Nonlinear interactions and chaotic dynamics of suspended cables with three-to-one internal resonances

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Abstract

Nonlinear planar oscillations of suspended cables subjected to external excitations with three-to-one internal resonances are investigated. At first, the Galerkin method is used to discretize the governing nonlinear integral–partial-differential equation. Then, the method of multiple scales is applied to obtain the modulation equations in the case of primary resonance. The equilibrium solutions, the periodic solutions and chaotic solutions of the modulation equations are also investigated. The Newton–Raphson method and the pseudo-arclength path-following algorithm are used to obtain the frequency/force–response curves. The supercritical Hopf bifurcations are found in these curves. Choosing these bifurcations as the initial points and applying the shooting method and the pseudo-arclength path-following algorithm, the periodic solution branches are obtained. At the same time, the Floquet theory is used to determine the stability of the periodic solutions. Numerical simulations are used to illustrate the cascades of period-doubling bifurcations leading to chaos. At last, the nonlinear responses of the two-degree-of-freedom model are investigated.

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1. Introduction

During the past decades, long-span cable structures are applied widely throughout the world, such as in cable-supported bridges, guyed masts and ocean mooring systems. Therefore, it is very important to understand the dynamics of cable structures in many engineering fields.

Because of their importance, a lot of literatures have been published on the nonlinear dynamics of the cables. Much of these literatures are summarized in recent review articles (Rega, 2004a,b; Ibrahim, 2004). In general, the nonlinear dynamics of elastic suspended cables have been explored for a variety of phenomena through single-degree-of-freedom (SDOF) model. However, complete understanding the nonlinear dynamics of cables requires additional knowledge regarding the modal interactions due to internal resonances.

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In distributed-parameter systems, the nonlinearities of the structure may activate modal interactions because of the presence of internal resonances among different modes (Nayfeh, 2000). In the past decades, there has been considerable interest in the study of modal interactions of suspended cables subjected to harmonic excitations at primary resonances (Benedettini et al., 1986; Rao and Iyengar, 1991; Perkins, 1992; Lee and Perkins, 1993, 1995; Benedettini et al., 1995; Pakdemirli et al., 1995; Rega et al., 1999; Nayfeh et al., 2002; Zhang and Tang, 2002; Gattulli et al., 2004), at random excitations (Chang et al., 1996; Chang and Ibrahim, 1997; Ibrahim and Chang, 1999), and at complex excitations including wind excitations (Takahashi and Wang, 1990; Luongo and Piccardo, 1998; Martinelli and Perotti, 2001), fluid excitations (Kim and Perkins, 2002), by using the multi-degree-of-freedom (MDOF) models. Furthermore, Pilipchuk and Ibrahim (2002) introduced a special type of coordinate transformation to examine different regimes of nonlinear modal interactions of suspended cables.

In recent years, the wide applications of cable-stayed bridge systems have attracted increasing attention on the modal interactions of inclined sagged cables (Zhao et al., 2002; Nielsen and Kirkegaard, 2002; Srinil et al., 2003), and cable-stayed beams (Gattulli and Lepidi, 2003; Gattulli et al., 2005). These researches enrich the nonlinear dynamics of cable structures.

Most of the above studies focus on the dynamic interaction phenomena of suspended cables with one-to-one and/or two-to-one internal resonances. And little interest has been devoted to three-to-one internal resonances in suspended cables. The results of Lacarbonara and Rega (2003) showed that three-to-one internal resonances in suspended cables might be activated between the symmetric in-plane modes for a certain range of elasto-geometric parameters. Applying the direct approach, Zhao and Wang (2006) investigated the nonlinear response of suspended cables with small initial sag-to-span ratios in the case of three-to-one internal resonances, and the dynamic solutions of the modulation equations were also examined. However, their study is limited to the moderately taut cable. Moreover, only the equilibrium and dynamic solutions of the modulation equations are examined, and the numerical solutions of the equations of motion are not investigated.

In this paper, modal interactions and chaotic dynamics of a suspended cable subjected to primary resonance excitation in the case of three-to-one internal resonances are investigated. Because the MDOF Galerkin cable model is important for understanding the nonlinear dynamical behavior, the equation of motion is discretized by using the Galerkin method with symmetric in-plane modes. Then, the method of multiple scales is applied to obtain the second-order uniform asymptotic solutions for the cases of primary resonances of the symmetric mode.

