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On a risk model with stochastic premiums income and dependence between income and loss

Zhimin Zhang *, Hu Yang

Department of Statistics and Actuarial Science, Chongqing University, Chongqing, PR China

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ABSTRACT

Labbé and Sendova (2009) [9] consider a compound Poisson risk model with stochastic premiums income. In this paper, we extend their model by assuming that there exists a specific dependence structure among the claim sizes, interclaim times and premium sizes. Assume that the distributions of the premium sizes and interclaim times are controlled by the claim sizes. When the individual premium sizes are exponentially distributed, the Laplace transforms and defective renewal equations for the (Gerber–Shiu) discounted penalty functions are obtained. When the individual premium sizes have rational Laplace transforms, we show that the Laplace transforms for the discounted penalty functions can also be obtained.

1. Introduction

The classical compound Poisson risk model, also known as the Cramér–Lundberg model, is one of the most popular risk models in ruin theory. Many of its nice properties, especially the stationary and independent increments property, make the study of ruin problems mathematically treatable. The study of ruin measures, including the Gerber–Shiu discounted penalty function, has been made by many actuarial researchers, see e.g. [7,8,11,12]. Note that two of the common assumptions in the classical compound Poisson risk model are, respectively, the deterministic premium rate and the independence between claim sizes and interclaim times. Although such assumptions indeed simplify the study, they have been proved to be very restrictive in some applications.

Sometimes, the insurance company may have lump sums of income. In order to describe the stochastic income, Boucherie et al. [5] add a compound Poisson process with positive jumps to the Cramér–Lundberg model. The (non-)ruin probability for the risk model with stochastic premiums are studied in [4] and [13]. Assuming that the premium process is a Poisson process, Bao [2] and Bao and Ye [3] study the Gerber–Shiu function in the compound Poisson risk model and the delayed renewal risk model, respectively. Yang and Zhang [14] extend the compound Poisson risk model in [2] to a Sparre Andersen risk model with generalized Erlang(n) interclaim time distribution. Note that in the models studied in [2,3,14], the premium rate is unit and the claim sizes have lattice distribution. Thus, although the operational time is continuous, the aforementioned models can be treated as discrete time risk models. Labbe and Sendova [9] consider a risk model with stochastic premiums income, where both the premium size distribution and the claim size distribution are non-lattice. In their paper, both a defective renewal equation and an integral equation satisfied by the Gerber–Shiu function are derived and, in particular, the case when the premiums have Erlang(n) distribution is investigated in more depth.

* Corresponding author.
E-mail address: cquzzm@163.com (Z. Zhang).

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For the study of risk models with stochastic premiums income, the independence assumption is very common in the aforementioned papers. The drawback of the independence assumption is obvious since the claim sizes usually have an effect on the premium sizes and interclaim times (see e.g. [1,15]). In this paper, we will extend the risk model studied in [9] by adding a specific dependence structure among the claim sizes, interclaim times and premium sizes. The rest of this paper is organized as follows. In Section 2, the risk model with stochastic premiums income is introduced and the dependence structure is specified. In Section 3, we focus on the case when the premiums are exponentially distributed and, in this case, we show that the Laplace transforms and the defective renewal equations for the Gerber–Shiu functions can be obtained. In Section 4, we consider the case when the premium sizes have rational Laplace transforms, and we show that the Laplace transforms for the discounted penalty functions can also be obtained. Finally, we conclude this paper in Section 5 by making a summary of the main results and discussing possible extensions in the future work.

2. Model description and notation

We describe the surplus process of an insurance company by the following process

$$U(t) = u + \sum_{i=1}^{M(t)} X_i - \sum_{i=1}^{N(t)} Y_i, \tag{2.1}$$

where $u \geq 0$ is the initial surplus, $M(t)$ counting the number of individual premium income up to time t is a Poisson process with intensity $\lambda > 0$, and $\{X_i\}$ is a sequence of strictly positive random variables (r.v.'s) representing the individual premium amounts. $\sum_{i=1}^{N(t)} Y_i$ is an aggregate claims process, where $N(t)$ is a counting process denoting the number of claims up to time t with interclaim times $\{V_i\}$, and the claim amounts r.v.'s $\{Y_i\}$ are independent and identically distributed (i.i.d.) like a generic variable Y with distribution function $F(y) = Pr(Y \leq y)$, density f, mean μ and Laplace transform $\hat{f}(s) = \int_0^\infty e^{-sy} f(y) dy$.

In this paper, we assume that $\{X_i\}$ and $N(t)$ are both dependent on the individual claim size as follows: If the claim size Y_i is larger than or equal to a threshold B_i, then the time until the next claim, V_{i+1}, is exponentially distributed with mean $1/\lambda_1$, and the individual premium sizes have distribution function $F_1(\cdot)$, mean μ_1 and Laplace transform $\hat{f}_1(\cdot)$, otherwise V_{i+1} is exponentially distributed with mean $1/\lambda_2$ and the premium sizes have distribution function $F_2(\cdot)$, mean μ_2 and Laplace transform $\hat{f}_2(\cdot)$. Assume that $\{B_i\}$ independent of $\{Y_i\}$ is an i.i.d. sequence of r.v.'s distributed like a generic variable B with distribution function $B(\cdot)$. Finally, assume that the time until the first claim occurs, V_1, is exponentially distributed with mean either $1/\lambda_1$ or $1/\lambda_2$, and assume that the premium size distribution function is $F_i(\cdot)$ during the first interclaim time if V_1 is exponentially distributed with mean $1/\lambda_i$.

For $n \in \mathbb{N}^+$, let $T_n = \sum_{i=1}^n V_i$ be the time when the nth claim occurs. The surplus immediately after the nth claim epoch can be expressed as

$$\begin{aligned} U_n &:= u + \sum_{i=1}^{M(T_n)} X_i - \sum_{i=1}^{n} Y_i \\ &= u + \left(\sum_{i=1}^{M(T_1)} X_i \right) + \left(\sum_{i=M(T_1)+1}^{M(T_2)} X_i \right) + \cdots + \left(\sum_{i=M(T_{n-1})+1}^{M(T_n)} X_i \right) - \sum_{i=1}^{n} Y_i \\ &= u + \sum_{i=1}^{M(T_1)} X_i - \sum_{k=1}^{n-1} \left(Y_k - \sum_{i=M(T_k)+1}^{M(T_{k+1})} X_i \right) - Y_n. \end{aligned}$$

Since the $k+1$th interclaim time V_{k+1} and the premium sizes in between the kth and $k+1$th claim epochs are only controlled by the claim size Y_k and the threshold B_k, then

$$Y_k - \sum_{i=M(T_k)+1}^{M(T_{k+1})} X_i, \quad k = 1, 2, \ldots,$$

are i.i.d. and equal in distribution to $\left\{ Y_1 - \sum_{i=1}^{M(V_2)} X_i \right\}$. Let $Y_k - \sum_{i=1}^{M(V_{k+1})} X_i$, $k = 1, 2, \ldots$, be the i.i.d. copies of $\left\{ Y_1 - \sum_{i=1}^{M(V_2)} X_i \right\}$. Then we have

$$U_n \stackrel{D}{=} u + \sum_{i=1}^{M(V_1)} X_i - \sum_{k=1}^{n-1} \left(Y_k - \sum_{i=1}^{M(V_{k+1})} X_i \right) - Y_n, \tag{2.2}$$

where $\stackrel{D}{=}$ means equality in distribution. By the law of large numbers, we have

$$\begin{aligned}
\lim_{n\to\infty}\frac{U_n}{n} &= \lim_{n\to\infty}\frac{u+\sum_{i=1}^{M(V_1)}X_i-Y_n}{n}-\lim_{n\to\infty}\frac{\sum_{k=1}^{n-1}\left(Y_k-\sum_{i=1}^{M(V_{k+1})}X_i\right)}{n}\\
&= -\lim_{n\to\infty}\frac{\sum_{k=1}^{n-1}\left(Y_k-\sum_{i=1}^{M(V_{k+1})}X_i\right)}{n}\\
&= -E\left[Y_1-\sum_{i=1}^{M(V_2)}X_i\right]\\
&= Pr(Y\geq B)\frac{\lambda\mu_1}{\lambda_1}+Pr(Y<B)\frac{\lambda\mu_2}{\lambda_2}-\mu.
\end{aligned}$$

Thus, to guarantee that U_n has a positive drift, we assume throughout this paper that the following net profit condition holds

$$Pr(Y\geq B)\frac{\lambda\mu_1}{\lambda_1}+Pr(Y<B)\frac{\lambda\mu_2}{\lambda_2}-\mu>0. \tag{2.3}$$

Define the ruin time by $\tau=\inf\{t,U(t)<0\}$ and ∞ if $U(t)\geq 0$ for all $t\geq 0$. Let $w(x_1,x_2)$ be a nonnegative measurable function defined on $[0,\infty)\times(0,\infty)$. For $\delta\geq 0$, define

$$\phi(u)=E[\mathrm{e}^{-\delta\tau}w(U(\tau-),|U(\tau)|)I(\tau<\infty)|U(0)=u] \tag{2.4}$$

to be the Gerber–Shiu discounted penalty function, where $I(A)$ is the indicator function of event A, $U(\tau-)$ and $|U(\tau)|$ are, respectively, the surplus immediately before ruin and the deficit at ruin. Given that the first interclaim time is exponentially distributed with mean $1/\lambda_i$, the Gerber–Shiu function is denoted by $\phi_i(u)$. Assume that the regularity condition $\lim_{u\to\infty}\phi_i(u)=0$ holds, which is not very restrictive since many ruin measures of interest such as the ruin probability, the distributions of the surplus immediately before ruin and the deficit at ruin satisfy this condition. Throughout this paper, we will use a hat ˆ to designate the Laplace transform of a function.

