\mathbb{D}} \frac{|f(\lambda)|^p\, dA_\sigma(\lambda)}{|1-\bar{\lambda}z|^{2+\sigma}} \times \left(\int_{\mathbb{D}} \frac{dA_\sigma(\lambda)}{|1-\lambda\overline{w}|^q |1-\lambda\bar{z}|^{(2+\sigma)p}}\right)^{\frac{p}{q}}.$$

Taking $w = z$, and using the standard estimate, [16, Lemma 3.10]

$$\int_{\mathbb{D}} \frac{dA_c(\lambda)}{|1-z\bar{\lambda}|^{2+c+d}} \asymp \frac{1}{(1-|z|^2)^d}, \quad \text{if } d > 0,\ c > -1, \tag{2}$$

we get the desired conclusion. □

For $\lambda \in \mathbb{D}$, consider the test function

$$F_{\lambda,\beta}(z) = (1-\bar{\lambda}z)^{-(1+\beta)}, \quad z \in \mathbb{D}.$$

If $\beta \geqslant 0$ is chosen such that $\delta := \delta(p,\alpha,\beta) = 2+\beta-(2+\alpha)/p > 0$, by (2), we have

$$\|F_{\lambda,\beta}\|^p_{\mathcal{D}^p_\alpha} \asymp \big(1-|\lambda|^2\big)^{-p\delta}.$$

The following theorem unifies and generalizes the previously known results of MacCluer [3, Theorem 3.12], Tjani [12, Theorem 3.5] and Wirths and Xiao [13, Theorem 3.2] on Hardy, Besov and weighted Dirichlet spaces, respectively. The techniques required in the proof are known, for the completeness, we give here the proof.

Theorem 2.2. *Let* $p > 1$. *Suppose* $\varphi \in \mathcal{D}_\alpha^p$ *satisfies* $\varphi(\mathbb{D}) \subset \mathbb{D}$. *Fix* $\beta \geqslant 0$ *such that* $\delta := \delta(p, \alpha, \beta) = 2 + \beta - (2 + \alpha)/p > 0$. *Then*:

(a) C_φ *is bounded on* $\mathcal{D}_\alpha^p \iff \sup_{\lambda \in \mathbb{D}} (1 - |\lambda|^2)^\delta \|F_{\lambda,\beta} \circ \varphi\|_{\mathcal{D}_\alpha^p} < \infty$;
(b) C_φ *is compact on* $\mathcal{D}_\alpha^p \iff \lim_{|\lambda| \to 1} (1 - |\lambda|^2)^\delta \|F_{\lambda,\beta} \circ \varphi\|_{\mathcal{D}_\alpha^p} = 0$.

Proof. Without loss of generality we assume that $\varphi(0) = 0$. To prove (a), we observe that if C_φ is bounded, then

$$\|F_{\lambda,\beta} \circ \varphi\|_{\mathcal{D}_\alpha^p} = O\big(\big(1 - |\lambda|^2\big)^{-\delta}\big).$$

For the converse, it follows from Lemma 2.1 that, for $f \in \mathcal{D}_\alpha^p$,

$$\begin{aligned}
&\int_{\mathbb{D}} \big|\varphi'(z)\big|^p \big|f'\big(\varphi(z)\big)\big|^p \, dA_\alpha(z) \\
&\quad \leqslant C \int_{\mathbb{D}} \big|\varphi'(z)\big|^p \bigg(\int_{\mathbb{D}} \frac{|f'(\lambda)|^p}{|1 - \bar{\lambda}\varphi(z)|^{(2+\beta)p}} \, dA_{2p+\beta p-2}(\lambda) \bigg) \, dA_\alpha(z) \\
&\quad = C \int_{\mathbb{D}} \big|f'(\lambda)\big|^p \big(1 - |\lambda|^2\big)^{p\delta} \big\|(F_{\lambda,\beta} \circ \varphi)'\big\|_{p,\alpha}^p \, dA_\alpha(\lambda).
\end{aligned}$$

Therefore part (a) follows.