Frequency/force–response curves of the suspended cable are obtained by using the Newton–Raphson method and the pseudo-arclength path-following algorithm, whose stability is analysed. The effects of the elasto-geometric parameters on the response curves are also discussed. Furthermore, the relationship between the solutions of the modulation equations and those of Galerkin model is explored by using the fourth-order Runge–Kutta algorithm.

2. Problem formulation

Considering a suspended cable whose two supports are fixed, the cable is subjected to a distributed harmonic excitation, as shown in Fig. 1. Neglecting the bending, torsional and shear rigidities, and assuming that the suspended cable stretches in a quasi-static manner (Perkins, 1992) due to the fact that the transverse wave speed is much lower than the longitudinal wave speed, the nondimensional equation of in-plane motion can be expressed by the following partial-differential equation (Benedettini et al., 1995; Nayfeh et al., 2002):

\[ \ddot{w} + 2c \dot{w} - w'' - z(w'' + y'') \int_0^1 \left\{ y'w' + \frac{1}{2}w'^2 \right\} dx = F(x) \cos(\Omega t). \]

The boundary conditions are given by

\[ w(x, t) = 0, \quad \text{at} \ x = 0 \quad \text{and} \quad x = 1. \]

In Eq. (1), \( y(x) = 4fx(1 - x) \) is the initial parabolic shape of the suspended cable; \( f = b/l \) is the sag-to-span ratio; \( b \) is the cable sag; \( l \) is the span of the cable; \( z = EA/H = 8bEA/(mg^2) \) is the nondimensional stiffness parameter (Lacarbonara and Rega, 2003); \( m \) is the mass per unit length; \( E \) is the Young modulus; \( A \) is the
area of the cross section; \( g \) is the gravitational acceleration; \( c \) is the nondimensional damping coefficients; \( w \) is the in-plane displacement (nondimensionalized with respect to the span); the overdot indicates the differentiation with respect to the nondimensional time \( t \); the prime indicates the differentiation with respect to the nondimensional coordinate \( x \); \( F(x) \) describes the spatial distribution of the harmonic load; and \( \Omega \) is the nondimensional frequency of the harmonic load. In the following sections, we use the discretization approach to study the three-to-one internal resonances between the third/fourth and first symmetric modes of the suspended cable.

3. Perturbation analysis

Using the discretization approach and the direct approach, Lacarbonara et al. (2003) have obtained the general modulation equations and the uniform expansions of the displacement field related to three-to-one internal resonances in distributed-parameter systems with quadratic and cubic geometric nonlinearities. Similarly, following Lacarbonara et al. (2003), we use the discretization approach to obtain the modulation equations governing the nonlinear dynamic of suspended cables with three-to-one internal resonances. First, we express the in-plane displacement \( w(x,t) \) as

\[
w(x,t) = \sum_{k=1}^{\infty} u_k(t) \phi_k(x), \tag{3}
\]

where \( u_k(t) \) is the generalized coordinates. Because this study is restricted to the symmetric modal interactions, \( \phi_k(x) \) is the linear mode shape of the \( k \)th symmetric mode corresponding to the natural frequency \( \omega_k \) (see Appendix A). Substitution of Eq. (3) into Eq. (1) and application of the Galerkin method yield to

\[
\ddot{u}_k + \omega_k^2 u_k = -2 \mu_k u_k + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} A_{kij} u_i u_j + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{h=1}^{\infty} \Gamma_{kijh} u_i u_j u_h + f_k \cos(\Omega t), \tag{4}
\]

where \( k = 1, 2, 3, \ldots, \infty \), and \( A_{kij}, \Gamma_{kijh}, \mu_k \) and \( f_k \) are defined in Appendix A. In order to balance the nonlinearities, damping and resonances, we rescale the \( \mu_k \) and \( f_k \) as \( \epsilon^2 \mu_k \) and \( \epsilon^3 f_k \). Following the standard details of the multiple scales method (Nayfeh, 1981), we introduce the new independent time variables

\[
T_i = \epsilon^i t \quad (i = 0, 1, 2, \ldots).
\]

