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Implementation of a new algorithm of computation of the Poincaré–Liapunov constants

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Abstract

In the last years many papers giving different methods to compute the Poincaré–Liapunov constants have been published. In (Appl. Math. Warsaw 28 (2002) 17) a new method to compute recursively all the Poincaré–Liapunov constants as a function of the coefficients of the system for an arbitrary analytic system which has a perturbed linear center at the origin was given, and thus a theoretical answer to the classical center problem was given. The method also computes the coefficients of the Poincaré series as a function of the same coefficients. We describe its implementation in two different ways, by means of a computer algebra system and an algorithm in any computer language. If this second alternative is used, later it is necessary to translate the results so that they can be manipulated with a computer algebra system. We describe also how the availability of symbolic manipulation procedures has recently led to a significant progress in the resolution of the different problems related with the Poincaré–Liapunov constants as they are the central problems like the small-amplitude limit cycles.

MSC: primary 34C05; secondary 37G15; 68W30

Keywords: Poincaré–Liapunov constants; Computer algebra system; Gröbner basis; Small-amplitude limit cycles; The center problem

1. Introduction

Many models of nature use differential equation systems in the plane and with the qualitative theory of differential equations introduced by Poincaré, the behavior of these systems in the majority

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of the cases can be known. One of the problems that persists to control the behavior of those type of systems is to distinguish among a focus and or a center (the center problem). The resolution of this problem goes through computing the so called Poincaré–Liapunov constants. Therefore, to have a fast and easy method for the computation of such constants is of great use in studying this class of systems. Other very important problem is to determine systems that have centers at some singular points due to the fact that perturbations of these systems give rich bifurcations of limit cycles.

The second part of the 16th Hilbert problem concerns the qualitative theory of differential equations and it is the following. Consider systems of the form

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1}$$

where P and Q are polynomials and x and y are real unknown functions. Systems of form (1) are called *polynomial systems*. Among trajectories of a polynomial system one can single out some which correspond to isolated periodic solutions. These trajectories are called *limit cycles*. Let $\pi(P, Q)$ be the number of limit cycles of (1) and define

$$H_n = \sup\{\pi(P, Q); \ \partial P, \partial Q \leqslant n\}.$$

The question of the second part of the 16th Hilbert problem is the maximal possible number of limit cycles. Estimates are sought for H_n in terms of n, and their location. The first part of the problem deals with an estimation of number of ovals of an algebraic curve. Very important connections exist among both parts as the limit cycles of a system with a polynomial inverse integrating factor $V(x, y)$ correspond to ovals of the curve $V(x, y) = 0$, see [14]. Therefore, if we know an estimation for the number of limit cycles of system (1), we can know an estimation for the number of ovals of an algebraic curve if we control the degree of the polynomial inverse integrating factor $V(x, y)$ as a function of n. In the present paper we consider some questions that are related with the Poincaré–Liapunov constants. In the second part of the 16th Hilbert problem. In fact, there exists a whole area on this. It is misleading to think of it as a single problem. Its history and present status are described in detail in [23]. Much of the recent progress has been achieved by considering various kinds of bifurcation. One of them in which the Poincaré–Liapunov constants intervene is the limit cycles which bifurcate out of a critical point, and some of them are the so-called *small amplitude limit cycles*.

Very briefly, the position is that remarkably little is known about H_n. It has not even been established that they are finite, and it has been proved that a given polynomial system cannot have infinitely many limit cycles by Ecalle [9] and Ilyashenko [17]. The first major contribution was that of Bautin [2], who proved that $H_2 \geqslant 3$ and this work is classical in the theory of limit cycles bifurcations and his ideas have been very influential in the development of the subject. Afterwards, Landis and Petrovskii published two papers, in one of which it was suggested that $H_2 = 3$ and in the other precise bounds were given for H_n with $n \geqslant 3$. However, the proofs of these results were soon withdrawn, but nevertheless it appears to have been widely believed that $H_2 = 3$. It was until 1979 that the first examples of quadratic systems with at least four limit cycles appeared given by Shi Songling [24] and Cheng and Wang [7]. These developments stimulated renewed interest in 16th Hilbert's problem. Only very recently some lower bounds were also obtained in the case where P and Q are polynomials of degree three, in what follows *cubic systems*. It was shown by Żołądek that $H_3 \geqslant 11$. Considerable results in the direction to prove that H_2 is finite were obtained by Dumortier, Roussarie and Rousseau trying to investigate limit cycles which appear from singular trajectories, mainly, from a center or focus type equilibrium point or from a separatrix cycle.