(b) Assume that $\lim_{|\lambda| \to 1} (1 - |\lambda|^2)^\delta \|F_{\lambda,\beta} \circ \varphi\|_{\mathcal{D}_\alpha^p} = 0$. Let $(f_n)_n$ be a bounded sequence of $\mathcal{D}_\alpha^p$ such that $f_n \to 0$ uniformly on compact sets. Since $f_n' \to 0$ uniformly on compact sets, it follows from the proof of part (a) and the hypothesis that, for r close enough to 1,

$$\begin{aligned}
&\big\|C_\varphi(f_n)\big\|_{\mathcal{D}_\alpha^p}^p - \big|f_n(0)\big|^p \\
&\quad \leqslant \int_{r\mathbb{D}} \big|f_n'(\lambda)\big|^p \big(1 - |\lambda|^2\big)^{p\delta} \big\|(F_{\lambda,\beta} \circ \varphi)\big\|_{p,\alpha}^p \, dA_\alpha(\lambda) \\
&\qquad + \int_{\mathbb{D} \setminus r\mathbb{D}} \big|f_n'(\lambda)\big|^p \big(1 - |\lambda|^2\big)^{p\delta} \big\|(F_{\lambda,\beta} \circ \varphi)'\big\|_{p,\alpha}^p \, dA_\alpha(\lambda) \to 0, \quad n \to \infty,
\end{aligned}$$

and C_φ is compact. The converse is obvious. □

The following result is an immediate consequence of Theorem 2.2.

Corollary 2.3. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ such that $\varphi \in \mathcal{D}$.*

(a) *If $\sup_{n\geqslant 1} \mathcal{D}(\varphi^n) < \infty$, then C_φ is bounded.*
(b) *If $\lim_{n\to\infty} \mathcal{D}(\varphi^n) = 0$, then C_φ is compact.*

Proof. We consider the test function $F_{\lambda,0}$ with $\beta = \alpha = 0$ and $p = 2$. Both (a) and (b) follow from the following inequality:

$$\begin{aligned}\mathcal{D}(F_{\lambda,0} \circ \varphi) &\leqslant 2\big(1-|\lambda|^2\big)^2 \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\lambda|^2\varphi(z)|^2)^4}\, dA(z)\\ &\leqslant c\big(1-|\lambda|^2\big)^2 \sum_{n\geqslant 0} (n+1)^3|\lambda|^{2n} \int_{\mathbb{D}} \big|\varphi'(z)\big|^2\big|\varphi^n(z)\big|^2\, dA(z)\\ &= c\big(1-|\lambda|^2\big)^2 \sum_{n\geqslant 0}(1+n)|\lambda|^{2n}\mathcal{D}\big(\varphi^{n+1}\big)\\ &\leqslant c \limsup_{n\to\infty} \mathcal{D}\big(\varphi^{n+1}\big). \qquad \square\end{aligned}$$

Remark 2.4. The compactness criterion for C_φ in the Bloch space is equivalent to $\|\varphi^n\|_{\mathcal{B}} \to 0$ as was shown in [15] (see also [10,12]). In the case of the Hardy space H^2, however, we know that if C_φ is compact on H^2 then $\|\varphi^n\|_{\mathrm{H}^2} \to 0$ but the converse does not hold [3]. Note that as before in the proof of Corollary 2.3 ($\beta = 0$, $\alpha = 1$ and $p = 2$) if $\|\varphi^n\|_{\mathrm{H}^2} = o(1/\sqrt{n})$, then C_φ is compact on H^2.

3. Hilbert–Schmidt membership

In the case of the Hardy space H^2, one can completely describe the membership of C_φ in the Hilbert–Schmidt class in terms of the size of the level sets of the inducing map φ. Indeed, C_φ is Hilbert–Schmidt in H^2 if and only if

$$\sum_{n\geqslant 0} \|\varphi^n\|^2_{\mathrm{H}^2} = \int_{\mathbb{T}} \frac{|d\zeta|}{1-|\varphi(\zeta)|^2} < \infty.$$

Given an arbitrary measurable function f on $\mathbb{T}$, consider the associated distribution function m_f defined by

$$m_f(\lambda) = \big|\big\{\zeta \in \mathbb{T}\colon \big|f(\zeta)\big| > \lambda\big\}\big|, \quad \lambda > 0.$$