3. Gerber–Shiu analysis for exponential premium size distributions

In this section, we pay attention to the situation in which the premium sizes are exponentially distributed. Firstly, we start from a system of integral equations which holds for general premium size distributions.

Let W_1 be the time when the first premium income arrives. We have by the lack of memory of exponential distribution

$$\begin{aligned}
\phi_1(u) &= \int_0^\infty Pr(W_1<V_1,W_1\in\mathrm{d}t)\mathrm{e}^{-\delta t}\int_0^\infty\phi_1(u+x)\mathrm{d}F_1(x)\\
&\quad+\int_0^\infty Pr(V_1<W_1,V_1\in\mathrm{d}t)\mathrm{e}^{-\delta t}\left[\int_0^u\phi_1(u-y)Pr(y\geq B)\mathrm{d}F(y)\right.\\
&\quad\left.+\int_0^u\phi_2(u-y)Pr(y<B)\mathrm{d}F(y)+\int_u^\infty w(u,y-u)\mathrm{d}F(y)\right]\\
&= \int_0^\infty\lambda\mathrm{e}^{-(\lambda+\lambda_1+\delta)t}\mathrm{d}t\int_0^\infty\phi_1(u+x)\mathrm{d}F_1(x)\\
&\quad+\int_0^\infty\lambda_1\mathrm{e}^{-(\lambda+\lambda_1+\delta)t}\mathrm{d}t\left[\int_0^u\phi_1(u-y)B(y)\mathrm{d}F(y)+\int_0^u\phi_2(u-y)\overline{B}(y)\mathrm{d}F(y)+\omega(u)\right]\\
&= \frac{\lambda}{\lambda+\lambda_1+\delta}\int_0^\infty\phi_1(u+x)\mathrm{d}F_1(x)+\frac{\lambda_1}{\lambda+\lambda_1+\delta}\left[\int_0^u\phi_1(u-y)B(y)\mathrm{d}F(y)\right.\\
&\quad\left.+\int_0^u\phi_2(u-y)\overline{B}(y)\mathrm{d}F(y)+\omega(u)\right],
\end{aligned} \tag{3.1}$$

where $\omega(u)=\int_u^\infty w(u,y-u)\mathrm{d}F(y)$. Similarly, we have

$$\begin{aligned}
\phi_2(u) &= \frac{\lambda}{\lambda+\lambda_2+\delta}\int_0^\infty\phi_2(u+x)\mathrm{d}F_2(x)+\frac{\lambda_2}{\lambda+\lambda_2+\delta}\left[\int_0^u\phi_1(u-y)B(y)\mathrm{d}F(y)\right.\\
&\quad\left.+\int_0^u\phi_2(u-y)\overline{B}(y)\mathrm{d}F(y)+\omega(u)\right].
\end{aligned} \tag{3.2}$$

Let $\xi_1(y) = B(y)f(y)$, $\xi_2(y) = \overline{B}(y)f(y)$, and for $i = 1, 2$, let $A_i(u) = \int_0^\infty \phi_i(u+x)\mathrm{d}F_i(x)$. Taking Laplace transforms in (3.1) and (3.2) gives

$$\hat{\phi}_1(s) = \frac{\lambda}{\lambda+\lambda_1+\delta}\hat{A}_1(s) + \frac{\lambda_1}{\lambda+\lambda_1+\delta}\left[\hat{\phi}_1(s)\hat{\xi}_1(s) + \hat{\phi}_2(s)\hat{\xi}_2(s) + \hat{\omega}(s)\right], \tag{3.3}$$

$$\hat{\phi}_2(s) = \frac{\lambda}{\lambda+\lambda_2+\delta}\hat{A}_2(s) + \frac{\lambda_2}{\lambda+\lambda_2+\delta}\left[\hat{\phi}_1(s)\hat{\xi}_1(s) + \hat{\phi}_2(s)\hat{\xi}_2(s) + \hat{\omega}(s)\right]. \tag{3.4}$$

In the rest of this section, we assume that the individual premium sizes are exponentially distributed with distribution functions

$$F_1(x) = 1 - \mathrm{e}^{-\frac{x}{\mu_1}}, \qquad F_2(x) = 1 - \mathrm{e}^{-\frac{x}{\mu_2}}, \tag{3.5}$$

for $\mu_1, \mu_2 > 0$.

For notational convenience of later use, we introduce the Dickson–Hipp operator T_s defined on a real-valued function as follows

$$T_s h(x) = \int_x^\infty \mathrm{e}^{-s(y-x)}h(y)\mathrm{d}y, \quad x \geq 0,$$

where s is a nonnegative real number (or a complex number with nonnegative real part) such that above integral is convergent. It is easy to see that the Laplace transform of h can be expressed as $T_s h(0)$. The operator T_s is commutative, i.e. $T_s T_r = T_r T_s$, and furthermore

$$T_s T_r h(x) = \frac{T_s h(x) - T_r h(x)}{r - s}, \quad r \neq s.$$

For more properties of the Dickson-Hipp operator, we refer the readers to [6,10].

3.1. Laplace transforms

For $Re(s) > \frac{1}{\mu_i}$, we have by changing the order of integration

$$\begin{aligned}
\hat{A}_i(s) &= \int_0^\infty \mathrm{e}^{-su}\int_0^\infty \phi_i(u+x)\frac{1}{\mu_i}\mathrm{e}^{-\frac{x}{\mu_i}}\mathrm{d}x\mathrm{d}u \\
&= \int_0^\infty\int_0^\infty \mathrm{e}^{-su}\phi_i(u+x)\frac{1}{\mu_i}\mathrm{e}^{-\frac{x}{\mu_i}}\mathrm{d}u\mathrm{d}x \\
&= \frac{1}{\mu_i}\int_0^\infty\int_x^\infty \mathrm{e}^{-s(u-x)}\phi_i(u)\mathrm{d}u\mathrm{e}^{-\frac{x}{\mu_i}}\mathrm{d}x \\
&= \frac{1}{\mu_i}T_s T_{\frac{1}{\mu_i}}\phi_i(0) \\
&= \frac{1}{\mu_i}\frac{T_s\phi_i(0) - T_{\frac{1}{\mu_i}}\phi_i(0)}{\frac{1}{\mu_i} - s} \\
&= \frac{\hat{\phi}_i(s) - \hat{\phi}_i\left(\frac{1}{\mu_i}\right)}{1-\mu_i s}.
\end{aligned}$$

Combining above result with (3.3) and (3.4), we obtain

$$\left[1 - \frac{\lambda}{(\lambda+\lambda_1+\delta)(1-\mu_1 s)} - \frac{\lambda_1\hat{\xi}_1(s)}{\lambda+\lambda_1+\delta}\right]\hat{\phi}_1(s) - \frac{\lambda_1\hat{\xi}_2(s)}{\lambda+\lambda_1+\delta}\hat{\phi}_2(s)$$
$$= \frac{\lambda_1\hat{\omega}(s)}{\lambda+\lambda_1+\delta} - \frac{\lambda\hat{\phi}_1\left(\frac{1}{\mu_1}\right)}{(\lambda+\lambda_1+\delta)(1-\mu_1 s)}, \tag{3.6}$$

$$\left[1 - \frac{\lambda}{(\lambda+\lambda_2+\delta)(1-\mu_2 s)} - \frac{\lambda_2\hat{\xi}_2(s)}{\lambda+\lambda_2+\delta}\right]\hat{\phi}_2(s) - \frac{\lambda_2\hat{\xi}_1(s)}{\lambda+\lambda_2+\delta}\hat{\phi}_1(s)$$
$$= \frac{\lambda_2\hat{\omega}(s)}{\lambda+\lambda_2+\delta} - \frac{\lambda\hat{\phi}_2\left(\frac{1}{\mu_2}\right)}{(\lambda+\lambda_2+\delta)(1-\mu_2 s)}. \tag{3.7}$$

Obviously, by analytic extension, (3.6) and (3.7) still hold for all s in the right half complex plane except the points $1/\mu_1$ and $1/\mu_2$.