In Section 2 we explain the algorithmic procedure for computing the Poincaré–Liapunov constants which enables us to solve some of the mentioned problems. The implementation of this algorithm is described in Section 3. In Section 4 we explain the basic idea of the bifurcation of limit cycles out of critical point and see on that consists the center problem. Finally, in Section 5 we have concentrated on the computational problems that arise using a computer algebra system to solve the above problems.

2. The calculation of the Poincaré–Liapunov constants

Consider two-dimensional autonomous systems of differential equations of the form

$$\dot{x} = -y + X(x,y), \quad \dot{y} = x + Y(x,y), \tag{2}$$

where the nonlinearities are $X(x,y) = \sum_{s=2}^{\infty} X_s(x,y)$ and $Y(x,y) = \sum_{s=2}^{\infty} Y_s(x,y)$ with $X_s(x,y) = \sum_{k=0}^{s} a_k^s x^k y^{s-k}$ and $Y_s(x,y) = \sum_{k=0}^{s} b_k^s x^k y^{s-k}$ and a_k^s and b_k^s are arbitrary real coefficients.

To compute the Poincaré–Liapunov constants, we follow the classical procedure of Poincaré [21] which developed an important technique that consists in finding a formal power series of the form

$$H(x,y) = \sum_{n=2}^{\infty} H_n(x,y), \tag{3}$$

where $H_2(x,y) = (x^2+y^2)/2$, and for each n, $H_n(x,y) = \sum_{k=0}^{n} C_k^n x^k y^{n-k}$ such that the derivative of H along the solutions of system (2) satisfies

$$\dot{H} = \sum_{k=2}^{\infty} V_{2k}(x^2+y^2)^k, \tag{4}$$

where V_{2k} are called the *Poincaré–Liapunov constants*.

In [16] it is proved that we can always determine C_k^n and V_{2k} from a_k^s and b_k^s, but C_k^n are not unique and consequently neither V_{2k}. Therefore, the Poincaré's formal series is not unique. Poincaré [21] proved, by acotation, that there exists one which is convergent for polynomial systems, and Liapunov [18] generalized Poincaré's theorem to analytic systems. In [6] Chazy demonstrated using the theorem of analytical dependence respect to the initial parameters that there exists one which is convergent choosing adequately the arbitrary parameters that appear in the construction of Poincaré's series.

As it has been said the V_{2k} and Poincaré's formal series are not unique, but for polynomial systems we have uniqueness for the V_{2k} in the sense of the following theorem due to Shi Songling [26].

Theorem 1. *Let* $\mathbf{A}$ *be the ring of real polynomials whose variables are the coefficients of the polynomial differential system. Given a set of Poincaré–Liapunov constants* $V_1, V_2, \ldots, V_i$, *let* $\mathbf{J}_{k-1}$ *be the ideal of* $\mathbf{A}$ *generated by* $V_1, V_2, \ldots, V_{k-1}$. *If* $V_1', V_2', \ldots, V_i'$ *is another set of Poincaré–Liapunov constants, then* $V_k \equiv V_k' \bmod (\mathbf{J}_{k-1})$.