It then follows that C_φ is in the Hilbert–Schmidt class of H^2 if and only if

$$\int_{\mathbb{T}} \frac{|d\zeta|}{1-|\varphi(\zeta)|^2} = \int_1^\infty m_{(1-|\varphi|^2)^{-1}}(\lambda)\, d\lambda \asymp \int_0^1 \frac{|E_\varphi(s)|}{(1-s)^2}\, ds < \infty.$$

It was shown by Gallardo-Gutiérrez and González [8, Main Theorem] that there is a mapping φ taking $\mathbb{D}$ to itself such that C_φ is compact in H^2, and that the level set $E_\varphi(1)$ has Hausdorff dimension equal to one. Recall that the Hausdorff dimension of E:

$$d(E) = \inf\{\alpha\colon\ \Lambda_\alpha(E) = 0\}$$

where $\Lambda_\alpha(E)$ is the α-Hausdorff measure of E given by

$$\Lambda_\alpha(E) = \lim_{\epsilon\to 0} \inf\left\{\sum_i |\Delta_i|^\alpha\colon\ E \subset \bigcup_i \Delta_i,\ |\Delta_i| < \epsilon\right\}.$$

Given $E \subset \mathbb{T}$ and $t > 0$, let us write

$$E_t = \{\zeta \in \mathbb{T}\colon\ d(\zeta, E) \leqslant t\}$$

where d denotes the arclength distance and $|E_t|$ denotes the Lebesgue measure of E.

Let E be a closed subset of $\mathbb{T}$ with $|E_t| = O((\log(e/t))^{-3})$ and E has Hausdorff dimension one (such examples can be given by generalized Cantor sets [2]). Let $\omega(t) = (\log(e/t))^{-2}$, and consider the outer function $f_{\omega,E}$ such that its radial limit $f^*_{\omega,E}$ is given by

$$\left|f^*_{\omega,E}(\zeta)\right| = e^{-w(d(\zeta,E))}, \quad \text{a.e. on } \mathbb{T}.$$

Since ω satisfies the Dini condition

$$\int_0 \frac{\omega(t)}{t}\,dt < \infty,$$

it follows that $f_{\omega,E} \in A(\mathbb{D}) := \mathrm{Hol}(\mathbb{D}) \cap C(\overline{\mathbb{D}})$, disc algebra (see [9, pp. 105–106]) and so $E_{f_{\omega,E}}(1) = E$. On the other hand

$$\int_{\mathbb{T}} \frac{|d\zeta|}{1 - |f_{\omega,E}(\zeta)|^2} \asymp \int_{\mathbb{T}} \frac{|d\zeta|}{\omega(d(\zeta, E))} \asymp \int_0 |E_t| \frac{\omega'(t)}{\omega(t)^2}\,dt$$

(see [4, Proposition A.1] for the last equality). Since the last integral converges, C_φ is a Hilbert–Schmidt operator in H^2.

We have the following more precise result.

Theorem 3.1. *Let E be a closed subset of $\mathbb{T}$ with Lebesgue measure zero. There exists a mapping $\varphi : \mathbb{D} \to \mathbb{D}$, $\varphi \in A(\mathbb{D})$, such that C_φ is a Hilbert–Schmidt operator on* H^2 *and that $E_\varphi(1) = E$.*

Proof. The proof is based a well-known construction of peak functions in the disc algebras. Let $\mathbb{T} \setminus E = \bigcup_{n\geqslant 1}(e^{ia_n}, e^{ib_n})$. For $t \in (a_n, b_n)$, we define

$$g(e^{it}) = \tau_n \frac{(b_n - a_n)^{1/2}}{[(b_n - a_n)^2 - (2t - (b_n + a_n))^2]^{1/4}},$$

where $(\tau_n)_n \subset (0, \infty)$ will be chosen later, and $g(e^{it}) := +\infty$ if $e^{it} \in E$.

