Large planar graphs with given diameter and maximum degree

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Abstract

We consider the problem of determining the maximum number of vertices in a planar graph with given maximum degree $A$ and diameter $k$. This number has previously been exactly determined when $k = 2$. We show here that when $k = 3$, the number is roughly between $4.5A$ and $8A$. We also show that in general the number is $\Theta(A^{3/2})$ for any fixed value of $k$.

1. Introduction

Let $G$ be a planar graph on $n$ vertices with maximum degree $A$ and diameter $k$. What is the maximum number of vertices that $G$ can have? This is to be viewed as a contribution in the general area of the construction of large graphs with given diameter and maximum degree (see, for instance [1-3]). Indeed, planarity is a natural restriction in many applications. When $k = 2$, it has been shown [5] that $n \leq \lfloor \frac{3}{2} A \rfloor + 1$ (for $A \geq 8$), and constructions are given (for all $A \geq 8$) of planar graphs with maximum degree $A$ and diameter two, containing precisely $\lfloor \frac{3}{2} A \rfloor + 1$ vertices.

For $k \geq 3$, it appears to be significantly more difficult to obtain the exact maximum number of vertices. We concentrate mainly on the case $k = 3$, and first give a construction for planar graphs with maximum degree $A$ and diameter three, containing

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\[ \frac{3}{2} |A| - 3 \] vertices. We then prove an upper bound of \( 8A + 12 \) on the maximum number of vertices in a planar graph with maximum degree \( A \) and diameter three.

For general values of \( k \geq 3 \), we can obtain a trivial lower bound on the maximum number of vertices by simply constructing a \( A \)-regular tree of height \( \lceil k/2 \rceil \), yielding a graph with \( \Omega(A^{\lceil k/2 \rceil}) \) vertices. A trivial upper bound of \( O(A^k) \) is obtained similarly, by constructing a \( A \)-regular tree with height \( k \). However, using a result of Lipton and Tarjan [6], we show that for fixed diameter \( k \), the maximum number of vertices in a planar graph with maximum degree \( A \) and diameter \( k \) is \( O(A^{k/2}) \).

2. Preliminaries

For a graph \( G \), we denote by \( V(G) \) and \( E(G) \), respectively, the vertex set and edge set of \( G \). We only need to consider connected and finite graphs. If \( v \) is a vertex of \( G \), then \( d(v) \) denotes the degree of \( v \) in \( G \), and \( A = A(G) \) denotes the maximum degree of vertices in \( G \). If \( x \) and \( y \) are two vertices of \( G \), the distance from \( x \) and \( y \), denoted \( d(x, y) \), is the length (the number of edges) of a shortest path from \( x \) to \( y \) in \( G \). The diameter of \( G \) is the maximum value of \( d(x, y) \) for all pairs \( x, y \in V(G) \). The radius of \( G \) is the minimum integer \( r \) such that there exists an \( r \in V(G) \) with \( d(r, y) \leq r \) for all \( y \in V(G) \).

Suppose that \( G \) is a graph and that \( X \) and \( Y \) are disjoint nonempty subsets of \( V(G) \). The set \( X \) is said to be completely connected to \( Y \) if each vertex of \( X \) has at least one neighbour in \( Y \). A vertex \( x \in X \) is said to be of distance \( k \) from \( Y \) if \( k = \min \{ d(x, y) : y \in Y \} \). A vertex \( x \in X \) has a private neighbour in \( Y \) with respect to \( X \) if there exists a vertex \( y \in Y \) whose only neighbour in \( X \) is \( x \).

By a plane graph, we mean a planar graph together with an embedding of the graph in the plane. From the Jordan Curve Theorem, we know that a cycle \( C \) in a plane graph separates the plane into two regions, the inside and the outside. We call these the two sides of the cycle \( C \). Vertices on different sides of a cycle \( C \) are said to be separated by \( C \). Throughout this paper, we often make implicit use of these facts.

The following lemma establishes a useful property of cutsets in graphs with diameter three.

Lemma 1. Let \( G \) be a graph of diameter three, and let \( A, B, C \) be a partition of \( V(G) \) into nonempty subsets such that there are no edges joining vertices of \( A \) with vertices of \( B \), i.e., \( C \) is a cutset. Then each vertex of \( A \cup B \) is of distance at most two from \( C \). Moreover, either \( A \) or \( B \) is completely connected to \( C \).

Proof. A path from \( a \in A \) to \( b \in B \) must contain at least one vertex of \( C \), so that both \( a \) and \( b \) are of distance at most two from \( C \). Furthermore, if neither \( A \) nor \( B \) is completely connected to \( C \), then there is a vertex \( a \in A \) and a vertex \( b \in B \) with \( d(a, b) \geq 4 \). \( \Box \)
We begin with a construction for planar graphs with maximum degree $\Delta$ and diameter three that results in graphs with $\left\lfloor \frac{3}{2} \Delta \right\rfloor - 3$ vertices. The general structure of these graphs is indicated in Fig. 1. The circles represent independent sets, with $|X| = \Delta - 3$, $|Y| = \left\lfloor (\Delta - 2)/2 \right\rfloor$, and $|Z| = \left\lceil (\Delta - 2)/2 \right\rceil$. Every vertex in each of these sets has precisely two neighbours, as indicated in the figure. It is easy to verify that these planar graphs have $\left\lfloor \frac{3}{2} \Delta \right\rfloor - 3$ vertices, diameter three, and maximum degree $\Delta$.

3. An upper bound

**Theorem 2.** Any planar graph with maximum degree $\Delta$ and diameter three contains at most $8\Delta + 12$ vertices.

**Proof.** It suffices to prove this statement for plane graphs. Let $G$ be a plane graph on $n$ vertices with maximum degree $\Delta$ and diameter three.

**Assumption 1.** $n > 20$.

We may make this assumption since $\Delta \geq 1$ implies that $8\Delta + 12 \geq 20$, so if $n \leq 20$, the result is trivially true.

