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On the representation theory of Galois and atomic topoi

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Abstract

In this paper we consider Galois theory as it was interpreted by Grothendieck in SGA1 (Lecture Notes in Mathematics 224 (1971)) and SGA4 (Lecture Notes in Mathematics 269 (1972)) and later extended by Joyal–Tierney in Memoirs of AMS 151 (1984). Grothendieck conceived Galois theory as the axiomatic characterization of the classifying topos of a progroup in terms of a representation theorem for pointed Galois Topoi. Joyal–Tierney extended this to the axiomatic characterization of the classifying topos of a localic group in terms of a representation theorem for pointed Atomic Topoi.

Classical Galois theory corresponds to discrete groups (the point is essential), and the representation theorem can be proved by elementary category-theory. This was developed by Barr–Diaconescu (Cahiers Top. Geo. Diff. cat 22-23 (1981) 301). Grothendieck theory corresponds to progroups or prodiscrete localic groups (the point is *proessential*, a concept we introduce in this paper), and the representation theorem is proved by inverse limit of topoi techniques. This was developed by Moerdijk (Proc. Kon. Nederl. Akad. van Wetens. Series A 92 (1989)). Joyal–Tierney theory corresponds to general localic groups (the point is a general point), and the representation theorem is proved by descent techniques. It can also be proved by the methods of localic Galois theory developed by Dubuc (Advances in Mathematics 175 (2003)).

Joyal–Tierney also consider the case of a general localic groupoid (in particular, it includes unpointed Atomic Topoi), which needs a sophisticated change of base. Bunge (Category Theory '91, CMS Conf. Proc. 13 (1992)) and Kennison (J. Pure Appl. Algeb. 77 (1992)) consider in particular the case of prodiscrete groupoids, and develop an unpointed Grothendieck theory.

We consider these contributions, make an original description, development and survey of the whole theory (but do not touch the representation of cohomology aspects), and present our own results.

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0. Introduction

The notion of a (pointed) *Galois pretopos* ("catégorie Galoisienne") was considered originally by Grothendieck in [11] in connection with the fundamental group of a scheme. In that paper *Galois theory is conceived as the axiomatic characterization of the classifying pretopos of a profinite group G*. The fundamental theorem takes the form of a representation theorem for Galois pretopos (see [10] for the explicit interpretation of this work in terms of filtered unions of categories—the link to filtered inverse limits of topoi—and its relation to classical Galois's Galois theory). An important motivation was pragmatical. The fundamental theorem is tailored to be applied to the category of etal coverings of a connected locally noetherian scheme pointed with a geometric point over an algebraically closed field. We quote: "Cette équivalence permet donc de interpréter les opérations courantes sur des revêtements en terms des opérations analogues dans $\mathscr{B}G$, i.e. en terms des opérations évidentes sur des ensembles finis où G opére". Later, in collaboration with Verdier [1, Ex IV], he considers the general notion of pointed Galois Topos in a series of commented exercises (specially Ex IV, 2.7.5). There, specific guidelines are given to develop the theory of classifying topoi of *progroups*. It is stated therein that Galois topoi correspond exactly, as categories, to the full subcategories generated by *locally constant objects* in connected locally connected topoi (this amounts to the construction of Galois closures), and that they classify progroups. In [18], Moerdiejk developed this program under the light of the *localic group* concept. He proves the fundamental theorem (in a rather sketchy way, Theorem 3.2 loc.cit.) in the form of a characterization of pointed Galois topoi as the classifying topoi of *prodiscrete localic groups*.

In Appendix A we develop the theory of locally constant objects as defined in [1, Ex. IX]. For the notion of Galois topos discussed here see Definition A.2.1. We take from [6] the idea of presenting the topos of objects split by a cover as a push-out topos. We show how the existence of Galois closures follows automatically by the fact that this topos has essential points.

Connected groupoids are considered already in [11] because of the lack of a canonical point. The groupoid whose objects are all the points and with arrows the natural transformations, imposes itself as the natural mathematical object to be considered (although all the information is already in any one of its vertex groups). The theory is developed with groups for the sake of simplicity, but the appropriate formulation of the groupoid version is not straightforward (see [11, V 5]).

On general grounds, the association of a localic groupoid to the set of points of a topos is evident by means of an enrichment over localic spaces of the categories of set-valued functors. Localic spaces are formal duals of locales, and it is not evident how this enrichment can be made in a way that furnish a manageable theory for the sometimes unavoidable work in the category of locales. Generalizing the construction in [9] of the localic group of automorphisms of a set-valued functor we develop this enrichment in Section 2.1, and in Section 2.2 we construct the localic groupoid of points. The objects are the points of the topos. The hom-sets are (in general pointless) localic spaces. This construction is adequate for the representation theorems only in presence of enough points in the topos.

We develop in detail the pointed theory in Section 3.3, where we bring into consideration the localic groupoid of all the points. We establish the fundamental theorem in the form of a characterization of Galois topoi with (at least one, and thus enough) points as the classifying topoi of *connected groupoids with discrete space of objects and prodiscrete localic spaces of hom-sets*. We also introduce the concept of *proessential point*, show how to construct Galois closures with this, and prove a new characterization of pointed Galois topoi.

Grothendieck and Verdier always assume the existence of enough points arguing that in all the meaningful examples the points are there. Their thoughts on pointless topoi are revealed in [1] Ex IV 6.4.2 where they write: "on peut cependant, "en faisant expres" construire des topos qui n'ont pas suffisamment de points". However, with present hindsight, and as it was first and long ago stressed by Joyal, we can argue that unpointed theories are justified.

The theories in [1,18] are localic only at the level of the fundamental groupoid arrows. Fundamental groupoids of Galois topoi *loose their objects by the same reason that they loose their arrows* (namely, some co-filtered inverse limits of sets become empty, see Section 3.4). It seems natural then to develop a theory which is localic also at the level of objects.

Bunge in [6] (see also [8]) develop an unpointed theory for Galois topoi following the inverse limit techniques implicit in [11,1] and made explicit in [18]. Around the same time, Kennison [15] also developed an unpointed theory with a different approach. They both prove the fundamental theorem under the form of a Galois topoi characterization as the classifying topoi of *prodiscrete localic groupoids*.

Joyal–Tierney Galois theory (see below) is behind Bunge development of the unpointed theory of Galois topoi. However, this theory follows by inverse limit techniques directly from the theory of classifying topoi of discrete groups (or groupoids), which is a very simple and elementary case of Joyal–Tierney theorems. We show in Section 3.4 how the pointed theory of Galois topoi *can as well be developed* in an unpointed way along the same lines of [18,6,8]. We show that the localic groupoid in the fundamental theorem, even in the unpointed case, can be considered to be the groupoid of (may be phantom) points of the topos.

The unpointed theory also applies in the presence of points, but it yields an slightly different groupoid than the pointed theory. We compare these groupoids in Section 3.5.

In their seminal paper on Galois theory, [14] (after Grothendieck's [11]), Joyal and Tierney bring new light into the subject. *Galois theory is conceived by interpreting the fundamental theorem as an statement that says that a given geometric morphism of topoi is of effective descent* (namely, the point involved in the classical and Grothendieck Galois theories). They prove that *any open surjection is of effective descent.* It follows an unpointed theory of representation for a completely arbitrary topos in terms of localic groupoids, which culminates with their fundamental theorem, Chapter VIII 3, Theorem 2, which states that any topos is the classifying topos of a localic groupoid. This theorem needs the construction of a localic cover and sophisticated change of base techniques, and we think it describes different phenomena from the one that concerns Galois topoi, *either pointed or unpointed.* The reader interested in Joyal–Tierney theory of classifying topoi of localic groupoids should also consult [17].

The representation theorem of pointed connected atomic topoi, [14, Chapter VIII 3. Theorem 1]], is however closely related to the representation theorem of pointed Galois topoi and classical Galois theory. It follows because any point of a connected atomic topos is an open surjection, thus a geometric morphism of effective descent.

In [9] we developed what we call *localic Galois theory* and prove therein this result in a closer manner to classical Galois theory, independently of descent techniques (and of Grothendieck's inverse limit techniques as well). This theorem shows that pointed connected atomic topoi classify *connected localic groupoids with discrete space of objects*. The groupoid in the theorem, as it is the case for Galois topoi, is the localic groupoid of points. We recall all this in Section 3.6.

Of course, Theorem 2 (loc. cit.) applies to an arbitrary, (even connected but maybe pointless) atomic topos, but it is a different theorem. The localic groupoid is not canonically associated and cannot be considered to be (as far as we can imagine) a groupoid of (may be phantom) points of the topos. Furthermore, when applied to atomic topoi with enough points (one for each connected component suffices) it does not yield the localic groupoid of points. It would be interesting to have a theorem which, in presence of a point, yields Theorem 1. We still not know how to define the groupoid of phantom points for a general atomic topos (as we do for a general Galois topos). An *unpointed localic Galois theory* (compressing the unpointed prodiscrete Galois theory as well as the pointed localic Galois theory) is yet to be developed.

1. Background, terminology and notation

In this section we recall some topos and locale theory that we shall explicitly need, and in this way fix notation and terminology. We also include some inedit proofs when it seems necessary. Our terminology concerning spaces and locales follows Joyal–Tierney [14], except that we define *localic space* to be the formal dual of a *locale*, although we omit very often the qualification "localic" and just write "space". Instead of saying *spatial group* we say *localic group*, and the same for groupoids. We do not distinguish notationally a localic space from its corresponding locale.

We denote $\mathscr{S}$ the topos of sets, and all topoi are supposed to be Grothendieck topoi over $\mathscr{S}$. We nonetheless think that all results in this paper hold as well for $\mathscr{S}$ an arbitrary base Grothendieck topos, albeit, a few of them suitably reformulated to avoid the use of choice.

1.1. Filtered inverse limits of topoi

We recall here the fundamental result on filtered inverse limits of topoi, which consists on the construction of the site for such kind of limit. Inverse limits of topoi have been extensively considered in SGA4, VI, where a fully detailed 2-categorical treatment is developed. Consider a filtered system of sites and morphisms of sites (continuous flat functors) and the induced system of topoi as shown in the following diagram (where the vertical arrows ε are the associate sheaf functor):

$$\begin{array}{ccccccc}
\mathscr{C}_\alpha & \xrightarrow{T_{\alpha\beta}} & \mathscr{C}_\beta & \cdots\longrightarrow\mathscr{C} & \mathscr{C}_\alpha & \xrightarrow{T_\alpha} & \mathscr{C} \\
\downarrow \varepsilon & & \downarrow \varepsilon & \downarrow \varepsilon & \downarrow \varepsilon & & \downarrow \varepsilon \\
\mathscr{C}_\alpha^\sim & \xrightarrow{t_{\alpha\beta}^*} & \mathscr{C}_\beta^\sim & \cdots\longrightarrow\mathscr{C}^\sim & \mathscr{C}_\alpha^\sim & \xrightarrow{t_\alpha^*} & \mathscr{C}^\sim
\end{array}$$

The diagram $\mathscr{C}_\alpha \xrightarrow{T_\alpha} \mathscr{C}$ is the filter colimit of the categories $\mathscr{C}_\alpha$, and the category $\mathscr{C}$ is furnished with the coarsest topology that makes the inclusions T_α continuous. The resulting site is called *the inverse limit site*. It is shown in [1] that the inclusions are flat (here is where the filterness condition plays a key role), and thus they are morphisms of sites. With this at hand, the next theorem follows immediately from SGA4, Ex. IV, 4.9.4.

Theorem 1.1.1. (SGA4, Ex VI, 8.2.11). *In the situation described above, the following formula $Lim_\alpha(\mathscr{C}_\alpha)^\sim = (Colim_\alpha \mathscr{C}_\alpha)^\sim$ holds. That is, in the diagram above, the bottom row consists of the inverse image functors of a filtered inverse limit of topoi.*

The interested reader will profit also consulting [16], where many ubiquitous and important preservation properties of filtered inverse limits are stated and proved. There, a construction of the inverse limit site (Theorem 3.1 loc. cit.) is developed in the style of the classical construction of the *p-adic numbers*. It is straightforward to check that this construction (made for inverse limit sequences) can be as easily done for general filtered systems, a fact that has its own independent interest. Then, all results in [16] can be derived directly for general filtered inverse limits in the same way that for sequences.

1.2. Basic facts on posets and locales

We think of locale theory as a reflection of topos theory (with the poset $2 = \{0, 1\}$ playing the role of the category $\mathscr{S}$ of sets), as well as that of a theory of generalized topological spaces.

We consider a *poset* as a category, and in this vein a *partial order* is a reflexive and transitive relation, not necessarily antisymmetric. We shall refer to the elements of a poset as *objects*.

Given any poset D, the free inf-lattice on D, which we denote $\mathscr{D}(D)$, is furnished with a poset-morphism $D \xrightarrow{\eta} \mathscr{D}(D)$ which is generic in the sense that giving any inf-lattice H, composing with η defines an equivalence of posets $\mathscr{L}ex(\mathscr{D}(D), H) \xrightarrow{\simeq} \mathscr{P}os(D,H)$, where $\mathscr{L}ex(\mathscr{D}(D), H)$ and $\mathscr{P}os(D,H)$ indicate inf-preserving morphisms and poset-morphisms respectively.

We recall now a construction of $\mathscr{D}(D)$.

Proposition 1.2.1. *The objects of* $\mathscr{D}(D)$ *are in one to one correspondence with the finite subsets of* D. *Given a subset* $A = \{a_1, \dots, a_n\} \subset D$, *we denote* $[A] = [\langle a_1 \rangle, \dots, \langle a_n \rangle]$ *the corresponding object in* $\mathscr{D}(D)$. *The morphism* η *is defined by* $\eta(a) = [\langle a \rangle]$. *Given any other* $[B] = [\langle b_1 \rangle, \dots, \langle b_k \rangle]$, *the order relation is given by*

$$\frac{[\langle a_1 \rangle, \dots, \langle a_n \rangle] \leqslant [\langle b_1 \rangle, \dots, \langle b_k \rangle]}{\exists \sigma : \{1, \dots, k\} \to \{1, \dots, n\},\ a_{\sigma i} \leqslant b_i}.$$

A *locale* is a complete lattice in which finite infima distribute over arbitrary suprema. A morphism of locales $E \xrightarrow{f^*} H$ is defined as a function f^* preserving finite infima and arbitrary suprema (notice that we put automatically an upper star to indicate that these arrows are to be considered as inverse images of geometric maps).

Inf-lattices D are sites of definition for locales (rather than bases of opens). 2-valued *presheaves* $D^{\mathrm{op}} \to 2$ form a locale, $D^\wedge = 2^{D^{\mathrm{op}}}$. Given a Grothendieck (pre) topology on D, 2-valued *sheaves* also form a locale, denoted $D^\sim$. The associated sheaf defines a morphism of locales $D^\wedge \to D^\sim$, and this is a procedure in which quotients of locales are obtained. A *site* is, in this sense, a *presentation* of the locale of sheaves.

The basic fundamental result of this construction is the following:

Lemma 1.2.2. *The associated sheaf* $D \xrightarrow{\#} D^\sim$ *is a morphism of sites* (*preserves infima and sends covers into epimorphic families*) *into a locale which is generic*, *in the sense that giving any locale* H, *composing with* $\#$ *defines an equivalence of posets* $\mathscr{M}orph(D^\sim, H) \xrightarrow{\simeq} \mathscr{M}orph(D, H)$.

This lemma is just [1, IV 4.9.4] in the poset context.

A *localic space* is the formal dual of a locale. Thus, $E \xrightarrow{f^*} H$ defines a map or morphism of localic spaces from H to E, $H \xrightarrow{f} E$. Following [14], all these maps are called *continuous maps*.

The *open subspaces* of a localic space E correspond to the objects of the locale E [14, Chapter V, 2]. We shall identify (as an abuse of notation) the object $u \in E$ with the subspace defined by the quotient locale $E \to U$, $w \mapsto w \wedge u$, $U = \{v \mid v \leqslant u\}$. We abuse $u = U$ and indistinctly write $u \subset E$ or $u \in E$.

A *surjection* between localic spaces is a map whose inverse image reflects isomorphisms. It follows immediately from the preservation of infima that f^* is injective (up to isomorphisms). Thus, surjections are epimorphisms in the category of localic spaces. Furthermore, it also follows that f^* is full, in the sense that the implication $(f^* u \leqslant f^* v \Rightarrow u \leqslant v)$ holds.

A *localic monoid*, (resp. *localic group*) is a monoid object (resp. group object) in the category of localic spaces. A *morphism of monoids* (*or groups*) $H \xrightarrow{\varphi} G$ is a continuous map satisfying the usual identities. Actually, all this is given in practice by the inverse image maps between the corresponding locales satisfying the dual equations.

The *locale of relations* $lRel(X, Y)$ between two sets X, Y is the free locale on $X \times Y$. Recall that the free locale on a set S is constructed by taking presheaves on the free inf-lattice on S (the lattice of finite subsets with the dual order, see 1.2.1). If $\{(x_1, y_1) \ldots, (x_n, y_n)\} \subset X \times Y$, we write $[\langle x_1 \mid y_1 \rangle \ldots, \langle x_n \mid y_n \rangle]$ for the corresponding object in the inf-lattice and in the locale. Remark that this object is the finite infimun of the (x_i, y_i) (see [9] for details, there for the case $X = Y$).