Let

$$\chi_i(s) = 1 - \frac{\lambda}{(\lambda+\lambda_i+\delta)(1-\mu_i s)} - \frac{\lambda_i \hat{\hat{\xi}}_i(s)}{\lambda+\lambda_i+\delta}, \quad i = 1, 2,$$
$$h_1(s) = \frac{\lambda_1}{\lambda+\lambda_1+\delta} - \frac{\lambda\lambda_1}{(\lambda+\lambda_1+\delta)(\lambda+\lambda_2+\delta)(1-\mu_2 s)},$$
$$h_2(s) = \frac{\lambda_2}{\lambda+\lambda_2+\delta} - \frac{\lambda\lambda_2}{(\lambda+\lambda_1+\delta)(\lambda+\lambda_2+\delta)(1-\mu_1 s)}.$$

Solving (3.6) and (3.7) gives

$$\hat{\phi}_1(s) = \frac{h_1(s)\hat{\omega}(s) - \frac{\lambda\chi_2(s)\hat{\phi}_1\left(\frac{1}{\mu_1}\right)}{(\lambda+\lambda_1+\delta)(1-\mu_1 s)} - \frac{\lambda\lambda_1\hat{\xi}_2(s)\hat{\phi}_2\left(\frac{1}{\mu_2}\right)}{(\lambda+\lambda_1+\delta)(\lambda+\lambda_2+\delta)(1-\mu_2 s)}}{\chi_1(s)\chi_2(s) - \frac{\lambda_1\lambda_2\hat{\xi}_1(s)\hat{\xi}_2(s)}{(\lambda+\lambda_1+\delta)(\lambda+\lambda_2+\delta)}}, \tag{3.8}$$

$$\hat{\phi}_2(s) = \frac{h_2(s)\hat{\omega}(s) - \frac{\lambda\chi_1(s)\hat{\phi}_2\left(\frac{1}{\mu_2}\right)}{(\lambda+\lambda_2+\delta)(1-\mu_2 s)} - \frac{\lambda\lambda_2\hat{\xi}_1(s)\hat{\phi}_1\left(\frac{1}{\mu_1}\right)}{(\lambda+\lambda_1+\delta)(\lambda+\lambda_2+\delta)(1-\mu_1 s)}}{\chi_1(s)\chi_2(s) - \frac{\lambda_1\lambda_2\hat{\xi}_1(s)\hat{\xi}_2(s)}{(\lambda+\lambda_1+\delta)(\lambda+\lambda_2+\delta)}}. \tag{3.9}$$

To get $\hat{\phi}_1(s)$ and $\hat{\phi}_2(s)$, we still have to determine the unknown quantities $\hat{\phi}_1\left(\frac{1}{\mu_1}\right)$ and $\hat{\phi}_2\left(\frac{1}{\mu_2}\right)$. For this purpose, we consider the zeros of the common denominator of (3.8) and (3.9), i.e., the roots of the following equation

$$\chi_1(s)\chi_2(s) - \frac{\lambda_1\lambda_2\hat{\hat{\xi}}_1(s)\hat{\hat{\xi}}_2(s)}{(\lambda+\lambda_1+\delta)(\lambda+\lambda_2+\delta)} = 0. \tag{3.10}$$

Lemma 1. *For $\delta > 0$, Eq. (3.10) has exactly two roots, say $\rho_1(\delta)$, $\rho_2(\delta)$, in the right half complex plane, i.e. $Re(\rho_i(\delta)) > 0$ for $i = 1, 2$.*

Proof. It suffices to consider the following equation

$$\prod_{i=1}^{2}[\chi_i(s)(1-\mu_i s)] = \prod_{i=1}^{2}\left[\frac{\lambda_i(1-\mu_i s)\hat{\hat{\xi}}_i(s)}{\lambda+\lambda_i+\delta}\right],$$

which is equivalent to

$$\prod_{i=1}^{2}\left[1-\mu_i s - \frac{\lambda}{\lambda+\lambda_i+\delta}\right] = \left[1-\mu_1 s - \frac{\lambda}{\lambda+\lambda_1+\delta}\right]\frac{\lambda_2(1-\mu_2 s)\hat{\hat{\xi}}_2(s)}{\lambda+\lambda_2+\delta} + \left[1-\mu_2 s - \frac{\lambda}{\lambda+\lambda_2+\delta}\right]\frac{\lambda_1(1-\mu_1 s)\hat{\hat{\xi}}_1(s)}{\lambda+\lambda_1+\delta}. \tag{3.11}$$

Now we apply Rouché's theorem to prove this Lemma.

Let $r > 0$ be a sufficiently large number, and denote by $\mathcal{C}_r$ the contour containing the imaginary axis running from $-ir$ to ir and a semicircle with radius r running clockwise from ir to $-ir$. We show that for $s \in \mathcal{C}_r$, the module of the left hand side of (3.11) is strictly larger than that of the right hand side of (3.11).

Firstly, for s on the imaginary axis, we have

$$\frac{|\lambda_i - \lambda_i\mu_i s|}{|\lambda_i+\delta-(\lambda+\lambda_i+\delta)\mu_i s|} < 1, \quad i = 1, 2.$$

Secondly, for s on the semicircle, we have for $\forall\varepsilon > 0$

$$\frac{\left|\frac{1}{\mu_i} - s\right|}{\left|\frac{\lambda_i+\delta}{(\lambda+\lambda_i+\delta)\mu_i} - s\right|} < 1+\varepsilon$$

when r is sufficiently large. In particular, for $\varepsilon = \min\{\frac{\lambda+\delta}{\lambda_1}, \frac{\lambda+\delta}{\lambda_2}\}$, there exists $r_0 > 0$ such that when $r > r_0$, we have for $i = 1, 2$,

$$\frac{|\lambda_i - \lambda_i\mu_i s|}{|\lambda_i+\delta-(\lambda+\lambda_i+\delta)\mu_i s|} = \frac{\lambda_i}{\lambda+\lambda_i+\delta}\frac{\left|\frac{1}{\mu_i} - s\right|}{\left|\frac{\lambda_i+\delta}{(\lambda+\lambda_i+\delta)\mu_i} - s\right|} < \frac{\lambda_i}{\lambda+\lambda_i+\delta}(1+\varepsilon) \le 1.$$

Thus, when r is sufficiently large, we have

$$\begin{aligned}
&\left|\left[1-\mu_1 s-\frac{\lambda}{\lambda+\lambda_1+\delta}\right]\frac{\lambda_2(1-\mu_2 s)\hat{\xi}_2(s)}{\lambda+\lambda_2+\delta}+\left[1-\mu_2 s-\frac{\lambda}{\lambda+\lambda_2+\delta}\right]\frac{\lambda_1(1-\mu_1 s)\hat{\xi}_1(s)}{\lambda+\lambda_1+\delta}\right|\\
&\quad=\left|\prod_{i=1}^{2}\left[1-\mu_i s-\frac{\lambda}{\lambda+\lambda_i+\delta}\right]\right|\cdot\left|\frac{\lambda_2(1-\mu_2 s)\hat{\xi}_2(s)}{\lambda_2+\delta-(\lambda+\lambda_2+\delta)\mu_2 s}+\frac{\lambda_1(1-\mu_1 s)\hat{\xi}_1(s)}{\lambda_1+\delta-(\lambda+\lambda_1+\delta)\mu_1 s}\right|\\
&\quad\le\left|\prod_{i=1}^{2}\left[1-\mu_i s-\frac{\lambda}{\lambda+\lambda_i+\delta}\right]\right|\left(\frac{|\hat{\xi}_2(s)|\cdot|\lambda_2-\lambda_2\mu_2 s|}{|\lambda_2+\delta-(\lambda+\lambda_2+\delta)\mu_2 s|}+\frac{|\hat{\xi}_1(s)|\cdot|\lambda_1-\lambda_1\mu_1 s|}{|\lambda_1+\delta-(\lambda+\lambda_1+\delta)\mu_1 s|}\right)\\
&\quad<\left|\prod_{i=1}^{2}\left[1-\mu_i s-\frac{\lambda}{\lambda+\lambda_i+\delta}\right]\right|\left(|\hat{\xi}_2(s)|+|\hat{\xi}_1(s)|\right)\\
&\quad\le\left|\prod_{i=1}^{2}\left[1-\mu_i s-\frac{\lambda}{\lambda+\lambda_i+\delta}\right]\right|\left(\hat{\xi}_2(0)+\hat{\xi}_1(0)\right)\\
&\quad=\left|\prod_{i=1}^{2}\left[1-\mu_i s-\frac{\lambda}{\lambda+\lambda_i+\delta}\right]\right|.
\end{aligned}$$

Note that both sides of (3.11) are analytic for s inside $\mathcal{C}_r$. Then by Rouché's theorem, we know that Eq. (3.11) has the same number of roots as the following equation inside $\mathcal{C}_r$

$$\prod_{i=1}^{2}\left[1-\mu_i s-\frac{\lambda}{\lambda+\lambda_i+\delta}\right]=0.$$

Obviously, the above equation has two roots, say $s_i:=\frac{\lambda_i+\delta}{(\lambda+\lambda_i+\delta)\mu_i}$, $i=1,2$, inside $\mathcal{C}_r$. Then Eq. (3.11) has also two roots inside $\mathcal{C}_r$. Finally, letting $r\to\infty$ completes the proof. □

Remark 1. Denote the root with the smaller module by $\rho_1(\delta)$. Then it is readily seen that $\lim_{\delta\to 0+}\rho_1(\delta)=0$. In the sequel, we assume that these two roots are distinct and denote them by ρ_1 and ρ_2 for simplicity.