Construct a breadth-first spanning tree, $T$, of $G$ rooted at a vertex $r$ of minimum degree in $G$ (so $d(r) \leq 5$). Since $G$ has diameter three, $d(r, y) \leq 3$ for all $y \in V(G)$. Furthermore, since $T$ is a breadth-first spanning tree, the distance from $r$ to any vertex is the same in $G$ as in $T$. Note that if all $d(r, y) \leq 2$, then it is easy to see that $n \leq 5\Delta + 1 < 8\Delta + 12$. Therefore we may make the following assumption:
Assumption 2. max \{d(r, y) : y \in V(G)\} = 3.

Let G' be a triangulation of G. For each edge xy \in E(G') - E(T), let C_{xy} denote the unique cycle in \(T \cup \{xy\}\), and, whenever convenient, also the vertex set of this cycle. Since \(T\) has radius three, \(|C_{xy}| \leq 7\). Let \(A_{xy}\) and \(B_{xy}\) denote sets of vertices on the two sides of \(C_{xy}\). Then \(A_{xy}, B_{xy}, C_{xy}\) is a partition of \(V(G)\), so \(|A_{xy}| + |B_{xy}| + |C_{xy}| = n\). Define \(D_{xy}\) to be the set of vertices of distance two in \(G\) from \(C_{xy}\).

Remark 1. If \(C_{xy}\) is a cutset for \(G'\), then it is also a cutset for \(G\). Thus, from the proof of Lemma 1, we know that not both \(A_{xy} \cap D_{xy}\) and \(B_{xy} \cap D_{xy}\) are nonempty. Therefore if \(D_{xy}\) is nonempty, it is a subset of either \(A_{xy}\) or \(B_{xy}\).

Assumption 3. Let \(xy \in E(G') - E(T)\). If \(A_{xy}\) and \(B_{xy}\) are both nonempty, then, without loss of generality, \(A_{xy}\) is completely connected to \(C_{xy}\) and every vertex of \(B_{xy}\) is of distance at most two from \(C_{xy}\). On the other hand, if one of \(A_{xy}\) and \(B_{xy}\) is empty, then, without loss of generality, \(A_{xy} = \emptyset\) and every vertex of \(B_{xy}\) is of distance at most three from \(C_{xy}\).

Remark 2. Whenever we mention the number of neighbours of a particular vertex, we mean this to be with respect to only the edges of \(G\), not those of \(G'\). Since the edges of \(C_{xy} - \{xy\}\) are edges of \(T\), and hence of \(G\), vertices \(x\) and \(y\) of \(C_{xy}\) each have at most \(\Delta - 1\) neighbours in \(V(G) - C_{xy}\) (in \(G\)), and every other vertex of \(C_{xy}\) has at most \(\Delta - 2\) neighbours in \(V(G) - C_{xy}\). Also note that if \(r \in C_{xy}\), it has at most four or three neighbours in \(V(G) - C_{xy}\), according as \(r = x, y\) or not.

We now suppose that \(xy \in E(G') - E(T)\) is an edge for which \(A_{xy}\) and \(B_{xy}\) are both nonempty.

Lemma 3. If \(|D_{xy}| \leq \Delta - 2\), then \(n \leq 7\Delta - 2\).

Proof. Assume first that \(r \in C_{xy}\) and \(r \neq x, y\). Then \(r\) has at most three neighbours in \(A_{xy} \cup B_{xy}\). Vertices \(x\) and \(y\) each have at most \(\Delta - 1\) neighbours in \(A_{xy} \cup B_{xy}\), and each vertex in \(C_{xy} - \{r, x, y\}\) has at most \(\Delta - 2\) neighbours in \(A_{xy} \cup B_{xy}\). Since \(|C_{xy}| \leq 7\), there are at most

\[
3 + 2(\Delta - 1) + 4(\Delta - 2) = 6\Delta - 7
\]

vertices of distance one from \(C_{xy}\). Since there are at most \(\Delta - 2\) vertices of distance two from \(C_{xy}\),

\[
n - |C_{xy}| = |A_{xy} \cup B_{xy}| \leq (6\Delta - 7) + (\Delta - 2) = 7\Delta - 9,
\]

and so \(n \leq 7\Delta - 2\). If \(r \notin C_{xy}\) then \(|C_{xy}| \leq 5\) and if \(r = x\) or \(r = y\) then \(|C_{xy}| \leq 4\), so we still have \(n \leq 7\Delta - 2\). \(\Box\)
Lemma 4. If $|D_{xy}| > \Delta - 2$, then at most three vertices of $C_{xy}$ are of distance two from all vertices of $D_{xy}$.

Proof. Suppose that (at least) four vertices of $C_{xy}$, say $v_1, v_2, v_3, v_4$, are of distance two from all vertices of $D_{xy}$. Let $d \in D_{xy}$, and suppose $db_1v_1$ is a path of length two from $d$ to $v_1$. Then a path of length two from $d$ to $v_3$ contains either $b_1$ or a vertex $b_2 \neq b_1$.

Case 1: Suppose $db_1v_3$ is a path of length two from $d$ to $v_3$. Then $b_1v_1$ and $b_1v_3$ are both edges of $G$, and without loss of generality, $G'$ contains the subgraph in Fig. 2. Clearly, any vertex of $D_{xy}$ must be adjacent to $b_1$ in order to be of distance two from both $v_2$ and $v_4$, and $b_1$ must be adjacent to at least one of $v_2$ and $v_4$, so $|D_{xy}| \leq \Delta - 3$.

Case 2: Suppose $db_2v_3$ is a path of length two from $d$ to $v_3$, $b_2 \neq b_1$; without loss of generality, $G'$ contains the subgraph in Fig. 3. It is clear that any vertex of $D_{xy} - \{d\}$ must be adjacent to either $b_1$ or $b_2$ in order to be of distance two from both $v_2$ and $v_4$, and so at least one of $b_1$ and $b_2$ is adjacent to each of $v_2$ and $v_4$. Thus, if every vertex of $D_{xy}$ is adjacent to both $b_1$ and $b_2$, then $|D_{xy}| \leq \Delta - 2$, a contradiction.

