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Numerical study of singularly perturbed differential–difference equation arising in the modeling of neuronal variability

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ABSTRACT

The objective of this paper is to present a numerical study of a class of boundary value problems of singularly perturbed differential difference equations (SPDDE) which arise in computational neuroscience in particular in the modeling of neuronal variability. The mathematical modeling of the determination of the expected time for the generation of action potential in the nerve cells by random synaptic inputs in dendrites includes a general boundary-value problem for singularly perturbed differential difference equation with shifts. The problem considered in this paper exhibit turning point behavior which add to the complexity in the construction of numerical approximation to the solution of the problem as well as in obtaining theoretical estimates on the solution. Exponentially fitted finite difference scheme based on Il'in-Allen-Southwell fitting is used on a specially designed mesh. Some numerical examples are given to validate convergence and computational efficiency of the proposed numerical scheme. Effect of the shifts on the layer structure is illuminated for the considered examples.

1. Introduction and problem formulation

Singularly perturbed differential–difference equations are used to model a large variety of practical phenomena in many areas of sciences such as, in the study of human pupil light reflex [1], mathematical biology [2], a variety of models of physiological processes and diseases [3], control theory [4], study of bistable devices [5], etc. There are many cases where they provide best and only realistic simulation of the observed phenomena.

The determination of the expected time for the generation of action potential in the nerve cells by random synaptic inputs in the dendrites can be modeled as a first-exit time problem. Many advanced models of nerve membrane potential in the presence of random synaptic input [6–8]. Stein [9] gave a differential–difference equation model incorporating stochastic effects due to neuron excitation and later [10] he generalized the model to deal with the distribution of postsynaptic potential amplitudes. Johannesma [11] and Tuckwell [12] included the reversal potentials into account. Various other models for neuronal activity have been proposed and many are discussed in Holden's book [8].

Stein's model [9] contains assumptions concerning random synaptic inputs that excitatory impulses occur randomly and arrive according to the Poisson process $\pi(f_e, t)$, each event of which leads to an instantaneous increase in depolarization $V(t)$ by a_e, whereas inhibitory impulses arrive at event times in a second Poisson process $\pi(f_i, t)$, which is independent of $\pi(f_e, t)$ and causes $V(t)$ to decrease by a_i. The neuron fires an impulse when $V(t)$ reaches or exceeds a threshold value r units. After each neuronal firing, there is a refractory period of duration t_0, during which impulses have no effect and the membrane depolarization $V(t)$ is reset to zero. At time $t > t_0$, each impulse produces unit depolarization. For sub-threshold levels, the depolarization decays exponentially among impulses with time constant μ. The depolarization in Stein's model

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is a continuous time, continuous state space Markov process whose sample paths have discontinuities of the first kind. The time of first passage to level at or above a threshold of r units is the expected time for the generation of action potential in the nerve cells and its determination is very difficult because the resulting equations are differential difference equations.

Investigation of boundary value problems for singularly perturbed linear second-order differential–difference equations was initiated by Lange and Miura in a series of papers [13–15] to name a few. They [14] presented mathematical model of determination of expected time for the generation of action potential in the nerve cells by random synaptic inputs in the dendrites. If there are inputs distributed as a Poisson process with exponential decay between inputs as well as inputs that can be modeled as a Wiener process with variance parameter σ and drift parameter μ, the problem for expected first-exit time u, given initial membrane potential $x \in (x_1, x_2)$, can be formulated as [14]

$$\frac{\sigma^2}{2}u''(x) + (\mu - x)u'(x) + \lambda_e u(x + a_e) + \lambda_i u(x - a_i) - (\lambda_e + \lambda_i)u(x) = -1, \tag{1.1}$$

with boundary condition $u(x) \equiv 0, x \notin (x_1, x_2)$ where the values $x = x_1$ and $x = x_2$ corresponds to the inhibitory reversal potential and to the threshold value of membrane potential for action potential generation, respectively. Here the first-order derivative term $-xu'(x)$ corresponds to exponential decay between synaptic inputs. The undifferentiated terms correspond to excitatory and inhibitory synaptic inputs modeled as a Poisson process with mean rates λ_e and λ_i, respectively, and produce jumps in the membrane potential of amounts a_e and a_i respectively, which are small quantities and could be dependent on voltage.

Lange and Miura [14] considered two model problems based upon the above biological problem, one in which a small negative shift (delay) is present in the convection term and the other in which positive (advance) as well as negative shift is present in the reaction term. But in both the problems they excluded the occurrence of the turning point term $(\mu - x)u'(x)$ and left it for future study. They used extension of the method of matched asymptotic expansions developed for ODEs for finding a solution of such types of boundary value problems. Kadalbajoo and Sharma [16] gave a numerical treatment of the SPDDE's but their study is also limited to the case when the convection coefficient has the same sign throughout the domain and the case of the turning point, i.e., point where the convection coefficient vanishes in the domain is still to be investigated. Later, many authors [17–19] used numerical techniques to solve SPDDE's but their approach is limited to the case of non-turning point problems where the shifts are of $o(\varepsilon)$.

The authors are unaware of any previous attempts to construct numerical solutions for SPDDE's exhibiting turning points. We consider the second model problem of Lange and Miura [14] but include the turning point which was not treated there and are concerned with the investigation of SPDDE's with the tuning points and bigger shifts.

In this paper, we consider the following SPDDE on $\Omega = (-1, 1)$ with positive as well as negative shifts in the reaction term

$$\begin{aligned} &L^{\varepsilon,\delta,\eta}y \equiv \varepsilon y''(x) + a(x)y'(x) - b(x)y(x) + c(x)y(x - \delta) + d(x)y(x + \eta) = f(x), \\ &y(x) = \phi(x) \quad -1 - \delta \le x \le -1 \\ &y(x) = \gamma(x) \quad 1 \le x \le 1 + \eta, \end{aligned} \tag{1.2}$$

where δ, η are delay and advance arguments respectively, $a(x), b(x), f(x), c(x), d(x), \gamma(x), \phi(x)$ are sufficiently smooth functions. When the shifts are zero (i.e., $\delta = 0, \eta = 0$), solution of the corresponding ordinary differential equation exhibits a layer behavior or turning point behavior depending upon the coefficient of the convection term, i.e., whether $a(x)$ does not change sign or changes sign in the domain. The layer will be on the left or the right end of the domain depending on the sign of the coefficient of convection term, i.e., according to $a(x) < 0$ or $a(x) > 0$ on the $\bar{\Omega} = [-1, 1]$. The points of the domain where $a(x) = 0$ are known as turning points. The presence of the turning point results in a boundary or interior layer in the solution of the problem and is more difficult to handle as compared to the non-turning point case. In this paper we consider the case in which the turning point results into interior layer in the solution of the problem.

We consider the problem (1.2) with the following assumptions

$$a(0) = 0, \qquad a'(0) > 0, \tag{1.3}$$

$$b(x) \ge b_0 > 0 \quad x \in \bar{\Omega}, \tag{1.4}$$

$$b(x) - c(x) - d(x) > 0, \qquad c(x) \ge 0, \qquad d(x) \ge 0 \quad \forall x \in \bar{\Omega} = [-1, 1]. \tag{1.5}$$

To ensure that there is no other turning point in the region $[-1, 1]$ it is assumed that

$$|a'(x)| \ge \frac{|a(0)|}{2}, \quad x \in \bar{\Omega}. \tag{1.6}$$

Under above assumptions the given turning point problem possesses a unique solution exhibiting an interior layer at $x = 0$ whose nature depends upon the value of the shifts arguments. Smoothness of the solution depends upon the parameter $\beta = b(0)/a'(0)$.

2. Continuous problem

Eq. (1.2) can be written as

$$L^{\varepsilon,\delta,\eta}y \equiv \begin{cases} \varepsilon y''(x) + a(x)y'(x) - b(x)y(x) + d(x)y(x+\eta) = f(x) - c(x)\phi(x-\delta) & \text{if } -1 < x \le -1+\delta \\ \varepsilon y''(x) + a(x)y'(x) - b(x)y(x) + c(x)y(x-\delta) + d(x)y(x+\eta) = f(x) & \text{if } -1+\delta < x < 1-\eta \\ \varepsilon y''(x) + a(x)y'(x) - b(x)y(x) + c(x)y(x-\delta) = f(x) - \gamma(x+\eta) & \text{if } 1-\eta \le x < 1. \end{cases} \tag{2.1}$$

Let C be a generic positive constant independent of ε and N, which may take different values at different places, $\|.\|$ is maximum norm. For $y(x)$ to be a smooth solution of (1.2) it must be continuous on $[-1, 1]$ and continuously differentiable on $(-1, 1)$.

