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Some Considerations for Linear Integrodifferential Equations

WOLFGANG DESCH

Universität Graz, Institut für Mathematik, Brandhofgasse 18, A-8010 Graz, Austria

RONALD GRIMMER*

Department of Mathematics, Southern Illinois University, Carbondale, Illinois 62901

AND

WILHELM SCHAPPACHER†

Universität Graz, Institut für Mathematik, Elisabethstrasse 16, A-8010 Graz, Austria

Submitted by Kenneth L. Cooke

1. INTRODUCTION

Let $(X, \|.\|)$ be a real or complex Banach space. Our objective is to study the linear integrodifferential equation

$$\begin{aligned} &\frac{d}{dt} x(t) = Ax(t) + \int_0^t B(t-s)\, x(s)\, ds + f(t), \qquad t \geqslant 0, \\ &x(0) = x_0. \end{aligned} \tag{1}$$

The basic approach used here is to treat (1) as a perturbation of another integrodifferential equation or of the equation

$$\begin{aligned} \frac{d}{dt} x(t) &= Ax(t), \qquad t \geqslant 0, \\ x(0) &= x_0. \end{aligned}$$

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We obtain a number of fundamental connections between the solutions of the corresponding Cauchy problems. In fact, our existence theorem is perhaps best viewed as a perturbation theorem, obtaining solutions of an integrodifferential equation if the kernel is an appropriate perturbation of a kernel of an integrodifferential equation for which the existence problem has been previously settled. This result is quite flexible and should allow wide application in conjunction with other existence theorems. We also obtain some representation formulas which will yield the semigroup generated by A as the limit of products involving the resolvent operator associated with (1). These results can then be used to obtain results concerning invariant sets. This is motivated by the study of positivity by Clement and Nohel [2] but should have a number of additional applications.

Throughout this paper we assume that A and $B(.)$ satisfy the following conditions:

(H1) A is a densely defined, closed linear operator in X. Hence $D(A)$ endowed with the graph norm $|x| = \|x\| + \|Ax\|$ is a Banach space which will be denoted by $(Y, |.|)$.

(H2) $(B(t); t \geqslant 0)$ is a family of linear operators on X so that $B(t)$ is continuous when regarded as a linear map from $(Y, |.|)$ into $(X, \|.\|)$ for almost all $t \geqslant 0$. Moreover, there is a locally integrable function $b: \mathbb{R}^+ \to \mathbb{R}^+$ so that $B(t)y$ is measurable and $\|B(t)y\| \leqslant b(t)|y|$ for all $y \in Y$ and $t \geqslant 0$.

(H3) For any $y \in Y$, the map $t \to B(t)y$ belongs to $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^+, X)$ and

$$\left\| \frac{d}{dt} B(t)y \right\| \leqslant b(t)|y|.$$

DEFINITION. Let $x_0 \in Y$. A solution $x(.)$ of (1) is a function that belongs to $C([0, \infty), Y) \cap C^1([0, \infty), X)$ so that $x(0) = x_0$ and (1) is satisfied for all $t \geqslant 0$.

The definition above leads naturally to our definition of a resolvent operator:

DEFINITION. A family $\{R(t); t \geqslant 0\}$ of continuous linear operators on X is called a resolvent (to Eq. (1)), iff

(R1) $R(0) = I$, the identity map on X,

(R2) for all $x \in X$, the map $t \to R(t)x$ is a continuous function $[0, \infty) \to X$,

(R3) for all $t \geqslant 0$, $R(t)$ is a continuous linear operator on Y, and for all $y \in Y$, the map $t \to R(t)y$ belongs to $C([0, \infty), Y) \cap C^1([0, \infty), X)$ and satisfies

$$\text{(R3, i)} \quad \frac{d}{dt} R(t)\, y = AR(t)\, y + \int_0^t B(t-s)\, R(s)\, y\, ds$$

and

$$\text{(R3, ii)} \quad \frac{d}{dt} R(t)\, y = R(t)\, Ay + \int_0^t R(t-s)\, B(s)\, y\, ds.$$

For a general discussion of resolvent operators and their most important properties we refer to [5–8]. We note, however, that the resolvent is unique if it exists [5, 7].