We may now assume that there exist vertices $d_1$ and $d_2$ in $D_{xy} - \{d\}$ such that $d_1$ is adjacent to $b_1$ but not $b_2$, and $d_2$ is adjacent to $b_2$ but not $b_1$. Let $P$ denote the path in $C_{xy}$ from $v_1$ to $v_3$ containing $v_4$, and consider the cycle $C^* = v_1b_1db_2v_3 \cup P$. We consider two cases: either $d_1$ and $d_2$ are on the same side of $C^*$ or they are on different sides.
(a) Suppose \( d_1 \) and \( d_2 \) lie on different sides of \( C^* \); without loss of generality, \( d_1 \) and \( v_2 \) both lie inside \( C^* \), as shown in Fig. 4. Then the cycle \( C^* \) separates \( d_2 \) from \( v_2 \), \( d \) and \( d_2 \) are both of distance two from \( C_{xy} \), and \( d_2 \) is not adjacent to \( b_1 \). Thus a path of length two from \( d_2 \) to \( v_2 \) contains \( b_2 \), so \( b_2v_2 \in E(G) \), and \( G' \) contains the subgraph shown in Fig. 4.

Let \( P' \) be the path in \( C_{xy} \) from \( v_1 \) to \( v_2 \), not containing \( v_3 \) and \( v_4 \), and let \( C' = v_1b_1db_2v_2 \cup P' \). Then \( C' \) separates \( d_1 \) from \( v_3 \), \( d_1 \) and \( d \) are both of distance two from \( C_{xy} \), and \( d_1 \) is not adjacent to \( b_2 \). Thus, a path of length two from \( d_1 \) to \( v_3 \) contains \( b_1 \), so \( b_1 \) is adjacent to both \( v_1 \) and \( v_3 \), giving us Case 1.

(b) Suppose \( d_1 \) and \( d_2 \) lie on the same side of \( C^* \); without loss of generality, \( d_1 \) and \( d_2 \) both lie inside \( C^* \), as shown in Fig. 5. Consider a path of length two from \( d_2 \) to \( v_1 \). If such a path contains \( b_2 \), then \( b_2 \) is adjacent to both \( v_1 \) and \( v_3 \), giving us Case 1, and since \( d_2 \) is not adjacent to \( b_1 \), such a path cannot contain \( b_1 \). Furthermore \( d_2 \) is of distance two from \( C_{xy} \). Thus a path of length two from \( d_2 \) to \( v_1 \) contains a vertex \( b_3 \neq b_1, b_2 \), and \( G' \) contains the subgraph shown in Fig. 5.

Let \( C' = v_1b_1db_2d_2v_3v_1 \), and consider a path of length two from \( d_1 \) to \( v_3 \). Since \( C' \) separates \( d_1 \) from \( v_3 \), such a path must contain a vertex of \( C' \). If such a path contains
then \( b_1 \) is adjacent to both \( v_1 \) and \( v_3 \), again giving us Case 1, and since \( d_1 \) and \( b_2 \) are not adjacent, such a path cannot contain \( b_2 \). Furthermore, since \( d_1 \) and \( d \) are both of distance two from \( C_{xy} \), the only possibility is for \( b_3 \) to be adjacent to both \( d_1 \) and \( v_3 \). However, this again gives us Case 1, and completes the proof. □

**Corollary 5.** If \( |D_{xy}| > \Delta - 2 \), then at most three vertices of \( C_{xy} \) have private neighbours in \( A_{xy} \) with respect to \( C_{xy} \).

**Proof.** Suppose \( a_1, \ldots, a_k \), respectively, are private neighbours in \( A_{xy} \) of \( v_1, \ldots, v_k \in C_{xy} \) with respect to \( C_{xy} \). The cycle \( C_{xy} \) separates \( a_i \) from \( D_{xy} \), so in order for \( a_i \) to be of distance at most three from each vertex of \( D_{xy} \), \( v_i \) must be of distance two from each vertex of \( D_{xy} \). By Lemma 4, \( k \leq 3 \). □

**Lemma 6.** If \( |D_{xy}| > \Delta - 2 \), then \( |A_{xy}| \leq 4\Delta - 1 \).

**Proof.** We consider three cases, according to the number of vertices in \( C_{xy} \).

(i) Suppose that \( |C_{xy}| \leq 5 \). Then, since at most three vertices of \( C_{xy} \) have private neighbours in \( A_{xy} \), it is clear that we can find a subset \( I \) of \( C_{xy} \) with size at most four, such that \( A_{xy} \) is completely connected to \( I \). Since \( I \) has a maximum number of neighbours in \( A_{xy} \) when both \( x \) and \( y \) belong to \( I \),

\[ |A_{xy}| \leq 2(\Delta - 1) + 2(\Delta - 2) = 4\Delta - 6. \]

(ii) Suppose \( |C_{xy}| = 6 \). Then \( r \in C_{xy} \), and has at most three neighbours in \( A_{xy} \). Since at most three vertices of \( C_{xy} \) have private neighbours in \( A_{xy} \), we can find a subset \( I \) of \( C_{xy} \) with \( r \in I \) and \( |I| \leq 5 \), such that \( A_{xy} \) is completely connected to \( I \). The subset \( I \) has a maximum number of neighbours in \( A_{xy} \) when both \( x \) and \( y \) belong to \( I \), so

\[ |A_{xy}| \leq 3 + 2(\Delta - 1) + 2(\Delta - 2) - 4\Delta - 3. \]

(iii) Finally, suppose \( |C_{xy}| = 7 \).

**Claim 1.** If the number of vertices on \( C_{xy} \) with private neighbours in \( A_{xy} \) with respect to \( C_{xy} \) is at most two, then \( |A_{xy}| \leq 4\Delta - 5 \).