Wraith in an inspiring paper [19] defines the locales of functions and of bijections between two sets X and Y by considering the appropriate generators and relations. In our context these relations become covers in the free inf-lattice on $X \times Y$.

The *locale of functions* $lFunc(X, Y)$ from X to Y, is the locale of sheaves for the topology that forces a relation to be a function ([19,9]). It is generated by the following covers (u: univalued) and (e: everywere defined):

$$\text{(u)}\ \emptyset \rightarrow [\langle z \mid x \rangle, \langle z \mid y \rangle] \quad (\textit{each}\ z \in X,\ x \neq y \in Y)$$

$$\text{(e)}\ [\langle z \mid x \rangle] \rightarrow\ 1,\ x \in Y \quad (\textit{each}\ z \in X)$$

The *locale of bijections* $lBij(X, Y)$ is determined if we add the covers which force a function to be bijective (i: injective) and (s: surjective):

$$\text{(i)}\ \emptyset \rightarrow [\langle x \mid z \rangle, \langle y \mid z \rangle] \quad (\textit{each}\ x \neq y \in X,\ z \in Y)$$

$$\text{(s)}\ [\langle x \mid z \rangle] \rightarrow 1,\ x \in X \quad (\textit{each}\ z \in Y)$$

We denote $lAut(X) = lBij(X, X)$.

Given any sets X, Y, corresponding to the basic covers the following equations hold in the locale $lFunc(X, Y)$:

$$\text{(u)}\ [\langle z \mid x \rangle, \langle z \mid y \rangle] = 0 \quad (\textit{each}\ x \neq y,\ z), \quad \text{(e)}\ \bigvee_x [\langle z \mid x \rangle] = 1 \quad (\textit{each}\ z)$$

Two additional equations hold in $lBij(X, Y)$:

$$\text{(i)}\ [\langle x \mid z \rangle, \langle y \mid z \rangle] = 0 \quad (\textit{each}\ x \neq y, z), \quad \text{(s)}\ \bigvee_x [\langle x \mid z \rangle] = 1 \quad (\textit{each}\ z)$$

Notice that we abuse notation and omit to indicate the associated sheaf morphism.

Given any set X, we consider now the group structure on the localic space $lAut(X)$. More generally, it is tedious but straightforward to check the following:

Proposition 1.2.3. *For any sets* X, Y, Z, *there are morphisms of locales*:

$$lFunc(X, Y) \xrightarrow{m^*} lFunc(X, Z) \otimes lFunc(Z, Y), \qquad lFunc(X, X) \xrightarrow{e^*} 2$$

defined on the generators by the following formulae:

$$m^*[\langle x \mid y \rangle] = \bigvee_z [\langle x \mid z \rangle] \otimes [\langle z \mid y \rangle], \qquad e^*[\langle x \mid y \rangle] = 1 \ \Leftrightarrow\ x = y.$$

These data satisfy the equations of an enrichment of the category of sets $\mathscr{S}$ *over the category of localic spaces*, *which we shall denote* $l\mathscr{S}$. *In particular*, *for each set* X, *the localic space* $lFunc(X,X)$ *is a localic monoid.*

The above formulae together with $\imath^*[\langle x \mid y\rangle] = [\langle y \mid x\rangle]$ *also define morphisms of locales*:

$$lBij(X, Y) \xrightarrow{m^*} lBij(X, Z) \otimes lBij(Z, Y), \quad lBij(X, X) \xrightarrow{e^*} 2$$

$$lBij(X, Y) \xrightarrow{\imath^*} lBij(Y, X)$$

which determine an structure of localic groupoid on the (*discrete*) *set of all sets. In particular*, *for each set* X, *the localic space* $lAut(X)$ *is a localic group.*

1.3. The classifying topos of a localic groupoid

Following [19] we now define group actions in terms of the (sub base) generators of the localic group $lAut(X)$:

Definition 1.3.1. Given a localic group G and a set X, an *action* of G on X is a continuous morphism of localic groups $G \xrightarrow{\mu} lAut(X)$. It is completely determined by the value of its inverse image on the generators, $X \times X \xrightarrow{\mu^*} G$. By definition, the following equations hold:

$$m^*\mu^* = (\mu^* \otimes \mu^*)m^*, \quad \mu^*\imath^* = \imath^*\mu^*, \quad e^*\mu^* = e^*$$

We say that the action is *transitive* when for all $x \in X$, $y \in X$, $\mu^*[\langle x \mid y\rangle] \neq 0$.

Given a localic group G, a G-*set* is a set furnished with an action of G.

Definition 1.3.2. Given two G-sets X, Y, $X \times X \xrightarrow{\mu^*} G$, $Y \times Y \xrightarrow{\mu^*} G$, a morphism of G-sets is a function $X \xrightarrow{f} Y$ such as $\mu^*[\langle x \mid y\rangle] \leqslant \mu^*[\langle f(x) \mid f(y)\rangle]$. This defines a category $\mathscr{B}G$ furnished with an underlying set functor $\mathscr{B}G \to \mathscr{S}$ into the category of sets.

Definition 1.3.3. Given a localic group G acting on a set X, and an element $x \in X$, the open subgroup of G, informally described as $\{g \in G \mid gx = x\}$, is defined to be the object $lFix(x) = \mu^*[\langle x \mid x\rangle]$ in the locale G.

Given a morphism between two localic groups $G \xrightarrow{t} H$, and an action of H in a set X, $X \times X \xrightarrow{\mu^*} H$, the composite $X \times X \xrightarrow{\mu^*} H \xrightarrow{t^*} G$ defines an action of G on X. This defines a functor, that we denote $\mathscr{B}(t)^*$, $\mathscr{B}H \to \mathscr{B}G$ (clearly commuting with the underlying sets), and all these assignments are functorial in the appropriate sense.

The transitive G-sets are the connected objects of $\mathscr{B}G$. We shall denote $t\mathscr{B}G$ the full subcategory of nonempty transitive G-sets.

Proposition 1.3.4. *The category* $t\mathscr{B}G$ *is an small category*, *which together with the underlying set functor satisfies* (1) (i) (ii) (iii) (iv) *in Proposition* 3.1.1. *The topos of* *sheaves for the canonical topology is $\mathscr{B}G$, which is then a pointed connected atomic topos. The canonical point, that we denote u, has the inverse image given by the underlying set functor.*

Proof. This is Proposition 8.2 in [9]. □

Proposition 1.3.5. *Given a morphism of localic groups $G \xrightarrow{t} H$, the functor $\mathscr{B}H \xrightarrow{\mathscr{B}(t)^*} \mathscr{B}G$ is the inverse image of a morphism of pointed topoi. If the morphism is a surjection, then, given any transitive H-set X, $\mathscr{B}(t)^*(X)$ is a transitive G-set, and the functor $t\mathscr{B}H \xrightarrow{\mathscr{B}(t)^*} t\mathscr{B}G$ is full and faithful.*

Proof. The first assertion is straightforward and stated in, for example, [18]. The second assertion is immediate: $0\langle\mu^*[\langle x \mid y\rangle]$, then $t^*\mu^*[\langle x \mid y\rangle]$ cannot be equal to 0 since t^* reflects isomorphisms by definition. Finally, let X, Y be any two H-sets and $X \xrightarrow{f} Y$ a morphism for the G-actions, that is, $t^*\mu^*[\langle x \mid y\rangle] \leqslant t^*\mu^*[\langle fx \mid fy\rangle]$. Since inverse images of surjections between locales are full, it follows that $\mu^*[\langle x \mid y\rangle] \leqslant \mu^*[\langle fx \mid fy\rangle]$. □

Given any localic groupoid $\mathscr{G}$, the category of *discrete $\mathscr{G}$ spaces* is defined in a standard way in [14, VIII, 3], and proved therein to be a topos, denoted $\mathscr{B}\mathscr{G}$ (see also [17, 5.2]).

Consider the enrichment $l\mathscr{S}$ of the category of sets over the category of localic spaces, 1.2.3. It is straightforward to check the following:

Proposition 1.3.6. *Given any localic groupoid with discrete set of objects, the category $\mathscr{B}\mathscr{G}$ can be defined as the (ordinary) category of enriched functors $\mathscr{G} \to l\mathscr{S}$ and natural transformations. $\mathscr{B}\mathscr{G} = l\mathscr{S}^{\mathscr{G}}$. In turn, it is straightforward to define these data in the style of Definitions* 1.3.1 *and* 1.3.2. *For each object of the groupoid there is a corresponding evaluation functor $\mathscr{B}\mathscr{G} \to \mathscr{S}$, and these functors (collectively) reflect isomorphisms.*

A localic groupoid with discrete set of objects is said to be *connected* if for each pair of objects p, q, the localic space $\mathscr{G}[p,q]$ is nonempty (equivalently, in the notation of [17], if the morphism $G_1 \xrightarrow{(d_0,d_1)} G_0 \times G_0$ is a surjection). A connected localic groupoid may not be connected as an ordinary groupoid since the localic spaces $\mathscr{G}[p,q]$ can be pointless (see Example 3.5.2).

It is possible to check with the methods of [9] that the tops $\mathscr{B}\mathscr{G}$ is atomic, and that if the groupoid is connected, it is equivalent to $\mathscr{B}G_p$, where $G_p = \mathscr{G}[p,p]$ is any one of its vertex localic groups (notice that the first assertion follows from the second (and (1.3.4), since a sum of atomic topoi is atomic). We omit to do all this in print, and invoke [17] as a proof.

Proposition 1.3.7. *Given any localic groupoid $\mathscr{G}$ with discrete set of objects, the topos $\mathscr{B}\mathscr{G}$ is an atomic topos with enough points (with inverse images given by the* *evaluation functors*). *If the groupoid is connected, then it is a connected topos, equivalent to the classifying topos of any one of its vertex localic groups.*

Proof. It is stated in [17, 4.5 c] that it is an atomic topos. Clearly, it has enough points. The second statement follows immediately from [17, 5.15 (v)] considering the inclusion morphism. □

1.4. Classifying topos and filtered inverse limits

In this subsection we study the behavior of the classifying topos regarding filtered inverse limits of localic groups, see [18], and groupoids. The reader should be aware that for the applications to the representation of Galois topoi, only the particular case of localic limits of discrete groups (or groupoids) is necessary.

Consider a localic group G an open subgroup $u \subset G$. In [18, 1.2], the quotient localic space $G \xrightarrow{\rho} G/u$ is defined as usual in group theory, working formally in the category of localic spaces considered as the formal dual of the category of locales. Then, it is proved that it is a discrete localic space, and that the localic group G has an "obvious" transitive action in the set $Z = [G, 2]$ of its points [18, 2.3]. Furthermore, $u = lFix(z_0)$ for the point $z_0 \in Z$ defined by the composite $e^* \circ \rho^*$ (where $G \xrightarrow{e^*} 2$ is the (co) unit of G). It follows from [9] Proposition 7.9 that any transitive action is of this form.

It is also stated in [18, 2.4] that given any (co) filtered inverse limit of localic groups $G_\alpha \xleftarrow{t_\alpha} G$, the subgroups of G of the form $t_\alpha^*(w)$ for some open subgroup $w \subset G_\alpha$ form a cofinal system of open subgroups of G, in the sense that given any open subgroup $u \subset G$, there exists α and an open subgroup $w \subset G_\alpha$ such that $t_\alpha^*(w) \leqslant u$ (this is due to the fact that the objects of G of the form $t_\alpha^*(w)$ for some α and $w \in G_\alpha$ generate G, as it can be seen, for example, by Theorem 1.1.1 in the context of posets and locales).

We prove now a generalization of a classical result in the theory of profinite topological groups.

Proposition 1.4.1. *Consider a* (*co*) *filtered inverse limit diagram* $G_\alpha \xleftarrow{t_\alpha} G$ *of localic groups with surjective transition morphisms* $G_\beta \xleftarrow{t_{\alpha\beta}} G_\beta$. *Then, given any transitive* G-*set* X, $X \times X \xrightarrow{\mu^*} G$, *the action* μ *factors through some* G_α-*action in the following sense*:

There exists α, *a* G_α-*set* Z, $Z \times Z \xrightarrow{\mu^*} G_\alpha$, *an epimorphism of* G-*sets* $\mathscr{B}(t_\alpha)^*(Z) \xrightarrow{f} X$ (*given by a surjective function* $Z \xrightarrow{f} X$), *and a factorization as follows*:

$$\begin{array}{ccc} Z \times Z & \xrightarrow{f \times f} & X \times X \\ \Big\downarrow \mu^* & \leqslant & \Big\downarrow \mu^* \\ G_\alpha & \xrightarrow{t^*} & G \end{array}$$

Proof. Choose any $x_0 \in X$, and consider the open subgroup $lFix(x_0) = \mu^*[\langle x_0 \mid x_0 \rangle] \in G$. By the remarks preceding this proposition, there exists α, a G_α-set Z, $Z \times Z \xrightarrow{\mu^*} G_\alpha$, and an element $z_0 \in Z$ such that $t_\alpha^* lFix(z_0) = t_\alpha^* \mu^*[\langle z_0 \mid z_0 \rangle] \leqslant lFix(x_0)$. But $t_\alpha^* \mu^*[\langle z_0 \mid z_0 \rangle]$ is the subgroup $lFix(z_0)$ for the action $\mathscr{B}(t_\alpha)^*(Z)$. Since the projection t_α is surjective (see [14]), this action is transitive (cf 1.3.5). The proof finishes then by [9, 7.9]. □

We can improve now a little over [18], where the following theorem is proved in the case of open surjections.

Theorem 1.4.2. *Given a (co)filtered diagram of localic groups and surjective localic group morphisms, and its inverse limit*:

$$G_\alpha \xleftarrow{t_{\alpha\beta}} G_\beta \cdots \longleftarrow G$$

the induced diagram of topoi and topoi morphisms:

$$\mathscr{B}(G_\alpha) \xleftarrow{\mathscr{B}(t_{\alpha\beta})} \mathscr{B}(G_\beta) \cdots \longleftarrow \mathscr{B}G$$

is also an inverse limit diagram.

Proof. We have the situation described in the following diagram:

$$\begin{array}{ccccccc}
t\mathscr{B}G_\alpha & \xhookrightarrow{\mathscr{B}(t_{\alpha\beta})^*} & t\mathscr{B}G_\beta & \cdots\hookrightarrow & \mathscr{C} & \hookrightarrow & t\mathscr{B}G \\
\downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\varepsilon} \\
\mathscr{B}G_\alpha & \xrightarrow{\mathscr{B}(t_{\alpha\beta})^*} & \mathscr{B}G_\beta & \cdots\longrightarrow & \mathscr{C}^\sim & \longrightarrow & \mathscr{B}G
\end{array}$$

Here $\mathscr{C}$ is the inverse limit site (as a category $\mathscr{C}$ is the filtered colimit), and in the top row all functors are full and faithful by Proposition 1.3.5. By (1.1.1) above $\mathscr{C}^\sim$ is the inverse limit of the topoi $\mathscr{B}G_\alpha$. Notice that the topology in $\mathscr{C}$ is induced by the canonical topology of $t\mathscr{B}G$. Then, by 1.4.1, it follows from the comparison lemma [1, Exposé III, 4] that the arrow $\mathscr{C}^\sim \to \mathscr{B}G$ is an equivalence. □

We comment that the corresponding theorem for filtered inverse limits of *discrete groupoids* has been stated and proved with do care by Kennison [15, 4.18]. In the result's statement it is necessary to assume that transition morphisms are *composably onto*, (see [15]). This takes care of the necessary surjectivity of the system at the level of arrows. In view of 1.3.6, the second statement in 1.3.7, and 1.4.2 above, a similar theorem for filtered inverse limits of localic groupoids with discrete sets of objects seems plausible. Abusing rigor, one could say that a corresponding result in the case of arbitrary localic groupoids also holds, but we do not know of any clear proof in print, and it remains an *open problem* to us.

2. Enrichment of set valued functor categories over localic spaces

In this section we do a brief review of the salient features of the construction and properties given in [9] of the locale of automorphism of a set-valued functor. We develop the more general case of natural transformations between two functors. We establish the whole 2-categorical specifications in some cases, and prove some new results. We introduce the localic groupoid of points of a topos, and study its behavior regarding filtered inverse limits. It is pertinent to remark that only the particular case of a system where the points are representable (thus the groupoids discrete, but not their limits) is necessary for the applications to the representation of Galois topoi.

2.1. The localic functor category

Given any category $\mathscr{C}$ and any set-valued functor $F:\mathscr{C}\to\mathscr{S}$, recall that the diagram of F, which we denote Γ_F, is the category whose objects are the elements of the disjoint union of the sets FX, $X\in\mathscr{C}$. That is to say, pairs (x,X) where $x\in FX$. The arrows $(x,X)\xrightarrow{f}(y,Y)$ are maps $X\xrightarrow{f}Y$ such that $F(f)(x)=y$. Given any (z,Z) in Γ_F, there is a natural transformation $[Z,-]\xrightarrow{z^*}F$ defined as follows: given $h\in[Z,X]$, $z^*(h)=F(h)(z)$, and the resulting diagram is a colimit cone (indexed by Γ_F).

Associated with Γ_F, we define a poset, which we denote D_F, identifying all arrows in each hom-set of category Γ_F.