The basic result for the considerations to follow is

PROPOSITION 1 (Integration by Parts Formula). *Let $T(.)$ be a family of linear operators so that* (R1)–(R3) *hold. Let $y \in W^{1,1}([a, b], Y)$ with $0 \leqslant a < b < \infty$.*

$$\int_a^b T(t)\, y'(t)\, dt \qquad \text{and} \qquad \int_a^b T'(t)\, y(t)\, dt$$

exist and

$$\int_a^b T(t)\, y'(t)\, dt + \int_a^b T'(t)\, y(t)\, dt = T(b)\, y(b) - T(a)\, y(a).$$

Proof. If y is piecewise linear, this formula follows by an easy computation. For general $y \in W^{1,1}([a, b], Y)$ we consider a sequence of piecewise linear functions y_n that converge to y. Making use of Lebesgue's dominated convergence theorem the result follows (cf. also [4]).

Remark. We shall frequently use Proposition 1 with $T(.) = R(.)$.

We next present some basic properties of $R(.)$:

PROPOSITION 2. *Assume that* (H1) *and* (H2) *hold. Let $R(.)$ be a family of continuous linear operators on X satisfying* (R1), (R2), (R3) *and* (R3,i). *Then we have*:

(a) *For any $f \in W^{1,1}_{\mathrm{loc}}([0, \infty), X)$ the function*

$$x(t) = \int_0^t R(t-s)\, f(s)\, ds$$

is a solution of (1) *with $x_0 = 0$.*

(b) *If $x = 0$ is the unique solution of* (1) *with $x_0 = 0$ and $f = 0$, then $R(.)$ also satisfies* (R3,ii), *i.e., $R(.)$ is a resolvent.*

(c) *Let* $R(.)$ *be a resolvent of* (1). *If there is a function* $x \in C([0, \infty), Y) \cap C^1([0, \infty), X)$, *so that*

$$x'(t) = Ax(t) + \int_0^t B(t-s)\, x(s)\, ds + f(t), \qquad x(0) = x_0,$$

for some $f \in C([0, \infty), X)$, *then*

$$x(t) = R(t)\, x_0 + \int_0^t R(t-s) f(s)\, ds.$$

Proof. First, let $f \in W^{1,1}_{\mathrm{loc}}([0, \infty), Y)$. It is easily seen that $x(t) = \int_0^t R(t-s) f(s)\, ds$ belongs to $C([0, \infty), Y) \cap C^1([0, \infty), X)$. Moreover,

$$\begin{aligned} \frac{d}{dt} x(t) &= \frac{d}{dt} \int_0^t R(t-s) f(s)\, ds \\ &= R(t) f(0) + \int_0^t R(t-s) f'(s)\, ds. \end{aligned}$$

By Proposition 1 the right-hand side equals

$$\begin{aligned} &R(t) f(0) + \int_0^t R'(t-s) f(s)\, ds - R(t) f(0) + R(0) f(t) \\ &\quad = f(t) + \int_0^t AR(t-s) f(s)\, ds + \int_0^t \int_0^{t-s} B(t-s-u)\, R(u) f(s)\, du\, ds \\ &\quad = f(t) + Ax(t) + \int_0^t B(u) \int_0^{t-u} R(t-u-s) f(s)\, ds\, du \\ &\quad = f(t) + Ax(t) + \int_0^t B(u)\, x(t-u)\, du. \end{aligned}$$

This calculation also shows that

$$\begin{aligned} Ax(t) = R(t) f(0) + \int_0^t R(s) f'(t-s)\, ds - f(t) \\ - \int_0^t B(s)\, x(t-s)\, ds. \end{aligned}$$

Consequently, there is a constant M independent of f, so that

$$|x(t)| \leqslant M\, \|f\|_{W^{1,1}([0,t],X)} + \int_0^t b(s)\, |x(t-s)|\, ds.$$

By Gronwall's inequality we infer that

$$|x(t)| \leqslant \kappa(t)\, \|f\|_{W^{1,1}([0,t],X)}$$

for $\kappa(t) = M \exp(\int_0^t b(s)\, ds)$. This in turn implies that for fixed $T > 0$ there is a constant N so that

$$\|x\|_{C([0,T],Y)} \leqslant N\, \|f\|_{W^{1,1}([0,T],X)},$$

and

$$\|x\|_{C^1([0,T],X)} \leqslant N\, \|f\|_{W^{1,1}([0,T],X)}.$$

Taking now an arbitrary $f \in W^{1,1}([0,T],X)$, we approximate f by a sequence $\{f_n\} \subset W^{1,1}([0,T],Y)$. Using standard techniques, we obtain (a). In order to verify (b), we put for any $x \in Y$

$$y(t) = x + \int_0^t R(s)\, Ax\, ds + \int_0^t \int_0^s R(s-u)\, B(u)\, x\, du\, ds.$$