The operator $L^{\varepsilon,\delta,\eta}y$ satisfies the following minimum principle.

Lemma 2.1. *Let $\pi(x)$ be a smooth function satisfying $\pi(-1) \ge 0, \pi(1) \ge 0$. Then $L^{\varepsilon,\delta,\eta}\pi(x) \le 0, \forall x \in \Omega$ implies that $\pi(x) \ge 0, \forall x \in \bar{\Omega}$.*

Proof. If possible, suppose that there is a point $x^* \in [-1, 1]$ such that $\pi(x^*) = \min_{x\in\bar{\Omega}} \pi(x)$ and $\pi(x^*) < 0$. From the given conditions it is clear that $x^* \notin \{-1, 1\}$. Also $\pi'(x^*) = 0$ and $\pi''(x^*) \ge 0$.

We have the following cases:

(a) $-1 < x^* \le -1+\delta$

In this case we have,

$$\begin{aligned} L^{\varepsilon,\delta,\eta}\pi(x^*) &= \varepsilon\pi''(x^*) + a(x^*)\pi'(x^*) - b(x^*)\pi(x^*) + d(x^*)\pi(x^*+\eta) \\ &= \varepsilon\pi''(x^*) + a(x^*)\pi'(x^*) - [b(x^*) - d(x^*)]\pi(x^*) + d(x^*)[\pi(x^*+\eta) - \pi(x^*)] \\ &> 0 \end{aligned}$$

(b) $-1+\delta < x^* < 1-\eta$

$$\begin{aligned} L^{\varepsilon,\delta,\eta}\pi(x^*) &= \varepsilon\pi''(x^*) + a(x^*)\pi'(x^*) - b(x^*)\pi(x^*) + c(x^*)\pi(x^*-\delta) + d(x^*)\pi(x^*+\eta) \\ &= \varepsilon\pi''(x^*) + a(x^*)\pi'(x^*) - [b(x^*) - d(x^*) - c(x^*)]\pi(x^*) + d(x^*)[\pi(x^*+\eta) - \pi(x^*)] \\ &\quad + c(x^*)[\pi(x^*-\delta) - \pi(x^*)] \\ &> 0 \end{aligned}$$

(c) $1-\eta \le x^* < 1$

$$\begin{aligned} L^{\varepsilon,\delta,\eta}\pi(x^*) &= \varepsilon\pi''(x^*) + a(x^*)\pi'(x^*) - b(x^*)\pi(x^*) + c(x^*)\pi(x^*-\delta) \\ &= \varepsilon\pi''(x^*) + a(x^*)\pi'(x^*) - [b(x^*) - c(x^*)]\pi(x^*) + c(x^*)[\pi(x^*-\delta) - \pi(x^*)] \\ &> 0. \end{aligned}$$

Combining the above three cases we get a contradiction to the assumption $L^{\varepsilon,\delta,\eta}\pi(x) < 0$. Hence, $\pi(x) \ge 0,\ \forall x \in \bar{\Omega}$. □

Lemma 2.2. *Let $y(x)$ be solution of the boundary value problem (1.2) and $b(x) - c(x) - d(x) \ge K > 0,\ x \in \Omega$. Then we have*

$$|y(x)| \le \|f\|K^{-1} + C\max\{\|\phi\|, \|\gamma\|\}. \tag{2.2}$$

Proof. Consider the barrier function $\Psi^{\pm}(x)$ as

$$\Psi^{\pm}(x) = \|f\|K^{-1} + C\max\{\|\phi\|, \|\gamma\|\} \pm y(x),$$

we find that

$$\Psi^{\pm}(-1) \ge 0, \qquad \Psi^{\pm}(1) \ge 0.$$

We have, for $-1 < x \le -1+\delta$

$$\begin{aligned} L^{\varepsilon,\delta,\eta}\Psi^{\pm}(x) &= -b(x)\left[\|f\|K^{-1} + C\max\{\|\phi\|, \|\gamma\|\}\right] + d(x)\left[\|f\|K^{-1} + C\max\{\|\phi\|, \|\gamma\|\}\right] \\ &\quad \pm [f(x) - c(x)\phi(x-\delta)] \\ &= -\left[(b(x) - d(x))\|f\|K^{-1} \pm f(x)\right] - [(b(x) - d(x))C\max\{\|\phi\|, \|\gamma\|\}] \\ &< 0, \end{aligned}$$

for $-1+\delta < x < 1-\eta$

$$\begin{aligned} L^{\varepsilon,\delta,\eta}\Psi^{\pm}(x) &= -b(x)\left[\|f\|K^{-1} + C\max\{\|\phi\|, \|\gamma\|\}\right] + d(x)\left[\|f\|K^{-1} + C\max\{\|\phi\|, \|\gamma\|\}\right] \\ &\quad + c(x)\left[\|f\|K^{-1} + C\max\{\|\phi\|, \|\gamma\|\}\right] \pm f(x) \\ &= -\left[(b(x) - c(x) - d(x))\|f\|K^{-1} \pm f(x)\right] - [(b(x) - c(x) - d(x))C\max\{\|\phi\|, \|\gamma\|\}] \\ &< 0, \end{aligned}$$

for $1-\eta \le x < 1$

$$\begin{aligned} L^{\varepsilon,\delta,\eta}\Psi^{\pm}(x) &= -b(x)\left[\|f\|K^{-1} + C\max\{\|\phi\|, \|\gamma\|\}\right] + c(x)\left[\|f\|K^{-1} + C\max\{\|\phi\|, \|\gamma\|\}\right] \\ &\quad \pm [f(x) - d(x)\gamma(x+\eta)] \\ &= -\left[(b(x)-c(x))\|f\|K^{-1} \pm f(x)\right] - [(b(x)-c(x))C\max\{\|\phi\|, \|\gamma\|\} \pm d(x)\gamma(x+\eta)] \\ &< 0. \end{aligned}$$

Therefore, we get $L^{\varepsilon,\delta,\eta}\Psi^{\pm}(x) < 0,\ x \in \Omega$ and this implies $\Psi^{\pm}(x) \ge 0,\ x \in \bar{\Omega}$ by Lemma 2.1. □

Lemma 2.1 proves that the solution of the problem (1.2) is unique and since the problem under consideration is linear, existence is implied by its uniqueness. Lemma 2.2 gives a bound on the solution.

Now, if $[p, q]$ is a subinterval of $[-1, 1]$ which does not contain the turning point, then we have following bound on the solution and its derivatives

Theorem 2.3. *Let $a(x), b(x), c(x), d(x), f(x) \in C^m[-1, 1]$ where m is a positive integer, $|a| > \xi$, $\|a\| = \kappa$. Then, there exist positive constants C and ξ such that for $a(x) < 0$ on $[p, q]$, the solution $y(x)$ of the problem (1.2) satisfies*

$$|y^{(k)}(x)| \le C\left[1 + \varepsilon^{-k}\exp\left(\frac{-\xi(q-x)}{\varepsilon}\right)\right] \quad \textit{for } k = 1, \ldots, m+1$$

and for $a(x) > 0$ on $[p, q]$ we have

$$|y^{(k)}(x)| \le C\left[1 + \varepsilon^{-k}\exp\left(\frac{-\xi(x-p)}{\varepsilon}\right)\right] \quad \textit{for } k = 1, \ldots, m+1.$$

Proof. Let us consider the case $a(x) < 0$ and the case $a(x) > 0$ can be proved analogously. Problem (1.2) can be written as

$$\varepsilon y''(x) - k(x)y' = h(x) \tag{2.3}$$

where $k(x) = -a(x)$, $h(x) = f(x) + b(x)y(x) - c(x)y(x-\delta) - d(x)y(x+\eta)$.