Given any category $\mathscr{C}$ and any pair of set-valued functors $F:\mathscr{C}\to\mathscr{S}$, $G:\mathscr{C}\to\mathscr{S}$, a *natural relation* between F and G is a relation $R\subset F\times G$ in the functor category. That is, it is a family of relations RX on $FX\times GX$, $X\in\mathscr{C}$, such that given any arrow $X\xrightarrow{f}Y$ in $\mathscr{C}$, $(Ff\times Gf)RX\subset RY$. In other terms, it is a family of functions $FX\times GX\xrightarrow{\phi_X}2$ such that $\phi_X\leqslant\phi_Y\circ(Ff\times Gf)$. It is clear that if a natural relation is functional, then it is a natural transformation.

In [9] the locale of natural relations from F to G is constructed and characterized as follows:

Consider the composite of the diagonal functor $\mathscr{C}\to\mathscr{C}\times\mathscr{C}$ with $F\times G$, which we denote $F\Delta G$, $(F\Delta G)(X)=FX\times GX$. Consider the poset $D_{F\Delta G}$ whose objects are the disjoint union of the sets $FX\times GX$, $X\in\mathscr{C}$. The order relation is given by the following rule:

$$\frac{(X,(x_0,x_1))\leqslant(Y,(y_0,\ y_1))}{\exists X\xrightarrow{f}Y\ \ F(f)(x_0)=y_0,G(f)(x_1)=y_1}.$$

Consider then the free inf-lattice $\mathscr{D}(D_{F\Delta G})$ on this poset (see 1.2.1). The locale of presheaves on this lattice is the locale of natural relations from F to G. By introducing in $\mathscr{D}(D_{\Delta FG})$ the appropriate covers, we construct the (quotient) locales of natural transformations and natural bijections.

Given an object X, a pair $(x_0, x_1) \in FX \times GX$ and a finite subset $A \subset D_{F\Delta G}$, we denote

$$[(X, \langle x_0 \,|\, x_1 \rangle), A] = [(X, \langle x_0 \,|\, x_1 \rangle)] \wedge [A]$$

the corresponding object in $\mathscr{D}(D_{F\Delta G})$.

For each $X \in \mathscr{C}$ there is a function $FX \times GX \xrightarrow{\lambda_X} \mathscr{D}(D_{F\Delta G})$ defined by $\lambda_X(x_0, x_1) = [(X, \langle x_0 \,|\, x_1 \rangle)]$.

Proposition 2.1.1.

(1)

(1.1) *The locale $lRel(F,G) = \mathscr{D}(D_{\Delta FG})^\wedge$ of natural relations from F to G is the locale of presheaves on $\mathscr{D}(D_{\Delta FG})$ (we consider this inf-lattice as a site with the empty topology).*

(1.2) *The locale $lFunc(F,G) = \mathscr{D}(D_{\Delta FG})^\sim$ of natural transformations from F to G is the locale of sheaves for the topology on $\mathscr{D}(D_{\Delta FG})$ which forces a natural relation to be functional. This topology is generated by the following basic covers*: (u: *univalued*) *and* (e: *everywere defined*).

(u) $\emptyset \to [(X, \langle z \,|\, x \rangle), (X, \langle z \,|\, y \rangle)]$ *(each X, and each $\in FX$, $x \neq y \in GX$)*

(e) $[(X, \langle z \,|\, x \rangle)] \to 1,\ x \in GX$ *(each X and each $z \in FX$)*

(1.3) *The locale $lBij(F,G) = \mathscr{D}(D_{\Delta FG})^\sim$ of natural bijections is constructed if we add the following covers*: (i: *injective*) *and* (s: *surjective*)

(i) $\emptyset \to [(X, \langle x \,|\, z \rangle), (X, \langle y \,|\, z \rangle)]$ *(each X, and each $x \neq y \in FX$, $z \in GX$)*

(s) $[(X, \langle x \,|\, z \rangle)] \to 1,\ x \in FX$ *(each X and each $z \in GX$)*

The points of these locales are exactly natural relations, natural transformations, and natural bijections respectively.

(2)

(2.1) *The inf-lattice $H = \mathscr{D}(D_{\Delta F})$, together with the functions $\phi_X = \lambda_X$ satisfy the following condition*:

For each $X \in \mathscr{C}$, there is a function $FX \times GX \xrightarrow{\phi_X} \mathscr{D}(D_{F\Delta G})$, such that for each $X \xrightarrow{f} Y$, $\phi_X \leqslant \phi_Y \circ (F(f) \times G(f))$.

(2.2) *The site defined in* 1.2 *satisfy in addition,*

The following families are coverings:

(u) $\emptyset \to \phi_X(z,x) \wedge \phi_X(z,y)$ *(each X, and each $z \in FX$, $x \neq y \in GX$)*

(e) $\phi_X(z,x) \to 1,\ x \in GX$ *(each X and each $z \in FX$)*

(2.3) *The site defined in* 1.3 *satisfy in addition,*

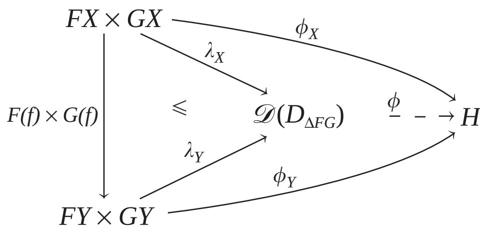
The following families are also coverings:

(i) $\emptyset \to \phi_X(x,z) \wedge \phi_X(y,z)$ *(each X, and each $x \neq y \in FX$, $z \in GX$)*

(s) $\phi_X(x,z) \to 1,\ x \in FX$ *(each X and each $z \in GX$)*

(3) *These sites have, and therefore are characterized by, the following universal property*:

For any other such data, $FX \times GX \xrightarrow{\phi_X} H$, *there is a unique morphism of sites* ϕ (*as indicated in the diagram below*):



such that $\phi \circ \lambda_X = \phi_X$. *If* H *is a locale, then there is a unique morphism of locales* $\mathscr{D}(D_{\Delta FG})^{\sim} \xrightarrow{\phi} H$ *such that* $\phi \circ \#\lambda_X = \phi_X$

Proof. (1) follows immediately from (2) and (3). (2) and (3) by construction of the poset $D_{\Delta FG}$ and the fact that $\mathscr{D}(D_{\Delta FG})$ is the free inf-lattice on this poset. The last statement follows then from Lemma 1.2.2. □

Notice that here (unlike in the case of functions between sets) we do not abuse the notation and indicate the associated sheaf morphisms $\mathscr{D}(D_{\Delta FG}) \to lFunc(F,G)$ and $\mathscr{D}(D_{\Delta FG}) \to lBij(F,G)$ with the symbol $'\#'$.

Given a natural transformation $F \xrightarrow{\sigma} G$, the corresponding point $lFunc(F,G) \xrightarrow{\sigma^*} 2$ is characterized by

$$\sigma^*\#[(X, \langle x_0 \,|\, x_1 \rangle)] = 1 \quad \Leftrightarrow \quad \sigma X(x_0) = x_1$$

Next, we prove the localic version of Yoneda's Lemma.

Lemma 2.1.2. *Given any set valued functor* $F : \mathscr{C} \to \mathscr{S}$, *and any object* $A \in \mathscr{C}$, *functions* $[A,X] \times FX \xrightarrow{\phi_X} FA$ *defined by*

Given $A \xrightarrow{x} X$, $y \in FX$: $\phi_X(x,y) = \{a \in FA \,|\, Fx(a) = y\}$

induce an isomorphism of locales $\phi : lFunc([A,-],F) \xrightarrow{\cong} FA$ (*where FA denotes the discrete locale on the set FA*).

Proof. Given $X \xrightarrow{f} Y$, the equation $\phi_X \leqslant \phi_Y \circ (F(f) \times G(f))$ is immediate to verify. It also becomes clear after inspection that the fact that Fx is a function implies that the covering Conditions 2.2 in Proposition 2.1.1 are satisfied. Thus, it follows there is a (unique) morphism of locales $\phi : lFunc([A,-],F) \xrightarrow{\cong} FA$ such that $\phi \circ \#\lambda_X = \phi_X$.

We define a morphism of locales λ in the other direction by:

$$\textit{Given } I \subset FA: \quad \lambda(I) = \bigvee_{a \in I} \#\lambda_A(id_A, a) = \bigvee_{a \in I} \#[(A, \langle id_A \mid a \rangle)]$$

The reader can check that conditions (1.2.u) and (1.2.e) in 2.1.1 imply that λ preserves "$\wedge$" and "1", respectively. Since it clearly preserves "$\bigvee$", we have that λ is a morphism of locales. Using this, we now show that λ is the inverse of ϕ.

Equation $\phi \circ \lambda = id$): It is enough to show for each $a \in FA$ that $\phi(\lambda\{a\}) = \{a\}$. But $\phi(\lambda\{a\}) = \phi\#\lambda_A(id_A, a) = \phi_A(id_A, a) = \{a\}$, which is clear.

Equation $\lambda \circ \phi = id$): It is enough to show for each $X \in \mathscr{C}$ the equation $\lambda \circ \phi_X = \#\lambda_X$. That is, for each $A \xrightarrow{x} X$ *and* $y \in FX$: $\lambda \circ \phi_X(x, y) = \#\lambda_X(x, y)$:

$$\lambda \circ \phi_X(x, y) = \lambda\{a \in FA \mid Fx(a) = y\} = \bigvee_{a \in FA \mid Fx(a) = y} \#[(A, \langle id_A \mid a \rangle)].$$

On the other hand, we have by 2.1.1 (1.2.e), $1 = \bigvee_{a \in FA} \#[(A, \langle id_A \mid a \rangle)]$. Taking infimun against $\#[(X, \langle x \mid y \rangle)]$ it follows

$$\#[(X, \langle x \mid y \rangle)] = \bigvee_{a \in FA} \#[(A, \langle id_A \mid a \rangle)] \wedge \#[(X, \langle x \mid y \rangle)].$$

But $\#[(A, \langle id_A \mid a \rangle)] \leqslant \#[(X, \langle x \mid Fx(a) \rangle)]$. Thus:

$$\#[(A, \langle id_A \mid a \rangle)] \wedge \#[(X, \langle x \mid y \rangle)] = \#[(A, \langle id_A \mid a \rangle)] \quad \text{if } Fx(a) = y$$

$$\#[(A, \langle id_A \mid a \rangle)] \wedge \#[(X, \langle x \mid y \rangle)] = 0 \quad \text{if } Fx(a) \neq y$$

(the second equation using Proposition 2.1.1 (1.2.u)). It follows then

$$\#\lambda_X(x,\ y) = \#[(X, \langle x \mid y \rangle)] = \bigvee_{a \in FA \mid Fx(a) = y} \#[(A, \langle id_A \mid a \rangle)].$$

This finishes the proof of the equation $\lambda \circ \phi = id$. □

As usual, given any two objects $A, B \in \mathscr{C}$, it follows there is an isomorphism of locales $\phi : lFunc([A, -], [B, -]) \xrightarrow{\cong} [B, A]$. We also have:

Lemma 2.1.3. *Given any category* $\mathscr{C}$ *and any two objects* $A, B \in \mathscr{C}$, *the functions* $[A, X] \times [B, X] \xrightarrow{\phi_X} Iso[B, A]$ *defined by*

$$\textit{Given } A \xrightarrow{x} X,\ A \xrightarrow{y} X: \quad \phi_X(x,\ y) = \{B \xrightarrow{a} A \mid a \textit{ iso and } xa = y\}$$

induce an isomorphism of locales $\phi : lBij([A, -], [B, -]) \xrightarrow{\cong} Iso[B, A]$ (*where* $Iso[B, A]$ *denotes the discrete locale on the set of isomorphisms from* B *to* A).

In particular, we have an isomorphism of locales $\phi : lAut([A, -]) \xrightarrow{\cong} Aut(A)^{\mathrm{op}}$ (*where* $Aut(A)$ *denotes the discrete locale on the set of automorphisms of* A, *and the* $'op'$ *indicates that there is a reversal of arrows in this last set*).

Proof. As in the proof of Lemma 2.1.2, the equation $\phi_X \leqslant \phi_Y \circ (F(f) \times G(f))$ is immediate to verify. We leave the reader to inspect that the two additional covering conditions 2.3 in Proposition 2.1.1 follow readily from the fact that the morphisms $B \xrightarrow{a} A$ are isomorphisms. Thus, it follows there is a (unique) morphism of locales $\phi : lBij([A,-],[B,-]) \to Iso[B,A]$ such that $\phi \circ \#\lambda_X = \phi_X$. We define a morphism of locales λ in the other direction as in 2.1.2:

$$\textit{Given } I \subset Iso[B,A]: \quad \lambda(I) = \bigvee_{a \in I} \#\lambda_A(id_A, a) = \bigvee_{a \in I} \#[(A, \langle id_A \mid a \rangle)]$$

As in 2.1.2 it is straightforward to check that λ is a morphism of locales. Using this, we now show that λ is the inverse of ϕ.

Equation $\phi \circ \lambda = id$): same proof that in 2.2.2.

Equation $\lambda \circ \phi = id$): as in 2.1.2 it is enough to show, for each $X \in \mathscr{C}$, $A \xrightarrow{x} X$ and $B \xrightarrow{y} X$, the equation $\lambda \circ \phi_X(x,y) = \#\lambda_X(x,y)$:

$$\lambda \circ \phi_X(x,y) = \lambda\{B \xrightarrow{a} A \mid a \ iso, \ xa = y\} = \bigvee_{B \xrightarrow{a} A \mid a \ iso, \ xa=y} \#[(A, \langle id_A \mid a \rangle)].$$

On the other hand, by the same reasoning as in 2.1.2 we have

$$\#\lambda_X(x,y) = \#[(X, \langle x \mid y \rangle)] = \bigvee_{B \xrightarrow{a} A \mid xa=y} \#[(A, \langle id_A \mid a \rangle)].$$

Thus, to finish the proof we have to show that if $B \xrightarrow{a} A$ is not an isomorphism, then $\#[(A, \langle id_A \mid a \rangle)] = 0$. We do this as follows:

Notice that $B \xrightarrow{a} A$ is an isomorphism if and only if it is an epimorphism and it has a left inverse $A \xrightarrow{x} B$, $xa = id_B$. Thus, that a is not an isomorphism means the following:

$$(\exists A \overset{y}{\underset{x}{\rightrightarrows}} X \mid x \neq y, \ xa = ya) \quad or \quad (\forall A \xrightarrow{x} B, \ xa \neq id_B).$$

Assume the first statement: By Proposition 2.1.2 (1.3.i) it follows:

$$\#[(A, \langle id_A \mid a \rangle)] \leqslant \#[(X, \langle x \mid xa \rangle), \ (X, \langle y \mid ya \rangle)] = 0.$$

Assume the second statement: By Proposition 2.1.2 (1.3.s),

$$1 = \bigvee_{A \xrightarrow{x} B} \#[(B, \langle x \mid id_B \rangle)].$$

Thus

$$\#[(A, \langle id_A \mid a \rangle)] = \bigvee_{A \xrightarrow{x} B} \#[(B, \langle x \mid id_B \rangle), (A, \langle id_A \mid a \rangle)].$$

But

$$\#[(B, \langle x \mid id_B \rangle), (A, \langle id_A \mid a \rangle)] \leqslant \#[(B, \langle x \mid id_B \rangle), (B, \langle x \mid xa \rangle)]$$

which is equal to 0 for all $A \xrightarrow{x} B$ by Proposition 2.1.2 (1.2.u). $\square$

Given any set valued functor $\mathscr{C} \xrightarrow{F} \mathscr{S}$, the localic space $lAut(F)$ is a localic group, and this group acts on each set FX.

More generally, given any two set valued functors $F, G : \mathscr{C} \to \mathscr{S}$ and any object $X \in \mathscr{C}$, the map $FX \times GX \xrightarrow{\lambda_X} \mathscr{D}(D_{F\Delta G})$ determines morphisms of locales $lRel(FX, GX) \xrightarrow{\lambda_X^*} lRel(F, G)$, $lFunc(FX, GX) \xrightarrow{\lambda_X^*} lFunc(F, G)$, and $lBij(FX, GX) \xrightarrow{\lambda_X^*} lBij(F, G)$, defined by $\lambda_X^*[\langle x_0 \,|\, x_1 \rangle] = \#[(X, \langle x_0 \,|\, x_1 \rangle)]$. These assertions follow since the map $FX \times GX \xrightarrow{\lambda_X} \mathscr{D}(D_{F\Delta G})$ sends covers into covers on the respective sites of definition.

It is straightforward to check from Proposition 1.2.3 the following:

Proposition 2.1.4. *For any set valued functors $F, G, H : \mathscr{C} \to \mathscr{S}$, there are morphisms of locales*:

$$lFunc(F, G) \xrightarrow{m^*} lFunc(F, H) \otimes lFunc(H, G), \qquad lFunc(F,) \xrightarrow{e^*} 2$$

defined on the generators by the following formulae:

$$m^*[X, \langle x \,|\, y \rangle] = (\lambda_X^* \otimes \lambda_X^*) m_X^*[\langle x \,|\, y \rangle], \qquad e^*[X, \langle x \,|\, y \rangle] = \lambda_X^* e_X^*[\langle x \,|\, y \rangle]$$

where m_X^* and e_X^* are the morphisms defined in Proposition 1.2.3.