**Proof of Claim 1.** If no vertices on \( C_{xy} \) have private neighbours in \( A_{xy} \) with respect to \( C_{xy} \), then each vertex of \( A_{xy} \) has at least two neighbours on \( C_{xy} \), and so

\[ 3 + 2(\Delta - 1) + 4(\Delta - 2) \geq 2|A_{xy}|. \]

Therefore \( |A_{xy}| \leq 3\Delta - \frac{7}{2} \).

If only one vertex \( j \in C_{xy} \) has private neighbours in \( A_{xy} \) with respect to \( C_{xy} \), then each vertex of \( A_{xy} \) minus the neighbours of \( j \) in \( A_{xy} \) has at least two neighbours on \( C_{xy} \) \( \{j\} \), and \( j \) has at most \( \Delta - 1 \) neighbours in \( A_{xy} \). Thus,

\[ 3 + (\Delta - 1) + 4(\Delta - 2) \geq 2(|A_{xy}| - (\Delta - 1)), \]

implying that \( |A_{xy}| \leq 7\Delta/2 - 4 \).
Finally, if two vertices $i$ and $j \in C_{xy}$ have private neighbours in $A_{xy}$ with respect to $C_{xy}$, then each vertex of $A_{xy}$ minus the neighbours of $i$ and $j$ in $A_{xy}$ has at least two neighbours on $C_{xy} - \{i,j\}$, and each of $i$ and $j$ has at most $\Delta - 1$ neighbours in $A_{xy}$. Therefore,

$$3 + 4(\Delta - 2) \geq 2(|A_{xy}| - 2(\Delta - 1)),$$

which implies that $|A_{xy}| \leq 4\Delta - \frac{9}{2}$. \qed

By the previous claim, we may assume that exactly three vertices of $C_{xy}$, say $w_1$, $w_2$, and $w_3$, have private neighbours in $A_{xy}$ with respect to $C_{xy}$. Notice that since $|C_{xy}| = 7$, we again have $r \in C_{xy}$. Let $I \subseteq C_{xy}$, $r \in I$, $|I| = 5$ such that $I$ contains all the vertices of $C_{xy}$ that are of distance two from all vertices of $D_{xy}$; in particular, $I$ contains all vertices of $C_{xy}$ that have private neighbours in $A_{xy}$ with respect to $C_{xy}$. Because of the way $I$ was chosen, $w_1, w_2, w_3 \in I$.

Let $s$ and $t$ denote the two vertices of $C_{xy} - I$. If every vertex of $A_{xy}$ is adjacent to some vertex of $I$, then

$$|A_{xy}| \leq 2(\Delta - 1) + 2(\Delta - 2) + 3 = 4\Delta - 3.$$

Suppose that there exist vertices of $A_{xy}$ not adjacent to vertices of $I$. Since $s$ and $t$ have no private neighbours, all such vertices must be adjacent to both $s$ and $t$, and not adjacent to any vertices of $I$. Let $q_1, \ldots, q_\ell \in A_{xy}$ be adjacent to $s$ and $t$, but not to any vertices of $I$. Then every vertex of $D_{xy}$ must be of distance exactly two from at least one of $s$ and $t$. Let $C_1$ and $C_2$ be the two cycles formed by taking the union of each of the two paths from $s$ to $t$ in $C_{xy}$ with the path $sq_1t$ and $sq_\ell t$, as shown in Fig. 6.

Suppose that $w_1, w_2, w_3 \in I$ have private neighbours $z_1, z_2$, and $z_3$, respectively, in $A_{xy}$ with respect to $C_{xy}$. There are two cases to consider: either $w_1, w_2$ and $w_3$ lie on the same $C_i$, or two of $w_1, w_2$, and $w_3$ lie on one $C_i$, and the third lies on $C_j$, $j \neq i$.

![Fig. 6.](image-url)
Case 1. Without loss of generality, we may assume that \( w_1 \in V(C_1) \) and that \( w_2, w_3 \in V(C_2) \), and that \( G' \) contains the subgraph shown in Fig. 7.

The vertices \( z_1 \) and \( z_2 \) are separated by both \( C_1 \) and \( C_2 \), so a path of length at most three from \( z_1 \) to \( z_2 \) must contain vertices from both \( C_1 \) and \( C_2 \). Since \( z_1 \) and \( z_2 \) are private neighbours of \( w_1 \) and \( w_2 \), respectively, with respect to \( C_{XY} \), such a path contains either both \( w_1 \) and \( w_2 \), or it contains \( q_1, \ldots, q_l \).

Suppose a path of length three from \( z_1 \) to \( z_2 \) contains \( w_1w_2 \), and let \( C^* \) be the cycle formed by the union of the edge \( w_1w_2 \) with the path in \( C_{xy} \) from \( w_1 \) to \( w_2 \) containing \( w_3 \) (see Fig. 8). All vertices of \( D_{xy} \) lie outside \( C_{xy} \), and must also lie outside \( C^* \) (i.e. the side of \( C^* \) not containing \( s \)). Since every vertex of \( D_{xy} \) is of distance at least two from vertices \( w_1, w_2, w_3 \), and \( t \) of \( C^* \), it follows that \( s \) is of distance at least three from every vertex of \( D_{xy} \). Also every vertex of \( D_{xy} \) is of distance two from either \( s \) or \( t \), implying that every vertex of \( D_{xy} \) is of distance two from \( t \). This contradicts our choice of \( I \), since \( I \) already contains all vertices of \( C_{xy} \) of distance two from all vertices of \( D_{xy} \).
Therefore, a path of length at most three from $z_1$ to $z_2$ contains $q_1, \ldots, q_t$, implying that $l \leq 2$, and that there are at most two vertices of $A_{xy}$ not adjacent to any vertex in $I$. Since the number of vertices of $A_{xy}$ adjacent to vertices of $I$ is at most

$$3 + 2(\Delta - 1) + 2(\Delta - 2) = 4\Delta - 3,$$

we have $|A_{xy}| \leq 4\Delta - 1$.