The solution of the above equation is given by

$$y(x) = y_p(x) + K_1 + K_2\int_x^q \exp\left[-\varepsilon^{-1}(K(q) - K(t))\right]dt, \tag{2.4}$$

where $y_p(x) = -\int_x^q u(t)dt$, $u(t) = \int_x^q \varepsilon^{-1}h(t)\exp\left[-\varepsilon^{-1}(K(t) - K(x))\right]dt,\ x \le t$, $K(x) = \int k(x)dx$. Now, $K(t) - K(x) \ge \xi(t-x)$ which implies $\exp\left[-\varepsilon^{-1}(K(t) - K(x))\right] \le \exp\left[-\varepsilon^{-1}\xi(t-x)\right]$. Therefore

$$|z(x)| \le C\varepsilon^{-1}\int_x^q \exp\left[-\varepsilon^{-1}(K(t) - K(x))\right]dt \le C \tag{2.5}$$

and if $y(p) = d_1, y(q) = d_2$ are the taken as boundary conditions, then using this in (2.4) gives $K_1 = d_2$ and $K_2 \le C\varepsilon^{-1}$.

Differentiating equation (2.4) once we get

$$y'(x) = u(x) - K_2\exp\left[-\varepsilon^{-1}(K(q) - K(x))\right] \tag{2.6}$$

$$\begin{aligned} |y'(x)| &\le |u(x)| + K_2\exp\left[-\varepsilon^{-1}(K(q) - K(x))\right] \\ &\le C\left[1 + \varepsilon^{-1}\exp\left(\frac{-\xi(q-x)}{\varepsilon}\right)\right] \end{aligned} \tag{2.7}$$

which is the required estimate. The result for the case $a(x) < 0$ can be proved analogously. □

The above theorem gives us bounds on the derivatives of the solution of the problem (1.2) outside the turning point region. Next theorem gives us bounds on the derivatives of the solution in the neighborhood of the turning point.

Theorem 2.4. *Let, $a(x), b(x), c(x), d(x), f(x) \in C^m[-1, 1]$, β being a non-integer such that $\beta_l < \beta < \beta_s$, where $\beta_l < 1 < \beta_s$. Then, there exist a constant C depending on $S(m) = \{\|a\|_2, \|b\|_1, \|c\|_1, \|d\|_1, \|f\|_1, b_0, \beta_l, \beta_s, \|\phi\|, \|\gamma\|, \|a\|_m, \|b\|_m, \|c\|_m, \|d\|_m, \|f\|_m, m\}$ such that, for the solution $y(x)$ of the problem (1.2) we have*

$$|y^{(k)}(x)| \le C(|x| + \varepsilon^{1/2})^{\beta-k}, \quad k = 1, \ldots, m+1.$$

Proof. Eq. (1.2) can be written as

$$\varepsilon y''(x) + a'(0)xy' - b(0)y = g(x) \tag{2.8}$$

where $g(x) = f(x) + [xa'(0) - a(x)]y'(x) + [b(x) - b(0)]y(x) - c(x)y(x-\delta) - d(x)y(x+\eta)$.

Dividing both sides of the above equation by $b(0)$ and taking into account the fact that $b(0)$ is bounded between b_0 and $\|b\|_1$ we get the following equation

$$\varepsilon/b(0)y''(x) + \frac{a'(0)}{b_0}xy' - y = g(x)/b(0). \tag{2.9}$$

Now, it can be easily seen that the priori estimates are not affected by replacing ε by $\bar{\varepsilon} = \varepsilon/b(0)$ because $(x^2+\bar{\varepsilon})^{(\beta-k)/2} \le C(x^2+\varepsilon)^{(\beta-k)/2}$ and $I(x,\bar{\varepsilon},\beta) \le CI(x,\varepsilon,\beta)$, where $I(x,\varepsilon,\beta) = \int_{x^2+\varepsilon}^{6} s^{(-\beta-1)/2}\,ds$, k being a positive integer and neither is the relevant behavior of $g(x)$. Therefore, Eq. (2.9) can be reduced to the study of the following problem

$$\varepsilon y''(x) + \alpha xy' - y = g(x), \quad \alpha = 1/\beta. \tag{2.10}$$

Assuming $|a(x)| \le C|x|,\ |c(x)| \le C|x|,\ |d(x)| \le C|x|$ and following the approach of Berger et al. [20], the required estimates can be obtained. □

Note: For $\beta < 1$ Theorem 2.4 holds good and if we have $\beta > 1$ then β can be written as $\beta = \lambda + s, 0 < \lambda < 1$. In this case the estimates obtained are

$$\begin{aligned} &|y^{(k)}(x)| \le C \quad \text{for } k = 1,\ldots,s \\ &|y^{(k)}(x)| \le C(|x| + \varepsilon^{1/2})^{\beta-k} I(x,\varepsilon,\lambda), \quad \text{for } k = s+1,\ldots,m+1. \quad \square \end{aligned} \tag{2.11}$$

3. Description of the numerical scheme

To construct the discrete counterpart of the problem (1.2) we consider an exponentially fitted finite difference scheme [21] on a specially designed mesh. The presence of shifts and the turning point make the problem (1.2) difficult to deal with. To deal with the shifts and the turning point, the mesh is designed in such a way that the term containing delay/advance and the turning point lies at the mesh point after the discretization. Also, to deal with the turning point, the forward difference is used in the first derivative term if $a(x) > 0$ whereas the backward difference is used if $a(x) < 0$, $x \in \bar{\Omega}^N = \{-1 = x_0 < x_1 < x_2, \ldots, x_N = 1\}$, $x_i = -1 + ih$ where $i = 0,\ldots,N$, $h = 2/N$. Thus, the difference scheme for the boundary value problem (1.2) is given by

$$\begin{aligned} &L^N Y_i = \varepsilon\rho_i D_+D_-Y_i + a_iD_*Y_i - b_iY_i + d_iY_{i+p} + c_iY_{i-m} = f_i, \quad x_i \in \Omega^N \\ &Y_i = \phi_i, \quad i = -m, -m+1, \ldots, 0 \\ &Y_i = \gamma_i \quad i = n, n+1, \ldots, n+p \end{aligned} \tag{3.1}$$

where, $\rho_i = \rho(x_i), a_i = a(x_i),\ b_i = b(x_i), c_i = c(x_i),\ d_i = d(x_i), f_i = f(x_i), \rho_i = \frac{a_ih}{2\varepsilon}\coth\left(\frac{a_ih}{2\varepsilon}\right), D_+D_-Y_i = (Y_{i+1} - 2Y_i + Y_{i-1})/h^2, D_+Y_i = (Y_{i+1} - Y_i)/h, D_-Y_i = (Y_i - Y_{i-1})/h, x_{N_1}$ is the turning point and

$$D_* = \begin{cases} D_+Y_i & \text{if } a_i > 0 \\ D_-Y_i & \text{if } a_i < 0. \end{cases}$$

On simplification the discrete problem (3.1) gives

$$L^NY_i = \begin{cases} \text{for } i = 0,1,\ldots,m & \varepsilon\rho_iD_+D_-Y_i + a_iD_-Y_i - b_iY_i + d_iY_{i+p} = f_i - c_i\phi_{i-m} \\ \text{for } i = m+1,\ldots,N_1-1 & \varepsilon\rho_iD_+D_-Y_i + a_iD_-Y_i - b_iY_i + c_iY_{i-m} + d_iY_{i+p} = f_i \\ \text{for } i = N_1,\ldots,N-p-1 & \varepsilon\rho_iD_+D_-Y_i + a_iD_+Y_i - b_iY_i + c_iY_{i-m} + d_iY_{i+p} = f_i \\ \text{for } i = N-p,\ldots,N & \varepsilon\rho_iD_+D_-Y_i + a_iD_+Y_i - b_iY_i + c_iY_{i-m} = f(i) - d_i\gamma_{i+p} \end{cases} \tag{3.2}$$

$$\begin{aligned} &Y_0 = \phi_0 \\ &Y_N = \gamma_N. \end{aligned} \tag{3.3}$$

Corresponding to the continuous problem we have a discrete minimum principle and bound on the discrete solution.