Obviously,

$$\frac{d}{dt} y(t) = R(t)\, Ax + \int_0^t R(t-s)\, B(s)\, x\, ds.$$

Rewriting y as

$$y(t) = x + \int_0^t R(t-s)\, Ax\, ds + \int_0^t R(t-s) \int_0^s B(u)\, x\, du\, ds$$

and making use of part (a) we obtain

$$\begin{aligned} \frac{d}{dt}(y(t)-x) &= A(y(t)-x) + \int_0^t B(t-s)(y(s)-x)\, ds \\ &\quad + Ax + \int_0^t B(t-s)\, x\, ds \\ &= Ay(t) + \int_0^t B(t-s)\, y(s)\, ds. \end{aligned}$$

Thus $z(t) = y(t) - R(t)\, x$ satisfies

$$\frac{d}{dt} z(t) = Az(t) + \int_0^t B(t-s)\, z(s)\, ds, \qquad z(0) = 0.$$

Hence $z(t)=0$, $y(t)=R(t)x$, and

$$\frac{d}{dt}R(t)x=R(t)Ax+\int_0^t R(t-s)B(s)x\,ds$$

as claimed.

Regarding (c), cf. [7, Theorem 2.3].

As a consequence we obtain

COROLLARY. *Let A be a closed, densely defined linear operator in X, $B(t)=0$ for all $t\geqslant 0$, and $R(t)$ be a resolvent for* (1). *Then $R(t)$ is a C_0-semigroup with infinitesimal generator A.*

Proof. By Proposition 2 the solutions of (1) are unique and hence the semigroup property holds. Applying the result of Phillips [10, p. 108] we obtain that $R(.)$ is a C_0-semigroup.

Obviously, Y is invariant under $R(.)$. Therefore the core theorem (see, for instance, [3, p. 8] or [9]) implies that the closure of A is the infinitesimal generator of $R(.)$. As A is closed itself, it is the generator.

Remark. We note that the function $\int_0^t R(t-s)f(s)\,ds$ has been previously shown to be a solution of (1) with $x_0=0$ when $f\in C([0,\infty),Y)$ (cf. [5] or [7]). Obtaining Proposition 2(a) without any smoothness conditions on $B(t)$ yields the desired parallel with results obtained for differential equations in a Banach space.

2. AN EXISTENCE RESULT

As already pointed out, we want to treat (1) as a perturbation problem. To do so, we consider a problem of the kind

$$\frac{d}{dt}x(t)=Ax(t)+\int_0^t B(t-s)x(s)\,ds+\int_0^t C(t-s)x(s)\,ds \quad \text{for} \quad t\geqslant 0;\, x(0)=x_0. \tag{2}$$

To begin with, we integrate the variation of constants formula (Proposition 2) by parts.

LEMMA 1. *Suppose A satisfies* (H1), *$B(.)$ satisfies* (H2) *and $C(.)$ satisfies* (H2) *and* (H3). *Let $R(.)$ be a resolvent operator for* (1) *and $S(.)$ be a resolvent operator for* (2). *Then*

$$S(t)x-R(t)x=\int_0^t R(t-s)Q(s)x\,ds$$

where the operators Q are defined by

$$Q(s)\,x = \int_0^s C'(s-u) \int_0^u S(v)\,x\,dv\,du + C(0) \int_0^s S(u)\,x\,du.$$

The operators $Q(.)$ are uniformly bounded on bounded intervals, and for each $x \in X$, $Q(.)\,x$ belongs to $C([0, \infty), X)$.