As it will be seen later on the origin is a center if and only if all the V_i's are zero. Let $\mathbf{J} = (V_1, V_2, \ldots)$ be the ideal of $\mathbf{A}$ generated by all the V_i's. For polynomials systems, using the Hilbert's basis theorem, $\mathbf{J}$ is finitely generated; i.e. there exist $B_1, B_2, \ldots, B_q$ in $\mathbf{J}$ such that $\mathbf{J} = (B_1, B_2, \ldots, B_q)$ because $\mathbf{A}$ is Noetherian. Such a set of generators is called a basis of $\mathbf{J}$.

Notice that Hilbert's basis theorem assures us the existence of a generators basis, but it does not provide us a constructive method to find it. The existent methods to solve this problem are based in the Buchberger's algorithm to find a Gröebner basis, but it is only applicable for very simple cases. Therefore, it is a problem of computational algebraic nature due to the appearance, already for simple systems, of massive Poincaré–Liapunov constants that are polynomials with rational coefficients and efficient algorithms do not exist that allow to determine simple groups of generators. One of the main difficulties comes ultimately on the decomposition in prime numbers of a big integer number. Therefore resolution of computational problem goes to have efficient algorithms that work with big integers and in decomposition into primes numbers of big numbers, a classical problem in computational mathematics. On the other hand there are recursive methods for the determination of these Poincaré–Liapunov constants and the development of the algebraic manipulators has allowed to approach the calculation of the first constants.

Different algorithms to compute the Poincaré–Liapunov constants exist. The technique used by Bautin [2] is based on computing the derivatives of the return map from a nonlinear system of recursive differential equations. The original Bautin's method is effectively costly in computer time because it involves computations of indefinite/definite integrals. There is another algorithm which involves the solution of a system of linear equations for the coefficients of H_n in terms of the coefficients of X_s, Y_s and H_k for $k = 2, \ldots, n-1$; see for instance [19,20]. Another method is to construct a Poincaré's formal power series in polar coordinates and the Poincaré–Liapunov constants can be computed from recursive linear formulas as definite integrals of trigonometric polynomials (see, for example, [1,4,5]). In [10] the authors give a survey of different ways to compute the Poincaré–Liapunov constants. Using a method based on the use of the Runge–Kutta–Fehlberg methods and the use of Richardson's extrapolation in [13]—an analytic-numerical method of computation of the Poincaré–Liapunov constants is given. Another algorithm to compute the Poincaré–Liapunov constants is developed in [12,11] where the method is based on the calculation of the successive derivatives of the first return map associated with the perturbations of some planar Hamiltonian systems. An important generalization of this last method is given in [27].

We present a formula developed in [16], to compute the Poincaré–Liapunov constants and the Poincaré series for general systems (1) as a recurrence form following the ideas of Shi Songling in [25] where he found the same expression for the Poincaré–Liapunov constants, but he did not find the recursive relation with the Poincaré series to establish a method to compute them. The advantages of this method are that in all the process the unique calculations are products and sums without indefinite/definite integrals as in most of the others methods and consequently it is very easy and its implementation on a computer is optimizable. Others methods display this advantage, this is certainly the case of the successive derivatives approach, see for instance [21,12]. But our method gives simultaneously the Poincaré–Liapunov constants and the Poincaré series. Knowing the Poincaré series is very useful for applications, for example, the study of systems that have a polynomial first integral, which have a finite Poincaré series. We planned to study in next works if it allows us to obtain new theoretical results.

Theorem 2. *The Poincaré–Liapunov constants of system* (2) *are*

$$V_n = \frac{\sum_{l=0}^{n/2} (n-(2l+1))!!(2l-1)!! d_{2l}^n}{\sum_{l=0}^{n/2} (n-(2l+1))!!(2l-1)!! \begin{pmatrix} n/2 \\ l \end{pmatrix}}, \quad n = 4, 6, 8, \ldots,$$

where $d_k^n = \sum_{m=1}^{n-2} \sum_{l=0}^{m+1} (l a_{k-l+1}^{n-m} + (m+1-l) b_{k-l}^{n-m}) C_l^{m+1}$, $n \geqslant 3$, $k = 0, \ldots, n$, *with* $a_k^s = b_k^s = 0$ *for* $k < 0$ *or* $k > s$, $C_0^2 = C_2^2 = 1/2$ *and* $C_1^2 = 0$, *and*