**Case 2.** We may now assume, without loss of generality, that $w_1, w_2$ and $w_3$ all lie on $C_1$, as shown in Fig. 9. Since every vertex of $D_{xy}$ must be of distance exactly two from $w_1, w_2, w_3$ and at least one of $s$ and $t$, an analogous argument to that used in Lemma 4 can be used to show that $|D_{xy}| \leq \Delta - 2$, contradicting our assumption that $|D_{xy}| > \Delta - 2$.

Therefore, $|A_{xy}| \leq 4\Delta - 1$. □

**Corollary 7.** If $|D_{xy}| > \Delta - 2$ and $|B_{xy}| \leq n/2$, then $n \leq 8\Delta + 12$.

**Proof.** We know from Lemma 6 that if $|D_{xy}| > \Delta - 2$, then $|A_{xy}| \leq 4\Delta - 1$. Since $|B_{xy}| \leq n/2$ and $|C_{xy}| \leq 7$,

$$|A_{xy}| = n - |B_{xy}| - |C_{xy}| \geq \frac{n}{2} - 7.$$

Therefore,

$$\frac{n}{2} - 7 \leq |A_{xy}| \leq 4\Delta - 1,$$

implying that $n \leq 8\Delta + 12$. □

Thus far we have shown that if some $xy \in E(G') - E(T)$ with $A_{xy}$ and $B_{xy}$ both nonempty has $D_{xy} \leq \Delta - 2$, then by Lemma 3, $n \leq 7\Delta - 2 < 8\Delta + 12$. On the other hand, if for some $xy \in E(G') - E(T)$ with $A_{xy}$ and $B_{xy}$ both nonempty, the set $D_{xy}$ has
size greater than $\Delta - 2$ but $|B_{xy}| \leq n/2$ (where $D_{xy} \subseteq B_{xy}$), then by Corollary 7, $n \leq 8\Delta - 12$. In either case, our proof is complete. We may therefore make the following assumption:

**Assumption 4.** For each $xy \in E(G') - E(T)$ with $A_{xy}$ and $B_{xy}$ both nonempty, $|D_{xy}| > \Delta - 2$, and $|B_{xy}| > n/2$.

**Remark 3.** Since $G'$ is a simple graph, $|C_{xy}| \geq 3$. If both $A_{xy}$ and $B_{xy}$ are nonempty, then by Assumption 4, $|B_{xy}| > n/2$, so that $|A_{xy}| < n/2 - 3$, i.e., the completely connected side contains less than $n/2 - 3$ vertices.

For each edge $xy \in E(G') - E(T)$, let $I_{xy}$ and $O_{xy}$ denote, respectively, the set of vertices inside $C_{xy}$ and the set of vertices outside $C_{xy}$, i.e. $\{I_{xy}, O_{xy}\} = \{A_{xy}, B_{xy}\}$. Choose an edge in $E(G') - E(T)$ for which

$$\max\{|I_{xy}|, |O_{xy}|\}$$

is minimized, and, subject to this, having the least number of faces of $G'$ on the same side of $C_{xy}$ as $\max\{|I_{xy}|, |O_{xy}|\}$. Denote this edge by $uv$. Without loss of generality, $I_{uv} = A_{uv}$ and $O_{uv} = B_{uv}$, since we may always redraw $G'$ and invert the inside and outside of $C_{uv}$. Then $A_{uv}$ is the set of vertices inside $C_{uv}$, $B_{uv}$ is the set of vertices outside $C_{uv}$, and $D_{uv} \subseteq B_{uv}$.

**Lemma 8.** If $|B_{uv}| > n/2$, then $G'$ has the structure shown in Fig. 10, where $z$ is the vertex in the triangular face with $w$ outside $C_{uv}$, and $uz$ and $zv$ are both edges in $E(G') - E(T)$. Note that $z$ may or may not be a vertex of $C_{uv}$. Furthermore, $A_{uv}$, $I_{uz}$, $I_{zv}$, $O_{uz}$, and $O_{zv}$ are all nonempty.

![Fig. 10.](image-url)
**Proof.** Let \( z \) denote the vertex in the triangular face with \( uv \) outside \( C_{uv} \) (in \( G' \)).

(i) If \( uz \) and \( vz \) are both edges of \( C_{uv} \), then \( C_{uv} = uvzu \) is a face outside \( C_{uv} \), so \( B_{uv} = \emptyset \), a contradiction.

(ii) If exactly one of \( uz \), \( vz \) is an edge of \( C_{uv} \), then without loss of generality \( uz \in E(C_{uv}) \) and \( vz \notin E(C_{uv}) \), as shown in Fig. 11. Clearly, \( vz \notin E(T) \). Notice that \( O_{zu} = B_{uv} \), so \( |O_{zu}| = |B_{uv}| > n/2 \). Since there are fewer faces outside \( O_{zu} \) than outside \( B_{uv} \), this contradicts our choice of \( uu \).

(iii) If neither \( uz \) nor \( vz \) is an edge of \( C_{uv} \), then it is clear that \( uz \) and \( vz \) are not both edges of \( T \). If precisely one of \( uz \), \( vz \in E(T) \), then without loss of generality \( uz \in E(T) \) and \( vz \notin E(T) \), as shown in Fig. 12. In this case, it is clear that \( z \notin V(C_{uv}) \). Notice that \( O_{zu} \subset B_{uv} \) and \( I_{zu} = A_{uv} \). Thus

\[
|O_{zu}| < |B_{uv}| \quad \text{and} \quad |I_{zu}| = |A_{uv}|
\]

But, \( n = |A_{uv}| + |B_{uv}| + |C_{uv}| \), and \( |C_{uv}| \geq 3 \) while \( |B_{uv}| > n/2 \), implying that

\[
|A_{uv}| < \frac{n}{2} - 3.
\]
Therefore

\[ \max \{|A_{uv}|, |B_{uv}|\} = |B_{uv}| > \max \{|I_{zw}|, |O_{zw}|\}, \]

contradicting our choice of \( uv \).