Then, we have the differential operators

\[
\frac{d}{dt} = D_0 + \epsilon D_1 + \epsilon^2 D_2 + \cdots, \quad \frac{d^2}{dt^2} = D_0^2 + 2 \epsilon D_0 D_1 + \epsilon^2 (D_1^2 + 2D_0 D_2) + \cdots, \tag{6}
\]

where \( D_n = \partial / \partial T_n \). Because the resonant terms appear at the third order (Lacarbonara et al., 2003), we can seek the uniform expansions of \( u_k \) in the following form:

\[
u_k(t) = au_{k1}(T_0, T_2) + \epsilon^2 u_{k2}(T_0, T_2) + \epsilon^3 u_{k3}(T_0, T_2) + \cdots. \tag{7}
\]
It is observed that $D_t u_{ki} \equiv 0$. Substituting Eq. (7) into Eq. (4) and equating the coefficients of like power of $\varepsilon$ in the left- and right-hand side of the equation, the following differential equations can be obtained:

Order $\varepsilon^1$:
\[
D_t^2 u_{k1} + \omega_k^2 u_{k1} = 0;
\]
(8)

Order $\varepsilon^2$:
\[
D_t^2 u_{k2} + \omega_k^2 u_{k2} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} A_{kij} u_{i1} u_{j1};
\]
(9)

Order $\varepsilon^3$:
\[
D_t^2 u_{k3} + \omega_k^2 u_{k3} = -2\mu_k D_0 u_{k1} - 2D_2 D_0 u_{k1} + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} A_{kij} (u_{i1} u_{j2} + u_{i2} u_{j1}) + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \Gamma_{kij} u_{i1} u_{j1} u_{k3}
+ f_k \cos(\Omega t).
\]
(10)

The solution in the complex form of Eq. (8) can be written as
\[
u_{k1} = A_k(T_2) e^{i\omega_k T_0} + cc,
\]
(11)
where $cc$ indicates the complex conjugate of the preceding terms. Next, we substitute Eq. (11) into Eq. (9), and obtain
\[
D_t^2 u_{k2} + \omega_k^2 u_{k2} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} A_{kij} (\overline{A_i A_j} e^{i(\omega_i + \omega_j) T_0} + A_i \overline{A_j} e^{i(\omega_i - \omega_j) T_0}) + cc,
\]
(12)
where the overbar indicates the complex conjugate. The solution of Eq. (12) can be expressed as
\[
u_{k2} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ A_{kij} A_{imn} + A_{kij} A_{imm} \right\} \left( \frac{A_i A_m A_n}{\omega_j^2 - (\omega_m + \omega_n)} e^{i(\omega_i + \omega_m + \omega_n) T_0} + \frac{A_i A_m \overline{A_n}}{\omega_j^2 - (\omega_m - \omega_n)} e^{i(\omega_i + \omega_m - \omega_n) T_0} 
+ \frac{A_i \overline{A_m} A_n}{\omega_j^2 - (\omega_m - \omega_n)} e^{i(\omega_i + \omega_m - \omega_n) T_0} + \frac{A_i \overline{A_m} \overline{A_n}}{\omega_j^2 - (\omega_m + \omega_n)} e^{i(\omega_i + \omega_m + \omega_n) T_0} \right) \}
+ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Gamma_{kijm}
\times \left( A_i A_m e^{i(\omega_i + \omega_m) T_0} + A_i \overline{A_m} e^{i(\omega_i - \omega_m) T_0} + A_i \overline{A_m} e^{i(\omega_i - \omega_m) T_0} + A_i \overline{A_m} e^{i(\omega_i + \omega_m) T_0} \right)
+ f_k \cos(\Omega t).
\]
(13)

Substituting Eqs. (11) and (13) into Eq. (10), we can obtain
\[
D_t^2 u_{k3} + \omega_k^2 u_{k3} = -2i\omega_k (D_2 A_k + \mu_k A_k) e^{i\omega_k T_0} + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \times \left\{ \left( A_{kij} A_{imn} + A_{kij} A_{imm} \right) \left( \frac{A_i A_m A_n}{\omega_j^2 - (\omega_m + \omega_n)} e^{i(\omega_i + \omega_m + \omega_n) T_0} + \frac{A_i A_m \overline{A_n}}{\omega_j^2 - (\omega_m - \omega_n)} e^{i(\omega_i + \omega_m - \omega_n) T_0} 
+ \frac{A_i \overline{A_m} A_n}{\omega_j^2 - (\omega_m - \omega_n)} e^{i(\omega_i + \omega_m - \omega_n) T_0} + \frac{A_i \overline{A_m} \overline{A_n}}{\omega_j^2 - (\omega_m + \omega_n)} e^{i(\omega_i + \omega_m + \omega_n) T_0} \right) \}
+ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Gamma_{kijm}
\times \left( A_i A_m e^{i(\omega_i + \omega_m) T_0} + A_i \overline{A_m} e^{i(\omega_i - \omega_m) T_0} + A_i \overline{A_m} e^{i(\omega_i - \omega_m) T_0} + A_i \overline{A_m} e^{i(\omega_i + \omega_m) T_0} \right)
+ f_k \cos(\Omega t).
\]
(14)