$$C_k^n = \frac{\sum_{l=0}^{(k-1)/2} (n-(2l+1))!!(2l-1)!! \left(d_{2l}^n - \begin{pmatrix} n/2 \\ l \end{pmatrix} V_n \right)}{(n-k)!!k!!}, \quad n \geqslant 3, \; k = 1, 3, 5, \ldots,$$

$$C_k^n = -\frac{\sum_{l=k/2}^{[(n-1)/2]} (n-(2l+2))!!(2l)!! d_{2l+1}^n + \lambda_n}{(n-k)!!k!!}, \quad n \geqslant 3, \; k = 0, 2, 4, \ldots,$$

where λ_n *are arbitrary constants and* V_n and λ_n *are zero for* n *odd.*

The method works as follows. From the first terms of the Poincaré series (3), i.e. $C_0^2 = C_2^2 = 1/2$ and $C_1^2 = 0$ it is possible to calculate d_k^3 for $k = 0, 1, 2, 3$ and from here C_k^3 for $k = 0, 1, 2, 3$. Therefore the next step is to calculate d_k^4 for $k = 0, 1, 2, 3, 4$ and finally we obtain V_4 and C_k^4 for $k = 0, 1, 2, 3, 4$. The process continues in an analogous way.

2.1. The quadratic and cubic homogeneous perturbations

We are going to apply the above expressions, for quadratic and cubic homogeneous perturbations, i.e. systems with a linear center perturbed by quadratic polynomials, in what follows, *quadratic systems*, and cubic homogeneous polynomials, respectively. For quadratic systems all a_k^s and b_k^s are zero except a_0^2, a_1^2, a_2^2 and b_0^2, b_1^2, b_2^2. Therefore, in the expression

$$d_k^n = \sum_{m=1}^{n-2} \sum_{l=0}^{m+1} (l a_{k-l+1}^{n-m} + (m+1-l) b_{k-l}^{n-m}) C_l^{m+1}, \tag{5}$$

we have $n - m = 2$; i.e. $m = n - 2$, and the previous expression takes the form

$$d_k^n = \sum_{l=0}^{n-1} (l a_{k-l+1}^2 + (n-1-l) b_{k-l}^2) C_l^{n-1}.$$

Taking into account that the subindex of a_{k-l+1} must be $k - l + 1 = 0, 1, 2$ and the subindex of b_{k-l} must be $k - l = 0, 1, 2$, we have that $l = k + 1$, $l = k$, $l = k - 1$ and $l = k$, $l = k - 1$, $l = k - 2$, respectively, with $0 \leqslant l \leqslant n - 1$. Then d_k^n is

$$\begin{aligned} d_k^n = {} & (k+1) a_0^2 C_{k+1}^{n-1} + (k a_1^2 + (n-1-k) b_0^2) C_k^{n-1} \\ & + ((k-1) a_2^2 + (n-k) b_1^2) C_{k-1}^{n-1} + (n+1-k) b_2^2 C_{k-2}^{n-1}, \end{aligned}$$

and the restriction $0 \leqslant l \leqslant n - 1$ implies that $C_l^{n-1} = 0$ if it is not satisfied.