Since $\hat{\phi}_i(s)$, $i=1,2$, are analytic for $Re(s)\ge 0$, ρ_1 and ρ_2 must also be zeros of the numerators of (3.8) and (3.9). Both cases give the following equations for $\hat{\phi}_1\left(\frac{1}{\mu_1}\right)$ and $\hat{\phi}_2\left(\frac{1}{\mu_2}\right)$,

$$\frac{\lambda\chi_2(\rho_i)\hat{\phi}_1\left(\frac{1}{\mu_1}\right)}{(\lambda+\lambda_1+\delta)(1-\mu_1\rho_i)}+\frac{\lambda\lambda_1\hat{\xi}_2(\rho_i)\hat{\phi}_2\left(\frac{1}{\mu_2}\right)}{(\lambda+\lambda_1+\delta)(\lambda+\lambda_2+\delta)(1-\mu_2\rho_i)}=h_1(\rho_i)\hat{\omega}(\rho_i),\quad i=1,2. \tag{3.12}$$

After solving (3.12), we can obtain $\hat{\phi}_1\left(\frac{1}{\mu_1}\right)$ and $\hat{\phi}_2\left(\frac{1}{\mu_2}\right)$, and accordingly, $\hat{\phi}_1(s)$ and $\hat{\phi}_2(s)$ can be determined.

Example 1. We give a numerical example to show how to find the ruin probabilities when the thresholds and the claim sizes are exponentially distributed with

$$B(x)=1-e^{-0.5x},\qquad F(y)=1-e^{-y}.$$

Set $\lambda=1$, $\lambda_1=0.4$, $\lambda_2=0.5$, $\mu_1=0.5$, $\mu_2=1$. It is easy to check that the net profit condition (2.3) holds under the above settings. Let $\delta=0$, $w\equiv 1$. Then the Gerber–Shiu function $\phi_i(u)$ $(i=1,2)$ reduces to the ruin probability $\psi_i(u)$.

Eq. (3.10) becomes

$$\begin{aligned}
&\left[1-\frac{1}{1.4(1-0.5s)}-\frac{0.4}{1.4}\left(\frac{1}{s+1}-\frac{1}{s+1.5}\right)\right]\left[1-\frac{1}{1.5(1-s)}-\frac{0.5}{1.5(s+1.5)}\right]\\
&\quad=\frac{0.2}{1.4\times 1.5}\left(\frac{1}{s+1}-\frac{1}{s+1.5}\right)\frac{1}{s+1.5}.
\end{aligned}$$

Solving above equation gives four roots, 0, 0.523009204, −0.270554613, −1.514359353. Then solving (3.12) gives $\hat{\phi}_1\left(\frac{1}{\mu_1}\right)=0.333621$, $\hat{\phi}_2\left(\frac{1}{\mu_2}\right)=0.541487$. Finally, the inversion of the Laplace transforms (3.8) and (3.9) yields

$$\begin{aligned}
\psi_1(u)&=0.7487227223e^{-0.270554613u}+0.01359317324e^{-1.514359353u},\\
\psi_2(u)&=0.6815162964e^{-0.270554613u}+0.01280823418e^{-1.514359353u}.
\end{aligned}$$

Fig. 1 shows the behaviors of $\psi_1(u)$ and $\psi_2(u)$.

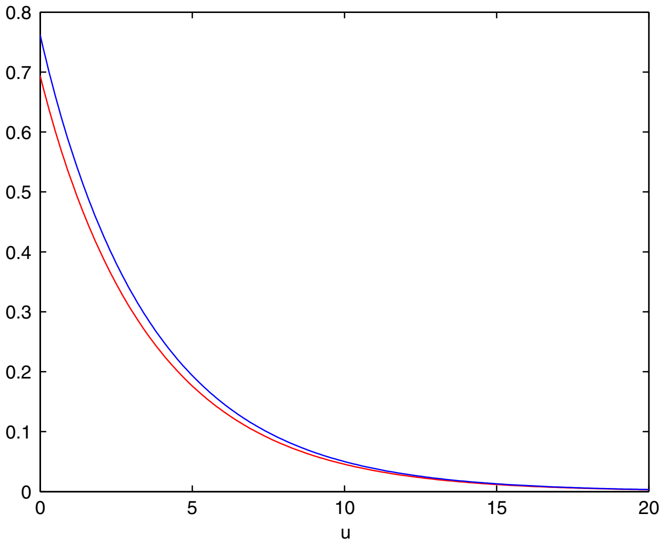

Fig. 1. Ruin probabilities $\psi_1(u)$ (the blue curve) and $\psi_2(u)$ (the red curve). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

3.2. Defective renewal equations

The main goal of this subsection is to derive defective renewal equations for $\phi_1(u)$ and $\phi_2(u)$.
Let

$$\begin{aligned} H(s) &= (1-\mu_1 s)\left(\frac{\lambda_2+\delta}{\lambda+\lambda_2+\delta}-\mu_2 s\right)\frac{\lambda_1\hat{\xi}_1(s)}{\lambda+\lambda_1+\delta}+(1-\mu_2 s)\left(\frac{\lambda_1+\delta}{\lambda+\lambda_1+\delta}-\mu_1 s\right)\frac{\lambda_2\hat{\xi}_2(s)}{\lambda+\lambda_2+\delta} \\ &= h_{11}\hat{\xi}_1(s)+h_{12}s\hat{\xi}_1(s)+h_{13}s^2\hat{\xi}_1(s)+h_{21}\hat{\xi}_2(s)+h_{22}s\hat{\xi}_2(s)+h_{23}s^2\hat{\xi}_2(s), \end{aligned}$$

where

$$\begin{aligned} h_{11} &= \frac{\lambda_1(\lambda_2+\delta)}{(\lambda+\lambda_1+\delta)(\lambda+\lambda_2+\delta)}, \qquad h_{21} = \frac{\lambda_2(\lambda_1+\delta)}{(\lambda+\lambda_1+\delta)(\lambda+\lambda_2+\delta)}, \\ h_{12} &= -\left[\frac{(\lambda_2+\delta)\mu_1}{\lambda+\lambda_2+\delta}+\mu_2\right]\frac{\lambda_1}{\lambda+\lambda_1+\delta}, \qquad h_{22} = -\left[\frac{(\lambda_1+\delta)\mu_2}{\lambda+\lambda_1+\delta}+\mu_1\right]\frac{\lambda_2}{\lambda+\lambda_2+\delta}, \\ h_{13} &= \frac{\lambda_1\mu_1\mu_2}{\lambda+\lambda_1+\delta}, \qquad h_{23} = \frac{\lambda_2\mu_1\mu_2}{\lambda+\lambda_2+\delta}. \end{aligned}$$

Then multiplying the common denominator of (3.8) and (3.9) by $(1-\mu_1 s)(1-\mu_2 s)$ gives

$$\begin{aligned} &(1-\mu_1 s)(1-\mu_2 s)\left(\chi_1(s)\chi_2(s)-\frac{\lambda_1\lambda_2\hat{\xi}_1(s)\hat{\xi}_2(s)}{(\lambda+\lambda_1+\delta)(\lambda+\lambda_2+\delta)}\right) \\ &\quad = \left(\frac{\lambda_1+\delta}{\lambda+\lambda_1+\delta}-\mu_1 s\right)\left(\frac{\lambda_2+\delta}{\lambda+\lambda_2+\delta}-\mu_2 s\right)-H(s). \end{aligned}$$

Obviously,

$$l(s) := \left(\frac{\lambda_1+\delta}{\lambda+\lambda_1+\delta}-\mu_1 s\right)\left(\frac{\lambda_2+\delta}{\lambda+\lambda_2+\delta}-\mu_2 s\right)-\mu_1\mu_2(s-\rho_1)(s-\rho_2)$$

is a polynomial of degree 1, and it satisfies

$$l(\rho_i) = H(\rho_i), \quad i = 1, 2,$$

due to Lemma 1. By Lagrange interpolation formula, we have

$$l(s) = \frac{s-\rho_2}{\rho_1-\rho_2}H(\rho_1)+\frac{s-\rho_1}{\rho_2-\rho_1}H(\rho_2).$$

Using the above results, we obtain

$$\begin{aligned}(1-\mu_1 s)(1-\mu_2 s)&\left(\chi_1(s)\chi_2(s)-\frac{\lambda_1\lambda_2\hat{\xi}_1(s)\hat{\xi}_2(s)}{(\lambda+\lambda_1+\delta)(\lambda+\lambda_2+\delta)}\right)\\ &=\mu_1\mu_2(s-\rho_1)(s-\rho_2)+\frac{s-\rho_2}{\rho_1-\rho_2}H(\rho_1)+\frac{s-\rho_1}{\rho_2-\rho_1}H(\rho_2)-H(s)\\ &=(s-\rho_1)(s-\rho_2)\left(\mu_1\mu_2-\frac{\frac{H(s)-H(\rho_2)}{s-\rho_2}-\frac{H(s)-H(\rho_1)}{s-\rho_1}}{\rho_2-\rho_1}\right).\end{aligned}\tag{3.13}$$