In order to describe the nearness of $\omega_r$ $(r = 3, 4)$ to $3\omega_1$ and $\Omega$ to either $\omega_1$ or $\omega_r$, we introduce the detuning parameters $\sigma_1$ and $\sigma_2$ defined by
\[
\omega_r = 3\omega_1 + \varepsilon^3 \sigma_1, \quad \Omega = \omega_1 + \varepsilon^3 \sigma_2 \quad (i = 1, r).
\]
(15)
Substituting Eq. (15) into Eq. (14) and eliminating the secular terms, we can obtain the following equations:

\[-2i\omega_1(A'_1 + \mu_1A_1) + A_1 \sum_{j=1}^{\infty} S_{j1}A_jA_j + S_{1}A_{2}^{2}A_{3}e^{i\alpha T_2} + \frac{f_1}{2} \delta_{1} e^{i\beta T_2} = 0, \]

\[-2i\omega_1(A'_r + \mu_rA_r) + A_r \sum_{j=1}^{\infty} S_{jr}A_jA_j + S_{r}A_{r}^{2}A_{r}e^{-i\alpha T_2} + \frac{f_r}{2} \delta_{r} e^{-i\beta T_2} = 0, \]

when \( k \neq 1, r \)

\[-2i\omega_k(A'_k + \mu_kA_k) + A_k \sum_{j=1}^{\infty} S_{kj}A_jA_j = 0, \]

where

\[
S_{kj} = \left\{ \begin{array}{ll}
\sum_{n=1}^{\infty} \left[ (A_{kn} + A_{jk})(\frac{2A_{nkk}}{\omega_n^2} + \frac{A_{nkk}}{\omega_n^2 - 4\omega_n^2}) \right] + 3\Gamma_{kkk} & k = j; \\
\sum_{n=1}^{\infty} \left[ (A_{kn} + A_{jk})\frac{2A_{njj}}{\omega_n^2} + (A_{kn} + A_{jk})(A_{nkj} + A_{jjk}) \right] \\
\times \left( \frac{1}{\omega_n^2 - (\omega_k + \omega_j)^2} + \frac{1}{\omega_n^2 - (\omega_k - \omega_j)^2} \right) + 2(\Gamma_{kjk} + \Gamma_{jkk} + \Gamma_{kjj}) & k \neq j,
\end{array} \right.
\]

\[S_{r} = \sum_{n=1}^{\infty} \left[ (A_{r1n} + A_{rn1})\frac{A_{n11}}{\omega_n^2 - 4\omega_1^2} \right] + \Gamma_{r111}. \]

From Eqs. (16) to (18), we can know that, except the amplitudes of the first and rth symmetric mode, other amplitudes will die out due to the presence of damping. Thus, the solvability conditions reduce to a two-dimensional system. Looking at the expressions of the coefficients \( S_{kj} \), we can note that \( S_{kj} \) is the summation of the contributions of all the symmetric modes. Because only the symmetric modes contribute to the coefficients (Arafat and Nayfeh, 2003), the values of the coefficients \( S_{kj} \) are equivalent to those obtained with the direct approach (Lacarbonara et al., 2003). Moreover, the symmetric condition (i.e., \( S_1 = 3S_2 \)) and the resonance-dependent condition (i.e., \( S_1 = 3S_2 \)) can also be obtained (Lacarbonara et al., 2003; Zhao and Wang, 2006). According to the numerical results and conclusions of Arafat and Nayfeh (2003) and Gattulli et al. (2004), we can infer that, if more than four symmetric modes are retained in the discretization procedure, the coefficients \( S_{kj} \) should converge onto the ones obtained by the direct approach (Zhao and Wang, 2006).