These data satisfy the equations of an enrichment of the functor category $\mathscr{S}^{\mathscr{C}}$ over the category of localic spaces, which we denote $lFunc(\mathscr{S}^{\mathscr{C}})$. Clearly the evaluation functor $\mathscr{S}^{\mathscr{C}} \xrightarrow{ev_X} \mathscr{S}$, $ev_X(F) = FX$ becomes a functor for the enriched structures (*by their very definition*). *This defines a* (*ordinary*) *functor into the* (*ordinary*) *category of enriched functors, which we denote μ^*, $X \mapsto ev_X$, $\mathscr{C} \xrightarrow{\mu^*} l\mathscr{S}^{lFunc(\mathscr{S}^{\mathscr{C}})}$.*

The above formulae together with $\imath^*[X, \langle x \,|\, y \rangle] = \lambda_X^* \imath_X^*[\langle x \,|\, y \rangle]$ define also morphisms of locales:

$$lBij(F, G) \xrightarrow{m^*} lBij(F, H) \otimes lBij(H, G), \qquad lBij(F, F) \xrightarrow{e^*} 2$$

$$lBij(F, G) \xrightarrow{\imath^*} lBij(G, F)$$

which determine a structure of localic groupoid on the (*discrete*) *set of objects of $\mathscr{S}^{\mathscr{C}}$, which we denote $lBij(\mathscr{S}^{\mathscr{C}})$. As before, the evaluation functor becomes enriched and defines an* (*ordinary*) *functor into the* (*ordinary*) *category of enriched functors, which we also denote μ^*, $X \mapsto ev_X$, $\mathscr{C} \xrightarrow{\mu^*} l\mathscr{S}^{lBij(\mathscr{S}^{\mathscr{C}})}$.*

Given a functor $\mathscr{C} \xrightarrow{T} \mathscr{H}$, any two set valued functors $F, G : \mathscr{H} \to \mathscr{S}$, and any object $X \in \mathscr{C}$, there are morphisms of locales $lRel(FT, G) \xrightarrow{T^*} lRel(F, G)$, $lFunc(FT, GT) \xrightarrow{T^*} lFunc(F, G)$, and $lBij(FT, GT) \xrightarrow{T^*} lBij(F, G)$, induced by morphisms of sites $\mathscr{D}(D_{FT\Delta GT}) \xrightarrow{T^*} \mathscr{D}(D_{F\Delta G})$ defined on the generators by the following formula: $T^*[X, \langle x \,|\, y \rangle] = [(TX, \langle x \,|\, y \rangle)]$ (that is, $T^* \lambda_X = \lambda_{TX}$). It is straightforward to check in 2.1.1 that this map send covers into covers. Furthermore, it is also straightforward to check that these data determine enriched functors $lFunc(\mathscr{S}^{\mathscr{H}}) \xrightarrow{T^*} lFunc(\mathscr{S}^{\mathscr{C}})$, $lBij(\mathscr{S}^{\mathscr{H}}) \xrightarrow{T^*} lBij(\mathscr{S}^{\mathscr{C}})$, for the (co)-structure defined in 2.1.4. Finally, it is clear that all this is contravariantly functorial in the variable $\mathscr{C}$. In all, this finishes the proof of (compare with 2.3.3):

Proposition 2.1.5. *The assignments of the localic category $lFunc(\mathscr{S}^{\mathscr{C}})$ and of the localic groupoid $lBij(\mathscr{S}^{\mathscr{C}})$ (with discrete set of objects) are contravariantly functorial on $\mathscr{C}$, into the category of localic categories and localic groupoids respectively.*

It follows immediately:

Proposition 2.1.6. *The assignments of the categories of enriched functors together with the functor μ^* in 2.1.4 are functorial in $\mathscr{C}$ in such a way that μ^* becomes a natural transformation, $\mathscr{C} \xrightarrow{\mu^*} l\mathscr{S}^{lFunc(\mathscr{S}^{\mathscr{C}})}$, $\mathscr{C} \xrightarrow{\mu^*} l\mathscr{S}^{lBij(\mathscr{S}^{\mathscr{C}})}$.*

2.2. The localic groupoid of points of a topos

Given any topos $\mathscr{E}$, consider a site $\mathscr{C}$ such that $\mathscr{E} = \mathscr{C}^{\sim}$. The usual equivalence between the category of points of the site (that is, set valued flat and continuous functors) and the category of (inverse images of) points of the topos of sheaves $\mathscr{E} = \mathscr{C}^{\sim}$, induces an enriched structure in the category and in the groupoid of points of $\mathscr{E}$. We define the *localic groupoid of points* of a topos $\mathscr{E} = \mathscr{C}^{\sim}$ (which will be meaningful in general only when the topos has sufficiently many points) to be $l\mathscr{P}oints(\mathscr{E}) \subset lBij(\mathscr{S}^{\mathscr{C}})^{op}$ (where "$\subset$" indicates the enriched structure induced on the (full) subgroupoid whose objects are the points of the site, changing the variance as in [1]). Notice that given any two points f, g, we can define directly this localic groupoid setting $l\mathscr{P}oints(\mathscr{E})[f, g] = lBij(g^*, f^*)$. The given definition does not add rigor, but the reason to do so is that it makes sense over an arbitrary base topos $\mathscr{S}$. In the same way we define the *localic category of points*.

In this terminology and notation, from Propositions 2.1.5 and 2.1.6 it follows (recall that morphisms of topoi go on the other way to the inverse image functors):

Proposition 2.2.1. *Let $\mathscr{E}$ be any topos. Then the assignment of the groupoid $l\mathscr{P}oints(\mathscr{E})$ is functorial in $\mathscr{E}$, into the category of localic groupoids (with discrete set of objects) and morphisms of localic groupoids. Furthermore, there is a geometric morphism of topoi μ, $\mathscr{B}(l\mathscr{P}oints(\mathscr{E})^{op}) \xrightarrow{\mu} \mathscr{E}$, whose inverse image is given by $\mu^*(X) = ev_X$, and μ is natural in $\mathscr{E}$.*

Proof. The only point that needs some care is the existence of the geometric morphism μ. Consider any site $\mathscr{C}$, $\mathscr{E} = \mathscr{C}^{\sim}$. Clearly we have $\mathscr{C} \xrightarrow{\mu^*} \mathscr{B}(l\mathscr{P}oints(\mathscr{E})^{op})$. Notice that the family of points of $\mathscr{B}(l\mathscr{P}oints(\mathscr{E})^{op})$ corresponding to evaluation functors is surjective (1.3.6). Then, it readily follows that μ^* is flat and continuous. $\square$

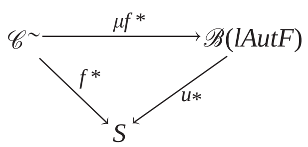
2.3. The localic group of a pointed topos and filtered inverse limits

We are interested in the functor $l\mathscr{P}oints$ behavior regarding filtered inverse limits, but we left it for another occasion (or the interested reader) the development of the general theory (however, see comment 3.5.1). We now develop in detail the particular case of pointed topoi and the localic group of automorphism of the point. We also take care of some necessary 2-categorical aspects, and for this purpose we first review the results of the previous section in this particular case.

We have in particular for each point $\mathscr{S} \xrightarrow{f} \mathscr{E}$, with $\mathscr{E} = \mathscr{C}^{\sim}$, $\mathscr{C} \xrightarrow{F} \mathscr{S}$, $F = f^*|_{\mathscr{C}}$, the following:

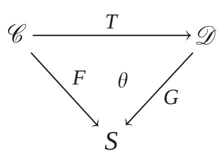
Proposition 2.3.1. *For any set valued functor* F, $\mathscr{C} \xrightarrow{F} \mathscr{S}$, *the localic space* $lAut(F)$ *is a localic group which has an action on the set* FX *for each* X, $FX \times FX \xrightarrow{\mu^*} lAut(F)$, *given by*: $\mu^*(x_0, x_1) = \#[(X, \langle x_0 \,|\, x_1 \rangle)]$, *and given any arrow* $X \xrightarrow{f} Y$, *the function* $FX \xrightarrow{F(f)} FY$ *becomes a morphism of actions* (*see* [9], 4.8). *This defines a lifting of* F, *which we denote* μF, $\mathscr{C} \xrightarrow{\mu F} \mathscr{B}(lAutF)$.

Proposition 2.3.2. *Let* $F : \mathscr{C} \to \mathscr{S}$ *be any pointed site* (*that is*, F *is a flat and continuous functor inducing a point of the topos* $\mathscr{S} \xrightarrow{f} \mathscr{C}^{\sim}$). *Then, the lifting of* F *defined in Proposition* 2.3.1 *induces a morphism of topoi, which we denote* μf, *commuting with the points*:



Proof. Notice that the canonical point of $\mathscr{B}(lAutF)$ is a surjection. Then, it readily follows then that the lifting $\mathscr{C} \xrightarrow{\mu F} \mathscr{B}(lAutF)$ is flat and continuous. $\square$

Proposition 2.3.3. *Given any two set valued functors related as in the diagram*



$, F \xrightarrow{\theta} GT$ (*isomorphism*).

(1) *There is a morphism of localic groups* $lAut(G) \xrightarrow{lAut(T)} lAut(F)$ (*induced by a morphism of sites as described below*).

(2) *There is a morphism of sites (which we denote in the same way)* $\mathscr{D}(D_{\Delta F}) \stackrel{lAut(T)}{\longrightarrow} \mathscr{D}(D_{\Delta G})$ *defined by*

$$aut(T)\,[(X, \langle x_0 \,|\, x_1 \rangle)] = [(TX, \langle \theta X(x_0) \,|\, \theta X(x_1)\rangle)].$$

(3) *If the functor T has a left adjoint S, $id \stackrel{\eta}{\longrightarrow} TS$, $ST \stackrel{\epsilon}{\longrightarrow} id$, then there is a natural transformation $G \stackrel{\sigma}{\longrightarrow} FS$ which defines a morphism of inf-lattices* $\mathscr{D}(D_{\Delta F}) \stackrel{aut(S)}{\longrightarrow} \mathscr{D}(D_{\Delta G})$ *by the formula;*

$$aut(S)\,[(X, \langle x_0 \,|\, x_1 \rangle)] = [(SX, \langle \sigma X(x_0) \,|\, \sigma X(x_1)\rangle)].$$

Furthermore, $aut(S)$ is left adjoint to $aut(T)$.

Proof. (1) The map between the localic spaces is given by the morphism of sites defined in (2). It easily follows from this definition and the definition of the (co)group structure (2.3.1, 2.1.4) that the inverse image preserves this (co)structure.

(2) We shall use 2.1.1. Let $aut(T)^*_X : FX \times FX \to \mathscr{D}(D_{\Delta G})$ be defined by

$$aut(T)_X(x_0, x_1) = \lambda_{TX}((\theta X \times \theta X)(x_0, x_1)) = [(TX, \langle \theta X(x_0) \,|\, \theta X(x_1)\rangle)].$$

Given $X \stackrel{f}{\longrightarrow} Y$, by naturality of θ it follows $aut(T)_X \leqslant t_Y \circ (F(f) \times F(f))$. So, it remains to check condition (ii) in (2) of 2.1.1. But this immediately follows since θX is a bijection (for the two basic empty covers use injectivity, and for the two basic covers of 1 use surjectivity)

(3) We shall use 2.1.1. Let $G \stackrel{\sigma}{\longrightarrow} FS$ be the composite $G \stackrel{G\eta}{\longrightarrow} GTS \stackrel{\theta^{-1}}{\longrightarrow} FS$. Then, as before, define $aut(S)_X : GX \times GX \to \mathscr{D}(D_{\Delta F})$ by

$$aut(S)_X(x_0, x_1) = \lambda_{SX}((\sigma X \times \sigma X)(x_0, x_1)) = [(SX, \langle \sigma X(x_0) \,|\, \sigma X(x_1)\rangle)].$$

It remains to prove that $aut(S)_X$ is left adjoint to $aut(T)_X$. This follows because ϵ and η actually define arrows in the poset $\mathscr{D}(D_{\Delta F})$. That this is the case amounts to the validity of the equations $F\epsilon \circ \sigma T \circ \theta = id$, and $\theta S \circ \sigma = G\eta$. We verify this now:

$$(F \stackrel{\theta}{\longrightarrow} GT \stackrel{\sigma T}{\longrightarrow} FST \stackrel{F\epsilon}{\longrightarrow} F) = (F \stackrel{\theta}{\longrightarrow} GT \stackrel{G\eta T}{\longrightarrow} GTST \stackrel{\theta^{-1}ST}{\longrightarrow} FST \stackrel{F\epsilon}{\longrightarrow} F)$$

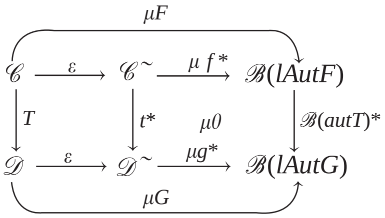
$$=(F \stackrel{\theta}{\longrightarrow} GT \stackrel{G\eta T}{\longrightarrow} GTST \stackrel{GT\epsilon}{\longrightarrow} GT \stackrel{\theta^{-1}}{\longrightarrow} F) = (F \stackrel{\theta}{\longrightarrow} GT \stackrel{\theta^{-1}}{\longrightarrow} F) = (F \stackrel{id}{\longrightarrow} F).$$

The first equality by definition of σ, the second by naturality, the third by the triangular equation of the adjointness, and the fourth is obvious.

$$(G \stackrel{\sigma}{\longrightarrow} FS \stackrel{\theta S}{\longrightarrow} GTS) = (G \stackrel{G\eta}{\longrightarrow} GTS \stackrel{\theta^{-1}S}{\longrightarrow} FS \stackrel{\theta S}{\longrightarrow} GTS) = (G \stackrel{G\eta}{\longrightarrow} GTS).$$

The first equality by definition of σ, and the second is obvious. □

Proposition 2.3.4. *In the situation of Proposition* 2.3.3, *assume* $\mathscr{C}$ *and* $\mathscr{D}$ *are sites, and* T *a morphism of sites. Then, there is a natural isomorphism* $\mathscr{B}(autT)^* \circ \mu f^* \xrightarrow{\mu\theta} \mu g^* \circ t^*$ *as indicated in the following diagram*:



Proof. It is enough to define $\mu\theta$ as a natural transformation $\mathscr{B}(autT)^* \circ \mu F \xrightarrow{\mu\theta} \mu G \circ T$. In order to do this, just check that given any object $X \in \mathscr{C}$, the bijective function $FX \xrightarrow{\theta X} GTX$ is actually a morphism of actions. □

Triangles of set valued functors compose in the obvious way, and it is straightforward to check that the constructions in the two propositions above are functorial in the appropriate way. More precisely:

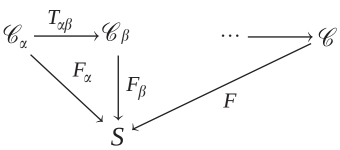
Proposition 2.3.5. *Given two triangles and its composition*:

fill respectively with natural isomorphisms ξ, θ, *and* κ (*where* $S = T \circ R$ *and* $\kappa = \theta R \circ \xi$).

Then, $lAut(S)^* = lAut(T)^* \circ lAut(R)^*$, *and* $\mu\kappa = \mu\theta r^* \circ \mathscr{B} lAut(T)^* \mu\xi$.

With this, we can state and prove the behavior regarding filtered inverse limits.

Proposition 2.3.6. *Consider a filtered system of set valued functors and its colimit as indicated in the diagram below*:



Assume the triangles fill with natural isomorphisms $\theta_{\alpha\beta} : F_\alpha \to F_\beta T_{\alpha\beta}$ *subject to the compatibility conditions* $\theta_{\alpha\gamma} = \theta_{\beta\gamma} T_{\alpha\beta} \circ \theta_{\alpha\beta}$ (*it follows there are also natural isomorphisms* $\theta_\alpha : F_\alpha \to FT_\alpha$, *where* $\mathscr{C}_\alpha \xrightarrow{T_\alpha} \mathscr{C}$ *are the inclusions into the colimit*).

Then:

(1) *The induced* (*by* 2.3.3. (1)) *cofiltered system of localic groups*:

$$lAut(F_\alpha) \xleftarrow{t_{\alpha\beta}} lAut(F_\beta) \cdots \longleftarrow lAut(F).$$

is a cofiltered inverse limit of localic groups.

(2) *The induced* (*by* 2.3.3. (2)) *filtered system of inf-lattices and site morphisms*:

$$\mathscr{D}(D_{\Delta F_\alpha}) \xrightarrow{t_{\alpha\beta}} \mathscr{D}(D_{\Delta F_\beta}) \cdots \longrightarrow \mathscr{D}(D_{\Delta F}).$$

is a filtered colimit of inf lattices, and the topology in $\mathscr{D}(D_{\Delta F})$ *is the coarsest that makes the arrows* $\mathscr{D}(D_{\Delta F_\alpha}) \xrightarrow{t_\alpha} \mathscr{D}(D_{\Delta F})$ *continuous.*

Proof. (1) Follows immediately from (2) by Lemma 1.2.2.