Lemma 3.1. *Suppose $\pi_0 \ge 0$ and $\pi_N \ge 0$. Then, $L^N\pi_i \le 0$ for all $i = 1,2,\ldots,N-1$ implies that $\pi_i \ge 0$ for all $i = 0,\ldots,N$.*

Proof. Let $k \in \{0,1,\ldots,N\}$ such that $\pi_k = \min_{0\le i\le N}\pi_i$. Let us assume $\pi_k < 0$, then $k \notin \{0,N\}$ and we have $\pi_{k+1} - \pi_k \ge 0, \pi_k - \pi_{k-1} \le 0$. Now, the following cases arise:

Case 1: $k \in \{1,\ldots,m\}$. In this case, we have

$$\begin{aligned} L^N\pi_k &= \varepsilon\rho_k\left[\frac{\pi_{k+1} - 2\pi_k + \pi_{k-1}}{h^2}\right] + a_k\left[\frac{\pi_k - \pi_{k-1}}{h}\right] - b_k\pi_k + d_k\pi_{k+p} \\ &= \varepsilon\rho_k\left[\frac{(\pi_{k+1} - \pi_k) - (\pi_k - \pi_{k-1})}{h^2}\right] + a_k\left[\frac{\pi_k - \pi_{k-1}}{h}\right] - (b_k - d_k)\pi_k + d_k(\pi_{k+p} - \pi_k) \\ &> 0. \end{aligned}$$

Case 2: $k \in \{m+1, \ldots, N_1 - 1\}$.
Here,

$$L^N \pi_k = \varepsilon \rho_k \left[\frac{(\pi_{k+1} - \pi_k) - (\pi_k - \pi_{k-1})}{h^2} \right] + a_k \left[\frac{\pi_k - \pi_{k-1}}{h} \right] - b_k \pi_k + c_k \pi_{k-m} + d_k \pi_{k+p} > 0.$$

Case 3: $k \in \{N_1, \ldots, N - p - 1\}$

$$L^N \pi_k = \varepsilon \rho_k \left[\frac{(\pi_{k+1} - \pi_k) - (\pi_k - \pi_{k-1})}{h^2} \right] + a_k \left[\frac{\pi_{k+1} - \pi_k}{h} \right] - b_k \pi_k + c_k \pi_{k-m} + d_k \pi_{k+p} > 0.$$

Case 4: $k \in \{N - p, \ldots, N - 1\}$

$$L^N \pi_k = \varepsilon \rho_k \left[\frac{(\pi_{k+1} - \pi_k) - (\pi_k - \pi_{k-1})}{h^2} \right] + a_k \left[\frac{\pi_{k+1} - \pi_k}{h} \right] - b_k \pi_k + c_k \pi_{k-m} > 0.$$

Thus, for $k \in \{1, \ldots, N-1\}$ the above four cases lead to a contradiction to the hypothesis $L^N \pi_k \le 0$. Therefore, $\pi_k \ge 0$ and since k was chosen arbitrarily we have $\pi_i \ge 0$ for all $i = 0, 1, \ldots, N$. □

Lemma 3.2. *The solution Y of the discrete problem (3.2) with the boundary conditions (3.3) satisfy*

$$\|Y\| \le \|f\| k^{-1} + C \max(\|\phi\|, \|\gamma\|)$$

where $k = \min_{0 \le i \le N} \{(b(x_i) - c(x_i) - d(x_i))\}$ *and* $\|.\|$ *is maximum norm.*

Proof. Consider two barrier functions

$$\pi^{\pm} = \|f\| k^{-1} + C \max(\|\phi\|, \|\gamma\|) \pm Y_i$$

then

$$\begin{aligned} \pi_0^{\pm} &= \|f\| k^{-1} + C \max(\|\phi\|, \|\gamma\|) \pm Y_0 \\ &= \|f\| k^{-1} + C \max(\|\phi\|, \|\gamma\|) \pm \phi_0 \\ &\ge 0 \end{aligned}$$

and

$$\begin{aligned} \pi_N^{\pm} &= \|f\| k^{-1} + C \max(\|\phi\|, \|\gamma\|) \pm Y_N \\ &= \|f\| k^{-1} + C \max(\|\phi\|, \|\gamma\|) \pm \gamma_N \\ &\ge 0. \end{aligned}$$

Now, the following four cases arise

(a) $1 < i \le m$

$$\begin{aligned} L^N \pi_i^{\pm} &= \varepsilon D_+ D_- \pi_i^{\pm} + a_i D_- \pi_i^{\pm} - b_i \pi_i^{\pm} + d_i \pi_{i+p}^{\pm} \\ &= -b_i \left[\|f\| k^{-1} + C \max(\|\phi\|, \|\gamma\|) \right] + d_i \left[\|f\| k^{-1} + C \max(\|\phi\|, \|\gamma\|) \right] \pm L^N Y_i \\ &= [-(b_i - d_i)\|f\| k^{-1} \pm f_i] - (b_i - d_i) C \max(\|\phi\|, \|\gamma\|) \pm c_i \phi_{i-m} \\ &\le 0, \end{aligned}$$

(b) $m + 1 \le i \le N_1 - 1$

$$\begin{aligned} L^N \pi_i^{\pm} &= \varepsilon D_+ D_- \pi_i^{\pm} + a_i D_- \pi_i^{\pm} - b_i \pi_i^{\pm} + c_i \pi_{i-m}^{\pm} + d_i \pi_{i+p}^{\pm} \\ &= -b_i \left[\|f\| k^{-1} + C \max(\|\phi\|, \|\gamma\|) \right] + d_i \left[\|f\| k^{-1} + C \max(\|\phi\|, \|\gamma\|) \right] \\ &\quad + c_i \left[\|f\| k^{-1} + C \max(\|\phi\|, \|\gamma\|) \right] \pm L^N Y_i \\ &= [-(b_i - c_i - d_i)\|f\| k^{-1} \pm f_i] - (b_i - c_i - d_i) C \max(\|\phi\|, \|\gamma\|) \\ &\le 0, \end{aligned}$$

(c) $N_1 \le i \le N-p-1$

$$\begin{aligned} L^N\pi_i^{\pm} &= \varepsilon D_+D_-\pi_i^{\pm} + a_iD_+\pi_i^{\pm} - b_i\pi_i^{\pm} + c_i\pi_{i-m}^{\pm} + d_i\pi_{i+p}^{\pm} \\ &= -b_i\left[\|f\|k^{-1} + C\max(\|\phi\|, \|\gamma\|)\right] + d_i\left[\|f\|k^{-1} + C\max(\|\phi\|, \|\gamma\|)\right] \\ &\quad + c_i\left[\|f\|k^{-1} + C\max(\|\phi\|, \|\gamma\|)\right] \pm L^N Y_i \\ &= [-(b_i - c_i - d_i)\|f\|k^{-1} \pm f_i] - (b_i - c_i - d_i)C\max(\|\phi\|, \|\gamma\|) \\ &\le 0, \end{aligned}$$

(d) $N-p \le i < N$

$$\begin{aligned} L^N\pi_i^{\pm} &= \varepsilon D_+D_-\pi_i^{\pm} + a_iD_+\pi_i^{\pm} - b_i\pi_i^{\pm} + c_i\pi_{i-m}^{\pm} \\ &= -b_i\left[\|f\|k^{-1} + C\max(\|\phi\|, \|\gamma\|)\right] + c_i\left[\|f\|k^{-1} + C\max(\|\phi\|, \|\gamma\|)\right] \pm L^N Y_i \\ &= [-(b_i - c_i)\|f\|k^{-1} \pm f_i] - b_iC\max(\|\phi\|, \|\gamma\|) \pm d_i\gamma_{i+p} \\ &\le 0. \end{aligned}$$

From the above four cases we obtain $L^N\pi_i \le 0$ for $i = 1, \dots, N-1$ and using Lemma 3.1 this gives us $\pi_i \ge 0$ for $i = 0, \dots, N$. □

Now, the following theorem gives bounds on the truncation error.

Theorem 3.3. *If Y is the solution of the discrete problem (3.2) corresponding to the solution y of the problem (1.2) then, the truncation error is estimated by*

$$|\tau_i^N| = |L^N y_i - L^N Y_i| = |L^N y_i - L^{\varepsilon,\delta,\eta} y_i| \le \mathrm{Ch}^{\min(\beta,1)}, \quad y_i = y(x_i),\ 0 \le i \le N.$$

Proof. We will prove the result for the case $a(x) \ge 0$ and the case $a(x) < 0$ can be proved analogously. Now,

$$\begin{aligned} |\tau_i^N| &= |L^N y_i - L^{\varepsilon,\delta,\eta} y_i| \\ &= |\varepsilon_i^0 D_+D_-y_i - \varepsilon y_i'' + a(x_i)D_+y_i - a(x_i)y_i'|, \quad (\text{where } \varepsilon_i^0 = \varepsilon\rho_i) \\ &= |\varepsilon_i^0 D_+D_-y_i - \varepsilon y_i'' + a(x_i)(D_+y_i - y_i') + \varepsilon_i^0 y_i'' - \varepsilon_i^0 y_i''| \\ &\le |\varepsilon_i^0|\,|D_+D_-y_i - y_i''| + |\varepsilon_i^0 - \varepsilon|\,|y_i''| + |a(x_i)|\,|D_+y_i - y_i'|. \end{aligned} \tag{3.4}$$