Recalling the properties of the Dickson–Hipp operator, we have for $k, i = 1, 2$,

$$\begin{aligned}\frac{\hat{\xi}_k(s)-\hat{\xi}_k(\rho_i)}{s-\rho_i}&=-T_sT_{\rho_i}\xi_k(0),\\ \frac{s\hat{\xi}_k(s)-\rho_i\hat{\xi}_k(\rho_i)}{s-\rho_i}&=\frac{s\hat{\xi}_k(s)-\rho_i\hat{\xi}_k(s)+\rho_i\hat{\xi}_k(s)-\rho_i\hat{\xi}_k(\rho_i)}{s-\rho_i}=\hat{\xi}_k(s)-\rho_iT_sT_{\rho_i}\xi_k(0),\\ \frac{s^2\hat{\xi}_k(s)-\rho_i^2\hat{\xi}_k(\rho_i)}{s-\rho_i}&=\frac{s^2\hat{\xi}_k(s)-\rho_i^2\hat{\xi}_k(s)+\rho_i^2\hat{\xi}_k(s)-\rho_i^2\hat{\xi}_k(\rho_i)}{s-\rho_i}\\ &=(s+\rho_i)\hat{\xi}_k(s)-\rho_i^2T_sT_{\rho_i}\xi_k(0).\end{aligned}$$

And accordingly, for $k = 1, 2$

$$\begin{aligned}\frac{\frac{\hat{\xi}_k(s)-\hat{\xi}_k(\rho_2)}{s-\rho_2}-\frac{\hat{\xi}_k(s)-\hat{\xi}_k(\rho_1)}{s-\rho_1}}{\rho_2-\rho_1}&=\frac{T_sT_{\rho_1}\xi_k(0)-T_sT_{\rho_2}\xi_k(0)}{\rho_2-\rho_1}=T_sT_{\rho_1}T_{\rho_2}\xi_k(0),\\ \frac{\frac{s\hat{\xi}_k(s)-\rho_2\hat{\xi}_k(\rho_2)}{s-\rho_2}-\frac{s\hat{\xi}_k(s)-\rho_1\hat{\xi}_k(\rho_1)}{s-\rho_1}}{\rho_2-\rho_1}&=\frac{\rho_1T_sT_{\rho_1}\xi_k(0)-\rho_2T_sT_{\rho_2}\xi_k(0)}{\rho_2-\rho_1}\\ &=\frac{\rho_1T_sT_{\rho_1}\xi_k(0)-\rho_1T_sT_{\rho_2}\xi_k(0)+\rho_1T_sT_{\rho_2}\xi_k(0)-\rho_2T_{\rho_2}\xi_k(0)}{\rho_2-\rho_1}\\ &=\rho_1T_sT_{\rho_1}T_{\rho_2}\xi_k(0)-T_sT_{\rho_2}\xi_k(0),\\ \frac{\frac{s^2\hat{\xi}_k(s)-\rho_2^2\hat{\xi}_k(\rho_2)}{s-\rho_2}-\frac{s^2\hat{\xi}_k(s)-\rho_1^2\hat{\xi}_k(\rho_1)}{s-\rho_1}}{\rho_2-\rho_1}&=\frac{\rho_2\hat{\xi}_k(s)-\rho_1\hat{\xi}_k(s)}{\rho_2-\rho_1}-\frac{\rho_2^2T_sT_{\rho_2}\xi_k(0)-\rho_1^2T_sT_{\rho_1}\xi_k(0)}{\rho_2-\rho_1}\\ &=\hat{\xi}_k(s)-\frac{\rho_2^2T_sT_{\rho_2}\xi_k(0)-\rho_1^2T_sT_{\rho_2}\xi_k(0)+\rho_1^2T_sT_{\rho_2}\xi_k(0)-\rho_1^2T_sT_{\rho_1}\xi_k(0)}{\rho_2-\rho_1}\\ &=\hat{\xi}_k(s)-(\rho_1+\rho_2)T_sT_{\rho_2}\xi_k(0)+\rho_1^2T_sT_{\rho_1}T_{\rho_2}\xi_k(0).\end{aligned}$$

Then

$$\begin{aligned}\hat{G}(s)&:=\frac{\frac{H(s)-H(\rho_2)}{s-\rho_2}-\frac{H(s)-H(\rho_1)}{s-\rho_1}}{\rho_2-\rho_1}\\ &=\sum_{k=1}^{2}\left[h_{k1}\frac{\frac{\hat{\xi}_k(s)-\hat{\xi}_k(\rho_2)}{s-\rho_2}-\frac{\hat{\xi}_k(s)-\hat{\xi}_k(\rho_1)}{s-\rho_1}}{\rho_2-\rho_1}+h_{k2}\frac{\frac{s\hat{\xi}_k(s)-\rho_2\hat{\xi}_k(\rho_2)}{s-\rho_2}-\frac{s\hat{\xi}_k(s)-\rho_1\hat{\xi}_k(\rho_1)}{s-\rho_1}}{\rho_2-\rho_1}\right.\\ &\quad\left.+\;h_{k3}\frac{\frac{s^2\hat{\xi}_k(s)-\rho_2^2\hat{\xi}_k(\rho_2)}{s-\rho_2}-\frac{s^2\hat{\xi}_k(s)-\rho_1^2\hat{\xi}_k(\rho_1)}{s-\rho_1}}{\rho_2-\rho_1}\right]\\ &=\sum_{k=1}^{2}\Big[h_{k1}T_sT_{\rho_1}T_{\rho_2}\xi_k(0)+h_{k2}\left(\rho_1T_sT_{\rho_1}T_{\rho_2}\xi_k(0)-T_sT_{\rho_2}\xi_k(0)\right)\\ &\quad+h_{k3}\left(\hat{\xi}_k(s)-(\rho_1+\rho_2)T_sT_{\rho_2}\xi_k(0)+\rho_1^2T_sT_{\rho_1}T_{\rho_2}\xi_k(0)\right)\Big].\end{aligned}\tag{3.14}$$

As for the numerator of (3.8), we multiply it by $(1-\mu_1 s)(1-\mu_2 s)$ to obtain

$$\begin{aligned} M_1(s) &:= (1-\mu_1 s)(1-\mu_2 s)\left[h_1(s)\hat{\omega}(s) - \frac{\lambda\chi_2(s)\hat{\phi}_1\left(\frac{1}{\mu_1}\right)}{(\lambda+\lambda_1+\delta)(1-\mu_1 s)} - \frac{\lambda\lambda_1\hat{\xi}_2(s)\hat{\phi}_2\left(\frac{1}{\mu_2}\right)}{(\lambda+\lambda_1+\delta)(\lambda+\lambda_2+\delta)(1-\mu_2 s)}\right] \\ &= \tau_{11}(s)\hat{\omega}(s) + \tau_{12}(s)\hat{\xi}_2(s) - \frac{\lambda\hat{\phi}_1\left(\frac{1}{\mu_1}\right)}{\lambda+\lambda_1+\delta}\left(\frac{\lambda_2+\delta}{\lambda+\lambda_2+\delta} - \mu_2 s\right), \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} \tau_{11}(s) &= \frac{\lambda_1(1-\mu_1 s)(1-\mu_2 s)}{\lambda+\lambda_1+\delta} - \frac{\lambda\lambda_1(1-\mu_1 s)}{(\lambda+\lambda_1+\delta)(\lambda+\lambda_2+\delta)}, \\ \tau_{12}(s) &= \frac{\lambda\lambda_2\hat{\phi}_1\left(\frac{1}{\mu_1}\right)(1-\mu_2 s) - \lambda\lambda_1\hat{\phi}_2\left(\frac{1}{\mu_2}\right)(1-\mu_1 s)}{(\lambda+\lambda_1+\delta)(\lambda+\lambda_2+\delta)}. \end{aligned}$$

Note that $\frac{\lambda\hat{\phi}_1\left(\frac{1}{\mu_1}\right)}{\lambda+\lambda_1+\delta}\left(\frac{\lambda_2+\delta}{\lambda+\lambda_2+\delta} - \mu_2 s\right)$ is a polynomial of degree 1 satisfying

$$\frac{\lambda\hat{\phi}_1\left(\frac{1}{\mu_1}\right)}{\lambda+\lambda_1+\delta}\left(\frac{\lambda_2+\delta}{\lambda+\lambda_2+\delta} - \mu_2\rho_i\right) = \tau_{11}(\rho_i)\hat{\omega}(\rho_i) + \tau_{12}(\rho_i)\hat{\xi}_2(\rho_i), \quad i=1,2,$$

due to the fact that ρ_1, ρ_2 are also zeros of the numerator of (3.8). By employing Lagrange interpolation, we find

$$\begin{aligned} \frac{\lambda\hat{\phi}_1\left(\frac{1}{\mu_1}\right)}{\lambda+\lambda_1+\delta}\left(\frac{\lambda_2+\delta}{\lambda+\lambda_2+\delta} - \mu_2 s\right) &= \frac{s-\rho_2}{\rho_1-\rho_2}\left[\tau_{11}(\rho_1)\hat{\omega}(\rho_1) + \tau_{12}(\rho_1)\hat{\xi}_2(\rho_1)\right] \\ &\quad + \frac{s-\rho_1}{\rho_2-\rho_1}\left[\tau_{11}(\rho_2)\hat{\omega}(\rho_2) + \tau_{12}(\rho_2)\hat{\xi}_2(\rho_2)\right], \end{aligned}$$