Proof. By Proposition 2(a) we conclude that the operators

$$I(t)\,x = \int_0^t S(s)\,x\,ds$$

map X into Y, and for all $x \in X$, $I(.)\,x$ is a continuous function from $[0, \infty)$ into $(Y, |.|)$. Consequently, $\{I(t)\}$ is a family of continuous linear operators from $(X, \|.\|)$ into $(Y, |.|)$ that are uniformly bounded for t in a bounded set. Hence

$$Q(t)\,x = \int_0^t C'(t-s)\,I(s)\,x\,ds + C(0)\,I(t)\,x$$

is uniformly bounded for t in bounded intervals and continuous in t for fixed $x \in X$. For $x \in Y$, $S(.)\,x \in C([0, \infty), Y) \cap C^1([0, \infty), X)$ and $S(.)\,x$ satisfies (2) for $t \geqslant 0$. Applying Proposition 2(c) and Proposition 1, we obtain

$$\begin{aligned} &S(t)\,x - R(t)\,x \\ &\quad = \int_0^t R(t-s) \int_0^s C(s-u)\,S(u)\,x\,du\,ds \\ &\quad = \int_0^t R(t-s) \left[C(0) \int_0^s S(u)\,x\,du + \int_0^s C'(s-u) \int_0^u S(v)\,x\,dv\,du \right] ds. \end{aligned}$$

By continuous extension, the claim holds for all $x \in X$.

We now state our main theorem which concerns the existence of resolvent operators. This result can be used in conjunction with the results of [6], [7] or [8] to obtain the existence of a resolvent operator when the kernel is not continuous in t. In particular, this theorem is a perturbation theorem to be used with other results.

THEOREM 1. *Let A be a closed, densely defined operator in X, and $B(.)$*

and $C(.)$ be families of linear operators satisfying (H2) *and* (H2), (H3), *respectively. Then* (1) *admits a* (*unique*) *resolvent iff* (2) *does.*

Proof. By symmetry it is sufficient to assume that (1) admits a resolvent $R(.)$ and to show that (2) does too. We shall construct the resolvent for (2) by the variation of constants formula given in Lemma 1, using a contraction argument. We fix $T>0$ and consider the function space $\mathscr{X}=\{x\in C([0,T],X)\colon \int_0^{\cdot} x(s)\,ds\in C([0,T],Y)\}$. It is clear from Lemma 1 and Proposition 2(a) that for a resolvent $S(.)$ of (2) the function $x(t)=S(t)\,x_0$ is a fixed point of the map $\mathscr{E}$:

$$(\mathscr{E}x)(t)=R(t)\,x_0 + \int_0^t R(t-s)\left[C(0)\int_0^s x(u)\,du+\int_0^s C'(s-u)\int_0^u x(v)\,dv\,du\right]ds.$$

By hypothesis, there is a constant $M>0$, so that

$$\|R(t)\|\leqslant M^{1/2},\qquad \left|\int_0^t R(s)\,x\,ds\right|\leqslant M^{1/2}\,\|x\|,$$

$$\|C(0)\,y\|\leqslant M^{1/2}\,|y|,\qquad \int_0^t b(s)\,ds\leqslant M^{1/2}$$

for all $x\in X$, $y\in Y$, $0\leqslant t\leqslant T$. We norm $\mathscr{X}$ by

$$|||x|||=\sup_{0\leqslant t\leqslant T}\max\left(e^{-4Mt}\,\|x(t)\|,e^{-4Mt}\left|\int_0^t x(s)\,ds\right|\right).$$

Then we obtain for $x,z\in\mathscr{X}$, $0\leqslant t\leqslant T$, putting $g=x-z$,

$$\begin{aligned}
&e^{-4Mt}\,\|\mathscr{E}x(t)-\mathscr{E}z(t)\|\\
&\quad=e^{-4Mt}\left\|\int_0^t R(t-s)\left[C(0)\int_0^s g(u)\,du+\int_0^s C'(s-u)\int_0^u g(v)\,dv\,du\right]ds\right\|\\
&\quad\leqslant e^{-4Mt}\int_0^t M^{1/2}\left(M^{1/2}e^{4Ms}\,|||g|||+\int_0^s b(s-u)\,e^{4Mu}\,|||g|||\,du\right)ds\\
&\quad\leqslant\left(M\int_0^t e^{4M(s-t)}\,ds+M^{1/2}\int_0^t e^{4M(u-t)}\int_0^{t-u} b(s)\,ds\,du\right)|||g|||\\
&\quad\leqslant 2M\int_0^t e^{4M(s-t)}\,ds\,|||g|||\leqslant\frac{1}{2}\,|||x-z|||.
\end{aligned}$$