However, numerical simulation of the four DOF cable model may lead to some unexpected phenomena (i.e., the hardening behavior, the period-doubling instability) in the primary resonance region when \( \lambda \approx 1.51n \) (Wang and Zhao, submitted for publication). Therefore, for the convenience of the numerical simulation in Section 6, we only consider the case of \( n = 1, r \) in this study.

For the case of primary resonance of the symmetric in-plane mode, we can introduce the polar transformation \( A_j = \frac{1}{2} a_j e^{i\beta_j}, j = 1, r \), where \( a_j \) and \( \beta_j \) are the undetermined real functions of \( T_2 \), and can be determined by imposing the solvability conditions. Substituting this transformation into Eqs. (16) and (17) and separating the real and imaginary parts, we can obtain the following linear equations:

\[ a'_1 = -\mu_1 a_1 + \frac{S_1}{8\omega_1} a_1^2 a_1 \sin \gamma_1 + \frac{f_1}{2\omega_1} \delta_1 \sin \gamma_2, \]

\[ a'_2 = -\mu_2 a_2 - \frac{S_r}{8\omega_2} a_2^2 a_2 \sin \gamma_2 + \frac{f_r}{2\omega_2} \delta_r \sin \gamma_2, \]

\[ a_1 \beta'_1 = -\frac{S_{11}}{8\omega_1} a_1^2 a_1 - \frac{S_{1r}}{8\omega_1} a_2^2 a_2 - \frac{S_{1}}{8\omega_1} a_2^2 a_1 \cos \gamma_1 - \frac{f_1}{2\omega_1} \delta_1 \cos \gamma_2, \]

\[ a_r \beta'_r = -\frac{S_{r1}}{8\omega_r} a_2^2 a_r - \frac{S_{rr}}{8\omega_r} a_1^3 - \frac{S_{r}}{8\omega_r} a_1^3 \cos \gamma_1 - \frac{f_r}{2\omega_r} \delta_r \cos \gamma_2, \]

where \( \gamma_1 = \beta_1 - 3\beta r + \sigma_1 T_2, \gamma_2 = \sigma_2 T_2 - \beta_1 \delta_1 - \beta_r \delta_r \). Alternatively, if we can express the \( A_j \) in the Cartesian form, the following Cartesian form of the modulation equations can be obtained:
where \( v_1 = \sigma_2, v_2 = 3\sigma_2 - \sigma_1 \) for the case of primary resonance of the first symmetric mode, and \( v_1 = (\sigma_1 + \sigma_2)/3, v_2 = \sigma_2 \) for the case of primary resonance of the rth symmetric mode. So we can obtain the following expansion of the cable’s displacement for the case of primary resonance:

\[
w(x,t) = a_1 \cos(\omega_1 t + \beta_1)\phi_1(x) + a_r \cos(\omega_r t + \beta_r)\phi_r(x) + \frac{1}{2} \{a^2_1 \cos(2(\omega_1 t + \beta_1))\psi_1(x) + \psi_2(x) + a^2_r \cos(2(\omega_r t + \beta_r))\psi_3(x) + \psi_4(x)\} + \cdots,
\]

where

\[
\psi_1(x) = \sum_{i=1,r} \frac{A_{i1}}{\omega^2_i - 4\omega^2_1} \phi_i(x); \quad \psi_2(x) = \sum_{i=1,r} \frac{A_{i2}}{\omega^2_i - 4\omega^2_2} \phi_i(x); \quad \psi_3(x) = \sum_{i=1,r} \frac{A_{i3}}{\omega^2_i - 4\omega^2_3} \phi_i(x); \quad \psi_4(x) = \sum_{i=1,r} \frac{A_{i4} + A_{i1}}{\omega^2_i - (\omega_r - \omega_1)^2} \phi_i(x).
\]

### 4. Equilibrium solutions and stability

This section contains details of the equilibrium solutions of the modulation equations for the chosen external and three-to-one internal resonance combination, where the excitation frequency is nearly equal to the natural frequency of the first and third/fourth symmetric mode. And the analysis is mainly focused on a moderately taut suspended cable (C1) and a slacker cable (C2), whose physical properties are reported in Table 1. The coefficients of the modulation equations and the corresponding internal detuning parameters \( \sigma_i \) are also shown in Table 1. It is observed from Table 1 that the coefficients of the nonlinear interaction term \( S_l \) in the case of three-to-one internal resonances between the third/fourth and first symmetric modes are nonzero. Hence, three-to-one internal resonances may be activated under some certain conditions. Furthermore, to obtain complex dynamics in this study, the damping coefficients and the excitation amplitudes are carefully chosen, where the damping coefficients considered are \( \mu_1 = 0.05, \mu_r = 0.05 \).