(2) Here is where the filtering condition is necessary. An object of $\mathscr{C}$ is a germ of objects. That is, it is a pair (X, α), with $X \in \mathscr{C}_\alpha$, two such pairs being considered equal if they become equal further on the system. An arrow between two germs is an arrow at some point in the system, two such arrows being considered equal if they become equal further on the system. From this it readily follows that the objects of the inf-lattice $\mathscr{D}(D_{\Delta F})$ are germs of objects, and that the order relation is what it should be. This shows that $\mathscr{D}(D_{\Delta F})$ is the filter colimit of the inf-lattices $\mathscr{D}(D_{\Delta F_\alpha})$. It is immediate that the covers that generate the topology in $\mathscr{D}(D_{\Delta F})$ are just the ones that generate the coarsest topology which makes the arrows t_α continuous. □

From Theorems 1.1.1 and 1.4.2 follows:

Proposition 2.3.7. *In the situation of Proposition* 2.3.6, *assume that each* $\mathscr{C}_\alpha$ *is a site, each* T_α *a morphism of sites, and* $\mathscr{C}$ *the inverse limit site* (*cf.* 1.1.1). *Assume furthermore that the transition morphisms* $lAut(T_{\alpha\beta})$ *given by Proposition* 2.3.3 *are surjections. Then, in the following diagram the two bottom rows are inverse limit diagrams of topoi* (*where we use also the notation in Propositions* 2.3.2 *and* 2.3.4).

$$\begin{array}{ccccccccc}
\mathscr{C}_\alpha & \xrightarrow{T_{\alpha\beta}} & \mathscr{C}_\beta & \cdots\longrightarrow & \mathscr{C} & & \mathscr{C}_\alpha & \xrightarrow{T_\alpha} & \mathscr{C} \\
\downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon} \\
\mathscr{C}_\alpha^\sim & \xrightarrow{t_{\alpha\beta}^*} & \mathscr{C}_\beta^\sim & \cdots\longrightarrow & \mathscr{C}^\sim & & \mathscr{C}_\alpha^\sim & \xrightarrow{t_\alpha^*} & \mathscr{C}^\sim \\
\downarrow{\scriptstyle f_\alpha^*} & \mu\theta_{\alpha\beta} & \downarrow{\scriptstyle f_\beta^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f_\alpha^*} & \mu\theta_\alpha & \downarrow{\scriptstyle f^*} \\
\mathscr{B}lAut(F_\alpha) & \xrightarrow{lAut(T_{\alpha\beta})^*} & \mathscr{B}lAut(F_\beta) & \cdots\longrightarrow & \mathscr{B}lAut(F) & & \mathscr{C}_\alpha & \xrightarrow{lAut(T_\alpha)^*} & \mathscr{C}
\end{array}$$

3. The fundamental theorems of Galois theory

The fundamental theorems of Galois theory are representation theorems for certain types of atomic topoi. We distinguish three cases in this paper: the discrete case, corresponding to the classical Galois theory, the prodiscrete case, corresponding to Grothendieck's Galois theory, and the general localic case, that we call localic Galois theory.

3.1. Pointed connected atomic sites

From the characterization of atomic sites given in [3] it is easy to check the following:

Proposition 3.1.1. *Let* $\mathscr{E}$ *be a topos with a point* $\mathscr{S} \xrightarrow{f} \mathscr{E}$, *and* $\mathscr{C} \subset \mathscr{E}$ *be a* (*small*) *full subcategory such that together with the canonical topology is a pointed site* $\mathscr{C} \xrightarrow{F} \mathscr{S}$ *for* $\mathscr{E}$, $\mathscr{E} = \mathscr{C}^{\sim}$, $F = f^*|_{\mathscr{C}}$. *Then*:

(1) *If* $\mathscr{E}$ *is a pointed connected atomic topos, a site as above can be chosen so that*:
   (i) *Every arrow* $Y \to X$ *in* $\mathscr{C}$ *is an strict epimorphism.*
   (ii) *For every* $X \in \mathscr{C}$ $FX \neq \emptyset$.
   (iii) F *preserves strict epimorphisms.*
   (iv) *The diagram of* F, Γ_F, *is a cofiltered category.*
(2) *Given any pointed site as in* (1), *the topos of sheaves is a pointed connected atomic topos.*

Condition (ii) *is equivalent to the connectedness of* $\mathscr{E}$. *The category* $\mathscr{C}$ *can be taken to be the full subcategory of non-empty connected objects, but not necessarily so.*

The following two propositions are easy to prove (see [9]).

Proposition 3.1.2. *The natural transformations* $[Z, -] \xrightarrow{z^*} F$ *are all injective, and the diagram of* F, Γ_F, *is a cofiltered poset.*

Proposition 3.1.3. *The functor* F *is faithful* (*and reflects isomorphisms*).

3.2. Discrete Galois theory

Discrete Galois theory corresponds exactly to Artin's interpretation of the classical Galois theory of roots of a polynomial with coefficients in a field. We call this theory *Galois' Galois theory*, and its fundamental theorem can be proved by elementary category methods (see [9]). The topos theoretical setting of this theory corresponds to the situation described in 3.1.1 when the diagram Γ_F of the functor F has a (co) final (i.e. initial) object, or, equivalently, the inverse image functor of the point is representable. This means (see 2.1.3) that the localic group $lAutF$ is isomorphic to the discrete group $Aut(A)^{\mathrm{op}}$, where $A \in \mathscr{C}$ is any representing object. In this case, the object A is a *universal covering* and the topos $\mathscr{E}$ in 3.1.1 is said to be *locally simple connected* (see [5], where this notion was first investigated in detail in the topos setting). Notice that since A is in $\mathscr{C}$, it is a cover, that is $A \to 1$ is an epimorphism (this is characteristic of the connected situation).

Proposition 3.2.1. *If A is a representing object of F, every arrow $X \xrightarrow{f} A$ is an isomorphism. In particular, every endomorphism of A is an isomorphism, $Aut(A) = [A,A]$, and if $G = [B,-]$ is any other representable point, $A \cong B$.*

Proof. By 3.1.1 (i) and (iii), it follows that there is $A \xrightarrow{g} X$ such that $fg = id$. Then, g is a monomorphism. Since by 3.1.1 (i) it is also a strict epimorphism, it follows that it is an isomorphism, and consequently so is f. □

Let $\vartheta : [A,-] \xrightarrow{\cong} F$, with $A \in \mathscr{C}$ be a representation of F, and let $a = \vartheta A(id_A) \in FA$. The pair (A,a) is an initial object in the diagram of F, and given any $x \in [A,X]$, $\vartheta X(x) = F(x)(a)$. We have:

Proposition 3.2.2. *The object A is a Galois object (see Section* A.2*) and every object X is A-split with the set $[A,X]$. We have $\mathscr{E} = Split(A) \xleftarrow{\cong} \mathscr{P}_A$ in such a way that the given point of $\mathscr{E}$ corresponds by the representing isomorphism ϑ with the canonical point of $\mathscr{P}_A$.*

Proof. Since A represents a point, it is connected. Notice now that it is enough to prove the statement for connected objects X. Let $\theta : \gamma^*[A,X] \times A \to X \times A$ be the arrow which corresponds under the adjunctions $\gamma^* \dashv \gamma_*$ and $(-) \times A \dashv (-)^A$ to the arrow $[A,X] \to [A, X \times A]$, defined by $x \mapsto (x, id_A)$. It can be seen that $F(\theta) : [A,X] \times FA \to FX \times FA$ is given by $(x,\ y) \mapsto (F(x)(a),\ y)$, that is $F(\theta) = (\vartheta X, id_{FA})$. Thus, $F(\theta)$ is a bijection, and the proof finishes by 3.1.3 (recall that by Assumption 3.1.1 (i) we already know that $A \to 1$ is a cover). □

Notice that since A represents a point, it is not only a Galois object, (thus a connected covering), but it is also projective, which means that it is a universal covering.

In this representable case the fundamental theorems of Galois theory can be easily established. Clearly every set $[A,X]$ has an action of the group $G = Aut(A)^{\mathrm{op}}$, thus the functor F lifts into the topos of G-sets $\mathscr{C} \xrightarrow{\mu F} \mathscr{B}G$. It is not difficult to prove the following (see [9, Section 1]):

Theorem 3.2.3. *For every object $X \in \mathscr{C}$ the action of the group $Aut(A)^{\mathrm{op}}$ on the set $[A,X]$ is transitive, and every arrow $A \xrightarrow{x} X$ in $\mathscr{C}$ is the categorical quotient by the action of the subgroup $\{h \in Aut(A) \mid xh = x\} \subset Aut(A)$ on A.*

From this theorem, by easy general categorical arguments, follows:

Theorem 3.2.4 (Fundamental theorem). *Let $\mathscr{E}$ be a pointed connected atomic topos $\mathscr{S} \xrightarrow{f} \mathscr{E}$, and $\mathscr{C} \subset \mathscr{E}$ be a pointed site $\mathscr{C} \xrightarrow{F} \mathscr{S}$ for $\mathscr{E}$, $\mathscr{E} = \mathscr{C}^{\sim}$, $F = f^*|_{\mathscr{C}}$ (in the sense of* 3.1.1 *above) such that the functor F is representable by an object $A \in \mathscr{C}$.*

Then, the lifting μF *of* F *lands in the subcategory of transitive G-sets,* $\mathscr{C} \xrightarrow{\mu F} t\mathscr{B}G$, *for the discrete group* $G = Aut(A)^{\mathrm{op}}$, *and the induced morphism of topoi* $\mathscr{B}G \xrightarrow{\mu f} \mathscr{E}$ *is an equivalence.*

From this theorem, or by the same elementary proof, the following groupoid version follows (compare with [1, Ex IV 7.6 d]):

Theorem 3.2.5 (Fundamental theorem). *Let* $\mathscr{E}$ *be an essentially-pointed connected atomic topos, and let* $\mathscr{G}$ *be its category of points* (*which is a connected discrete groupoid* A.1.6), $\mathscr{G} = \mathscr{P}oints(\mathscr{E})$. *Then, the canonical geometric morphism* $\mathscr{B}(\mathscr{G}^{\mathrm{op}}) = \mathscr{S}^{\mathscr{G}^{\mathrm{op}}} \xrightarrow{\mu} \mathscr{E}$ *is an equivalence.*

We use the variance convention of SGA4. Given any geometric morphism $\mathscr{E} \xrightarrow{f} \mathscr{F}$, clearly the induced morphism $\mathscr{B}(\mathscr{P}oints(\mathscr{E})^{\mathrm{op}}) \to \mathscr{B}(\mathscr{P}oints(\mathscr{F})^{\mathrm{op}})$ makes the square which expresses the naturality of μ commutative (here we have all ordinary categories, compare 2.1.6).

This situation is characterized in terms of exactness properties of the inverse image of the point. It is equivalent to the preservation of all limits by the inverse image functor f^*, or, equivalently, the point is essential (in the sense of [1]). For simplicity we shall assume that $\mathscr{C}$ is the full subcategory of all nonempty connected objects.

Proposition 3.2.6 (compare with [1, IV 7.6]). *In the situation of* 3.1.1, *assume that the functor* $\mathscr{E} \xrightarrow{f^*} \mathscr{S}$ *preserves all* (*small*) *limits* (*the point is essential*). *Then, the diagram* Γ_F *of the functor* $\mathscr{C} \xrightarrow{F} \mathscr{S}$ *has an initial object* (A, a), *and* F *is representable by* A.

Proof. Let B be the limit $B = Lim_{(X,x) \in \Gamma_F} X$ taken in $\mathscr{E}$. By assumption, the canonical morphism $FB \to Lim_{(X,x) \in \Gamma_F} FX$ is a bijection. Let $a \in FB$ be the unique element corresponding under this bijection to the tuple $(x)_{(X,x) \in \Gamma_F} \in Lim_{(X,x) \in \Gamma_F} FX$, and let A be the connected component of B so that $a \in FA$. The verification of the statement in the theorem is standard and straightforward. □

Corollary 3.2.7. *A pointed connected atomic topos is a locally simply connected Galois topos if and only if the point is essential.*

3.3. Prodiscrete Galois theory

Grothendieck's Galois theory corresponds to the situation described in 3.1.1 when the Galois objects are (co) cofinal in the diagram Γ_F of the functor F. Then, by means of inverse limit techniques the fundamental theorem can be proved by reducing it to the representable (or discrete) case. This yields a prodiscrete localic group as the localic group of automorphisms of the point. This is the method introduced and developed by Grothendieck in SGA1 [11] to treat the profinite case (see also [13,10] for a detailed and elementary description of all this). Later, in a series of commented exercises in SGA4 [1] he gave guidelines to treat the general prodiscrete case by means of locally constant sheaves and progroups. The key result in these developments is the construction of the Galois closure. In [18] Moerdijk developed this program using prodiscrete localic groups instead of progroups, and gave a rather sketchy proof of the fundamental theorem (Theorem 3.2 loc. cit.). Prodiscrete localic groups and their classifying topoi are completely equivalent to strict (in the sense that transition morphisms are surjective) progroups and their classifying topoi, as it was first observed by Tierney in lectures at Columbia University, and later stated in print independently by Moerdijk in [18]. This result has been generalized to groupoids by Kennison [15, 4.18 (see Section 1.4)].

Pointed Galois topoi are given by pointed atomic sites (as explicitly described in 3.1.1) so that pairs (A,a), $a \in FA$, with A a Galois object, are (co) cofinal in the diagram Γ_F of F.

Let (A,a) be an object in the diagram of F, with A a Galois object. Let $\mathscr{C}_A$ be the full subcategory of $\mathscr{C}$ defined by:

$$X \in \mathscr{C}_A \;\Leftrightarrow\; [A,X] \xrightarrow{a^*} FX,\; a^*(h) = Fh(a),\; \textit{is a bijection}.$$

Proposition 3.3.1. *An object $X \in \mathscr{C}$ is in $\mathscr{C}_A$ if and only if there exist an arrow $A \to X$.*

Proof. Consider an arrow $A \xrightarrow{x} X$. We have to see that $X \in \mathscr{C}_A$. We have the commutative diagram:

$$\begin{array}{ccc} [A,A] & \xrightarrow{a^*} & FA \\ \downarrow{\scriptstyle x_*} & & \downarrow{\scriptstyle F(x)} \\ [A,X] & \xrightarrow{a^*} & FX \end{array}$$

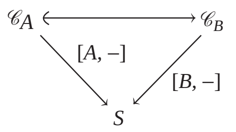
The bottom row is a bijection since it is already injective (3.1.2) and $F(x)$ is surjective. □

Notice that it follows that $\mathscr{C}$ is the filtered union (indexed by the Galois objects in Γ_F) of the full subcategories $\mathscr{C}_A$.

By definition $A \in \mathscr{C}_A$, and the restriction of the functor F to $\mathscr{C}_A$ is naturally isomorphic to $[A,-]$. Theorem 3.2.3 gives:

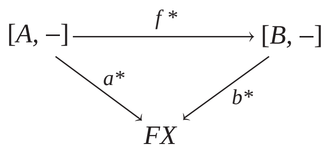
Proposition 3.3.2. *The pair $\mathscr{C}_A$, $[A,-]$ defines an atomic site with a representable point. The induced morphism $\mathscr{B}(lAutA^{\mathrm{op}}) \to \mathscr{E}_A$ is an equivalence (where $\mathscr{E}_A$ denotes the topos of sheaves on $\mathscr{C}_A$).*

Proposition 3.3.3. *Given a morphism $(B,b) \xrightarrow{f} (A,a)$ in Γ_F with A and B Galois objects, there is a (full) inclusion of categories $\mathscr{C}_A \subset \mathscr{C}_B$. This gives rise to a triangle* *as follows*:



$$f^*: [A,-] \xrightarrow{\cong} [B,-].$$

Proof. Let $X \in \mathscr{C}_A$. We have the commutative diagram:



Arrow b^* is a bijection since it is already injective (3.1.2) and, by assumption, a^* is a bijection. Thus $X \in \mathscr{C}_B$. Clearly it follows that f^* is also a bijection. □

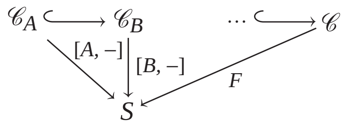
In Lemma 2.1.3 we established an isomorphism $lAut([A,-]) \xrightarrow{\cong} Aut(A)^{\mathrm{op}}$. We shall explicitly describe now how the morphism $lAut([B,-]) \longrightarrow lAut([A,-])$ defined in Proposition 2.3.3 is induced by a morphism $Aut(B) \xrightarrow{\varphi} Aut(A)$.

Let $l \in Aut(B)$, and let $h \in Aut(A)$ be the unique morphism such that $f^* \circ h^* = l^* \circ f^*$ (recall that f^* is an isomorphism). Define $\varphi(l) = h$. Then, $\varphi(l) \circ f = f \circ l$ (since f is an epimorphism, $\varphi(l)$ is characterized by this equation). Let $[(X, \langle x \mid y \rangle)]$ (with $A \xrightarrow{x} X$, $A \xrightarrow{y} X$) be a generator of the locale $lAut([A,-])$. Under the isomorphism with $Aut(A)$ it corresponds to the open set $\{h \mid xh = y\}$ (where $A \xrightarrow{h} A$). Then, $\varphi^{-1}\{h \mid xh = y\} = \{l \mid x\varphi(l) = y\}$. Since f is an epimorphism, this set is equal to $\{l \mid x\varphi(l)f = yf\} = \{l \mid xfl = y\} = \{l \mid f^*(x)l = y\}$, which corresponds to $[(X, \langle f^*x \mid f^*y \rangle)]$. This shows that the morphism of Proposition 2.3.3 corresponds to φ as defined above. Now, from $\varphi(l) \circ f = f \circ l$ it follows $F(\varphi(l)) \circ Ff = Ff \circ Fl$. Since $Ff(b) = a$ we have $F(\varphi(l))(a) = Ff \circ Fl(b)$, that is $a^*(\varphi(l)) = Ff \circ b^*(b)$ (this equation also characterizes $\varphi(l)$). Thus, $\varphi = (a^*)^{-1} \circ Ff \circ b^*$.