Similarly, we show

$$\left| \int_0^t (\mathscr{E}x(s) - \mathscr{E}z(s))\,ds \right| e^{-Mt}$$

$$= e^{-4Mt} \left| \int_0^t \int_0^s R(s-u) \left[C(0) \int_0^u g(v)\,dv \right.\right.$$

$$\left.\left. + \int_0^u C'(u-v) \int_0^v g(w)\,dw\,dv \right] du\,ds \right|$$

$$= e^{-4Mt} \left| \int_0^t \left(\int_0^{t-u} R(s)\,ds \right) \left[C(0) \int_0^u g(v)\,dv \right.\right.$$

$$\left.\left. + \int_0^u C'(u-v) \int_0^v g(w)\,dw\,dv \right] du \right|$$

$$\leqslant \frac{1}{2} |||x - z|||$$

by the same estimate. Consequently, $|||\mathscr{E}x - \mathscr{E}z||| \leqslant \frac{1}{2} |||x - z|||$, which implies that $\mathscr{E}$ admits a unique fixed point x in $\mathscr{X}$.

Defining $S(t)\,x_0 = x(t)$, it is obvious that $S(t)\,x_0$ is a continuous function of t and $S(0)\,x_0 = x_0$, i.e., $S(.)$ satisfies (R1) and (R2). Given $x_0, x_1 \in X$, we construct $\mathscr{E}_0, \mathscr{E}_1$ as above and obtain

$$|||\mathscr{E}_0(0) - \mathscr{E}_1(0)||| = |||R(.)(x_0 - x_1)||| \leqslant M^{1/2} \|x_0 - x_1\|.$$

Therefore, by standard contraction arguments, the fixed points $S(.)\,x_0$, $S(.)\,x_1$ satisfy

$$|||S(.)\,x_0 - S(.)\,x_1||| \leqslant 2M^{1/2} \|x_0 - x_1\|.$$

Consequently the operators $S(.)$ are bounded.

Now let x_0 belong to Y. We consider the subspace $\mathscr{X}_1 = C([0, T], Y)$ of $\mathscr{X}$, normed by

$$|||x|||_1 = \sup_{0 \leqslant t \leqslant T} e^{-4Mt} |x(t)|,$$

with the same constant M as above. For $x \in \mathscr{X}_1$, Proposition 1 yields

$$x(t) = R(t)\,x_0 + \int_0^t R(t-s) \int_0^s C(s-u)\,x(u)\,du\,ds.$$

Applying Proposition 1 once more, we have

$$x(t) = R(t)\,x_0 + \int_0^t \left(\int_0^{t-s} R(u)\,du\right)\left[\int_0^s C'(s-u)\,x(u)\,du + C(0)\,x(s)\right] ds.$$

The same arguments as above show that $\mathscr{E}$ is a contraction on $\mathscr{X}_1$ with respect to $|||.|||_1$. Therefore the fixed point $x = S(.)\,x_0$ lies in $\mathscr{X}_1$.

Since $\int_0^t C(t-s)\,x(s)\,ds$ belongs to $W^{1,1}([0,T],X)$, we may apply Proposition 2(a) to the fixed point equation

$$x(t) = R(t)\,x_0 + \int_0^t R(t-s)\int_0^s C(s-u)\,x(u)\,du\,ds$$

and obtain that $x \in C^1([0,T],X)$ and

$$x'(t) = Ax(t) + \int_0^t B(t-s)\,x(s)\,ds + \int_0^t C(t-s)\,x(s)\,ds.$$

Consequently $S(.)$ satisfies (R3,i). By Proposition 2(b) and the uniqueness of the fixed point of $\mathscr{E}$ (with $x_0 = 0$) we also obtain (R3,ii).

COROLLARY. *Let A and $B(.)$ satisfy* (H1), (H2) *and* (H3). *Then* (1) *admits a resolvent operator iff A generates a C_0-semigroup.*

Proof. This is obtained from Theorem 1 by replacing C by B and B by 0.