Note that

$$\int_0^{2\pi} g\bigl(e^{it}\bigr)^2\,dt = \sum_{n\geqslant 1} \tau_n^2(b_n - a_n)\int_{a_n}^{b_n} \frac{dt}{[(b_n-a_n)^2 - (2t-(b_n+a_n))^2]^{1/2}}$$

$$= \frac{1}{2}\sum_{n\geqslant 1}\tau_n^2(b_n - a_n)\int_{-1}^{1}\frac{du}{[1-u^2]^{1/2}}$$

$$= \frac{\pi}{2}\sum_{n=1}^{\infty}\tau_n^2(b_n - a_n).$$

Since $\sum_{n=1}^{\infty}(b_n - a_n) = 2\pi$, there exists a sequence $(\tau_n)_n$ such that

$$\lim_{n\to+\infty}\tau_n = +\infty \quad \text{and} \quad \sum_{n=1}^{\infty}\tau_n^2(b_n - a_n) < \infty.$$

Let U denote the harmonic extension of g on the unit disc given by

$$U\bigl(re^{i\theta}\bigr) = \frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{|e^{it} - re^{i\theta}|^2}g\bigl(e^{it}\bigr)\,dt = \sum_{n\in\mathbb{Z}}\widehat{g}(n)r^{|n|}e^{in\theta}.$$

Since $\tau_n \to \infty$, one can easily verify that $\lim_{t\to\theta} g(e^{it}) = +\infty$, for $e^{i\theta} \in E$. Hence, $\lim_{r\to 1^-} U(re^{i\theta}) = +\infty$, for $e^{i\theta} \in E$.

Let V be the harmonic conjugate of U, with $V(0) = 0$. It is given by

$$V\bigl(re^{i\theta}\bigr) = \sum_{n\neq 0}\frac{n}{|n|}\widehat{g}(n)r^{|n|}e^{in\theta}.$$

Now, since g is a C^1 function on $\mathbb{T}\setminus E$, we see that the holomorphic function $f = U + iV$ is continuous on $\overline{\mathbb{D}}\setminus E$. Knowing that $\lim_{r\to 1^-} U(re^{it}) = +\infty$, for $e^{it} \in E$, we get that $\varphi = \frac{f}{f+1} \in A(\mathbb{D})$, disc algebra, and $E_\varphi(1) = E$. Finally

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{dt}{1 - |\varphi(e^{it})|^2} = \frac{1}{2\pi}\int_0^{2\pi}\frac{(U(e^{it})+1)^2 + V^2(e^{it})}{(U(e^{it})+1)^2 - U^2(e^{it})}\,dt$$

$$\leqslant \frac{1}{2\pi}\int_0^{2\pi}\bigl(U\bigl(e^{it}\bigr)+1\bigr)^2 + V^2\bigl(e^{it}\bigr)\,dt$$

$$\leqslant 1 + 2\sum_{n\in\mathbb{Z}}\bigl|\widehat{g}(n)\bigr|^2,$$

which shows that C_φ is Hilbert–Schmidt because $g \in L^2(\mathbb{T})$. $\square$

Let E be a closed subset of the unit circle $\mathbb{T}$. Fix a non-negative function $w \in C^1(0,\pi]$ such that

$$\int_{\mathbb{T}} w\big(d(\zeta, E)\big)|d\zeta| < \infty,$$

where d denotes the arclength distance. Now, let $f_{w,E}$ be the outer function given by

$$\big|f^*_{w,E}(\zeta)\big| = e^{-w(d(\zeta,E))}, \quad \text{a.e. on } \mathbb{T}. \tag{3}$$

The following lemma gives an estimate for the Dirichlet integral of $f_{w,E}$ in terms of w and the distance function on E. The proof is based on Carleson's formula, and can be achieved by slightly modifying the arguments used in [5, Theorem 4.1].

Lemma 3.2. *Assume that the function ω is nondecreasing and $\omega(t^\gamma)$ is concave for all $\gamma > 2$. Then*

$$\mathcal{D}(f_{w,E}) \asymp \int_{\mathbb{T}} \omega'\big(d(\zeta, E)\big)^2 e^{-2w(d(\zeta,E))}\, d(\zeta, E)\, |d\zeta|.$$

Since the sequence $\{z^n/\sqrt{n+1}\}_{n=0}^{\infty}$ is an orthonormal basis of $\mathcal{D}$, the operator C_φ is Hilbert–Schmidt on the Dirichlet space if and only if

$$\frac{1}{\pi}\int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}\, dA(z) = \sum_{n\geqslant 1} \frac{\mathcal{D}(\varphi^n)}{n} < \infty.$$