Thus we may assume that neither \( uz \) nor \( zv \) is an edge of \( T \) (so \( z \) may or may not be a vertex of \( C_{uv} \)), and so \( G' \) has the structure shown in Fig. 10. In this case, \( I_{uz} \subseteq B_{uv} \), and there are fewer faces inside \( C_{uz} \) than outside \( C_{uv} \). Thus, if \( |I_{uz}| > |O_{uz}| \), the edge \( uz \) would have been chosen instead of \( uv \). Therefore,

\[ |I_{uz}| \leq |O_{uz}|, \]

and similarly,

\[ |I_{zw}| \leq |O_{zw}|. \]

If either \( |O_{uz}| < |B_{uw}| \) or \( |O_{zw}| < |B_{uv}| \), then \( uz \) or \( zv \), respectively, would have been chosen instead of \( uv \). Therefore,

\[ |O_{uz}| \geq |B_{uw}| > \frac{n}{2} \quad \text{and} \quad |O_{zw}| \geq |B_{uv}| > \frac{n}{2}, \]

so \( O_{uz} \) and \( O_{zw} \) are both nonempty.

Suppose that \( A_{uv} = \emptyset \). Then \( O_{uz} \cap O_{zw} = \emptyset \), so that

\[ |O_{uz} \cup O_{zw}| = |O_{uz}| + |O_{zw}| > \frac{n}{2} + \frac{n}{2} = n, \]

a contradiction. Therefore, \( A_{uv} \) is not empty.

Let \( Q \) denote the vertices of \( C_{uz} \cap C_{zw} - C_{uv} \). Then \( B_{uv} = I_{uz} \cup Q \cup I_{zw} \), so if \( I_{uz} = \emptyset \), then \( B_{uv} = Q \cup I_{zw} \). Since \( Q \cap I_{zw} = \emptyset \), this implies that

\[ |B_{uv}| = |Q| + |I_{zw}|. \]

Also, recall that \( |O_{zw}| > n/2 \) and that \( n = |I_{zw}| + |O_{zw}| + |C_{zw}| \), so

\[ |I_{zw}| < n \frac{2}{2} - |C_{zw}|. \]

It is clear that \( Q \subseteq C_{zw} \), so \( |Q| < |C_{zw}| \). Therefore,

\[ \frac{n}{2} < |B_{uv}| = |Q| + |I_{zw}| < |Q| + \frac{n}{2} - |C_{zw}|, \]

implying that \( |C_{zw}| < |Q| \), a contradiction.

Therefore, \( I_{uz} \) is not empty. The same argument can be used to show that \( I_{zw} \neq \emptyset \), thus completing the proof of the lemma. \( \square \)
We will now show that $C_{uv}$ is, in fact, a cutset. Suppose that $C_{uv}$ is not a cutset in $G'$. Then by Assumption 3, $A_{uv} = \emptyset$, so that $n = |B_{uv}| + |C_{uv}|$. Since $|C_{uv}| \leq 7$, this implies that

$$|B_{uv}| = n - |C_{uv}| \geq n - 7.$$ 

By Assumption 1, $n > 20$, so $n - 7 > n/2$, and it now follows from Lemma 8 that $A_{uv} \neq \emptyset$, giving us a contradiction.

Thus $C_{uv}$ is a cutset for $G'$, and both $A_{uv}$ and $B_{uv}$ are nonempty. By Assumption 4 and Remark 3, $|D_{uv}| > \Delta - 2$, $|B_{uv}| > n/2$, and $|A_{uv}| < n/2 - 3$, and thus from Lemma 8, we deduce that $G'$ has the structure indicated in Fig. 10, where $A_{uv}$, $B_{uv}$, $I_{uz}$, $O_{uz}$, $I_{zv}$, and $O_{zv}$ are all nonempty. The partitions $I_{uz}$, $O_{uz}$, $C_{uz}$ and $I_{zv}$, $O_{zv}$, $C_{zv}$ of $V(G)$ satisfy the conditions of Lemma 1, so either $I_{uz}$ or $O_{uz}$ is completely connected to $C_{uz}$, and either $I_{zv}$ or $O_{zv}$ is completely connected to $C_{zv}$.

**Lemma 9.** $I_{uz}$ is completely connected to $C_{uz}$ and $I_{zv}$ is completely connected to $C_{zv}$; i.e. $I_{uz} = A_{uz}$, $I_{zv} = A_{zv}$, $O_{uz} = B_{uz}$, and $O_{zv} = B_{zv}$, and $G'$ has the form depicted in Fig. 13.

**Proof.** From the proof of Lemma 8, we know that

$$|O_{uz}| > n/2 \quad \text{and} \quad |O_{zv}| > n/2,$$

and from Remark 3, we know that the completely connected sides of each of $C_{uz}$ and $C_{zv}$ contain less than $n/2 - 3$ vertices. Therefore, $I_{uz}$ is completely connected to $C_{uz}$ and $I_{zv}$ is completely connected to $C_{zv}$. \qed

Let $P$ denote the vertices of $C_{uv} \cup C_{uz} \cup C_{zv}$. Since $C_{uv}$, $C_{uz}$, and $C_{zv}$ each have length at most seven, it is clear that $|P| \leq 10$. It is also easy to see that $A_{uv}$, $A_{uz}$, $A_{zv}$, $I_{uz}$, $O_{uz}$, $I_{zv}$, $O_{zv}$, $C_{uz}$, and $C_{zv}$ all remain nonempty. We leave the remaining details of the proof to the reader.
$P$ is a partition of $V(G)$. Since $A_{uv}, A_{uw},$ and $A_{uz}$ are each completely connected to $C_{uv},$ $C_{uw},$ and $C_{uz},$ respectively, it follows that $V(G) - P$ is completely connected to $P$.

**Remark 4.** Suppose $z \in C_{uv}$. Then $P = C_{uv}$, and hence every vertex of $V(G) - P = A_{uv} \cup B_{uv}$ is adjacent to at least one vertex of $C_{uv}$. This implies that $D_{uv} = \emptyset$, a contradiction. Thus, $z \notin C_{uv}$.