The equilibrium solution of the modulation equations corresponds to the periodic motion of the suspended cable. To determine the equilibrium solution, we can set \( a'_1 = a'_r = 0 \) and \( \gamma'_1 = \gamma'_2 = 0 \) in the polar form of the modulation equations or set \( p'_1 = q'_1 = p'_r = q'_r = 0 \) in the Cartesian form of the modulation equations. In either case, the resulting systems of four nonlinear equations can be solved by using the Newton–Raphson method. After the equilibrium solution \( x_0(x_0 = [a_1, \gamma_1, a_r, \gamma_2]^T \) or \( x_0 = [p_1, q_1, p_r, q_r]^T, T \) is the transpose) is determined, the stability of the equilibrium solution can then be assessed by applying the classical method of linearization (Nayfeh and Balachandran, 1995). If the equilibrium solution is nonzero, we can substitute a new solution \( x = x_0 + \Delta x \) (where \( \Delta x \) is a small disturbance of nontrivial solution) into the modulation equations. Then, expanding in a Taylor series about \( x_0 \), and retaining only the linear terms in the disturbance lead to

<table>
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<td>Natural frequencies and nonlinear interaction coefficients of the modulation equations</td>
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<td>Cable</td>
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<td>C1</td>
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<td>C2</td>
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where \([J_c]\), the matrix of the first partial derivatives, is called the Jacobian matrix, whose the eigenvalues will determine the stability of the equilibrium solution. If the real part of all the eigenvalues of the Jacobian matrix is negative, the equilibrium solutions is stable. If at least one eigenvalue of Jacobian matrix has positive real part, the equilibrium solution is unstable. However, the perturbed equations will not contain the terms \(\Delta y_1\) or \(\Delta y_2\) for trivial solution. Therefore, the stability of the trivial equilibrium solution cannot be assessed, and we should apply the Cartesian form of the modulation equations to determine the stability of the trivial equilibrium solution. Then, the pseudo-arclength path-following algorithm (Nayfeh and Balachandran, 1995) can be used to trace the solution branch. In the following frequency/force–response curves, the stable and unstable solutions are indicated respectively by solid and dashed line.

4.1. Primary resonance of the first symmetric mode

Fig. 2 illustrates the computed amplitudes for the first and third symmetric mode of the cable C1 as functions of the detuning parameter \(\sigma_2\) in the neighborhood of the primary resonance of the first symmetric mode with \(f_1 = 0.50\), where SN and HB represent the saddle-node point and Hopf bifurcation point respectively. The results shown in Fig. 2 exhibit a softening behavior for the frequency–response curves of the first symmetric mode, where the quadratic nonlinearity due to the initial shape dominates the response. Whereas, the frequency–response curves of the third symmetric mode exhibit a hardening behavior, where the cubic nonlinearity due to the stretching dominates the response. Referring to Fig. 2, when \(\sigma_2 < -3.12\) and \(\sigma_2 \in (0.91, 1.42)\), the modulation equations have multiple solutions, two stable solutions and one unstable solution. To which stable solution the suspended cable's response settles depends on the initial conditions. Whereas, the unstable solution does not exist in the cable's response. However, between SN2 and HB2, \(\sigma_2 \in (-0.01, 0.91)\), the system only has one stable solution. It is also observed that there does not exist any stable solution in the range \(\sigma_2 \in (-3.12, -0.01)\). Therefore, the modulation equations may exhibit very rich nonlinear dynamic phenomena such as period-double bifurcations, chaos, which will be discussed in the following section.

As the detuning parameter \(\sigma_2\) increases from small value, the higher stable solution of the first symmetric mode deceases, but the higher stable solution of the third symmetric mode increases. When \(\sigma_2\) is increased beyond \(-1.97\), one pair of complex conjugate eigenvalues of Jacobian matrix crosses the imaginary axis transversely; and according to the Hopf bifurcation theorem (Nayfeh and Balachandran, 1995), this gives rise to a periodic solution with period \(2\pi/|\beta|\), where \(\beta\) is the purely imaginary eigenvalue. And the stable solution

\[
\Delta \dot{x} = [J_c] \Delta x,
\]
branch loses its stability at HB1. As $\sigma_2$ increases further, the unstable solution branch regains its stability via a reverse Hopf bifurcation, indicated by HB2 in Fig. 2. At last, this solution branch experiences a saddle-node bifurcation at SN3.