For the Il'in-Allen-Southwell scheme the following result holds good [21]

$$|\varepsilon_i^0 - \varepsilon| \le \mathrm{Ch}(|a(x_i)| + h) \tag{3.5}$$

and as $|a(x_i)| \le \mathrm{Ch}$ in the turning point region this gives us $|\varepsilon_i^0 - \varepsilon| \le \mathrm{Ch}^2$. Using this and the Taylor series expansion we get

$$\begin{aligned} |a(x_i)|\,|D_+y_i - y_i'| &\le \mathrm{Ch}^2|y_i''| + O(h^3) \\ &\le \mathrm{Ch}^2(|x_i| + \varepsilon^{1/2})^{\beta-2}. \end{aligned} \tag{3.6}$$

For $\beta > 2$ we have $(|x_i| + \varepsilon^{1/2})^{\beta-2} < C$ and $\beta < 2$ implies $(|x_i| + \varepsilon^{1/2})^{\beta-2} < h^{\beta-2}$. Using this we obtain

$$|a(x_i)|\,|D_+y_i - y_i'| \le \mathrm{Ch}^{\min(\beta,2)} \tag{3.7}$$

and,

$$|\varepsilon_i^0 - \varepsilon|\,|y_i''| \le \mathrm{Ch}^{\min(\beta,2)}. \tag{3.8}$$

Now, we show that the above bound also holds for the first term in the expression for the truncation error. We have,

$$\begin{aligned} |\varepsilon_i^0|\,|D_+D_-y_i - y_i''| &\le C(\varepsilon^{1/2} + h)^2h^2|y_i^{iv}| \\ &\le C(\varepsilon^{1/2} + h)^2h^2(|x_i| + \varepsilon^{1/2})^{\beta-4}. \end{aligned}$$

Considering three cases, i.e., $|x_i| > h$, $|x_i| = h$, and $|x_i| = 0$ separately we get for each case

$$|\varepsilon_i^0|\,|D_+D_-y_i - y_i''| \le \mathrm{Ch}^{\min(\beta,2)}. \tag{3.9}$$

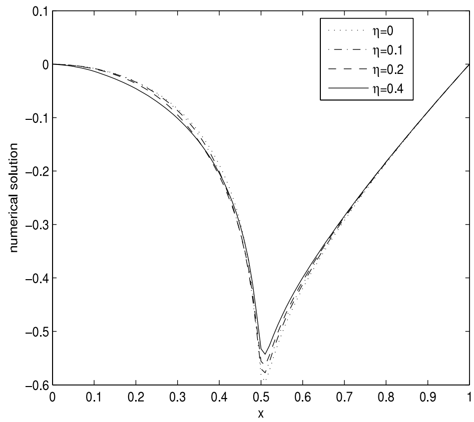

Fig. 1. The numerical solution for Example 1 when $\delta = 0$ ($\varepsilon = 0.0001$).

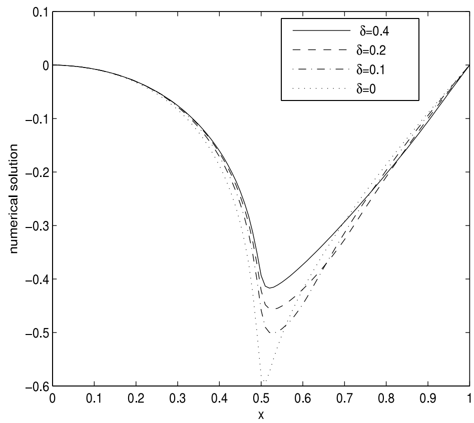

Fig. 2. The numerical solution for Example 1 when $\eta = 0$ ($\varepsilon = 0.0001$).

Also, away from the turning point region our solution is smooth and we have

$$|a(x_i)|\,|D_+y_i - y_i'| \le \mathrm{Ch} \tag{3.10}$$

$$|\varepsilon_i^0 - \varepsilon|\,|y_i''| \le \mathrm{Ch}(|a(x_i)| + h) \le \mathrm{Ch} \tag{3.11}$$

$$\begin{aligned}|\varepsilon_i^0|\,|D_+D_-y_i - y_i''| &\le C(\varepsilon + h)h^2|y^{iv}(x)| \\ &\le \mathrm{Ch}^2.\end{aligned} \tag{3.12}$$

Finally, combining the truncation error for the turning point region and outer region we get following bound on the truncation error

$$|\tau_i^N| \le \mathrm{Ch}^{\min(\beta,1)} \tag{3.13}$$

which is the desired bound. □

Remark. The consistency result give above together with the uniform stability result proves uniform convergence of the proposed numerical scheme.

4. Test examples and numerical results

In this section, we apply the proposed numerical scheme to some test problems. Since the exact solution for the considered problems are not available, the maximum absolute errors E_ε^N are evaluated using the double mesh principle [22] for the proposed numerical scheme

$$E_\varepsilon^N = \max_{x\in\bar{\Omega}^N} |v_j^N - v_j^{2N}|$$

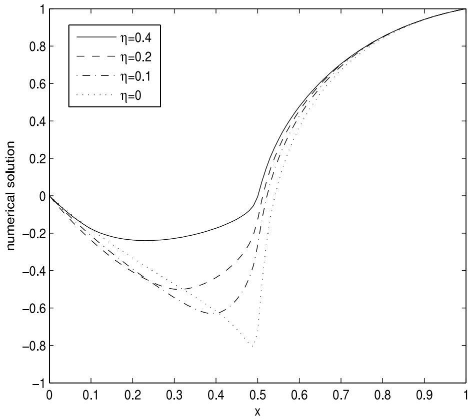

Fig. 3. The numerical solution for Example 2 when $\delta = 0$ ($\varepsilon = 0.0001$).

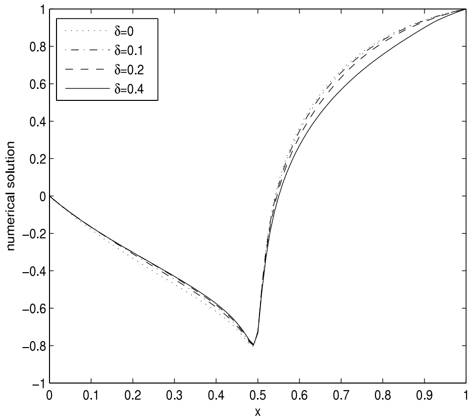

Fig. 4. The numerical solution for Example 2 when $\eta = 0$ ($\varepsilon = 0.0001$).

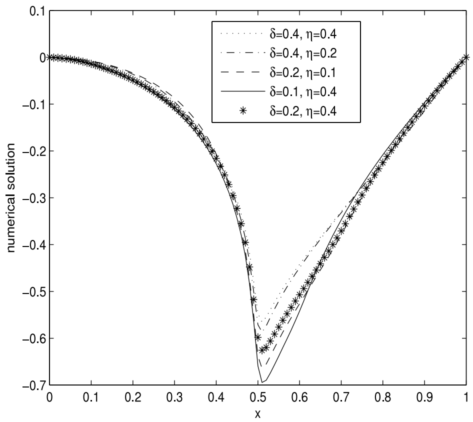

Fig. 5. The numerical solution for Example 1 when δ as well as $\eta \neq 0$ ($\varepsilon = 0.0001$).

where v_j^N and v_j^{2N} are the computed solutions by taking N and $2N$ points, respectively. The numerical rates of convergence are computed using the formula

$$r_N = \log_2(E_\varepsilon^N / E_\varepsilon^{2N}).$$

Example 1. Consider the problem on $x \in (0, 1)$ with the turning point at $x = 0.5$

$$\varepsilon y''(x) + 2(x - 0.5)[1 + 0.3121(x - 0.5)]y'(x) - [4/3 + 0.2764(x - 0.5)]y(x) + 0.2y(x - \delta) + 1/8y(x + \eta) = x$$
$$y(x) = 0, \quad -1 - \delta \le x \le -1, \qquad y(x) = 0, \quad 1 \le x \le 1 + \eta.$$

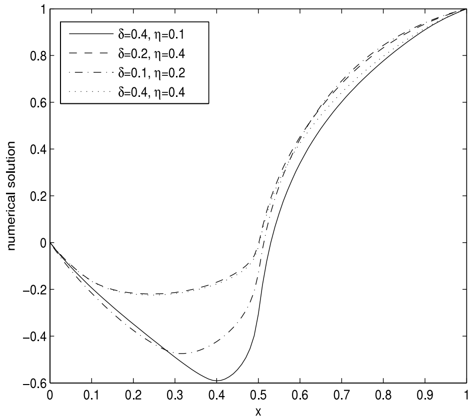

Fig. 6. The numerical solution for Example 2 when δ as well as $\eta \neq 0$ ($\varepsilon = 0.0001$).