For cubic homogeneous perturbations all a_k^s and b_k^s are zero except $a_0^3, a_1^3, a_2^3, a_3^3$ and $b_0^2, b_1^2, b_2^2, b_3^3$. Since $n-m=3$, i.e. $m=n-3$, expression (5) takes the form

$$d_k^n = \sum_{l=0}^{n-2} (l a_{k-l+1}^3 + (n-2-l) b_{k-l}^3) C_l^{n-2}.$$

Taking into account that the subindex of a_{k-l+1} must be $k-l+1=0,1,2,3$ and the subindex of b_{k-l} must be $k-l=0,1,2,3$, we have that $l=k+1$, $l=k$, $l=k-1$, $l=k-2$ and $l=k$, $l=k-1$, $l=k-2$, $l=k-3$, respectively, with $0 \leqslant l \leqslant n-2$. Then d_k^n is

$$\begin{aligned} d_k^n &= (k+1) a_0^3 C_{k+1}^{n-2} + (k a_1^3 + (n-2-k) b_0^3) C_k^{n-2} \\ &\quad + ((k-1) a_2^3 + (n-k-1) b_1^3) C_{k-1}^{n-2} \\ &\quad + ((k-2) a_3^3 + (n-k) b_2^3) C_{k-2}^{n-2} + (n+1-k) b_3^3 C_{k-3}^{n-2}, \end{aligned}$$

and the restriction $0 \leqslant l \leqslant n-2$ implies that $C_l^{n-2}=0$ if it is not satisfied.

The application to more general systems is based on finding the expression d_k^n and it is easy to see that contributions to d_k^n of each homogeneous term of the system are independent.

3. Implementation

The implementation of an algorithm can be approached in two different ways. On the one hand they can be used in the commercial versions of the algebraic manipulators as Axiom (the commercial version of Scratchpad), Maple, Mathematica, Reduce, Macsyma and specialized programs as Macaulay, Cocoa, Mas, Magma, Posso and Singular. However, these manipulators are not even powerful enough for very extensive calculations or the programming of the algorithms is not foreseen. On the other hand, these programs are of very general character and written in language LISP generally. They require the use of big computers or computers specially designed for their use that consume a great quantity of memory and a lot of time of CPU, which hinders their use considerably for certain problems. Another form of approaching these problems is by means of the use of algebraic manipulators specially designed for the resolution of concrete problems and not with a general purpose. Implementing the algorithm using a programming language and building a program specifies for the resolution of the concrete problem.

When we first became involved in computations relating to the center problem and small-amplitude limit cycles, we used the method that consists in constructing a Poincaré's formal power series in polar coordinates and the Poincaré–Liapunov constants that can be computed from recursive linear formulas as definite integrals of trigonometric polynomials, see [4,5]. The algorithm was written in C++ and was used to obtain results for linear centers perturbed by quartic and quintic homogeneous polynomials. The program's operation is controlled by a PSP (Poincaré series processor) command file running under the LINUX operating system, and different files identifier of LINUX are exploited to give the user a simple method of distinguishing between various files relating to a particular system of differential equations. Initially, the user is required to provide information in one file.

In < filename > LYCONFIG the user enters the degree of the polynomial system, the type of system, i.e., homogeneous or complete, the range of k for which V_{2k} is computed together with the optional relations between the coefficients of the polynomial system that we want to introduce.

The program is organized so that in the kth "round", the polynomial V_{2k} is computed. When the nominated terminal value of k is reached, different files are produced and their contents are described as follows. First, the initial value of k is 2. The program runs as far as round the terminal value of k and the Poincaré–Liapunov constants $V_4, V_6, \ldots, V_{2k}$ are stored in different files (< filenames >) PLCVk (Poincaré–Liapunov constant k) and if there are restrictions introduced in the file LYCONFIG, the constants are stored in (< filenames >) SPLCVk (simplified Poincaré–Liapunov constant k). The Poincaré series are stored in different files (< filenames >) TMPnCk (kth homogeneous part of the Poincaré series for a polynomial system of degree n). In this way, the calculations can be restarted at $k = k + 1$. In practice, the program is first run from $k = 2$ to $2r - 1$, then substitutions from $V_4, V_6, \ldots, V_{2r}$ are decided and the program is called again, but now the initial value of k is $2r$ with these relations between coefficients in the LYCONFIG. This is a valuable facility, for the appropriate substitutions cannot usually be seen in advance of knowing the first Poincaré–Liapunov constants. It is a matter of judgement how many Poincaré–Liapunov constants should be calculated before entering further substitutions. As a rough guide one would not normally compute more than two or three, and often only one. After the Poincaré–Liapunov constants are computed, it is possible to do their reduction using Mathematica by the program translation CONVERT, which give the Poincaré–Liapunov constants in the Mathematica format. The reduction procedure is heavily interactive. We have not sought to automate it, experience suggests that some of the information which we require later would be lost if we did. Like many other computer implementations, it has evolved with changes made in response to user requirements as well as the continuing efforts to improve its efficiency.