We see, in particular, that φ is then a surjective function. This proves the following proposition:

Proposition 3.3.4. *The transition morphism between the localic groups corresponding to a transition between two Galois objects in the diagram of F is a surjection.*

We have the situation described in the following diagram:



It follows from 1.4.2 that the localic group $lAut(F)$ is prodiscrete and it is the inverse limit of the induced filtered system of discrete groups $Aut(A)^{op}$

$$Aut(A)^{op} \leftarrow Aut(B)^{op} \cdots \leftarrow lAut(F).$$

Furthermore, since $\mathscr{C}$ is the filtered union (indexed by the Galois objects in Γ_F) of the full subcategories $\mathscr{C}_A$, and all the topologies are the canonical one, $\mathscr{C}$ is the inverse limit site. It follows from 3.2.3 that the lifting $\mathscr{C} \xrightarrow{\mu F} \mathscr{B}lAut(F)$ of F (2.3.2) lands in the subcategory of transitive G-sets. Furthermore, from 1.1.1, 3.3.4 and 2.3.7 it follows that both rows in the following diagram are filtered inverse limits of topoi (indexed by the Galois objects in Γ_F):

$$\begin{array}{ccccc}
\mathscr{E}_A & \longleftarrow & \mathscr{E}_B & \cdots \longleftarrow & \mathscr{E} \\
\cong\uparrow & & \cong\uparrow & & \uparrow \\
\mathscr{B}Aut(A^{op}) & \longleftarrow & \mathscr{B}Aut(B^{op}) & \cdots \longleftarrow & \mathscr{B}lAut(F)
\end{array}$$

Therefore the arrow $\mathscr{B}lAut(F) \to \mathscr{E}$ is also an equivalence. In conclusion, this finishes the proof of the following theorem:

Theorem 3.3.5 (Fundamental theorem). *Let $\mathscr{E}$ be a pointed Galois topos $\mathscr{S} \xrightarrow{f} \mathscr{E}$, and $\mathscr{C} \subset \mathscr{E}$ be a pointed site $\mathscr{C} \xrightarrow{F} \mathscr{S}$ for $\mathscr{E}$, $\mathscr{E} = \mathscr{C}^{\sim}$, $F = f^*|_{\mathscr{C}}$ (in the sense of* 3.1.1 *above). Then, the localic group $G = lAut(F)$ is prodiscrete, the lifting $\mathscr{C} \xrightarrow{\mu F} \mathscr{B}lAut(F)$ of F* (2.3.2) *lands in the subcategory of transitive G-sets, and the induced morphism of topoi $\mathscr{B}lAut(F) \xrightarrow{\mu f} \mathscr{E}$ is an equivalence.*

Recall that if $\mathscr{S} \xrightarrow{f} \mathscr{E}$ is the corresponding point of the topos, $F = f^*|_{\mathscr{C}}$, then $lAut(F) = lAut(f)^{op}$.

From this theorem follows a groupoid version:

Theorem 3.3.6 (Fundamental theorem). *Let $\mathscr{E}$ be a Galois topos with points (thus enough). Then the canonical geometric morphism $\mathscr{B}(\mathscr{G}^{op}) \to \mathscr{E}$ is an equivalence, where $\mathscr{G}$ is the localic groupoid of points $\mathscr{G} = l\mathscr{P}oints(\mathscr{E})$ defined in Section* 2.2, *and this groupoid has prodiscrete "hom"spaces (in particular, prodiscrete vertex localic groups).*

Proof. Let $\mathscr{S} \xrightarrow{f} \mathscr{E}$ be any point of $\mathscr{E}$, and consider the commutative diagram:

$$\begin{array}{ccc}
\mathscr{B}(lAut(f)^{op}) & \longrightarrow & \mathscr{B}(\mathscr{G}^{op}) \\
& \searrow \quad \swarrow & \\
& \mathscr{E} &
\end{array}$$

From the second statement of 1.3.7 and 3.3.5 it follows that the horizontal arrow and the left diagonal are equivalences. Thus the remaining arrow is so (compare with [11, V 5.8]). The last statement follows from 3.5.1 below. □

The reader should be aware that the groupoid in this theorem is not a prodiscrete localic groupoid.

We now characterize this situation in terms of exactness properties of the inverse image of the point. Theorem 3.3.8 below is inspired in a natural way of constructing a normal covering (which covers a given covering) in the classical topological theory of covering spaces. In fact, this theorem is an explicit construction of the Galois closure.

Grothendieck's theory corresponds to the case in which the point, although not necessarily essential, is such that the inverse image preserves certain infinite limits, namely, cotensors of connected objects. This is equivalent to the existence of Galois closure (that is, the Galois objects generate the topos), or to the fact that the localic group $lAut(p)$ is prodiscrete. We elaborate on this now.

Consider a pointed connected atomic topos $\mathscr{S} \xrightarrow{f} \mathscr{E}$ and a corresponding pointed site $\mathscr{C} \xrightarrow{F} \mathscr{S}$ as in 3.1.1. For simplicity we shall assume that $\mathscr{C}$ is the full subcategory of all nonempty connected objects. Recall that the topology is the canonical topology.

Definition 3.3.7. Let $\mathscr{E} \xrightarrow{\gamma} \mathscr{S}$ be any topos. We say that a point $\mathscr{S} \xrightarrow{p} \mathscr{E}$ of $\mathscr{E}$ is proessential if the inverse image preserves cotensors of connected objects. That is, given any connected object X and any set S, the canonical morphism:

$$p^*(X^{\gamma^* S}) = p^*\left(\prod_S X\right) \to \prod_S p^*X = (p^*X)^S.$$

is a bijection.

Notice that preservation of cotensors (of any object) is a much stronger condition which implies that the point is essential (see [4]).

Theorem 3.3.8. *In the situation of* 3.1.1, *assume that the point* $\mathscr{S} \xrightarrow{p} \mathscr{E}$, $F = p^*|_{\mathscr{C}}$ *is proessential. Then*, *the objects* (A, a) *with* A *a Galois object are cofinal in the diagram* Γ_F *of* F. *In fact*, *the following holds*: *Given any connected object* X, *there exists a Galois object* A, *and an element* $a \in FA$ *such that for all* $x \in FX$ *there exists an arrow* $(A, a) \xrightarrow{f} (X, x)$ *such that* $Ff(a) = x$. *Notice that in this context* f *is unique and a strict epimorphism* (3.1.2 *and* 3.1.1 (i)).

Proof. Let B be the cotensor $B = \prod_{FX} X$ taken in $\mathscr{E}$. By assumption, the canonical morphism $F(\prod_{FX} X) \to \prod_{FX} FX$ is a bijection. Let $a \in FB$ be the unique element corresponding under this bijection to the tuple $(x)_{x \in FX} \in \prod_{FX} FX$, and let A be the connected component of B such that $a \in FA$. Clearly, for each $x \in FX$ there is an arrow in Γ_F given by the projection $A \xrightarrow{\pi_x} X$, characterized by the equation $F\pi_x(a) = x$. We prove now that A, with the element $a \in FA$, is a Galois object. To this end we establish:

Lemma. *Given any $b \in FA$, there exists $A \xrightarrow{f_x} X$ such that $Ff_x(b) = x$.*

Clearly, from this it follows (by the universal property of the product) that there exists $A \xrightarrow{h} A$ such that $Fh(b) = a$. Let $c = Fh(a)$, and apply the lemma to this element $c \in FA$. It follows as before that there is $A \xrightarrow{g} A$ such that $Fg(c) = a$. Then, by 3.1.2 it must be $g \circ h = id$. So h is a monomorphism, and thus by 3.1.1 (i) it is an isomorphism. This shows that A is a Galois object. □

Proof of the lemma. Consider the action μ of $lAut(F)$ (2.3.2). Take any $x \in FX$. Then:

$$1 = \bigvee_{z \in FX} \mu^*[\langle z \mid x \rangle]$$

Since the action is transitive (3.6.1), taking the infimun against $\mu^*[\langle a \mid b \rangle]$ yields:

$$0 \neq \mu^*[\langle a \mid b \rangle] \leqslant \bigvee_{z \in FX} \mu^*[\langle a \mid b \rangle] \wedge \mu^*[\langle z \mid x \rangle]$$

It follows that there exists $z \in FX$ such that

$$0 \neq \mu^*[\langle a \mid b \rangle] \wedge \mu^*[\langle z \mid x \rangle]$$

Since π_z is a morphism of actions, we have:

$$\mu^*[\langle a \mid b \rangle] \leqslant \mu^*[\langle \pi_z a \mid \pi_z b \rangle] = \mu^*[\langle z \mid \pi_z b \rangle]$$

It follows:

$$0 \neq \mu^*[\langle z \mid \pi_z b \rangle] \wedge \mu^*[\langle z \mid x \rangle] = \mu^*([\langle z \mid \pi_z b \rangle] \wedge [\langle z \mid x \rangle])$$

Thus, $0 \neq [\langle z \mid \pi_z b \rangle] \wedge [\langle z \mid x \rangle])$, which implies $x = \pi_z b$. We set $f_x = \pi_z$. This finishes the proof of the lemma. □

On the other hand, given any pointed Galois topos, it is easy to see that the point is proessential. In fact, given any connected object X, take a Galois object A such that $X \in \mathscr{C}_A$. Any cotensor of X lives in $\mathscr{E}_A = \mathscr{C}_A^{\sim}$, and the result follows since the restriction of the inverse image to the full subcategory $\mathscr{C}_A$ determines an essential point of $\mathscr{E}_A$. We have:

Corollary 3.3.9. *A pointed connected atomic topos is a Galois topos if and only if the point is proessential.*

3.4. Unpointed prodiscrete Galois theory

Bunge [6] (see also [8]) developed an unpointed theory for Galois topoi based on Joyal–Tierney descent theory [14], and following Grothendieck's inverse limit techniques along the lines of the pointed theory of [18], necessarily in this case in terms of localic groupoids. Around the same time, Kennison [15] also developed an unpointed theory with a different approach, but we shall not elaborate on this theory here.

We now describe briefly the unpointed theory along the lines of [6,8, Section 2], and describe explicitly the fundamental groupoid as the localic groupoid of "points" (which may not be there !). We do this in a independent way of the representation theorems in [14]. We shall show that Grothendieck's theory of Sections 3.2 and 3.3 can as well be developed in a pointless way. We also think that it will interest the reader to see explicitly how the points, which are always there at the starting line (the topoi $Split(U)$ always have points), are lost along the way.

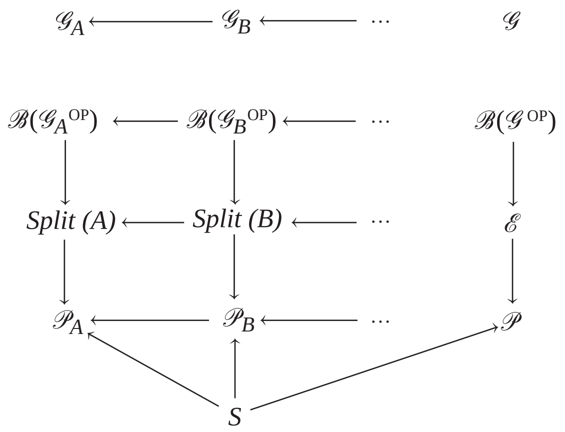
Consider a Galois topos as in Definition A.2.1. In Section 3.3 the point furnish a filtered poset Γ_F along which compute an inverse limit of pointed topoi. In the absence of the point we have to deal differently. Proposition A.2.6 is at the base of this development. Even though all the topoi in the system furnished by this proposition have points, the system is not a pointed system in the sense that there is no simultaneous choice of points commuting with the transition morphisms. In fact, such a choice is equivalent to a point of the inverse limit topos.

Consider now any connected locally connected topos $\mathscr{F}$ and the Galois topos $\mathscr{E} = GLC(\mathscr{F})$ (notice that $\mathscr{E}$ can be any Galois topos A.2.4). Given a morphism between Galois objects $A \xrightarrow{f} B$, the geometrical morphism $Split(A) \leftarrow Split(B)$ (with inverse image the full inclusion of categories) clearly induces a surjective function between the sets of points (for the surjectivity compare with 3.3.3). A point of $\mathscr{E}$ furnish a way of choosing a point (consistently with respect to the transition morphisms) on each topoi $Split(A)$, thus, it is exactly an element of the inverse limit of the sets of points of the topoi $Split(A)$. This inverse limit may be empty, but taken in the category of localic spaces it always defines a nontrivial prodiscrete localic space (since the projections are surjective [14, IV 4.2]) G_0, which is the space of (may be phantom) points of the inverse limit topos $\mathscr{E}$.

More over, there is induced a groupoid morphism $\mathscr{G}_A \leftarrow \mathscr{G}_B$ between the categories (which are discrete connected groupoids) of points, $\mathscr{G}_A = \mathscr{P}oints(Split(A))$ (compare 2.2.1). The inverse limit of this filtered diagram, taken in the category of localic groupoids, defines a *prodiscrete localic groupoid* (see [15, Definition 2.8]) $\mathscr{G}$ with the prodiscrete localic space G_0 as its localic space of "objects".

We shall say that $\mathscr{G}$ is the *localic groupoid of phantom points* of the Galois topos $\mathscr{E}$, and write $ph\mathscr{P}oints(\mathscr{E})$. The points (if any) of G_0 are exactly the points of the topos $\mathscr{E}$. On the other hand, there are also geometric morphisms between the push-out topoi $\mathscr{P}_A \leftarrow \mathscr{P}_B$, which are morphisms of pointed topoi for the canonical points (cf A.1.4 and A.1.6). Thus, there is always a consistent choice of points for the system of push-out topoi $\mathscr{P}_A$.

The whole situation we have at hand is synthesized in the following diagram:



The isolated first row is an inverse limit by definition. That the second row is an inverse limit means that the functor $\mathscr{B}$ commutes with the inverse limit of discrete groupoids which defines $\mathscr{G}$ in the first row. This is proved in [15, 4.18]. That the third row is an inverse limit is Proposition A.2.6. Finally, we define $\mathscr{P}$ as the mathematical object which makes the fourth row an inverse limit. In the previous considerations we already saw that everything commutes in the appropriate way, and this implies the existence of the point $\mathscr{S} \to \mathscr{P}$.

The vertical down arrows on the left of the dots are equivalences by 3.2.5 and A.1.2, respectively. It follows (using the horizontal rows) that the arrow $\mathscr{B}(\mathscr{G}^{\mathrm{op}}) \to \mathscr{E}$ is an equivalence. This finishes the proof of:

Theorem 3.4.1 (Fundamental theorem). *Let $\mathscr{F}$ be any connected locally connected topos, and $\mathscr{E}$ be the Galois topos $\mathscr{E} = GLC(\mathscr{F})$. Then the canonical geometric morphism $\mathscr{B}(\mathscr{G}^{\mathrm{op}}) \to \mathscr{E}$ is an equivalence, where $\mathscr{G}$ is the prodiscrete localic groupoid of (phantom) points $\mathscr{G} = ph\mathscr{P}oints(\mathscr{E})$ defined by the inverse limit above.*

Now, since each $Split(A)$ is equivalent to $\mathscr{P}_A$, $\mathscr{E}$ should be equivalent to $\mathscr{P}$, and it would follow then that the topos $\mathscr{E} = GLC(\mathscr{F})$ (and so any Galois topos) always has a point?

The problem here is that the system of push out topoi is not a filtered system and *can not be used as such* to define an inverse limit topos. Given $A \leqslant B$ in $GCov(\mathscr{E})$, the transition morphism $\mathscr{P}_A \leftarrow \mathscr{P}_B$, which now we shall denote p_f, depends on the arrow $A \xrightarrow{f} B$ which witnesses that $A \leqslant B$. The reader can easily verify this by direct inspection. However, there is no complete chaos here. Given any two arrows $f, g : A \to B$, it is also immediate to check by the same method that there is a (canonical) invertible natural transformation $p_f \cong p_g$, and that all these two-cells define a *biordered inversely bifiltered two system* (see [15] for this notion and further references) of topoi which is not inversely filtered. It is not known if the inverse limit (or bilimit) of such a thing is a topos, and even less what kind of topos if that were the case. The equivalences $Split(A) \to \mathscr{P}_A$ induce an arrow on the inverse limits (whatever it is $\mathscr{P}$) $\mathscr{E} \to \mathscr{P}$ which presumably will not have a pseudo inverse to compose with the point of $\mathscr{P}$ to give a point for $\mathscr{E}$.

3.5. Comparison between the pointed and unpointed theories

In the presence of points, both the pointed and the unpointed theories apply, but do not furnish the same groupoid in the fundamental theorem. We now study how the two constructions of localic groupoids are related. Namely, the localic groupoid $l\mathscr{P}oints(\mathscr{E})$ in Section 3.3, and the localic groupoid $ph\mathscr{P}oints(\mathscr{E})$ in Section 3.4. It turns out that both correspond to filtered inverse limits of discrete groupoids, but taken in different categories.