EXAMPLE. Consider the equation

$$\begin{aligned} x'(t) &= Ax(t) + \int_0^t (t-s)^{-1/2}\,Ax(s)\,ds, \\ x(0) &= x_0 \end{aligned} \tag{3}$$

where A generates an analytic semigroup and satisfies $\|(\lambda I - A)^{-1}\| \leqslant M/|\lambda|$, $\lambda = re^{i\theta}$, $r \geqslant 0$, $|\theta| < \theta_1$, where θ_1 is a constant greater than $3\pi/4$. One can then show, using [7, Theorem 3.1], that (3) has an analytic resolvent operator. It follows now from Theorem 1 that

$$\begin{aligned} x'(t) &= Ax(t) + \int_0^t [(t-s)^{-1/2} + C(t)]\,Ax(s)\,ds, \\ x(0) &= x_0 \end{aligned}$$

has a resolvent operator if $C(t) \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^+, \mathbb{R})$.

As an application we can clarify the connection between compactness properties of $T(.)$ and $R(.)$.

DEFINITION. A resolvent $R(.)$ is called compact iff $R(t)$ is a compact operator for all $t > 0$.

THEOREM 2. *Let A be the infinitesimal generator of a C_0-semigroup and $T(.)$ and $B(.)$ satisfy* (H2) *and* (H3). *Then the resolvent $R(.)$ for* (1) *is compact iff $T(.)$ is.*

Proof. Assume that $R(.)$ is compact. By Lemma 1 we have

$$T(t)\,x = R(t)\,x + \int_0^t R(t-s)\,Q(s)\,x\,ds$$

with

$$Q(s)\,x = -\int_0^s B'(s-u)\int_0^u T(v)\,x\,dv\,du - B(0)\int_0^s T(u)\,x\,du$$

and $\|Q(s)\| \leqslant M$ for $0 \leqslant s \leqslant t$. We choose K so that $\|R(s)\| \leqslant K$ for $0 \leqslant s \leqslant t$. Then

$$\begin{aligned} T(t)\,x = {} & R(h)\left[R(t-h)\,x + \int_0^{t-h} R(t-h-s)\,Q(s)\,x\,ds\right] \\ & - [R(h)\,R(t-h)\,x - R(t)\,x] \\ & - \int_0^{t-h} [R(h)\,R(t-h-s) - R(t-s)]\,Q(s)\,x\,ds \\ & + \int_{t-h}^t R(t-s)\,Q(s)\,x\,ds. \end{aligned} \tag{4}$$

In order to estimate these terms, we first show that the family $\{h^{-1}(R(h)\,R(s) - R(s+h)) : 0 \leqslant s \leqslant t, 0 < h < t\}$ is uniformly bounded. In fact, we have for all $x \in Y$,

$$\begin{aligned} & h^{-1}\,\|R(t+h)\,x - R(h)\,R(t)\,x\| \\ & \quad = h^{-1}\left\|\int_0^h R(h-s)\int_0^s B(s+h-u)\,R(u)\,x\,du\,ds\right\| \\ & \quad = h^{-1}\left\|\int_0^h R(h-s)\,W_h(s)\,x\,ds\right\| \end{aligned}$$

with

$$W_h(s)\,x = B(h)\int_0^s R(u)\,x\,ds + \int_0^s B'(s+h-u)\int_0^u R(v)\,x\,dv\,du.$$

Therefore, $h^{-1}\|R(h)R(t)-R(t+h)\| \leqslant L$ with

$$L = \sup\{\|R(s)\| \, \|W_h(s)\| : 0 \leqslant s \leqslant t, 0 \leqslant h \leqslant t\}.$$

Now we see from (3) that

$$\begin{aligned}\|T(t)x - R(h)T(t-h)x\| & \\ &\leqslant (hL + (t-h)hLM + hKM)\|x\| \leqslant h(L + tLM + KM)\|x\|.\end{aligned}$$

Hence the compact operators $R(h)T(t-h)$ converge to $T(t)$ in the operator norm as $h \to 0$, which implies that $T(t)$ is compact. The converse is proved by the same technique.

3. Representation Formulas

If $T(t)$ is a C_0-semigroup with infinitesimal generator A, then it is well known [10] that we have the representation formula

$$T(t)x = \lim_{n\to\infty}\left(I - \frac{t}{n}A\right)^{-n} x.$$

The Generalized Lax Equivalence Theorem [1] provides a complete description of all such representations. We are going to show that the analogous result does not hold for resolvents. In fact we have:

THEOREM 3. *Suppose A generates a C_0-semigroup $T(t)$ and A and $B(.)$ satisfy* (H1)–(H3). *If $R(t)$ is the resolvent for* (1), *then for all $x \in X$ and $t > 0$*

$$T(t)x = \lim_{n\to\infty} R\left(\frac{t}{n}\right)^n x,$$

the convergence being uniform on bounded intervals for fixed x.