**Lemma 10.** If $|P| \geq 8$, then $r \in P$.

**Proof.** Suppose that $r \notin P$. Then $C_{uv}, C_{uw},$ and $C_{uz}$ all have length at most five, in which case $|P| \leq 7$. □

**Lemma 11.**

$$n \leq \begin{cases} |P|(\Delta - 1) + 2 & \text{if } r \notin P, \\ |P|(\Delta - 1) - \Delta + 7 & \text{if } r \in P. \end{cases}$$

**Proof.** From the structure of $G'$ given in Lemma 9, and because $z \notin C_{uv}$ we see that exactly three vertices of $P$ each have at most $\Delta - 1$ neighbours in $V(G) - P$, one vertex of $P$ has at most $\Delta - 3$ neighbours in $V(G) - P$, and the remaining vertices of $P$ each have at most $\Delta - 2$ neighbours in $V(G) - P$. Since $V(G) - P$ is completely connected to $P$, this implies that, if $r \notin P$, then

$$n - |P| \leq 3(\Delta - 1) + (\Delta - 3) + (|P| - 4)(\Delta - 2) = |P|(\Delta - 2) + 2,$$

and so $n \leq |P|(\Delta - 1) + 2$.

If $r \in P$, then $r$ has at most two, three, or four neighbours, respectively, in $V(G) - P$, according as $r$ takes the place of a vertex of $P$ with at most $\Delta - 3$, $\Delta - 2$, or $\Delta - 1$ neighbours in $V(G) - P$. We consider only the case where $r$ takes the place of the vertex of $P$ with at most $\Delta - 3$ neighbours in $V(G) - P$; the other cases are analogous. In this case, $r$ has at most two neighbours in $V(G) - P$, so

$$n - |P| \leq 3(\Delta - 1) + 2 + (|P| - 4)(\Delta - 2) = |P|(\Delta - 2) - \Delta + 7,$$

and thus $n \leq |P|(\Delta - 1) - \Delta + 7$. □

**Corollary 12.** If $|P| \leq 9$, then $n \leq 8\Delta - 2$.

**Proof.** First suppose that $|P| \leq 7$. Then, from Lemma 11, we know that

$$n \leq \max \{ |P|(\Delta - 1) + 2, |P|(\Delta - 1) - \Delta + 7 \} \leq \max \{ 7(\Delta - 1) + 2, 7(\Delta - 1) - \Delta + 7 \} = \max \{ 7\Delta - 5, 6\Delta \} \leq 8\Delta - 2.$$
If $|P| \geq 8$, then from Lemma 10 we know that $r \in P$, and thus by Lemma 11
\[ n \leq |P|(\Delta - 1) - \Delta + 7 \leq 9(\Delta - 1) - \Delta + 7 = 8\Delta - 2. \]

The only case that remains is when $|P| = 10$. In this case $G'$ has the structure shown in Fig. 14.

Suppose that some vertex $q \in P - \{r\}$ has no private neighbours in $V(G) - P$ with respect to $P$. Then $V(G) - P$ is completely connected to $P - \{q\}$, $r$ has at most two neighbours in $V(G) - P$, $u$, $v$, and $z$ each have at most $\Delta - 1$ neighbours in $V(G) - P$, and each vertex of $P - \{r, u, v, z\}$ has at most $\Delta - 2$ neighbours in $V(G) - P$. The maximum number of vertices is attained when $q$ is a vertex in $P - \{r, u, v, z\}$. Thus,
\[ n - |P| \leq 2 + 3(\Delta - 1) + 5(\Delta - 2) = 8\Delta - 11, \]
and so $n \leq 8\Delta - 1$.

We may therefore assume that each vertex of $P - \{r\}$ has at least one private neighbour in $V(G) - P$ with respect to $P$. From now on (unless otherwise specified) "private neighbour" refers to "private neighbour with respect to $P". Let $d_1$ be a private neighbour of $w_2$; without loss of generality, $d_1 \in A_{uw}$. Suppose a private neighbour $x_3$ of $v$ lies in $A_{uv}$ (see Fig. 15).

Since $d_1$ and $x_3$ are private neighbours of $w_2$ and $v$, respectively, and because $d_1$ and $x_3$ are separated by both $C_{uv}$ and $C_{uw}$, it follows that the only path of length at most three from $d_1$ to $x_3$ is $d_1w_2ex_3$. This implies that $w_2v \in E(G)$, and thus $v$ is of distance two from $r$ in $G$. But since $v$ is of distance three from $r$ in $T$, this contradicts the fact that $T$ is a breadth-first spanning tree. Therefore, all private neighbours of $v$ lie in $A_{zv}$, in particular, $x_3 \in A_{zv}$. 
Fig. 15.

Fig. 16.

Now suppose that a private neighbour $d_2$ of $w_4$ lies in $A_{uz}$, and consider a path of length at most three from $d_2$ to $x_3$ (see Fig. 16). An analogous argument to that just given for $d_1$ and $x_3$ again gives us a contradiction, and hence $d_2 \in A_{uv}$. In fact, all private neighbours of $w_4$ lie in $A_{uv}$.

Let $x_1$ denote a private neighbour of $z$, $d_3$ a private neighbour of $w_6$, and $x_2$ a private neighbour of $u$. By repeating the previous argument again for the pairs $d_2$ and $x_1$, $x_1$ and $d_3$, $d_3$ and $x_2$, and finally for $x_2$ and $d_1$, it follows that all private neighbours of $z$ lie in $A_{uz}$, all private neighbours of $w_6$ lie in $A_{uz}$, all private neighbours of $u$ lie in $A_{uv}$, and all private neighbours of $w_2$ lie in $A_{uz}$. Thus, $G'$ contains the subgraph shown in Fig. 17.

Since $x_1$ and $d_2$ are private neighbours of $z$ and $w_4$, respectively, and $x_1$ and $d_2$ are separated by both $C_{uz}$ and $C_{uv}$, it follows that a path of length at most three from $x_1$ to
$d_2$ contains $w_4$ and some vertex $t_1$ of $A_{uz} - \{x_1, d_1\}$. This forces a path of length at most three from $d_1$ to $x_2$ to contain $t_1$, as shown in Fig. 18.