which together with (3.15) gives

$$\begin{aligned} M_1(s) &= \frac{s-\rho_2}{\rho_1-\rho_2}\left(\tau_{11}(s)\hat{\omega}(s) - \tau_{11}(\rho_1)\hat{\omega}(\rho_1)\right) + \frac{s-\rho_1}{\rho_2-\rho_1}\left(\tau_{11}(s)\hat{\omega}(s) - \tau_{11}(\rho_2)\hat{\omega}(\rho_2)\right) \\ &\quad + \frac{s-\rho_2}{\rho_1-\rho_2}\left(\tau_{12}(s)\hat{\xi}_2(s) - \tau_{12}(\rho_1)\hat{\xi}_2(\rho_1)\right) + \frac{s-\rho_1}{\rho_2-\rho_1}\left(\tau_{12}(s)\hat{\xi}_2(s) - \tau_{12}(\rho_2)\hat{\xi}_2(\rho_2)\right) \\ &= \frac{(s-\rho_1)(s-\rho_2)}{\rho_1-\rho_2}\left[\frac{\tau_{11}(s)-\tau_{11}(\rho_1)}{s-\rho_1}\hat{\omega}(s) - \tau_{11}(\rho_1)T_sT_{\rho_1}\omega(0)\right] \\ &\quad + \frac{(s-\rho_1)(s-\rho_2)}{\rho_2-\rho_1}\left[\frac{\tau_{11}(s)-\tau_{11}(\rho_2)}{s-\rho_2}\hat{\omega}(s) - \tau_{11}(\rho_2)T_sT_{\rho_2}\omega(0)\right] \\ &\quad \times \frac{(s-\rho_1)(s-\rho_2)}{\rho_1-\rho_2}\left[\frac{\tau_{12}(s)-\tau_{12}(\rho_1)}{s-\rho_1}\hat{\xi}_2(s) - \tau_{12}(\rho_1)T_sT_{\rho_1}\xi_2(0)\right] \\ &\quad + \frac{(s-\rho_1)(s-\rho_2)}{\rho_2-\rho_1}\left[\frac{\tau_{12}(s)-\tau_{12}(\rho_2)}{s-\rho_2}\hat{\xi}_2(s) - \tau_{12}(\rho_2)T_sT_{\rho_2}\xi_2(0)\right] \\ &= (s-\rho_1)(s-\rho_2)\left[\frac{\frac{\tau_{11}(s)-\tau_{11}(\rho_1)}{s-\rho_1} - \frac{\tau_{11}(s)-\tau_{11}(\rho_2)}{s-\rho_2}}{\rho_1-\rho_2}\hat{\omega}(s) - \frac{\tau_{11}(\rho_1)}{\rho_1-\rho_2}T_sT_{\rho_1}\omega(0) - \frac{\tau_{11}(\rho_2)}{\rho_2-\rho_1}T_sT_{\rho_2}\omega(0)\right. \\ &\quad \left. + \frac{\frac{\tau_{12}(s)-\tau_{12}(\rho_1)}{s-\rho_1} - \frac{\tau_{12}(s)-\tau_{12}(\rho_2)}{s-\rho_2}}{\rho_1-\rho_2}\hat{\xi}_2(s) - \frac{\tau_{12}(\rho_1)}{\rho_1-\rho_2}T_sT_{\rho_1}\xi_2(0) - \frac{\tau_{12}(\rho_2)}{\rho_2-\rho_1}T_sT_{\rho_2}\xi_2(0)\right] \\ &= (s-\rho_1)(s-\rho_2)\left[\frac{\lambda_1\mu_1\mu_2}{\lambda+\lambda_1+\delta}\hat{\omega}(s) - \frac{\tau_{11}(\rho_1)}{\rho_1-\rho_2}T_sT_{\rho_1}\omega(0) - \frac{\tau_{11}(\rho_2)}{\rho_2-\rho_1}T_sT_{\rho_2}\omega(0)\right. \\ &\quad \left. - \frac{\tau_{12}(\rho_1)}{\rho_1-\rho_2}T_sT_{\rho_1}\xi_2(0) - \frac{\tau_{12}(\rho_2)}{\rho_2-\rho_1}T_sT_{\rho_2}\xi_2(0)\right]. \end{aligned} \tag{3.16}$$

Similarly, after multiplying the numerator of (3.9) by $(1-\mu_1 s)(1-\mu_2 s)$ and applying some careful calculations, we can make the numerator of (3.9) into the following form

$$M_2(s) := (s-\rho_1)(s-\rho_2)\left[\frac{\lambda_2\mu_1\mu_2}{\lambda+\lambda_2+\delta}\hat{\omega}(s) - \frac{\tau_{21}(\rho_1)}{\rho_1-\rho_2}T_sT_{\rho_1}\omega(0) - \frac{\tau_{21}(\rho_2)}{\rho_2-\rho_1}T_sT_{\rho_2}\omega(0) - \frac{\tau_{22}(\rho_1)}{\rho_1-\rho_2}T_sT_{\rho_1}\xi_1(0) - \frac{\tau_{22}(\rho_2)}{\rho_2-\rho_1}T_sT_{\rho_2}\xi_1(0)\right], \tag{3.17}$$

where

$$\tau_{21}(s) = \frac{\lambda_2(1-\mu_1 s)(1-\mu_2 s)}{\lambda+\lambda_2+\delta} - \frac{\lambda\lambda_2(1-\mu_2 s)}{(\lambda+\lambda_1+\delta)(\lambda+\lambda_2+\delta)},$$

$$\tau_{22}(s) = \frac{\lambda\lambda_1\hat{\phi}_2\left(\frac{1}{\mu_2}\right)(1-\mu_1 s) - \lambda\lambda_2\hat{\phi}_1\left(\frac{1}{\mu_1}\right)(1-\mu_2 s)}{(\lambda+\lambda_1+\delta)(\lambda+\lambda_2+\delta)}.$$

Theorem 1. *Assume that the premium sizes are exponentially distributed with distribution functions given by (3.5). Then the Gerber–Shiu functions $\phi_1(u)$ and $\phi_2(u)$ satisfy the following defective renewal equations*

$$\phi_1(u) = \frac{1}{\mu_1\mu_2}\int_0^u \phi_1(u-x)G(x)\mathrm{d}x + B_1(u), \tag{3.18}$$

$$\phi_2(u) = \frac{1}{\mu_1\mu_2}\int_0^u \phi_2(u-x)G(x)\mathrm{d}x + B_2(u), \tag{3.19}$$

where

$$G(x) = \sum_{k=1}^{2}\left[h_{k1}T_{\rho_1}T_{\rho_2}\xi_k(x) + h_{k2}\left(\rho_1T_{\rho_1}T_{\rho_2}\xi_k(x) - T_{\rho_2}\xi_k(x)\right) + h_{k3}\left(\xi_k(x) - (\rho_1+\rho_2)T_{\rho_2}\xi_k(x) + \rho_1^2T_{\rho_1}T_{\rho_2}\xi_k(x)\right)\right],$$

$$B_1(u) = \frac{\lambda_1\omega(u)}{\lambda+\lambda_1+\delta} - \frac{1}{\mu_1\mu_2}\left[\frac{\tau_{11}(\rho_1)}{\rho_1-\rho_2}T_{\rho_1}\omega(u) + \frac{\tau_{11}(\rho_2)}{\rho_2-\rho_1}T_{\rho_2}\omega(u) + \frac{\tau_{12}(\rho_1)}{\rho_1-\rho_2}T_{\rho_1}\xi_2(u) + \frac{\tau_{12}(\rho_2)}{\rho_2-\rho_1}T_{\rho_2}\xi_2(u)\right],$$

$$B_2(u) = \frac{\lambda_2\omega(u)}{\lambda+\lambda_2+\delta} - \frac{1}{\mu_1\mu_2}\left[\frac{\tau_{21}(\rho_1)}{\rho_1-\rho_2}T_{\rho_1}\omega(u) + \frac{\tau_{21}(\rho_2)}{\rho_2-\rho_1}T_{\rho_2}\omega(u) + \frac{\tau_{22}(\rho_1)}{\rho_1-\rho_2}T_{\rho_1}\xi_1(u) + \frac{\tau_{22}(\rho_2)}{\rho_2-\rho_1}T_{\rho_2}\xi_1(u)\right].$$

Proof. By (3.13)–(3.17), we find that for $i = 1, 2$,

$$\hat{\phi}_i(s) = \frac{M_i(s)}{(s-\rho_1)(s-\rho_2)[\mu_1\mu_2 - \hat{G}(s)]} = \frac{\frac{M_i(s)}{\mu_1\mu_2(s-\rho_1)(s-\rho_2)}}{1-\frac{\hat{G}(s)}{\mu_1\mu_2}}.$$

An arrangement of the above equation leads to

$$\hat{\phi}_i(s) = \frac{1}{\mu_1\mu_2}\hat{\phi}_i(s)\hat{G}(s) + \frac{M_i(s)}{\mu_1\mu_2(s-\rho_1)(s-\rho_2)}.$$

Inverting the Laplace transforms in the above equation gives (3.18) and (3.19).