This concerns the preservation of the filtered inverse limit of topoi

$$Split(A) \leftarrow Split(B) \leftarrow \cdots \mathscr{E}$$

by the functor $l\mathscr{P}oints$ defined in 2.2.1. As we shall see, this inverse limit is preserved into the category of localic groupoids with discrete space of objects.

First, notice that since all the points of the essentially pointed topoi $Split(A)$ are representable, it follows from the localic Yoneda's Lemma 2.1.3 that the localic and the discrete groupoids of points are equivalent in this case. We have $\mathscr{G}_A = \mathscr{P}oints(Split(A)) \cong l\mathscr{P}oints(Split(A))$.

Given any two points f, g of $\mathscr{E}$, they are given by compatible (with respect to the transition morphisms) tuples $f = (f_A)$, $g = (g_A)$ of points of the topoi $Split(A)$. Consider the filtered system of discrete spaces (where $\mathscr{C}_A = Split(A)$).

$$l\mathscr{P}oints(\mathscr{C}_A)[f_A, g_A] \leftarrow l\mathscr{P}oints(\mathscr{C}_B)[f_B, g_B] \leftarrow \cdots l\mathscr{P}oints(\mathscr{E})[f, g]$$

Taking into account the proof of 2.2.1, with the same arguments as in the proof of Proposition 2.3.6 (where the case of one of the vertex localic groups is donned in detail), it follows that this diagram is an inverse limit diagram of localic spaces.

This shows that $l\mathscr{P}oints(\mathscr{E})$ is the inverse limit of the filtered system of discrete groupoids $\mathscr{G}_A \leftarrow \mathscr{G}_B \cdots$ in the category of localic groupoids with discrete space of objects, while $ph\mathscr{P}oints(\mathscr{E})$ is by definition the inverse limit of the same system in the category of all localic groupoids. It follows then that there is a comparison morphism of localic groupoid $l\mathscr{P}oints(\mathscr{E}) \to ph\mathscr{P}oints(\mathscr{E})$.

Notice that from the representation Theorems 3.4.1 and 3.3.6 follows that this morphism induces an equivalence between the classifying topoi.

Comment 3.5.1. In the arguing above it is given an sketch of the proof that the functor $l\mathscr{P}oints(\mathscr{E})$ preserves filtered limits of topoi (into the category of localic groupoids with discrete space of objects), generalizing 2.3.6.

Proposition 1.3.6 says in a way that the classifying topos of a localic groupoid with discrete space of objects is a rather simple construction, similar to the classifying topos of a discrete groupoid. Based on this, it can be proved that when the inverse limit topos $\mathscr{E}$ has points, the functor $\mathscr{B}$ preserves the filtered inverse limit of the system of localic groupoids $\mathscr{G}_A$ considered above (limit taken in the category of localic groupoids with discrete space of objects). Use the fact that all the points f_A of the topoi $Split(A)$ are representable by projective objects, and that this implies that the transition morphisms $\mathscr{P}oints(\mathscr{C}_A)[f_A, g_A] \leftarrow \mathscr{P}oints(\mathscr{C}_B)[f_B, g_B]$ are surjective (compare with 1.4.2). It follows that the topos $\mathscr{B}l\mathscr{P}oints(\mathscr{E})$ is the inverse limit of the system of topoi $\mathscr{B}(\mathscr{G}_A^{\mathrm{op}})$ (as it was the case for the topos $\mathscr{B}ph\mathscr{P}oints(\mathscr{E})$ by [15, 4.18]). This gives a proof of 3.3.6 along the lines of the proof of 3.4.1, and without the need of using 1.3.7. At the same time it shows directly why the comparison morphism between the two localic groupoids of points induces an equivalence of the classifying topoi.

Galois topos with points are connected but may have a non-connected groupoid of points. We finish this section with an example:

Example 3.5.2. In SGA4 IV 7.2.6, (d) it is said that there exists a strict progroup $H = (H_i)_{i \in I}$ such that the classifying topos $\mathscr{B}H$ has two nonisomorphic points. Equivalently, there is a prodiscrete localic group H such that $\mathscr{B}H$ has two nonisomorphic points. This implies that the groupoid of points $\mathscr{P}oints(\mathscr{B}H)$ is not connected. Its classifying topos $\mathscr{B}\mathscr{P}oints(\mathscr{B}H)$ can not be $\mathscr{B}H$. However, the localic groupoid of points $l\mathscr{P}oints(\mathscr{B}H)$ (which has discrete set of objects) is connected (in particular, the localic space of morphisms between any two points is nontrivial). We have $\mathscr{B}H \cong \mathscr{B}l\mathscr{P}oints(\mathscr{B}H)$ and $H \cong l\mathscr{P}oints(\mathscr{B}H)$

3.6. Localic Galois theory

In the previous sections we have developed the fundamentals of the Galois theory as given by Grothendieck's guidelines up to its natural end point, which is the representation theorems of Galois topoi 3.3.5 and 3.4.1. One aspect of these theorems is that they furnish an axiomatic characterization of the classifying topoi of prodiscrete localic groups and (connected) prodiscrete localic groupoids respectively. With the notion of localic group generalizing the notion of progroup, the natural end point of the theory is push forward into the representation theorem of pointed connected atomic topoi, which would be, in particular, an axiomatic characterization of the classifying topoi of general localic groups. This theorem is [14, Ex.VIII 3]. Theorem 1, and it still generalizes closely Grothendieck's Galois theory. In particular, the localic group in the statement is still the localic group of automorphisms of the point (or the localic groupoid of all points as defined in Section 2.2), and as such, it is canonically associated to the topos (and functorial).

We now recall the fundamental theorems of *localic Galois theory* established in [9], where the representation theorem of pointed connected atomic topoi is a consequence of a theory completely different to the Joyal–Tierney theory, and more akin in its methods to classical Galois theory (compare with Section 3.2)

Assume that the pair $\mathscr{C}$, $\mathscr{C} \xrightarrow{F} \mathscr{S}$, is a pointed connected atomic site in the sense explicitly described in 3.1.1 above. We have:

Theorem 3.6.1. *For every object $X \in \mathscr{C}$ the action of the localic group of automorphisms $lAut(F)$ on the set FX is transitive. That is, given any pair $(x_0, x_1) \in FX \times FX$, $\#[(X, \langle x_0 \mid x_1 \rangle)] \neq 0$ in $lAut(F)$.*

Theorem 3.6.2. *Lifting Lemma: Given any objects $X \in \mathscr{C}$, $Y \in \mathscr{C}$, and elements $x \in FX$, $y \in FY$, if $lFix(x) \leqslant lFix(y)$ in $lAut(F)$, then there exist a unique arrow $X \xrightarrow{f} Y$ in $\mathscr{C}$ such that $F(f)(x) = y$.*

More generally, the following rule holds in $lAut(F)$

$$\frac{\#[(X, \langle x_0 \mid x_1 \rangle)] \leqslant \#[(Y, \langle y_0 \mid y_1 \rangle)]}{\exists X \xrightarrow{f} Y \;\; F(f)(x_0) = y_0, F(f)(x_1) = y_1}$$

From 3.1.3, 3.6.1 and 3.6.2 follows by easy categorical arguments:

Theorem 3.6.3 (Fundamental theorem). *Let $\mathscr{E}$ be a pointed connected atomic topos $\mathscr{S} \xrightarrow{f} \mathscr{E}$, and $\mathscr{C} \subset \mathscr{E}$ be a pointed site $\mathscr{C} \xrightarrow{F} \mathscr{S}$ for $\mathscr{E}$, $\mathscr{E} = \mathscr{C}^{\sim}$, $F = f^*|_{\mathscr{C}}$ (in the sense of* 3.1.1 *above). Then, the lifting μF of F* (2.3.2) *lands in the subcategory of transitive G-sets, $\mathscr{C} \xrightarrow{\mu F} t\mathscr{B}G$, for the localic group $G = lAut(F)$, and the induced morphism of topoi $\mathscr{B}G \xrightarrow{\mu f} \mathscr{E}$ is an equivalence.*

Actually, considering all the points (and with exactly the same proof that Theorem 3.3.6) this theorem yields:

Theorem 3.6.4 (Fundamental theorem). *Let $\mathscr{E}$ be any pointed connected atomic topos. Then the canonical geometric morphism $\mathscr{B}(\mathscr{G}^{\mathrm{op}}) \to \mathscr{E}$ is an equivalence, where $\mathscr{G}$ is the localic groupoid of points $\mathscr{G} = l\mathscr{P}oints(\mathscr{E})$ defined in Section* 2.2.

4. Summary of the representation theorems and final conclusions

In this section we summarize an analysis of the results of this paper and make some comments on Joyal–Tierney generalization of Grothendieck's Galois theory and its relation to the Galois theory of Galois topoi. We also treat the nonconnected theory, which follows trivially from the connected case (as opposed to the groupoid formulation from the group formulation within the connected theory).

4.1. Comments on Joyal–Tierney Galois theory

In [14] Joyal–Tierney develop Galois theory in a new way. Classifying topoi are explicitly described as descent topoi. For them, the fundamental theorem of Galois theory states that (*) *open surjections are geometric morphisms of effective descent*.

The fundamental theorems of previous Galois theories follow because these theorems are statements about a point, and this point is an open surjection.

It also follows an unpointed theory of representation for a completely arbitrary topos in terms of localic groupoids, theorem Chapter VIII 3, Theorem 2, which states that any topos is the classifying topos of a localic groupoid. This theorem is dependent on change of base techniques. An important difference here with previous Galois theories is that the geometric morphism which is proved to be of effective descent *is not part or is not canonically associated to the data*. As a consequence, the groupoid is not associated to the topos in a functorial way. We think that part of Joyal–Tierney theory describes different phenomena from the one that concerns the Galois theory of Galois topoi, either pointed or unpointed.

In [14] one also finds the representation theorem for pointed atomic topoi Chapter VIII 3, Theorem 1. It is worth to notice that Theorem 1 as such is not a particular instance of Theorem 2. The proof of Theorem 2 goes in two steps. Step 1: construct an open spatial (or localic) cover, which is the part that does not corresponds to Galois theory. Step 2: using this cover construct the localic groupoid that proves the statement by the theorem (*) quoted above. In Theorem 1 only the second step is used, the first one is already part of the data (the given point is the cover), and as such it is canonical. The atomic topoi with enough points have a canonical open spatial cover, namely, the discrete localic space of all the points (only one is necessary if the topos is connected), and it can be seen that the construction in Step 2 yields the localic groupoid of points (defined in 2.2 above). The recipe given in Section 3 of Chapter VII for Step 1, applied to an atomic topos with enough points, does not yield the discrete cover given by the points.

We can say that pre-Joyal–Tierney Galois theory cover Step 2 (and thus it suffices to state and prove Theorem 1), as it has been shown in [9]. While Step 1 (and thus Theorem 2) goes beyond.

4.2. *The nonconnected theory*

Locally connected topoi are sums of connected locally connected ones. Generally, because of this, it is enough to prove results for the connected case. In [11, V 9]. the nonconnected theory is left to the reader ("Nous en laissons le detail au lecteur"). However, in [12], locally connected (but not connected) Galois topoi are considered under the name "*Topos Multigaloisiennes*". There the topos are supposed to have enough points.

Definition 4.2.1. A *Multigalois topos* is a locally connected topos generated by its Galois objects, or, equivalently, it is a sum of Galois topoi.

Let $\mathscr{E}$ be any locally connected topos. The following theorems follow by decomposing $\mathscr{E}$ as a sum of connected topoi, and proving the statements for connected topoi (which we indicate in between parenthesis). The implication chain is best understood if performed by the increasing cycle permutation.

From the results in Section 3.2 we have:

Theorem 4.2.2 (Discrete case). *The following are equivalent*:

(2) $\mathscr{E}$ *is a locally simply connected* (*Galois*) *multigalois topos*.
(3) $\mathscr{E}$ *is* (*connected*) *atomic with enough essential points*.
(4) *The canonical geometric morphism* $\mathscr{B}(\mathscr{G}) \xrightarrow{\mu} \mathscr{E}$ *is an equivalence*, *where* $\mathscr{G}$ *is the* (*connected*) *discrete groupoid of points* $\mathscr{G} = \mathscr{P}oints(\mathscr{E})$.
(5) $\mathscr{E}$ *is the classifying topos of any* (*connected*) *discrete groupoid*.

From the results in Section 3.3 (and A.2.5) we have:

Theorem 4.2.3 (Pointed prodiscrete case). *The following are equivalent*:

(1) $\mathscr{E}$ *has enough points and it is* (*connected*) *generated by its locally constant objects*.
(2) $\mathscr{E}$ *has enough points and it is a* (*Galois*) *multigalois topos*.
(3) $\mathscr{E}$ *is* (*connected*) *atomic with enough proessential points*.
(4) *The canonical geometric morphism* $\mathscr{B}(\mathscr{G}) \xrightarrow{\mu} \mathscr{E}$ *is an equivalence*, *where* $\mathscr{G}$ *is the* (*connected*) *localic groupoid of points* $\mathscr{G} = l\mathscr{P}oints(\mathscr{E})$, *which in this case has prodiscrete "hom-spaces"*.
(5) $\mathscr{E}$ *is the classifying topos of any* (*connected*) *localic groupoid with discrete space of objects and prodiscrete "hom-spaces"*.

From the results in Section 3.4 (and A.2.5) we have:

Theorem 4.2.4 (Unpointed prodiscrete case). *The following are equivalent*:

(1) $\mathscr{E}$ *is* (*connected*) *generated by its locally constant objects*.
(2) $\mathscr{E}$ *is a* (*Galois*) *multigalois topos*.
(4) *The canonical geometric morphism* $\mathscr{B}(\mathscr{G}) \xrightarrow{\mu} \mathscr{E}$ *is an equivalence*, *where* $\mathscr{G}$ *is the* (*connected*) *localic groupoid of phantom points* $\mathscr{G} = ph\mathscr{P}oints(\mathscr{E})$, *which is prodiscrete* (*by definition*).
(5) $\mathscr{E}$ *is the classifying topos of any* (*connected*) *prodiscrete localic groupoid*.

From the results in Section 3.6 we have:

Theorem 4.2.5 (Pointed localic case). *The following are equivalent*:

(3) $\mathscr{E}$ *is* (*connected*) *atomic with enough points*.

(4) *The canonical geometric morphism* $\mathscr{B}(\mathscr{G}) \xrightarrow{\mu} \mathscr{E}$ *is an equivalence*, *where* $\mathscr{G}$ *is the* (*connected*) *localic groupoid of points* $\mathscr{G} = l\mathscr{P}oints(\mathscr{E})$.

(5) $\mathscr{E}$ *is the classifying topos of any* (*connected*) *localic groupoid with discrete space of object*.

We see that the theorem that should be labeled *unpointed localic case* is missing. It should concern an arbitrary, (even connected but may be pointless) atomic topos. Of course [14, Chapter VIII 3]. Theorem 2 applies to such a topos, but it is a far too general theorem. The localic groupoid $\mathscr{G}$ such that $\mathscr{E} \cong \mathscr{B}\mathscr{G}$ is not identified and can not be considered to be (as far as we can imagine) the groupoid of points of the topos. In [17] Moerdiejk investigates Joyal–Tierney theorems and establishes that atomic topoi are characterized by the fact that the localic groupoid can be so chosen that the map $G_1 \xrightarrow{(d_0,d_1)} G_0 \times G_0$ is open ([17, 4.7 c]). However we are still far from any canonicity for the groupoid. We need a theorem (still unknown) which, in particular, in the presence of a point, yields [14, Chapter VIII 3. Theorem 1]]. We still not know how to define the groupoid of phantom points for a general atomic topos (as we do for a general Galois topos). The solution is not given by Joyal–Tierney's generalization of Grothendieck's Galois theory, and it is not its purpose either. An *unpointed localic Galois theory* (compressing the unpointed prodiscrete Galois theory as well as the pointed localic Galois theory) is yet to be developed.

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Appendix A. Galois theory of covering topoi

We shall denote by γ the structure morphism of any topos $\mathscr{E}$, $\mathscr{E} \xrightarrow{\gamma} \mathscr{S}$. A topos $\mathscr{E}$ is said to be *locally connected* if the inverse image functor γ^* is essential, that is, if it has itself a left adjoint denoted $\gamma_!$ (the set of connected components). A topos $\mathscr{E}$ is said to be *connected* if the inverse image functor γ^* is full and faithful. If $\mathscr{E}$ is connected and locally connected clearly $\gamma_!\gamma^* = id$. The reference for connected and locally connected topoi is [1], Exposé IV, 4.3.5, 4.7.4, 7.6 and 8.7. A geometric morphism $\mathscr{E} \to \mathscr{F}$ is said to be a *locally connected morphism* if the topos $\mathscr{E}$ considered as an $\mathscr{F}$*-Topos* is locally connected. This relative version was introduced in [5] under the name $\mathscr{F}$*-essential*, see also the appendix of [16]. Recall that a *connected atomic topos* is a connected, locally connected and boolean topos. For atomic topoi and atomic sites see Ref. [3], also [14].

A *covering* of a topos $\mathscr{E}$ is a geometric morphism of the form $\mathscr{E}/X \to \mathscr{E}$, with X a *locally constant object*.