Proof. In order to apply the Generalized Lax Equivalence Theorem [1] we show that $R(t/n)^n$ is stable, i.e., the operators $R(t/n)^n$ are uniformly bounded for $n \in \mathbb{N}$ and t in a bounded interval, and that for $x \in Y$

$$\lim_{h\to 0+} h^{-1}(R(h)x - x) = Ax.$$

The latter is true by (R3). To obtain stability, we assume without loss of generality that $\|T(t)\| \leqslant e^{\omega t}$ for some $\omega > 0$ and all $t > 0$, by renorming if necessary [10]. Let

$$Q(t)\,x = B(0) \int_0^t R(s)\,x\,ds + \int_0^t B'(t-s) \int_0^s R(u)\,x\,du\,ds.$$

By Lemma 1 (applied to problem (1) instead of (2)) we have that the operators $Q(t)$ are uniformly bounded for t in a bounded interval and that

$$R(t)\,x = T(t)\,x + \int_0^t T(t-s)\,Q(s)\,x\,ds.$$

Let $\alpha = \sup_{0 \leqslant t \leqslant T} \|Q(t)\|$. Then for $t \in [0, T]$ we have

$$\begin{aligned} \|R(t)\,x\| &= \left\| T(t)\,x + \int_0^t T(t-s)\,Q(s)\,x\,ds \right\| \\ &\leqslant \left(e^{\omega t} + \int_0^t e^{\omega(t-s)} \alpha\,ds \right) \|x\| \leqslant (1 + \alpha t)\,e^{\omega t}\,\|x\| \\ &\leqslant e^{(\omega+\alpha)t}\,\|x\|. \end{aligned}$$

Consequently, for $t \in [0, T]$ we have

$$\left\| R\left(\frac{t}{n}\right)^n x \right\| \leqslant e^{(\omega+\alpha)t}\,\|x\|.$$

An obvious consequence of this Theorem is a result which has bearing on a number of problems including positivity (cf. [2]).

THEOREM 4. *Suppose A generates a C_0-semigroup $T(t)$ and A and $B(.)$ satisfy* (H1)–(H3) *and $R(t)$ is the resolvent for* (1). *Let K be a closed subset of X such that $R(t)\,K \subset K$ for all $t > 0$. Then we have also $T(t)\,K \subset K$ for all $t > 0$.*

In passing we note that the converse is not true as is seen by the following simple example.

Let $X = \mathbb{R}$, $A = 2$, $B(t) = -2$ and $K = \mathbb{R}^+$. Then $R(t) = e^t(\cos t + \sin t)$ which is clearly not positive for all $t > 0$. Nevertheless, $T(t) = e^{2t}$ is positive and $T(t)\,K \subset K$. However, it is clear that $R(.)$ is positive if both $T(.)$ and $B(.)$ are positive.

We note that the approximation formula for a semigroup $T(t)$ is given in terms of

$$(\lambda - A)^{-1} = \hat{T}(\lambda)$$

where $\hat{T}$ denotes the Laplace transform of T:

$$\hat{T}(\lambda)\, x = \int_0^\infty e^{-\lambda t} T(t)\, x\, dt.$$

We are going to investigate a similar formula for

$$\rho(\lambda) = (\lambda I - A - \hat{B}(\lambda))^{-1}$$

which is formally $\hat{R}(\lambda)$.

We give next a condition which implies existence of $\rho(\lambda)$ on a half plane and the expected representation formula.

LEMMA 2. *Let A be the infinitesimal generator of a C_0-semigroup $T(.)$ satisfying $\|T(t)\| \leqslant e^{\omega t}$ with some $\omega > 1$. For all $t > 0$ let $B(t)$ be a bounded linear operator $Y \to X$ so that for all $x \in Y$, $B(.)\, x$ is measurable and $e^{-\gamma t} \|B(t)\, x\| \leqslant \beta\, |x|$ $(\gamma > 0, \beta > 0)$.*

Then we have:

(a) *For each $\lambda > \gamma$ and all $x \in Y$, $\hat{B}(\lambda)\, x$ exists and satisfies $\|\hat{B}(\lambda)\, x\| \leqslant \beta/(\lambda - \gamma)\, |x|$.*

(b) *For each $\lambda > 2 \max(\gamma, \omega, 8\beta)$,*

$$\rho(\lambda) = (\lambda I - A - \hat{B}(\lambda))^{-1}$$

exists and is a continuous linear operator on X with $\|\rho(\lambda)\| \leqslant 1/(\lambda - \kappa)$, where $\kappa = \omega + 8\beta$.