To show that (3.18) and (3.19) are defective renewal equations, we need to show that $\frac{1}{\mu_1\mu_2}\int_0^\infty G(x)\mathrm{d}x < 1$, or equivalently, $\frac{1}{\mu_1\mu_2}\hat{G}(0) < 1$. We have by (3.13)

$$\frac{\hat{G}(s)}{\mu_1\mu_2} = 1 - \frac{(1-\mu_1 s)(1-\mu_2 s)\left[\chi_1(s)\chi_2(s) - \frac{\lambda_1\lambda_2\hat{\xi}_1(s)\hat{\xi}_2(s)}{(\lambda+\lambda_1+\delta)(\lambda+\lambda_2+\delta)}\right]}{(s-\rho_1)(s-\rho_2)\mu_1\mu_2}.$$

Then for $s = 0$

$$\begin{aligned}\frac{\hat{G}(0)}{\mu_1\mu_2} &= 1 - \frac{1}{\rho_1\rho_2\mu_1\mu_2}\left[\chi_1(0)\chi_2(0) - \frac{\lambda_1\lambda_2\hat{\xi}_1(0)\hat{\xi}_2(0)}{(\lambda+\lambda_1+\delta)(\lambda+\lambda_2+\delta)}\right]\\ &= 1 - \frac{(\lambda_1+\delta)(\lambda_2+\delta) - (\lambda_1+\delta)\lambda_2\hat{\xi}_2(0) - (\lambda_2+\delta)\lambda_1\hat{\xi}_1(0)}{\rho_1\rho_2\mu_1\mu_2(\lambda+\lambda_1+\delta)(\lambda+\lambda_2+\delta)}\\ &< 1 - \frac{(\lambda_1+\delta)(\lambda_2+\delta) - (\lambda_1+\delta)(\lambda_2+\delta)(\hat{\xi}_1(0)+\hat{\xi}_2(0))}{\rho_1\rho_2\mu_1\mu_2(\lambda+\lambda_1+\delta)(\lambda+\lambda_2+\delta)} = 1.\end{aligned} \tag{3.20}$$

Now we consider the case $\delta = 0$. Firstly, setting $s = \rho_1(\delta)$ in (3.10) gives

$$\prod_{i=1}^{2}\left[(\lambda+\lambda_i+\delta) - \frac{\lambda}{1-\mu_i\rho_1(\delta)} - \lambda_i\hat{\xi}_i(\rho_1(\delta))\right] = \lambda_1\lambda_2\hat{\xi}_1(\rho_1(\delta))\hat{\xi}_2(\rho_1(\delta)).$$

Next, differentiating the above equation w.r.t. δ and then setting $\delta = 0$, we can obtain

$$\begin{aligned}\rho_1'(0) &= \frac{\lambda_2(1-\hat{\xi}_2(0)) + \lambda_1(1-\xi_1(0))}{\lambda_1\lambda_2(\hat{\xi}_1'(0)+\hat{\xi}_2'(0)) + \lambda\lambda_2\mu_1(1-\xi_2(0)) + \lambda\lambda_1\mu_2(1-\hat{\xi}_1(0))}\\ &= \frac{\frac{1}{\lambda_1}Pr(B\le Y) + \frac{1}{\lambda_2}Pr(B>Y)}{\frac{\lambda\mu_1}{\lambda_1}Pr(B\le Y) + \frac{\lambda\mu_2}{\lambda_2}Pr(B>Y) - \mu}\\ &> 0,\end{aligned}$$

where the last step follows from the net profit condition (2.3). Then taking the limit $\delta \to 0^+$ in (3.20) and applying L'Hôspital's rule, one obtains

$$\begin{aligned}\frac{\hat{G}(0)}{\mu_1\mu_2} &= 1 - \frac{1}{\rho_2(0)\mu_1\mu_2(\lambda+\lambda_1)(\lambda+\lambda_2)} \times \lim_{\delta\to 0^+}\frac{\delta^2 + (\lambda_1(1-\hat{\xi}_1(0)) + \lambda_2(1-\hat{\xi}_2(0)))\delta}{\rho_1(\delta)}\\ &= 1 - \frac{\lambda_1 Pr(B>Y) + \lambda_2 Pr(B\le Y)}{\rho_1'(0)\rho_2(0)\mu_1\mu_2(\lambda+\lambda_1)(\lambda+\lambda_2)}\\ &< 1.\end{aligned}$$

Thus, Eqs. (3.18) and (3.19) are defective renewal equations, and the proof is complete. □

We remark that the explicit analytic solutions to the defective renewal equations (3.18) and (3.19) can be obtained by compound geometric distributions — see e.g. [11] for reference.

4. Premium sizes with rational Laplace transforms

In this section, we consider the case when the premium sizes have rational Laplace transforms — i.e.

$$\hat{f}_i(s) = \frac{q_i(s)}{\prod_{j=1}^{m_i}(s+\lambda_{ij})^{n_{ij}}}, \quad i = 1, 2, \tag{4.1}$$

where $m_i, n_{ij} \in \mathbb{N}^+$ with $n_{i1} + n_{i2} + \cdots + n_{im_i} = k_i$, $\lambda_{i1}, \ldots, \lambda_{ij}$ with $\lambda_{ij_1} \ne \lambda_{ij_2}$ for $j_1 \ne j_2$ are (possibly complex) numbers with positive real parts, $q_i(s)$ satisfying $q_i(0) = \prod_{j=1}^{m_i}\lambda_{ij}^{n_{ij}}$ is a polynomial function of degree $k_i - 1$ or less. By partial fraction, we can rewrite (4.1) as

$$\hat{f}_i(s) = \sum_{j_1=1}^{m_i}\sum_{j_2=1}^{n_{ij_1}} q_{ij_1j_2}\left(\frac{\lambda_{ij_1}}{s+\lambda_{ij_1}}\right)^{j_2}, \tag{4.2}$$

where

$$q_{ij_1j_2} = \frac{1}{\lambda_{ij_1}^{j_2}(n_{ij_1}-j_2)!}\frac{\mathrm{d}^{n_{ij_1}-j_2}}{\mathrm{d}s^{n_{ij_1}-j_2}}\left\{\prod_{k=1,k\ne j_1}^{m_i}\frac{q_i(s)}{(s+\lambda_{ik})^{n_{ik}}}\right\}\Bigg|_{s=-\lambda_{ij_1}}.$$

Without loss of generality, we assume that λ_{ij}'s in (4.1) are positive real numbers. By analytic extension, (4.2) can be extended to the whole complex plane except the points $-\lambda_{ij}$'s. In the rest of the paper, we will still use the notation $\hat{f}_i(s)$ after such analytic extension.

(4.2) implies that F_1 and F_2 are mixtures of Erlangs — i.e. for $i = 1, 2$

$$F_i(x) = \sum_{j_1=1}^{m_i}\sum_{j_2=1}^{n_{ij_1}} q_{ij_1j_2}F_{ij_1j_2}(x),$$

where $F_{ij_1j_2}(x) = 1 - \sum_{k=0}^{j_2-1}\frac{(\lambda_{ij_1}x)^k}{k!}\mathrm{e}^{-\lambda_{ij_1}x}$ is an Eralng(j_2) distribution with parameter λ_{ij_1}. Let $X_{ij_1j_2}^{(1)}, \ldots, X_{ij_1j_2}^{(j_2)}$ be j_2 i.i.d. r.v.'s exponentially distributed with mean $1/\lambda_{ij_1}$. Then $X_{ij_1j_2}^{(1)} + \cdots + X_{ij_1j_2}^{(j_2)}$ has distribution function $F_{ij_1j_2}$. For $Re(s) > \max_j \lambda_{ij}$, we have

$$\begin{aligned}
\hat{A}_i(s) &= \int_0^\infty \mathrm{e}^{-su} \int_0^\infty \phi_i(u+x)\mathrm{d}F_i(x)\mathrm{d}u \\
&= \int_0^\infty \int_0^\infty \mathrm{e}^{-su}\phi_i(u+x)\mathrm{d}u\mathrm{d}F_i(x) \\
&= \sum_{j_1=1}^{m_i}\sum_{j_2=1}^{n_{ij_1}} q_{ij_1j_2} \int_0^\infty T_s\phi_i(x)\mathrm{d}F_{ij_1j_2}(x) \\
&= \sum_{j_1=1}^{m_i}\sum_{j_2=1}^{n_{ij_1}} q_{ij_1j_2} E[T_s\phi_i(X_{ij_1j_2}^{(1)} + \cdots + X_{ij_1j_2}^{(j_2)})] \\
&= \sum_{j_1=1}^{m_i}\sum_{j_2=1}^{n_{ij_1}} q_{ij_1j_2}\lambda_{ij_1} E\int_0^\infty T_s\phi_i(x + X_{j_1j_2}^{(2)} + \cdots + X_{ij_1j_2}^{(j_2)})\mathrm{e}^{-\lambda_{ij_1}x}\mathrm{d}x \\
&= \sum_{j_1=1}^{m_i}\sum_{j_2=1}^{n_{ij_1}} q_{ij_1j_2}\lambda_{ij_1} ET_sT_{\lambda_{ij_1}}\phi_i(X_{j_1j_2}^{(2)} + \cdots + X_{ij_1j_2}^{(j_2)}) \\
&\vdots \\
&= \sum_{j_1=1}^{m_i}\sum_{j_2=1}^{n_{ij_1}} q_{ij_1j_2}\lambda_{ij_1}^{j_2} T_sT_{\lambda_{ij_1}}^{j_2}\phi_i(0),
\end{aligned}$$