A.1. Locally constant objects

We recall now the definition of locally constant object in an arbitrary topos given in SGA4, Exposé IX (see also [7] where this definition is considered over an arbitrary base topos).

Definition A.1.1. An object X of a topos $\mathscr{E} \xrightarrow{\gamma} \mathscr{S}$ is said to be U-split, for a cover $U = \{U_i\}_{i \in I}$ (i.e. epimorphic family $U_i \to 1$), if it becomes constant on each U_i. That is, if there exists family of sets $\{S_i\}_{i \in I}$ and isomorphisms in $\mathscr{E}$, $\{\gamma^* S_i \times U_i \xrightarrow{\theta_i} X \times U_i\}_{i \in I}$ over U_i. We say that X is *locally constant* if X is U-split for *some* cover U in $\mathscr{E}$.

It is often convenient to identify a family with a function $\beta : S \to I$, $S = \sum_i S_i$. We abuse the language and write also U for the coproduct $U = \sum_i U_i$. Notice that there is a map $\zeta : U \to \gamma^* I = \sum_i 1$, and in this way a cover as above is given by such a map with $U \to 1$ epimorphic. In this notation the family of isomorphisms θ_i is the same thing as an isomorphism $\theta : \gamma^* S \times_{\gamma^* I} U \to X \times U$ over U.

When the topos is connected a classical (in the theory of topological coverings spaces) connectivity argument shows that all the sets S_i can be considered equal (see [4] for a proof of this in the topos context). If the topos is not connected this "single set" concept is clearly not equivalent (we can have a different set for each connected component of 1), and not the right one, as it has been observed in [2]. We comment to the reader interested in the relative theory over an arbitrary base topos that the connectivity argument depends on the excluded middle. Based on this, even when the topos is connected, in the relative case the "single set version" of the notion will not be equivalent to the "family version" of SGA4 or [7], even for connected topoi (unless the base topos is boolean). I thank here Kock for some fruitful conversations on the connectivity argument in [4], which led me to conjecture that if the argument is valid for any connected locally connected topos $\mathscr{E} \to \mathscr{S}$, then necessarily $\mathscr{S}$ has to be boolean.

(*) Assume now that $\mathscr{E}$ is locally connected. In this case, by enlarging I, we can always consider $I = \gamma_! U$. In fact, the connected components of $U = \sum_i U_i$ are the connected components of the U_i's. Repeat then the same set S_i for each connected component of U_i. The map $\zeta : U \to \gamma^* \gamma_! U$ results the unit of the adjunction $\gamma_! \dashv \gamma^*$ at U.

We consider now a locally connected topos. Given a cover U, the full subcategory $Split(U)$ of objects split by U is a topos (see [1,4]), in fact a quotient topos with inverse image given by the inclusion. Obviously the adjunction $\gamma_! \dashv \gamma^*$ restricts to $Split(U)$, and so this topos is locally connected. It is immediate to check that it is boolean (given $Z \hookrightarrow X$ in $Split(U)$), then $Z \times U \hookrightarrow X \times U$ has a complement and this implies that $Z \hookrightarrow X$ has one, (see [4]). Thus $Split(U)$ is an atomic topos, and clearly, if $\mathscr{E}$ is connected, so is $Split(U)$. These results are derived by elegant and simpler (but indirect, and less convincing in a way) arguments in [8].

Given any topos, the covers $U \to 1$ epi form a (co) filtered poset $Cov(\mathscr{E})$ if we define $U \leqslant V \Leftrightarrow \exists U \to V$. This poset has a small cofinal subset. In fact, as observed in [3], the irredundant sums of generators with global support, of which there is only a set, are cofinal.

If $U \leqslant V$, it is immediate to check that $Split(V) \subset Split(U)$, and that the inclusion is the inverse image of a geometric morphism of topoi. In this way we have a filtered inverse limit of topoi. Clearly the inverse limit site for this topos is the full subcategory of all connected locally constant objects of $\mathscr{E}$, and the topos, that we denote $GLC(\mathscr{E})$, as a category, is the full subcategory of objects generated by the locally constant objects. It follows that the inclusion is the inverse image of a geometric morphism, and $\mathscr{E} \to GLC(\mathscr{E})$ is a quotient topos. Again, the adjunction $\gamma_! \dashv \gamma^*$ obviously restricts to $GLC(\mathscr{E})$, and so this topos is locally connected. In the same way that for $Split(U)$ it is immediate to check that it is boolean, thus it is an atomic topos, and if $\mathscr{E}$ is connected, so is $GLC(\mathscr{E})$. We resume this situation in the following diagram:

$$Split(U) \leftarrow Split(V) \cdots \leftarrow GLC(\mathscr{E}) \leftarrow \mathscr{E},$$

where $GLC(\mathscr{E})$ is a filtered inverse limit of topoi indexed by the poset $Cov(\mathscr{E})$.

In [6] a push-out topos is considered in order to define categories of locally constant objects.

$$\begin{array}{ccc} \mathscr{E}/U & \xrightarrow{\varphi_U} & \mathscr{E} \\ \downarrow{\scriptstyle \rho_U} & & \downarrow{\scriptstyle \sigma_U} \\ \mathscr{S}/\gamma_! U & \xrightarrow{f_U} & \mathscr{P}_U \end{array}$$

where ρ_U and φ_U are given by $\rho_U^*(S \to \gamma_! U) = \gamma^* S \times_{\gamma^*\gamma_! U} U$ and $\varphi_U^*(X) = X \times U$. It is well known that the geometric morphism φ_U is locally connected, and it can be checked that the geometric morphism ρ_U is connected and locally connected. It follows that f_U is locally connected and that σ_U is connected locally connected (see [6, Lemma 2.3]).

Consider the construction of push outs of topoi. An object of $\mathscr{P}_U$ is a 3-tuple $\langle X, S \to \gamma_! U, \theta \rangle$, with $\theta : X \times U \to \gamma^* S \times_{\gamma^*\gamma_! U} U$ an isomorphism over U, and a morphism $\langle X, S \to \gamma_! U, \theta \rangle \to \langle X', S' \to \gamma_! U, \theta' \rangle$ is determined by a pair of morphisms $f : X \to X'$ and $\alpha : S \to S'$, the latter over $\gamma_! U$, compatible with the isomorphisms θ and θ'. The functor σ_U^* is the projection functor $\mathscr{P}_U \to \mathscr{E}$, which is then fully faithful (the reader can also check by direct inspection that $f : X \to X'$ determines $\alpha : S \to S'$ once we assume that X and X' are part of the data of U-locally constant objects in $\mathscr{E}$).

By considerations made above (*), the essential image of σ_U^* is the full subcategory $Split(U)$, thus we have:

Proposition A.1.2. *Given any locally connected topos $\mathscr{E}$ and a cover U, the push-out topos $\mathscr{P}_U$ is equivalent (as a category) via the full and faithful projection functor $\sigma_U^* : \mathscr{P}_U \xrightarrow{\cong} Split(U) \subset \mathscr{E}$ to the full subcategory $Split(U)$. Thus $\mathscr{P}_U$ and $Split(U)$ are equivalent topoi.*

The morphism f_U is actually a family of points indexed by $\gamma_! U$, and since it is locally connected, all these points are essential. The inverse image of f_U is given by $f_U^* \langle X, S \to \gamma_! U, \theta \rangle = S \to \gamma_! U$. We can prove directly:

Proposition A.1.3. *Given any locally connected topos $\mathscr{E}$ and a cover U, for each $i \in \gamma_! U$ the composite, denoted f_i, of the corresponding point of $\mathscr{S}/\gamma_! U$ with f_U is an essential point of $\mathscr{P}_U$.*

Proof. The inverse image of f_i is given by $f_i^* \langle X, S \to \gamma_! U, \theta \rangle = S_i$. It follows by the construction of inverse limits in the push-out topos that f_i^* preserves all inverse limits, that is, it is essential. □

Given any connected locally connected topos $\mathscr{E}$, it follows then from Proposition 3.2.6 that *all the results of Section* 3.2 *apply to the topoi* $\mathscr{P}_U$ *and* $Split(U)$. From Proposition 3.2.2 we have the following important fact (existence of Galois closure):

Proposition A.1.4. *Given any connected locally connected topos* $\mathscr{E}$ *and any cover* U, $Split(U) = Split(A)$, *and* $\mathscr{P}_U \cong \mathscr{P}_A$, *for* A *any representing object* (*necessarily a Galois object* (A.2), *thus in particular a connected cover*) *of one of the points* f_i.

Proposition A.1.5. *In the situation of Proposition* A.1.3, *any point* g *is isomorphic to some* f_i.

Proof. Let g be any point. Since the family $U_i \to 1$ is epimorphic it follows that there is (at least) one i with $g^*U_i \to 1$ epimorphic, thus $g^*U_i \neq \emptyset$. Given any object X split by U, since $g^*\gamma^* = id$ we have an isomorphism $S_i \times g^*U_i \xrightarrow{\cong} g^*X \times g^*U_i$ over g^*U_i. This clearly implies $g^*X \cong S_i$. □

It follows then from Proposition 3.2.1 that when the topos is connected, all the points f_i are isomorphic.

Furthermore, given any locally connected topos $\mathscr{E}$ and a connected cover U (notice that this forces $\mathscr{E}$ to be a connected topos) we have:

Proposition A.1.6. *Given any connected locally connected topos* $\mathscr{E}$ *and a connected cover* U, *the topos* $\mathscr{P}_U$ *has a canonical essential point* $f = f_U$, *and any other point is isomorphic to* f.

Here it is important to stress the fact that although there is a canonical (geometrical morphism) equivalence $Split(U) \xrightarrow{\cong} \mathscr{P}_U$, the topos $Split(U)$ *does not have a canonical point* since the equivalence does not have a canonical inverse.

A.2. Galois objects

Recall that a non-empty connected object A (called a molecule in [5]) in a topos $\mathscr{E}$ is said to be a *Galois object* if it is an $Aut(A)$-torsor. That is, if $A \to 1$ is an epimorphism, and the canonical morphism $A \times \gamma^*Aut(A) \to A \times A$ is an isomorphism. Clearly, any Galois object is a locally constant object. Notice also that non-empty connected objects in atomic topoi are *atoms* in the sense that the lattice of subobjects is 2.

After Grothendieck's "Categories Galoisiennes" of [11] and Moerdiejk "Galois topos" of [18], we state the following definition:

Definition A.2.1. A Galois topos is a connected locally connected topos generated by its Galois objects.

Notice that unlike [11,18] we do not require the topos to be pointed. Although all the applications concern pointed topoi, it is still interesting to notice that the basic theory can be developed without this assumption, as it has been shown in [6,8,15].

Since Galois objects are locally constant, clearly the canonical morphism gives an equality of topoi $\mathscr{E} = GLC(\mathscr{E})$. In particular, Galois topoi are atomic.

From the equation $\gamma^* f^* = \gamma^*$ and the fact that inverse image f^* of a connected geometric morphism $\mathscr{E} \xrightarrow{f} \mathscr{F}$ is a full and faithful left-exact functor, it immediately follows that an object A in $\mathscr{F}$ is an $Aut(A)$-torsor if and only if the object f^*A in $\mathscr{E}$ is an $Aut(f^*A)$-torsor. Furthermore, using now in addition that f^* preserves (in particular) binary coproducts, it easily follows that A is a nonempty connected object if and only if f^*A is so. In conclusion we have:

Proposition A.2.2. *Given any connected geometric morphism $\mathscr{E} \xrightarrow{f} \mathscr{F}$, an object $A \in \mathscr{F}$ is a Galois object if and only if $f^*A \in \mathscr{E}$ is so.*

Proposition A.2.3. *Any filtered inverse limit of Galois topoi and connected locally connected geometrical morphisms is a Galois topos.*

Proof. That the inverse limit topos is connected and locally connected is proved in [16]. Consider now the corresponding colimit of sites as in Section 1.1. By construction of the inverse limit site and the previous proposition it follows that the Galois objects generate the inverse limit topos. □

From this proposition and Proposition A.1.4 it follows:

Theorem A.2.4. *Given any connected locally connected topos $\mathscr{E}$, the topos $GLC(\mathscr{E})$ is a Galois topos.*

Notice that it follows the equality $GLC(GLC(\mathscr{E})) = GLC(\mathscr{E})$ (any locally constant object is split by a locally constant cover), fact which is not evident by definition.

Corollary A.2.5. *Given any connected locally connected topos $\mathscr{E}$, then $\mathscr{E}$ is a Galois topos if and only if $\mathscr{E}$ is generated by its locally constant objects if and only if $\mathscr{E} = GLC(\mathscr{E})$.*

Let $GCov(\mathscr{E})$ be the subposet of $Cov(\mathscr{E})$ whose objects are Galois objects (necessarily covers). Although *it is not cofinal*, it is also filtered. In fact, given two Galois objects A, B, consider the Galois object C such that $Split(A \times B) = Split(C)$ given by Proposition A.1.4. We have:

Proposition A.2.6. *Any Galois topos is the filtered inverse limit of the topoi $Split(A)$, $A \in GCov(\mathscr{E})$. The inverse limit site is the filtered union of the full subcategories $cSplit(A) \subset \mathscr{E}$ of connected objects split by A.*

Given any connected locally connected topos we can now synthesize the situation in the following diagram:

$$Split(A) \longleftarrow Split(B) \cdots \longleftarrow GLC(\mathscr{E}) \longleftarrow \mathscr{E}$$

where $GLC(\mathscr{E})$ is a filtered inverse limit of topoi indexed by the poset $GCov(\mathscr{E})$ whose objects are the Galois covers. $\mathscr{E}$ is a Galois topos if and only if the left-most arrow is the equality.

When $\mathscr{E}$ is a pointed topos, clearly $GLC(\mathscr{E})$ is also pointed, thus it follows that *all the results of Section* 3.3 *apply to the topos* $GLC(\mathscr{E})$.

The original definition of Galois object given in [11] was relative to a point of the topos. However, that point was surjective, and it is easy to check:

Proposition A.2.7. *Let* $\mathscr{E}$ *be a topos furnished with a surjective point* (*meaning the inverse image functor reflects isomorphisms*), $\mathscr{S} \xrightarrow{f} \mathscr{E}$. *Then, a non-empty connected object* A *is a Galois object if and only if there exists* $a \in f^*A$ *so that the map* $Aut(A) \xrightarrow{a^*} f^*A$, *defined by* $a^*(h) = f^*(h)(a)$ *is a bijection* (*the same holds then for any other element* $b \in f^*A$).

Notice that this characterization of Galois objects is word by word equal to the definition of normal extension in the classical Artin's interpretation of Galois theory.

References

[1] M. Artin, A. Grothendieck, J. Verdier, SGA 4, (1963–64), Lecture Notes in Mathematics, Vols. 269, 270, 305, Springer, Berlin, 1972.

[2] M. Barr, Abstract Galois theory II, J. Pure Appl. Algebra 25 (1982) 227–247.

[3] M. Barr, R. Diaconescu, Atomic toposes, J. Pure Appl. Algebra 17 (1980) 1–24.

[4] M. Barr, R. Diaconescu, On locally simply connected toposes and their fundamental groups, Cahiers Top. Geo. Diff. cat. 22-3 (1981) 301–304.

[5] M. Barr, R. Pare, Molecular toposes, J Pure Appl. Algebra 17 (1980) 127–152.

[6] M. Bunge, Classifying toposes and fundamental localic groupoids, in: R.A.G. Seely (Ed.), Category Theory '91, CMS Conf. Proc., Vol. 13, 1992, pp. 75–95.

[7] M. Bunge, S. Lack, A Van Kampen theorem for toposes, Preprint, 2001.

[8] M. Bunge, I. Moerdijk, On the construction of the Grothendieck fundamental group of a topos by paths, J. Pure Appl. Algebra 116 (1997) 99–113.

[9] E.J. Dubuc, Localic Galois theory, Advances in Mathematics, Vol. 175/1, 2003, 143–167.

[10] E.J. Dubuc, C.S. De La Vega, On the Galois theory of Grothendieck, Bol. Acad. Nac. Cienc. Cordoba 65 (2000) 111–136.

[11] A. Grothendieck, SGA1 (1960–1961), Springer Lecture Notes in Mathematics, Vol. 224, 1971.

[12] A. Grothendieck, in: Jean Malgoire (Ed.), La Longe Marche a travers la theorie de Galois (1980–81), Universite Montpellier II, 1995.

[13] P.T. Johnstone, Topos Theory, Academic Press, New York, 1977.

[14] A. Joyal, M. Tierney, An extension of the Galois Theory of Grothendieck, Memoirs of the American Mathematical Society, Vol. 151, 1984.

[15] J. Kennison, The fundamental localic groupoid of a topos, J. Pure Appl. Algebra 77 (1992) 67–86.

[16] I. Moerdijk, Continuous fibrations and inverse limits of toposes, Compositio Math. 58 (1986) 45–72.

[17] I. Moerdijk, The classifying topos of a continuous groupoid, Trans. Amer. Math. Soc. 310 (2) (1988) 629–668.

[18] I. Moerdijk, Prodiscrete groups and Galois toposes, Proc. Kon. Nederl. Akad. van Wetens. Series A 92-2 (1989) 219–234.

[19] G. Wraith, Localic Groups, Cahiers de Top. et Geom. Diff. Vol XXII-1 (1981) 61–66.