Table 1
Maximum pointwise error E_ε^N for $\delta = 0.2$, $\eta = 0.1$.

$\varepsilon \downarrow$	$n \rightarrow$ 100	200	400	800
Example 1				
1	4.387e−5	2.219e−5	1.116e−5	5.596e−6
10^{-1}	6.963e−4	3.54e−4	1.785e−4	8.962e−5
10^{-2}	1.802e−3	8.788e−4	4.337e−4	2.154e−4
10^{-4}	6.390e−3	3.230e−3	1.478e−3	6.707e−4
10^{-6}	6.784e−3	4.295e−3	2.708e−3	1.660e−3
10^{-8}	6.784e−3	4.295e−3	2.708e−3	1.704e−3
10^{-10}	6.784e−3	4.295e−3	2.708e−3	1.704e−3
10^{-12}	6.784e−3	4.295e−3	2.708e−3	1.704e−3
10^{-14}	6.784e−3	4.295e−3	2.708e−3	1.704e−3
10^{-16}	6.784e−3	4.295e−3	2.708e−3	1.704e−3
10^{-18}	6.784e−3	4.295e−3	2.708e−3	1.704e−3
Example 2				
1	7.018e−5	3.683e−5	1.885e−5	9.539e−6
10^{-1}	2.585e−3	1.295e−3	6.481e−4	3.242e−4
10^{-2}	8.054e−3	3.798e−3	1.838e−3	9.034e−4
10^{-4}	2.431e−2	1.263e−2	5.967e−3	2.781e−3
10^{-6}	2.492e−2	1.550e−2	9.549e−3	5.825e−3
10^{-8}	2.492e−2	1.550e−2	9.55e−3	5.868e−3
10^{-10}	2.492e−2	1.550e−2	9.55e−3	5.868e−3
10^{-12}	2.492e−2	1.550e−2	9.55e−3	5.868e−3
10^{-14}	2.492e−2	1.550e−2	9.55e−3	5.868e−3
10^{-16}	2.492e−2	1.550e−2	9.55e−3	5.868e−3
10^{-18}	2.492e−2	1.550e−2	9.55e−3	5.868e−3

Example 2. Consider the problem on $x \in (0, 1)$ with the turning point at $x = 0.5$

$$\varepsilon y''(x) + (x - 0.5)[3 + 4(x - 0.5)]y'(x) - 2y(x) + 4(x - 0.5)^2 y(x - \delta) + y(x + \eta) = 1$$
$$y(x) = 0, \quad 1 - \delta \le x \le -1, \qquad y(x) = 1, \quad 1 \le x \le 1 + \eta.$$

5. Discussion

Singularly perturbed differential–difference equations exhibiting turning point behavior and having positive as well as negative shifts in the reaction term are considered. An exponentially fitted finite difference scheme is constructed on a specially designed mesh. Numerical experiments are carried out to support the theoretical estimates and illustrate the effect of shifts on the layer behavior of the solution. (Tables 1–9) give maximum pointwise error for the considered examples for various values of δ and η and it is seen that the rate of convergence is independent of the value of the delay/advance argument.

Graphs of the solution are plotted to illustrate the effect of the shifts on the layer behavior of the solution in all the three possible cases, i.e., when (i) only a positive shift is present in the problem, (ii) only a negative shift is present in the problem,

Table 2
Maximum pointwise error E_ε^N for $\delta = 0.2,\ \eta = 0.2$.

$\varepsilon \downarrow$	$n \rightarrow$ 100	200	400	800
Example 1				
1	4.443e−5	2.25e−5	1.132e−5	5.677e−6
10^{-1}	7.212e−4	3.677e−4	1.857e−4	9.328e−5
10^{-2}	1.978e−3	9.608e−4	4.733e−4	2.348e−4
10^{-4}	7.395e−3	3.744e−3	1.729e−3	7.929e−4
10^{-6}	7.994e−3	5.085e−3	3.202e−3	1.956e−3
10^{-8}	7.994e−3	5.085e−3	3.202e−3	2.006e−3
10^{-10}	7.994e−3	5.085e−3	3.202e−3	2.006e−3
10^{-12}	7.994e−3	5.085e−3	3.202e−3	2.006e−3
10^{-14}	7.994e−3	5.085e−3	3.202e−3	2.006e−3
10^{-16}	7.994e−3	5.085e−3	3.202e−3	2.006e−3
10^{-18}	7.994e−3	5.085e−3	3.202e−3	2.006e−3
Example 2				
1	6.295e−5	3.301e−5	1.69e−5	8.548e−6
10^{-1}	2.226e−3	1.115e−3	5.581e−4	2.791e−4
10^{-2}	6.464e−3	3.037e−3	1.467e−3	7.202e−4
10^{-4}	1.82e−2	9.453e−3	4.441e−3	2.067e−3
10^{-6}	1.865e−2	1.150e−2	7.058e−3	4.303e−3
10^{-8}	1.865e−2	1.150e−2	7.058e−3	4.334e−3
10^{-10}	1.865e−2	1.150e−2	7.058e−3	4.334e−3
10^{-12}	1.865e−2	1.150e−2	7.058e−3	4.334e−3
10^{-14}	1.865e−2	1.150e−2	7.058e−3	4.334e−3
10^{-16}	1.865e−2	1.150e−2	7.058e−3	4.334e−3
10^{-18}	1.865e−2	1.150e−2	7.058e−3	4.334e−3

Table 3
Maximum pointwise error E_ε^N for $\delta = 0.2$, $\eta = 0.4$.

$\varepsilon \downarrow$	$n \rightarrow$ 100	200	400	800
Example 1				
1	4.425e−5	2.240e−5	1.127e−5	5.651e−6
10^{-1}	7.223e−4	3.681e−4	1.858e−4	9.333e−5
10^{-2}	1.761e−3	8.576e−4	4.229e−4	2.099e−4
10^{-4}	6.981e−3	3.532e−3	1.632e−3	7.502e−4
10^{-6}	7.558e−3	4.828e−3	3.047e−3	1.864e−3
10^{-8}	7.558e−3	4.828e−3	3.047e−3	1.912e−3
10^{-10}	7.558e−3	4.828e−3	3.047e−3	1.912e−3
10^{-12}	7.558e−3	4.828e−3	3.047e−3	1.912e−3
10^{-14}	7.558e−3	4.828e−3	3.047e−3	1.912e−3
10^{-16}	7.558e−3	4.828e−3	3.047e−3	1.912e−3
10^{-18}	7.558e−3	4.828e−3	3.047e−3	1.912e−3
Example 2				
1	5.127e−5	2.685e−5	1.374e−5	6.949e−6
10^{-1}	1.802e−3	9.020e−4	4.511e−4	2.256e−4
10^{-2}	4.901e−3	2.294e−3	1.104e−3	5.414e−4
10^{-4}	1.249e−2	6.557e−3	3.084e−3	1.410e−3
10^{-6}	1.281e−2	7.836e−3	4.801e−3	2.938e−3
10^{-8}	1.281e−2	7.836e−3	4.801e−3	2.957e−3
10^{-10}	1.281e−2	7.836e−3	4.801e−3	2.957e−3
10^{-12}	1.281e−2	7.836e−3	4.801e−3	2.957e−3
10^{-14}	1.281e−2	7.836e−3	4.801e−3	2.957e−3
10^{-16}	1.281e−2	7.836e−3	4.801e−3	2.957e−3
10^{-18}	1.281e−2	7.836e−3	4.801e−3	2.957e−3

(iii) both types of shifts are present in the problem. Figs. 1 and 3 shows the effect on the solution behavior for Examples 1 and 2, respectively, due to presence of the positive shift only (i.e., $\delta = 0$, $\eta \neq 0$). Figs. 2 and 4 are plotted for the test Examples 1 and 2, respectively, for the case when only a negative shift is present (i.e., $\eta = 0$, $\delta \neq 0$). The effect on the interior layer behavior of the solution by the presence of mixed type of shifts is illustrated by Figs. 5 and 6 for the test Examples 1 and 2,

Table 4
Maximum pointwise error E_ε^N for $\delta = 0.1$, $\eta = 0.1$.