Theorem 3.3. *Assume that the function ω is nondecreasing and $\omega(t^\gamma)$ is concave for some $\gamma > 2$. Then $C_{f_{w,E}}$ is in the Hilbert–Schmidt class in $\mathcal{D}$ if and only if*

$$\int_{\mathbb{T}} \frac{\omega'(d(\zeta, E))^2}{w(d(\zeta, E))^2}\, d(\zeta, E)\, |d\zeta| < \infty.$$

Proof. We first note that $f^n_{w,E} = f_{nw,E}$. Therefore, by Lemma 3.2, we have

$$\begin{aligned}
\int_{\mathbb{D}} \frac{|f'_{w,E}(z)|^2}{(1-|f_{w,E}(z)|^2)^2}\, dA(z) &= \sum_{n=1}^{\infty} \frac{\mathcal{D}(f_{nw,E})}{n} \\
&\asymp \int_{\mathbb{T}} \omega'\big(d(\zeta, E)\big)^2 d(\zeta, E) \sum_{n=1}^{\infty} n e^{-2nw(d(\zeta,E))}|d\zeta| \\
&\asymp \int_{\mathbb{T}} \frac{\omega'(d(\zeta, E))^2}{[1-e^{-2w(d(\zeta,E))}]^2}\, d(\zeta, E)\, |d\zeta|.
\end{aligned}$$

Since $1 - e^{-2w(d(\zeta,E))} \asymp w(d(\zeta, E))$, the result follows. □

Given a (Borel) probability measure μ on $\mathbb{T}$, we define its α-energy, $0 \leqslant \alpha < 1$, by

$$I_\alpha(\mu) = \sum_{n=1}^{\infty} \frac{|\widehat{\mu}(n)|^2}{n^{1-\alpha}}.$$

For a closed set $E \subset \mathbb{T}$, its α-capacity $\operatorname{cap}_\alpha(E)$ is defined by

$$\operatorname{cap}_\alpha(E) := 1/\inf\bigl\{I_\alpha(\mu)\colon\ \mu \text{ is a probability measure on } E\bigr\}.$$

If $\alpha = 0$, we simply note $\operatorname{cap}(E)$ and this means the logarithmic capacity of E.

The weak-type inequality for capacity [2] states that, for $f \in \mathcal{D}$ and $t \geqslant 4\|f\|_{\mathcal{D}}^2$,

$$\operatorname{cap}\bigl(\bigl\{\zeta\colon\ \bigl|f(\zeta)\bigr| \geqslant t\bigr\}\bigr) \leqslant \frac{16\|f\|_{\mathcal{D}}^2}{t^2}.$$

As a result of this inequality, we see that if $\liminf \|\varphi^n\|_{\mathcal{D}} = 0$, then $\operatorname{cap}(E_\varphi(1)) = 0$. Indeed, since $E_\varphi(1) = E_{\varphi^n}(1)$, the weak capacity inequality implies that

$$\operatorname{cap}\bigl(E_\varphi(1)\bigr) = \operatorname{cap}\bigl(E_{\varphi^n}(1)\bigr) \leqslant 16\bigl\|\varphi^n\bigr\|_{\mathcal{D}}^2.$$

Now let $n \to \infty$. Hence, in particular, if the operator C_φ is in the Hilbert–Schmidt class in $\mathcal{D}$, then $\operatorname{cap}(E_\varphi(1)) = 0$. This result was first obtained by Gallardo-Gutiérrez and González [6,7] using a completely different method. Theorems 3.4 and 3.6 give quantitative versions of this result.

Theorem 3.4. *If C_φ is a Hilbert–Schmidt operator in $\mathcal{D}$, then*

$$\int_0^1 \frac{\operatorname{cap}(E_\varphi(s))}{1-s} \log \frac{1}{1-s}\, ds < \infty. \tag{4}$$

Proof. Fix $\lambda \in \mathbb{T}$ and let

$$\varphi_\lambda(\zeta) = \log \operatorname{Re} \frac{1+\lambda\varphi(\zeta)}{1-\lambda\varphi(\zeta)}, \quad \zeta \in \mathbb{T}.$$

Since

$$\int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)^2|)^2}\, dA(z) < \infty,$$

it follows that $\varphi_\lambda \in \mathcal{D}(\mathbb{T})$, see [6], where

$$\mathcal{D}(\mathbb{T}) := \left\{ f \in L^2(\mathbb{T})\colon\ \|f\|_{\mathcal{D}(\mathbb{T})}^2 = \sum_{n\in\mathbb{Z}} \bigl|\widehat{f}(n)\bigr|^2\bigl(1+|n|\bigr) < \infty \right\}.$$