An analogous argument can be used to show that there exists a vertex $t_2 \in A_{uv} - \{x_2, d_2\}$ adjacent to $x_2$, $d_2$, $w_6$, and $v$, and there exists a vertex $t_3 \in A_{zu} - \{x_3, d_3\}$ adjacent to $x_3$, $d_3$, $w_2$, and $z$. Thus $G'$ contains the subgraph shown in Fig. 19.

Let $y_2$ denote a private neighbour of $w_3$. If $y_2$ lies in $A_{uz}$, then since $y_2$ and $x_3$ are private neighbours of $w_3$ and $v$, respectively, and are separated by both $C_{uz}$ and $C_{zu}$, a path of length at most three from $y_1$ to $x_3$ must contain $w_3$ and $v$. But $w_3$ and $v$ are separated by the cycle $ux_2t_2d_2w_4t_1u$. Therefore, $y_2 \in A_{uv}$ (see Fig. 20).

Since $A_{uz}$ and $A_{zu}$ are completely connected to $C_{uz}$ and $C_{zu}$, respectively, every vertex of $D_{uv}$ must be adjacent to at least one of $z$, $w_1$ or $w_2$. Consider a path of length
at most three from a vertex of $D_{uv}$ to $y_2$. Since $y_2$ is a private neighbour of $w_3$, such a path must contain $w_3$ and $t_1$, and hence every vertex of $D_{uv}$ is adjacent to $t_1$. Therefore, $|D_{uv}| \leq \Delta - 3$, since $u$, $w_3$ and $w_4$ are also neighbours of $t_1$. This contradicts our assumption that $|D_{uv}| > \Delta - 2$, and thus our initial assumption that every vertex of $P - \{r\}$ has a private neighbour in $V(G) - P$ is invalid. This completes the proof of the theorem. \( \square \)

4. Larger values of $k$

As mentioned in Section 1, we can easily construct a planar graph with maximum degree $\Delta$ and diameter $k$, containing $\Omega(\Delta^{[k/2]})$ vertices, for any given values of $\Delta$ and $k$. 
The following special case of a theorem of Lipton and Tarjan [6] allows us to show that for any fixed value of $k$, the maximum number of vertices in a planar graph with maximum degree $\Delta \geq 4$ and diameter $k$ is $O(\Delta^{k/2})$.

**Theorem 13** (Lipton and Tarjan [6]). Let $G$ be a planar graph on $n$ vertices containing a spanning tree of radius $r$. Then $V(G)$ can be partitioned into sets $A$, $B$, and $C$ such that no edges join vertices in $A$ with vertices in $B$, $|A| \leq \frac{2}{3} n$, $|B| \leq \frac{2}{3} n$, and $|C| \leq 2r + 1$.

**Corollary 14.** Let $G$ be a planar graph on $n$ vertices with maximum degree $\Delta \geq 4$ and diameter $k$. Then $n \leq (6k + 3)(2\Delta^{k/2} + 1)$.

**Proof.** Since $G$ has diameter $k$, $G$ certainly has a spanning tree of radius at most $k$. By Theorem 13, $V(G)$ can be partitioned into sets $A$, $B$, and $C$ such that $|A|, |B| \leq \frac{2}{3} n$, $|C| \leq 2k + 1$, and no edges join vertices in $A$ with vertices in $B$.

If some vertex $x \in A$ is of distance at least $\left\lceil \frac{k}{2} \right\rceil + 1$ from every vertex of $C$, and some vertex $y \in B$ is of distance at least $\left\lceil \frac{k}{2} \right\rceil + 1$ from every vertex of $C$, then the distance from $x$ to $y$ in $G$ is at least

$$2 \left\lceil \frac{k}{2} \right\rceil + 2 > k,$$

contradicting the fact that $G$ has diameter $k$.

Thus, without loss of generality, we may assume that each vertex of $A$ is of distance at most $\left\lfloor \frac{k}{2} \right\rfloor$ from every vertex of $C$, and thus

$$|A| \leq |C| \Delta + |C| \Delta (\Delta - 1) + \cdots + |C| \Delta (\Delta - 1)^{\frac{k}{2} - 1}$$

$$< (2k + 1) \frac{\Delta}{\Delta - 2} \Delta^{\frac{k}{2}}$$

$$\leq 2(2k + 1) \Delta^{\frac{k}{2}}$$

for $\Delta \geq 4$.

Since $|B| \leq 2n/3$ and $|C| \leq 2k + 1$, $|A| \geq n/3 - (2k + 1)$, so

$$\frac{n}{3} - (2k + 1) \leq |A| < 2(2k + 1) \Delta^{\frac{k}{2}}.$$  

Thus $n \leq (6k + 3)(2\Delta^{\frac{k}{2}} + 1)$. \qed

In the diameter three case, this theorem gives us an upper bound on the maximum number of vertices of $42\Delta + 21$. The arguments in the previous section improve this bound to $8\Delta + 12$, which significantly narrows the gap between the lower bound and the upper bound on the maximum number of vertices. (Recall that the construction described in Section 2 gives a lower bound of $\lceil \frac{9}{2} \Delta \rceil - 3$ on the maximum number of vertices.)

A simple construction showing a lower bound of $\Omega(\Delta^{\frac{k}{2}})$ on the maximum number of vertices in a planar graph of maximum degree $\Delta$ and diameter $k$ can be described as
follows: two complete \((A - 1)\)-branching trees of height \(\lceil k/2 \rceil\) are joined by identifying corresponding leaves. The lower bounds given by this basic construction can be improved by a small constant factor for most parameter values by various special constructions which employ this strategy of joining two \((A - 1)\)-trees as a basic building block in more complicated arrangements [4]. Narrowing the gap between the upper and lower bounds for \(k \geq 4\) remains an interesting and seemingly difficult open problem.

References