As seen in the previous section our investigations developed a more sophisticated approach [16] which we have initially implemented in `Mathematica 3.0` on a Pentium III with 450 MHz and 64 Mb RAM. Our current implementation of the algorithm, and that which we describe here, uses C++ on the same computer. The program's operation is also controlled by a PSP (Poincaré series processor) command file running under the LINUX operating system. Initially, the user is required to provide information. The program asks if the user wants to store the d_k^n and C_k^n in different files (< filenames >) Dk and Ck, respectively. After that the user enters the range of k for which V_{2k} is computed and the degree of the polynomial system. As the coefficients of the Poincaré–Liapunov constants are rational numbers, the implementation uses a library for doing number theory. The NTL library v. 3.6b, is freely available for research and educational purposes. The latest version of NTL is available at `www.shoup.net`. The output of the algorithm, the V_{2k}, is directly in Mathematica format. The obtained timings have been controlled by the function *Timing* of mathematica and by function *gettimeofday* of C++. The implementation versions are available to anyone who is interested. Please contact the authors.

We planned to study if our method is computationally more effective than the methods of others. Comparisons among these methods are very difficult because each method uses different coordinates system and therefore the number of terms of the coefficients of the Poincaré series and the Poincaré–Liapunov constants varies according to the coordinates used. Therefore, we present the following tables for quadratic and cubic homogeneous perturbations giving the times of calculation, the width in bytes and the number of terms for the methods developed in [27,4,5,16]. The computations of the

Table 1
Quadratic perturbations

Algorithm	Constants	Time	Width in bytes	Number of terms
Method [27]	$k=2$–4	1.17	57; 543; 2104	2; 14; 44
Maple V	$k=5$	4.34	6075	110
Method [4]	$k=2$–4	1.20	24; 204; 772	1; 7; 24
C++	$k=5$	14.45	2240	58
Method [16]	$k=2$–4	51.25	87; 1329; 7092	6; 56; 220
Mathematica	$k=5$	711.67	25413	628
Method [16]	$k=2$–4	2.24	87; 1329; 7092	6; 56; 220
C++	$k=5$	34.47	25413	628

Table 2
Cubic homogeneous perturbations

Algorithm	Constants	Time	Width in bytes	Number of terms
Method [27]	$k=2..6$	1.71	21; 60; 281; 1214; 2895	2; 2; 14; 30; 82
Complex c.	$k=7$	3.92	7540	150
Maple V	$k=8$	15.15	13555	302
Method [4]	$k=2..6$	1.09	10; 24; 163; 382; 1181	1; 1; 7; 14; 41
Polar c.	$k=7$	1.81	2427	74
C++	$k=8$	13.23	5306	151
Method [16]	$k=2..6$	197.62	39; 285; 1456; 4650; 13880	4; 16; 60; 160; 396
Cartesian c.	$k=7$	1516.15	36321	848
Mathematica	$k=8$	10697.4	85432	1716
Method [16]	$k=2..6$	2.64	39; 285; 1456; 4650; 13880	4; 16; 60; 160; 396
Cartesian c.	$k=7$	5.53	36321	848
C++	$k=8$	36.47	85432	1716

method developed in [27] have been implemented with Maple VR4 on a Workstation (SUN Ultra E-450) with three processor Pentinum II with 250 MHz and 256 Mb RAM (Tables 1, 2).