$\varepsilon \downarrow$	$n \rightarrow$			
	100	200	400	800
Example 1				
1	4.468e−5	2.264e−5	1.14e−5	5.717e−6
10^{-1}	7.09e−4	3.623e−4	1.831e−4	9.205e−5
10^{-2}	2.422e−3	1.179e−3	5.821e−4	2.891e−4
10^{-4}	9.517e−3	4.785e−3	2.215e−3	1.014e−3
10^{-6}	1.023e−2	6.486e−3	4.068e−3	2.477e−3
10^{-8}	1.023e−2	6.486e−3	4.068e−3	2.538e−3
10^{-10}	1.023e−2	6.486e−3	4.068e−3	2.538e−3
10^{-12}	1.023e−2	6.486e−3	4.068e−3	2.538e−3
10^{-14}	1.023e−2	6.486e−3	4.068e−3	2.538e−3
10^{-16}	1.023e−2	6.486e−3	4.068e−3	2.538e−3
10^{-18}	1.023e−2	6.486e−3	4.068e−3	2.538e−3
Example 2				
1	6.818e−5	3.581e−5	1.835e−5	9.285e−6
10^{-1}	2.731e−3	1.368e−3	6.849e−4	3.426e−4
10^{-2}	8.28e−3	3.911e−3	1.895e−3	9.319e−4
10^{-4}	2.484e−2	1.288e−2	6.088e−3	2.836e−3
10^{-6}	2.546e−2	1.581e−2	9.738e−3	5.938e−3
10^{-8}	2.546e−2	1.581e−2	9.739e−3	5.983e−3
10^{-10}	2.546e−2	1.581e−2	9.739e−3	5.983e−3
10^{-12}	2.546e−2	1.581e−2	9.739e−3	5.983e−3
10^{-14}	2.546e−2	1.581e−2	9.739e−3	5.983e−3
10^{-16}	2.546e−2	1.581e−2	9.739e−3	5.983e−3
10^{-18}	2.546e−2	1.581e−2	9.739e−3	5.983e−3

Table 5
Maximum pointwise error E_ε^N for $\delta = 0.1$, $\eta = 0.2$.

$\varepsilon \downarrow$	$n \rightarrow$			
	100	200	400	800
Example 1				
1	4.458e−5	2.258e−5	1.137e−5	5.701e−6
10^{-1}	7.002e−4	3.576e−4	1.807e−4	9.082e−5
10^{-2}	2.289e−3	1.112e−3	5.477e−4	2.717e−4
10^{-4}	9.031e−3	4.542e−3	2.11e−3	9.704e−4
10^{-6}	9.725e−3	6.183e−3	3.882e−3	2.364e−3
10^{-8}	9.725e−3	6.183e−3	3.882e−3	2.422e−3
10^{-10}	9.725e−3	6.183e−3	3.882e−3	2.422e−3
10^{-12}	9.725e−3	6.183e−3	3.882e−3	2.422e−3
10^{-14}	9.725e−3	6.183e−3	3.882e−3	2.422e−3
10^{-16}	9.725e−3	6.183e−3	3.882e−3	2.422e−3
10^{-18}	9.725e−3	6.183e−3	3.882e−3	2.422e−3
Example 2				
1	6.1e−5	3.202e−5	1.64e−5	8.299e−6
10^{-1}	2.352e−3	1.178e−3	5.896e−4	2.949e−4
10^{-2}	6.666e−3	3.137e−3	1.516e−3	7.448e−4
10^{-4}	1.844e−2	9.568e−3	4.501e−3	2.091e−3
10^{-6}	1.890e−2	1.16e−2	7.097e−3	4.322e−3
10^{-8}	1.890e−2	1.16e−2	7.097e−3	4.353e−3
10^{-10}	1.890e−2	1.16e−2	7.097e−3	4.353e−3
10^{-12}	1.890e−2	1.16e−2	7.097e−3	4.353e−3
10^{-14}	1.890e−2	1.16e−2	7.097e−3	4.353e−3
10^{-16}	1.890e−2	1.16e−2	7.097e−3	4.353e−3
10^{-18}	1.890e−2	1.16e−2	7.097e−3	4.353e−3

respectively. It is observed that interior layer is maintained but layer get shifted as delay/advance argument changes. Shifts in the layer depends upon the type of shift as well as on the value of the coefficients of the term containing delay/advance.

Table 6
Maximum pointwise error E_ε^N for $\delta = 0.1$, $\eta = 0.4$.

$\varepsilon \downarrow$	$n \rightarrow$ 100	200	400	800
Example 1				
1	4.44e−5	2.249e−5	1.131e−5	5.675e−6
10^{-1}	7.022e−4	3.584e−4	1.810e−4	9.097e−5
10^{-2}	2.044e−3	9.948e−4	4.905e−4	2.435e−4
10^{-4}	8.422e−3	4.220e−3	1.957e−3	9.018e−4
10^{-6}	9.093e−3	5.816e−3	3.668e−3	2.240e−3
10^{-8}	9.093e−3	5.816e−3	3.668e−3	2.297e−3
10^{-10}	9.093e−3	5.816e−3	3.668e−3	2.297e−3
10^{-12}	9.093e−3	5.816e−3	3.668e−3	2.297e−3
10^{-14}	9.093e−3	5.816e−3	3.668e−3	2.297e−3
10^{-16}	9.093e−3	5.816e−3	3.668e−3	2.297e−3
10^{-18}	9.093e−3	5.816e−3	3.668e−3	2.297e−3
Example 2				
1	4.933e−5	2.588e−5	1.325e−5	6.704e−6
10^{-1}	1.917e−3	9.595e−4	4.800e−4	2.400e−4
10^{-2}	5.138e−3	2.407e−3	1.16e−3	5.687e−4
10^{-4}	1.283e−2	6.753e−3	3.170e−3	1.448e−3
10^{-6}	1.316e−2	7.999e−3	4.884e−3	2.984e−3
10^{-8}	1.316e−2	7.999e−3	4.884e−3	3.002e−3
10^{-10}	1.316e−2	7.999e−3	4.884e−3	3.002e−3
10^{-12}	1.316e−2	7.999e−3	4.884e−3	3.002e−3
10^{-14}	1.316e−2	7.999e−3	4.884e−3	3.002e−3
10^{-16}	1.316e−2	7.999e−3	4.884e−3	3.002e−3
10^{-18}	1.3165e−2	7.999e−3	4.884e−3	3.002e−3

Table 7
Maximum pointwise error E_ε^N for $\delta = 0.4$, $\eta = 0.1$.

$\varepsilon \downarrow$	$n \rightarrow$ 100	200	400	800
Example 1				
1	4.387e−5	2.219e−5	1.116e−5	5.596e−6
10^{-1}	6.963e−4	3.54e−4	1.785e−4	8.961e−5
10^{-2}	1.801e−3	8.788e−4	4.337e−4	2.154e−4
10^{-4}	6.390e−3	3.230e−3	1.478e−3	6.707e−4
10^{-6}	6.784e−3	4.295e−3	2.708e−3	1.660e−3
10^{-8}	6.784e−3	4.295e−3	2.708e−3	1.704e−3
10^{-10}	6.784e−3	4.295e−3	2.708e−3	1.704e−3
10^{-12}	6.784e−3	4.295e−3	2.708e−3	1.704e−3
10^{-14}	6.784e−3	4.295e−3	2.708e−3	1.704e−3
10^{-16}	6.784e−3	4.295e−3	2.708e−3	1.704e−3
10^{-18}	6.784e−3	4.295e−3	2.708e−3	1.704e−3
Example 2				
1	7.375e−5	3.856e−5	1.971e−5	9.962e−6
10^{-1}	2.150e−3	1.077e−3	5.387e−4	2.694e−4
10^{-2}	6.97e−3	3.301e−3	1.600e−3	7.871e−4
10^{-4}	2.255e−2	1.175e−2	5.539e−3	2.574e−3
10^{-6}	2.312e−2	1.452e−2	9.0e−3	5.512e−3
10^{-8}	2.312e−2	1.452e−2	9.0e−3	5.552e−3
10^{-10}	2.312e−2	1.452e−2	9.0e−3	5.552e−3
10^{-12}	2.312e−2	1.452e−2	9.0e−3	5.552e−3
10^{-14}	2.312e−2	1.452e−2	9.0e−3	5.552e−3
10^{-16}	2.312e−2	1.452e−2	9.0e−3	5.552e−3
10^{-18}	2.312e−2	1.452e−2	9.0e−3	5.552e−3

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Table 8
Maximum pointwise error E_ε^N for $\delta = 0.4$, $\eta = 0.2$.