where $T_{\lambda_{ij_1}}^{j_2} = \underbrace{T_{\lambda_{ij_1}}\cdots T_{\lambda_{ij_1}}}_{j_2}$. Furthermore, by property 5 of the Dickson–Hipp operator in [10], we have

$$\begin{aligned}
\hat{A}_i(s) &= \sum_{j_1=1}^{m_i}\sum_{j_2=1}^{n_{ij_1}} q_{ij_1j_2}\lambda_{ij_1}^{j_2}\left(\frac{\hat{\phi}_i(s)}{(\lambda_{ij_1}-s)^{j_2}} - \sum_{j=1}^{j_2}\frac{T_{\lambda_{ij_1}}^{j}\phi_i(0)}{(\lambda_{ij_1}-s)^{j_2+1-j}}\right) \\
&= \hat{f}_i(-s)\hat{\phi}_i(s) - L_i(s),
\end{aligned} \tag{4.3}$$

where $L_i(s) = \sum_{j_1=1}^{m_i}\sum_{j_2=1}^{n_{ij_1}}\sum_{j=1}^{j_2}\frac{T_{\lambda_{ij_1}}^{j}\phi_i(0)}{(\lambda_{ij_1}-s)^{j_2+1-j}}$. By analytic extension, (4.3) holds for all s in the right half complex plane except the points λ_{ij}'s. In the rest of the paper, we will still use the notation $\hat{A}_i(s)$ after such analytic extension.

Plugging (4.3) into (3.3) and (3.4) gives

$$\left[1 - \frac{\lambda\hat{f}_1(-s)}{\lambda+\lambda_1+\delta} - \frac{\lambda_1\hat{\xi}_1(s)}{\lambda+\lambda_1+\delta}\right]\hat{\phi}_1(s) - \frac{\lambda_1\hat{\xi}_2(s)}{\lambda+\lambda_1+\delta}\hat{\phi}_2(s) = \frac{\lambda_1\hat{\omega}(s) - \lambda L_1(s)}{\lambda+\lambda_1+\delta}, \tag{4.4}$$

$$\left[1 - \frac{\lambda\hat{f}_2(-s)}{\lambda+\lambda_2+\delta} - \frac{\lambda_2\hat{\xi}_2(s)}{\lambda+\lambda_2+\delta}\right]\hat{\phi}_2(s) - \frac{\lambda_2\hat{\xi}_1(s)}{\lambda+\lambda_2+\delta}\hat{\phi}_1(s) = \frac{\lambda_2\hat{\omega}(s) - \lambda L_2(s)}{\lambda+\lambda_2+\delta}. \tag{4.5}$$

Solving (4.4) and (4.5) gives

$$\hat{\phi}_1(s) = \frac{\left[\lambda_1(\lambda+\lambda_2+\delta) - \lambda\lambda_1\hat{f}_2(-s)\right]\hat{\omega}(s) - \lambda\nu_2(s)L_1(s) - \lambda\lambda_1\hat{\xi}_2(s)L_2(s)}{\nu_1(s)\nu_2(s) - \lambda_1\lambda_2\hat{\xi}_1(s)\hat{\xi}_2(s)}, \tag{4.6}$$

$$\hat{\phi}_2(s) = \frac{\left[\lambda_2(\lambda+\lambda_1+\delta) - \lambda\lambda_2\hat{f}_1(-s)\right]\hat{\omega}(s) - \lambda\nu_1(s)L_2(s) - \lambda\lambda_2\hat{\xi}_1(s)L_1(s)}{\nu_1(s)\nu_2(s) - \lambda_1\lambda_2\hat{\xi}_1(s)\hat{\xi}_2(s)}, \tag{4.7}$$

where

$$\nu_i(s) = \lambda + \lambda_i + \delta - \lambda\hat{f}_i(-s) - \lambda_i\hat{\xi}_i(s), \quad i = 1, 2.$$

The common denominator of (4.6) and (4.7) is analytic for s in the right half complex plane except the poles λ_{ij}'s. To make it analytic for all $Re(s) \geq 0$, let $\Lambda_i(s) = \prod_{j=1}^{m_i}(s-\lambda_{ij})^{n_{ij}}$ and multiply both the numerators and denominators of (4.6) and (4.7) by $\Lambda_1(s)\Lambda_2(s)$. Then we obtain

$$\hat{\phi}_1(s) = \frac{Q_1(s)\hat{\omega}(s) - \lambda\nu_2(s)\Lambda_2(s)\Lambda_1(s)L_1(s) - \lambda\lambda_1\hat{\xi}_2(s)\Lambda_1(s)\Lambda_2(s)L_2(s)}{\nu_1(s)\nu_2(s)\Lambda_1(s)\Lambda_2(s) - \lambda_1\lambda_2\hat{\xi}_1(s)\hat{\xi}_2(s)\Lambda_1(s)\Lambda_2(s)}, \tag{4.8}$$

$$\hat{\phi}_2(s) = \frac{Q_2(s)\hat{\omega}(s) - \lambda\nu_1(s)\Lambda_1(s)\Lambda_2(s)L_2(s) - \lambda\lambda_2\hat{\xi}_1(s)\Lambda_2(s)\Lambda_1(s)L_1(s)}{\nu_1(s)\nu_2(s)\Lambda_1(s)\Lambda_2(s) - \lambda_1\lambda_2\hat{\xi}_1(s)\hat{\xi}_2(s)\Lambda_1(s)\Lambda_2(s)}, \tag{4.9}$$

where

$$Q_1(s) = \Lambda_1(s)\Lambda_2(s)\left[\lambda_1(\lambda+\lambda_2+\delta) - \lambda\lambda_1\hat{f}_2(-s)\right],$$
$$Q_2(s) = \Lambda_1(s)\Lambda_2(s)\left[\lambda_2(\lambda+\lambda_1+\delta) - \lambda\lambda_2\hat{f}_1(-s)\right].$$

From (4.8) and (4.9), we know that $\hat{\phi}_1(s)$ and $\hat{\phi}_2(s)$ can be obtained if we can determine $\Lambda_1(s)L_1(s)$ and $\Lambda_2(s)L_2(s)$. It is easily seen that $\Lambda_i(s)L_i(s)$ is a polynomial of degree $k_i - 1$, and then it can be expressed as

$$\Lambda_i(s)L_i(s) = \sum_{n=1}^{k_i} L_{in}s^{n-1}.$$

Consequently, we have to determine k_1+k_2 unknown coefficients L_{in}'s. For this purpose, we give without proof the following Lemma which can be proved by exactly the same technique used in Lemma 1.

Lemma 2. *The common denominator of (4.8) and (4.9) has exactly $k_1 + k_2$ zeros, say $\rho_1, \ldots, \rho_{k_1+k_2}$, in the right half complex plane.*

Assume that $\rho_1, \ldots, \rho_{k_1+k_2}$ are distinct. Sine $\hat{\phi}_1(s)$ and $\hat{\phi}_2(s)$ are analytic for $Re(s) \geq 0$, then $\rho_1, \ldots, \rho_{k_1+k_2}$ are also zeros of the numerators of (4.8) and (4.9). And both cases give the following $k_1 + k_2$ linear equations satisfied by L_{in}'s

$$\lambda\nu_2(\rho_i)\Lambda_2(\rho_i)\sum_{n=1}^{k_1} L_{1n}\rho_i^{n-1} + \lambda\lambda_1\hat{\xi}_2(\rho_i)\Lambda_1(\rho_i)\sum_{n=1}^{k_2} L_{2n}\rho_i^{n-1} = Q_1(\rho_i)\hat{\omega}(\rho_i), \quad i = 1, 2, \ldots, k_1+k_2. \tag{4.10}$$

After solving (4.10), we can obtain L_{in}'s. Then the Laplace transforms (4.8) and (4.9) are fully determined.

5. Conclusion

In this paper, we analyze the ruin problems in a risk model with stochastic premiums income and a dependence structure among the claim sizes, interclaim times and premium sizes. Some analytic techniques are applied to study the Gerber–Shiu functions. For exponential premium sizes, we show that the Laplace transforms and defective renewal equations for the Gerber–Shiu functions can be obtained. While for premiums with rational Laplace transforms, the Laplace transforms for the Gerber–Shiu functions are also obtained.

The model considered in this paper can be extended in the more general framework. For example, we can introduce an underlying Markov process to modulate the claim sizes, interclaim times and premiums. We can also add some diffusion processes to describe the stochastic volatility of the premiums income and claims loss. In particular, a mathematically treatable candidate for the diffusion volatility is the standard Brownian motion, and such extension will only lead to a little computation involvement.

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