4. Small-amplitude limit cycles and the center problem

In this case we consider systems in which the origin is a critical point of focus type, and show how to bifurcate limit cycles out of it. Thus we investigate systems of the form

$$\dot{x} = \lambda x - y + X(x, y), \quad \dot{y} = x + \lambda y + Y(x, y), \tag{6}$$

where the nonlinearities are $X(x, y) = \sum_{s=2}^{n} X_s(x, y)$ and $Y(x, y) = \sum_{s=2}^{n} Y_s(x, y)$ where X_s and Y_s are homogeneous polynomials of degree s. The linear part is in canonical form, and the stability of the origin is determined by the sign of λ. If $\lambda = 0$, the origin is a center for the linearized system, and is said to be a *fine focus* of the nonlinear system. In order to solve the problem of the stability at the origin of system (6), it is sufficient to consider the sign of the first Poincaré–Liapunov constant different from zero. The origin is a nonlinear center, i.e., there is an open neighborhood of the origin where all orbits are periodic except of course the origin, if and only if all Poincaré–Liapunov constants are zero. The idea is to perturb the coefficients arising in the X_s and Y_s so that limit cycles bifurcate out of the origin. Such limit cycles are said to be of *small amplitude*. The origin is said to be a fine focus of order k if V_{2k+2} is the first nonzero Poincaré–Liapunov constant. In this case at most k limit cycles can bifurcate from this fine focus, see for instance [3]. To maximize the number of limit cycles which can bifurcate, we start with a fine focus which is as close to being a center for the nonlinear system as possible. Therefore, to obtain the maximum number of limit cycles which can bifurcate from the origin for a given system, one has to find the maximum possible order of a fine focus.

Suppose that the origin is a fine focus of order k. The first step is to perturb the coefficients in X and Y so that $V_{2k} \neq 0$ with $V_{2l} = 0$ for $l < 2k$ and $V_{2k}V_{2k+2} < 0$; if this can be achieved, the stability of the origin is reversed, and a limit cycle Γ_1 bifurcates. Next, further perturbations are introduced so that $V_{2k-2}V_{2k} < 0$ with $V_{2l} = 0$ for $l < k - 2$. The stability of the origin is again reversed, and another limit cycle Γ_2 appears. Provided that V_{2k-2} is small enough, Γ_1 persists, and there are therefore two limit cycles. Proceeding in this way, k limit cycles bifurcate provided perturbations can be so arranged that $V_{2k}V_{2k+2} < 0$ for $1 \leqslant l \leqslant k$, see [25].

Since it is the first nonzero Poincaré–Liapunov constant that is of significance, what we really need are the non-zero expressions obtained by calculating each V_{2k} under the conditions $V_2 = \cdots = V_{2k-2} = 0$. It can happen that a reduced Poincaré–Liapunov constant is zero, in which case it does not contribute in the process of bifurcation of limit cycles. For a given class of systems, the aim is to maximize the number of limit cycles which can bifurcate from the origin. Thus, it is necessary to find k_1, the maximum possible order of a fine focus. This k_1 is characterized by the fact that the origin is a center if $V_{2k} = 0$ for $k \leqslant 1 + k_1$, but not if any of these constants is non-zero. In practice, one proceeds with the computation of the Poincaré–Liapunov constants until it appears that k_1 has been reached. Then, it is necessary to prove independently that the origin is a center. This is often difficult, and developing criteria for the existence of a center is a significant and substantive problem. Different techniques such as reversibility of the system, existence of a first integral or an integrating factor defined in a neighborhood of the critical point and existence of analytical changes to simplified systems are the most used ones.

5. Computational problems

Assisting the previous section, there are four phases to the procedure each one of them with concrete computational problems.