Setting $\Delta_\lambda := \{\zeta \in \mathbb{T}\colon\ |1-\lambda\varphi(\zeta)| \geqslant 1\}$, we see that

$$\left|\varphi_\lambda(\zeta)\right| \asymp \log\frac{1}{1-|\varphi(\zeta)|^2}, \quad \forall \zeta \in \Delta_\lambda.$$

Applying the strong capacity inequality [14, Theorem 2.2] to φ_λ, we get

$$\begin{aligned}
\infty > \|\varphi_\lambda\|^2_{\mathcal{D}(\mathbb{T})} &\geqslant c\int^{\infty} \operatorname{cap}\bigl\{\zeta \in \mathbb{T}\colon \left|\varphi_\lambda(\zeta)\right| > s\bigr\}\, ds^2 \\
&= c\int^{\infty} \operatorname{cap}\left\{\zeta \in \mathbb{T}\colon \left|\log\frac{1-|\varphi(\zeta)|^2}{|1-\lambda\varphi(\zeta)|^2}\right| > s\right\} ds^2 \\
&\geqslant c\int^{\infty} \operatorname{cap}\left\{\zeta \in \mathbb{T}\cap\Delta_\lambda\colon \left|\log\frac{1-|\varphi(\zeta)|^2}{|1-\lambda\varphi(\zeta)|^2}\right| > s\right\} ds^2 \\
&\geqslant c\int^{\infty} \operatorname{cap}\left\{\zeta \in \mathbb{T}\cap\Delta_\lambda\colon \log\frac{1}{1-|\varphi(\zeta)|^2} > 4s\right\} ds^2 \\
&\geqslant c_1\int^{1} \operatorname{cap}\bigl\{\zeta \in \mathbb{T}\cap\Delta_\lambda\colon \left|\varphi(\zeta)\right| > u\bigr\}\, d\left(\log\frac{1}{1-u}\right)^2.
\end{aligned}$$

Since $\mathbb{T} = \Delta_1 \cup \Delta_{-1}$, the subadditivity of the capacity implies that

$$\infty > \|\varphi_1\|^2_{\mathcal{D}(\mathbb{T})} + \|\varphi_{-1}\|^2_{\mathcal{D}(\mathbb{T})} \geqslant c_2\int^{1} \operatorname{cap}\bigl\{\zeta \in \mathbb{T}\colon \left|\varphi(\zeta)\right| > u\bigr\}\, d\left(\log\frac{1}{1-u}\right)^2,$$

and hence the theorem follows. □

Remark 3.5. Since $\{z^n/(1+n)^{\frac{1-\alpha}{2}}\}_{n=0}^{\infty}$ is an orthonormal basis in $\mathcal{D}_\alpha$, $\alpha \in (0,1)$, C_φ is a Hilbert–Schmidt operator in $\mathcal{D}_\alpha$ if and only if

$$\sum_{n=1}^{\infty}\frac{\mathcal{D}_\alpha(\varphi^n)}{n^{1-\alpha}} \asymp \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2+\alpha}}\, dA_\alpha(z) < \infty.$$

Therefore, for fixed $\lambda \in \mathbb{T}$, the function

$$\varphi_\lambda(\zeta) = \left(\operatorname{Re}\frac{1+\lambda\varphi(\zeta)}{1-\lambda\varphi(\zeta)}\right)^{-\alpha/2}, \quad \zeta \in \mathbb{T},$$

belongs to the weighted harmonic Dirichlet space

$$\mathcal{D}_\alpha(\mathbb{T}) := \left\{ f \in L^2(\mathbb{T})\colon\ \|f\|^2_{\mathcal{D}_\alpha(\mathbb{T})} = \sum_{n\in\mathbb{Z}} \left|\widehat{f}(n)\right|^2\bigl(1+|n|\bigr)^{1-\alpha} < \infty \right\}$$

(see [7]). Applying again the strong capacity inequality [14, Theorem 2.2] for $\mathcal{D}_\alpha$ to φ_λ, we get as before

$$\int_0^1 \frac{\mathrm{cap}_\alpha(E_\varphi(s))}{(1-s)^{1+\alpha}}\, ds < \infty.$$

The following theorem is the analogue of Proposition 3.1 for the Dirichlet space. It shows that condition (4) is optimal.