Proof. Part (a) is an easy exercise in analysis. To prove (b), we first note that

$$\|(\lambda I - A)^{-1}\| \leqslant (\lambda - \omega)^{-1}$$

and for all $x \in X$,

$$|(\lambda I - A)^{-1}\, x| \leqslant [\lambda/(\lambda - \omega) + 2]\, \|x\|.$$

Hence we obtain for $\lambda > 2 \max(\gamma, \omega, 8\beta)$

$$\|\hat{B}(\lambda)(\lambda I - A)^{-1}\| \leqslant \left(\frac{\beta}{\lambda - \gamma}\right)\left(\frac{3\lambda - 2\omega}{\lambda - \omega}\right) \leqslant \frac{8\beta}{\lambda} < 1.$$

Consequently, $(I - \hat{B}(\lambda)(\lambda I - A)^{-1})$ is invertible, and putting

$$\rho(\lambda) = (\lambda I - A)^{-1}[I - \hat{B}(\lambda)(\lambda I - A)^{-1}]^{-1}$$

we obtain

$$\|\rho(\lambda)\| \leqslant \frac{1}{\lambda-\omega}\frac{1}{1-8\beta/\lambda} \leqslant \frac{1}{\lambda-\omega-8\beta}.$$

So it remains to show that $\rho(\lambda)$ is the inverse of $(\lambda I - A - \hat{B}(\lambda))$. By definition of ρ, we see that $\rho(\lambda)\,x \in Y$ for all $x \in X$ and

$$\begin{aligned}
&(\lambda I - A - \hat{B}(\lambda))\,\rho(\lambda)\,x \\
&\quad = (\lambda I - A - \hat{B}(\lambda))(\lambda I - A)^{-1}[I - \hat{B}(\lambda)(\lambda I - A)^{-1}]^{-1}\,x \\
&\quad = [I - B(\lambda)(\lambda I - A)^{-1}][I - \hat{B}(\lambda)(\lambda I - A)^{-1}]^{-1}\,x = x.
\end{aligned}$$

Similarly for any $x \in Y$,

$$\begin{aligned}
&\rho(\lambda)(\lambda I - A - B(\lambda))\,x \\
&\quad = (\lambda I - A)^{-1}[I - \hat{B}(\lambda)(\lambda I - A)^{-1}]^{-1}[I - \hat{B}(\lambda)(\lambda I - A)^{-1}](\lambda I - A)\,x \\
&\quad = (\lambda I - A)^{-1}(\lambda I - A)\,x = x.
\end{aligned}$$

THEOREM 5. *With the above notation*

$$\lim_{n\to\infty}\left(\frac{n}{t}\rho\left(\frac{n}{t}\right)\right)^n x = T(t)\,x \qquad \textit{for all } t > 0,\ x \in X,$$

the convergence being uniform on bounded intervals.

Proof. We use the Generalized Lax Equivalence Theorem. Stability is obtained from the fact that

$$\left\|\left(\frac{n}{t}\rho\left(\frac{n}{t}\right)\right)^n\right\| \leqslant \left(\frac{n/t}{n/t-\kappa}\right)^n = \left(1-\frac{t}{n\kappa}\right)^{-n} \to e^{\kappa t},$$

whereas consistency follows from

$$\begin{aligned}
\lambda^2\,\|\rho(\lambda)\,x - (\lambda I - A)^{-1}\,x\| &= \lambda^2\,\|\rho(\lambda)\,\hat{B}(\lambda)(\lambda I - A)^{-1}\,x\| \\
&\leqslant \frac{\lambda^2\beta}{(\lambda-\kappa)(\lambda-\gamma)(\lambda-\omega)}\,|x| \to 0, \qquad \text{as} \quad \lambda \to \infty.
\end{aligned}$$

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