1. *Calculation of Poincaré–Liapunov constants*: In the calculation of this constants, very large expressions arise. It is here that computer algebra systems have proved so valuable. We have a recurrent formula to compute the polynomials and the limitations that it imposes on the computer.
2. *Reduction of Poincaré–Liapunov constants*: In the reduction of V_{2k} direct substitutions from the relations $V_2 = \cdots = V_{2k-2} = 0$ which involve rational functions of the coefficients arising in X and Y can be used. This contrasts with the formal calculation of a basis for the ideal generated by the Poincaré–Liapunov constants applying the Buchberger's Gröbner basis method or variations of

this one, where in that case all substitutions are polynomials. These methods are based on defining a division algorithm using some monomial ordering, see [8]. The division algorithm is used also to make the reduction of the Poincaré–Liapunov constants. Calculations based on Buchberger's algorithm can be done only for sufficiently simple polynomials with the program packages of a computer algebra system. Unfortunately, we deal with very massive polynomials and even if very powerful computers are used it is not possible to solve the ideal membership problem. There are variations of the Buchberger's algorithm taking into account some special properties of the Poincaré–Liapunov constants, see [22].

3. Establishing the value of k_1 by proving that the origin is center if $V_{2k}=0$ for $k \leqslant 1+k_1$.
4. Beginning with a fine focus of maximal order, finding a sequence of perturbations each of which reverses the stability of the origin.

6. Some results

We give a brief resumé of some of the results which have been obtained using the techniques described in this paper. Let $\tilde{H}$ denote the maximum number of limit cycles which can bifurcate out of a fine focus. It has long been known that $\tilde{H}=3$ for quadratic systems; this was shown by Bautin [2]. In [3], Blows and Lloyd proved that $\tilde{H}=5$ for cubic systems in which the quadratic terms are absent. For general cubic systems, the last result due to Żołądek, see [28], is $\tilde{H} \geqslant 11$.

Certain quartic and quintic systems have been recently investigated using the described method, see [15]. For systems of the form

$$\dot{x} = -y, \quad \dot{y} = x + Q_n(x, y), \tag{7}$$

where $Q_n(x, y)$ is homogeneous polynomial of degree n, for $n=4$ and $n=5$, it is proved that the maximum number of small-amplitude limit cycles which can bifurcate from the origin is at least four for system (7) with $n=4$ and five for system (7) with $n=5$.

Other type of systems which we have studied in detail are the so-called "homogeneous systems"; these systems are of the form

$$\dot{x} = -y + P_n(x, y), \quad \dot{y} = x + Q_n(x, y), \tag{8}$$

where $P_n(x, y)$ and $Q_n(x, y)$ are homogeneous polynomial of degree n. These type of systems have been studied using the method with polar coordinates for $n=4$ and 5 and we discuss about the number of small-amplitude limit cycles which can bifurcate from the origin for system (8) which is $\geqslant 7$ for $n=4$ and $\geqslant 9$ for $n=5$, see [4,5].

Much of activity has been concerned with systems of Liénard type

$$\dot{x} = y - F(x), \quad \dot{y} = -g(x). \tag{9}$$

The value of $\tilde{H}$ is obtained in a large number of cases for such systems, see [3]. The BiLiénard systems are systems of the form

$$\dot{x} = y - F(x), \quad \dot{y} = -x - G(y). \tag{10}$$

The case with $F(x) = a_2x^2 + a_3x^3 + a_4x^4$ and $G(y) = b_2y^2 + b_3y^3 + b_4y^4$ have been studied in [16] and the maximum number of small-amplitude limit cycles which can bifurcate from the origin is at least six, see [16].

Other systems recently investigated using the techniques described in this paper are the systems of the form

$$\dot{x} = y + xf(x, y), \quad \dot{y} = -x + yf(x, y). \tag{11}$$

This type of systems are called *uniformly isochronous centers* because they have an isochronous center at the origin and in polar coordinates (r, φ) the angle φ satisfies the equation $\dot{\varphi} = 1$. For system (11) the maximum number of small-amplitude limit cycles which can bifurcate from the origin is at least three when $f(x, y) = f_1(x, y) + f_3(x, y)$ where $f_i(x, y)$ are homogeneous polynomials of degree i, see [16].

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