Theorem 3.6. *Let $h : [1, +\infty[\to [1, +\infty[$ be a function such that $\lim_{x\to\infty} h(x) = +\infty$. Let E be a closed subset of $\mathbb{T}$ such that $\mathrm{cap}(E) = 0$. Then there is $\varphi \in A(\mathbb{D}) \cap \mathcal{D}$, $\varphi(\mathbb{D}) \subset \mathbb{D}$ such that*:

(1) $E_\varphi(1) = E$;
(2) C_φ *is in the Hilbert–Schmidt class in* $\mathcal{D}$;
(3) $\int^1 \frac{\mathrm{cap}(E_\varphi(s))}{1-s} \log \frac{1}{1-s} h(\frac{1}{1-s})\, ds = +\infty$.

Proof. Let $k(x) = h(e^x)$, there exists a continuous decreasing function ψ such that

$$\int^{+\infty} \psi(x)\, dx^2 < \infty \quad \text{and} \quad \int^{+\infty} \psi(x)k(x)\, dx^2 = \infty.$$

Set $\eta(t) = \psi^{-1}(\mathrm{cap}(E_t))$. We have

$$\int_0 \mathrm{cap}(E_t)\big|d\eta^2(t)\big| \asymp \int_0 \psi\big(\eta(t)\big)\big|d\eta^2(t)\big| \asymp \int^{+\infty} \psi(x)\, dx^2 < \infty$$

and

$$\int_0 \mathrm{cap}(E_t)h\big(e^{\eta(t)}\big)\big|d\eta^2(t)\big| \asymp \int_0 \psi\big(\eta(t)\big)k\big(\eta(t)\big)\big|d\eta^2(t)\big| \asymp \int^{+\infty} \psi(x)k(x)\, dx^2 = \infty.$$

Since

$$\int_0 \mathrm{cap}(E_t)\big|d\eta^2(t)\big| < \infty,$$

by [4, Theorem 5.1], there exists a function $f \in \mathcal{D}$ such that

$$\mathrm{Re}\, f(\zeta) \geqslant \eta\big(d(\zeta, E)\big) \quad \text{and} \quad \big|\mathrm{Im}\, f(\zeta)\big| < \pi/4, \quad \text{q.e. on } \mathbb{T}.$$

By harmonicity,

$$\big|\mathrm{Im}\, f(z)\big| < \pi/4, \qquad |z| < 1.$$

Now take

$$\varphi = \exp\bigl(-e^{-f}\bigr).$$

By a simple modification in the construction of f as in [1], we can suppose that $\varphi \in A(\mathbb{D})$. Hence $E_\varphi(1) = E$ and

$$\begin{aligned}\int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2} &\asymp \int_{\mathbb{D}} \frac{|f'(z)|^2 e^{-2\operatorname{Re} f(z)} e^{-2e^{-\operatorname{Re} f(z)}\cos(\operatorname{Im} f(z))}}{e^{-2\operatorname{Re} f(z)}\cos^2(\operatorname{Im} f(z))}\, dA(z) \\ &\leqslant \int_{\mathbb{D}} \bigl|f'(z)\bigr|^2 \exp\bigl(-\sqrt{2}e^{-\operatorname{Re} f(z)}\bigr)\, dA(z) \\ &\leqslant c \int_{\mathbb{D}} \bigl|f'(z)\bigr|^2\, dA(z) < \infty.\end{aligned}$$

Hence C_φ is in the Hilbert–Schmidt class. Finally, since

$$E_\varphi(s) \supseteq \bigl\{\zeta \in \mathbb{T}\colon\ \eta\bigl(d(\zeta, E)\bigr) \geqslant \log(1/1-s)\bigr\},$$

we get

$$\int_0 \operatorname{cap}\bigl(E_\varphi(s)\bigr) h(1/1-s)\, d\bigl(\log(1/1-s)\bigr)^2 \geqslant \int_0 \operatorname{cap}(E_t) h\bigl(e^{\eta(t)}\bigr)\bigl|d\eta^2(t)\bigr| = +\infty. \qquad \square$$

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