$\varepsilon \downarrow$	$n \rightarrow$ 100	200	400	800
Example 1				
1	4.377e−5	2.214e−5	1.113e−5	5.581e−6
10^{-1}	6.891e−4	3.501e−4	1.765e−4	8.861e−5
10^{-2}	1.700e−3	8.259e−4	4.068e−4	2.018e−4
10^{-4}	5.967e−3	3.032e−3	1.389e−3	6.317e−4
10^{-6}	6.417e−3	4.093e−3	2.592e−3	1.593e−3
10^{-8}	6.417e−3	4.093e−3	2.591e−3	1.635e−3
10^{-10}	6.417e−3	4.093e−3	2.591e−3	1.635e−3
10^{-12}	6.417e−3	4.093e−3	2.591e−3	1.635e−3
10^{-14}	6.417e−3	4.093e−3	2.591e−3	1.635e−3
10^{-16}	6.417e−3	4.093e−3	2.591e−3	1.635e−3
10^{-18}	6.417e−3	4.093e−3	2.591e−3	1.635e−3
Example 2				
1	6.645e−005	3.471e−5	1.773e−5	8.961e−6
10^{-1}	1.853e−003	9.278e−4	4.642e−4	2.321e−4
10^{-2}	5.601e−003	2.644e−3	1.28e−3	6.288e−4
10^{-4}	1.703e−002	8.894e−3	4.176e−3	1.936e−3
10^{-6}	1.747e−002	1.088e−2	6.713e−3	4.11e−3
10^{-8}	1.747e−002	1.088e−2	6.713e−3	4.139e−3
10^{-10}	1.747e−002	1.088e−2	6.713e−3	4.139e−3
10^{-12}	1.747e−002	1.088e−2	6.713e−3	4.139e−3
10^{-14}	1.747e−002	1.088e−2	6.713e−3	4.139e−3
10^{-16}	1.747e−002	1.088e−2	6.713e−3	4.139e−3
10^{-18}	1.747e−002	1.088e−2	6.713e−3	4.139e−3

Table 9
Maximum pointwise error E_ε^N for $\delta = 0.4$, $\eta = 0.4$.

$\varepsilon \downarrow$	$n \rightarrow$ 100	200	400	800
Example 1				
1	4.360e−5	2.204e−5	1.108e−5	5.556e−6
10^{-1}	6.911e−4	3.510e−4	1.769e−4	8.880e−5
10^{-2}	1.495e−3	7.284e−4	3.592e−4	1.783e−4
10^{-4}	5.653e−3	2.887e−3	1.324e−3	6.033e−4
10^{-6}	6.154e−3	3.954e−3	2.512e−3	1.546e−3
10^{-8}	6.154e−3	3.954e−3	2.512e−3	1.5867−3
10^{-10}	6.154e−3	3.954e−3	2.512e−3	1.587e−3
10^{-12}	6.154e−3	3.954e−3	2.512e−3	1.587e−3
10^{-14}	6.154e−3	3.954e−3	2.512e−3	1.587e−3
10^{-16}	6.154e−3	3.954e−3	2.512e−3	1.586e−3
10^{-18}	6.154e−3	3.954e−3	2.512e−3	1.587e−3
Example 2				
1	5.466e−5	2.85e−5	1.454e−5	7.347e−6
10^{-1}	1.495e−3	7.474e−4	3.737e−4	1.868e−4
10^{-2}	4.157e−3	1.957e−3	9.457e−4	4.644e−4
10^{-4}	1.211e−2	6.336e−3	2.981e−3	1.362e−3
10^{-6}	1.243e−2	7.748e−3	4.797e−3	2.951e−3
10^{-8}	1.243e−2	7.748e−3	4.797e−3	2.970e−3
10^{-10}	1.243e−2	7.748e−3	4.797e−3	2.970e−3
10^{-12}	1.243e−2	7.748e−3	4.797e−3	2.970e−3
10^{-14}	1.243e−2	7.748e−3	4.797e−3	2.970e−3
10^{-16}	1.243e−2	7.748e−3	4.797e−3	2.970e−3
10^{-18}	1.243e−2	7.748e−3	4.797e−3	2.970e−3

References

[1] A. Longtin, J. Milton, Complex oscillations in the human pupil light reflex with mixed and delayed feedback, Math. Biosci. 90 (1988) 183–199.
[2] J.D. Murray, Mathematical Biology I: An Introduction, 3rd ed., Springer-Verlag, Berlin, 2001.
[3] M.C. Mackey, L. Glass, Oscillations and chaos in physiological control systems, Science 197 (1977) 287–289.
[4] E.L. Els'gol'ts, Qualitative Methods in Mathematical Analysis, in: Translations of Mathematical Monographs, vol. 12, American Mathematical Society, Providence, RI, 1964.

[5] M.W. Derstein, H.M. Gibbs, F.A. Hopf, D.L. Kaplan, Bifurcation gap in a hybrid optical system, Phys. Rev. A 26 (1982) 3720–3722.

[6] J.P. Segundo, D.H. Perkel, H. Wyman, H. Hegstad, G.P. Moore, Input–output relations in computer simulated nerve cell: influence of the statistical properties, strength, number and inter-dependence of excitatory pre-dependence of excitatory pre-synaptic terminals, Kybernetik 4 (1968) 157–171.

[7] S.E. Fienberg, Stochastic models for a single neuron firing trains: a survey, Biometrics 30 (1974) 399–427.

[8] A.V. Holden, Models of the Stochastic Activity of Neurons, Springer-Verlag, New York, 1976.

[9] R.B. Stein, A theoretical analysis of neuronal variability, Biophys. J. 5 (1965) 173–194.

[10] R.B. Stein, Some models of neuronal variability, Biophys. J. 7 (1967) 37–68.

[11] P.I.M. Johannesma, Diffusion models of the stochastic activity of neurons, in: E.R. Caianello (Ed.), Neural Networks: Proceedings of the School on Neural Networks, Ravello, 1967, Springer-Verlag, New York, 1968, pp. 116–144.

[12] H.C. Tuckwell, Synaptic transmission in a model for stochastic neural activity, J. Theoret. Biol. 77 (1979) 65–81.

[13] C.G. Lange, R.M. Miura, Singular perturbation analysis of boundary value problems for differential–difference equations, SIAM J. Appl. Math. 42 (1982) 502–531.

[14] C.G. Lange, R.M. Miura, Singular perturbation analysis of boundary value problems for differential–difference equations. V. Small shifts with layer behavior, SIAM J. Appl. Math. 54 (1994) 249–272.

[15] C.G. Lange, R.M. Miura, Singular perturbation analysis of boundary value problems for differential–difference equations. VI. Small shifts with rapid oscillations, SIAM J. Appl. Math. 54 (1994) 273–283.

[16] M.K. Kadalbajoo, K.K. Sharma, Numerical treatment of a mathematical model arising from a model of neuronal variability, J. Math. Anal. Appl. 307 (2005) 606–627.

[17] M.K. Kadalbajoo, V.P. Ramesh, Numerical methods on Shishkin mesh for singularly perturbed delay differential equations with a grid adaptation strategy, Appl. Math. Comput. 188 (2007) 1816–1831.

[18] M.K. Kadalbajoo, K.K. Sharma, A numerical method based on finite difference for boundary value problems for singularly perturbed delay differential equations, Appl. Math. Comput. 197 (2008) 692–707.

[19] K.C. Patidar, K.K. Sharma, Uniformly convergent non-standard finite difference methods for singularly perturbed differential–difference equations with delay and advance, Internat. J. Numer. Methods Engrg. 66 (2006) 272–296.

[20] A.E. Berger, H. Han, R.B. Kellog, A priori estimates and analysis of a numerical method for a turning point problem, Math. Comp. 42 (166) (1984) 465–492.

[21] P.A. Farrell, Sufficient conditions for the uniform convergence of a difference scheme for a singularly perturbed turning point problem, SIAM J. Numer. Anal. 25 (3) (1988) 618–643.

[22] E.P. Doolan, J.J.H. Miller, W.H.A. Schilders, Uniform Numerical Methods for Problems with Initial and Boundary Layers, Boole Press, Dublin, 1980.