Next, the effects of the elasto-geometric parameter $\lambda$ on the frequency–response curves are investigated. In this case, we set $\lambda = 2.55\pi$, and the detuning parameter $\sigma_1 \approx 0.0$. The frequency–force curves of the cable C2 for the case of the primary resonance of the first symmetric mode with $f_1 = 0.50$ are shown in Fig. 3. Compared with Fig. 2, it is noted that the frequency–force curves are significantly different from the curves of the cable C1. Another two solution branches are observed in this case with one stable and other unstable, and they terminate at another saddle-node bifurcation point (SN1), which results in one relatively wide multi-value range. Moreover, it is also noted that, excepting for these branches, the amplitudes of the directly and indirectly excited modes decrease as the value of $\lambda$ increases. Another significant difference is the disappearance of Hopf bifurcation.

In Fig. 4, the force–response curves obtained for various excitation amplitudes are shown by setting the parameter $\sigma_2 = -2.0$. The force–response curves exhibit a relatively small multi-valued range due to saddle-node bifurcations at SN1 and SN2. Between SN1 and SN2, two stable solutions are separated by one unstable solution. As $f_1$ increases from 0.016, where two stable equilibrium solutions coexist, the larger stable equilibrium solution loses its stability via a Hopf bifurcation (indicated by HB1 in Fig. 4) with one pair of complex conjugate eigenvalues crossing the imaginary axis transversely from the left- to right-half plane.

Because the rigidities of the suspended cable are neglected, the value of total tension of the cable must be larger than zero. The positive tension can be used to verify the actual effectiveness of the asymptotic amplitudes. Moreover, the additional horizontal tensions are constant in every point of the cable. Therefore, only the nondimensional total horizontal tension $H_T$ is considered, defined as

$$H_T = 1 + \frac{EA}{H} \int_0^1 \left(y'w' + \frac{1}{2}w^2\right) dx = 1 + a \int_0^1 \left(y'w' + \frac{1}{2}w^2\right) dx,$$

where the quasi-static assumption is applied. Fig. 5 shows the time history of horizontal tensions of the cable C1 (Fig. 5a) and the cable C2 (Fig. 5b) for the case of primary resonance of the first symmetric mode. To illustrate the effects of the three-to-one internal resonances, the single-mode analysis results are also given. It is observed that the internal resonances result in a little positive tension drift, and the drift can be neglected in the cable C2. It is also very important to note that the horizontal tensions of the cable are positive values in all case. Therefore, the asymptotic amplitudes are reasonable.

![Fig. 3. Frequency–response curves of the cable C2 with $\Omega \approx \omega_1$ and $f_1 = 0.50$.](image-url)
4.2. Primary resonance of the third/fourth symmetric mode

For the case of the primary resonance of the higher symmetric in-plane mode, there are two possibilities. First \(a_1 = 0, a_r \neq 0\). Second, \(a_1 \neq 0, a_r \neq 0\). For the first case, \(a_r\) and \(\gamma_2\) are determined by

\[
\mu_r a_r = \frac{f_r}{2\omega_r} \sin \gamma_2,
\]

\[
\sigma_2 a_r = -\frac{S_{rr}}{8\omega_r} a_r^3 - \frac{f_r}{2\omega_r} \cos \gamma_2.
\]

Hence, the frequency–response equation is given by

\[
\sigma_2 = -\frac{S_{rr}}{8\omega_r} a_r^3 \pm \left( \frac{f_r^2}{4\omega_r^3 a_r^2} - \mu_r^2 \right)^{\frac{1}{2}},
\]

Also we can obtain \(a_r\) through the relation \(a_r = \sqrt{p_r^2 + q_r^2}\) by using the following equations:

\[
\mu_r p_r = -\sigma_2 q_r - \frac{1}{8\omega_r} \{ S_{rr}, q_r, (p_r^2 + q_r^2) \},
\]

\[
\mu_r q_r = \sigma_2 p_r - \frac{1}{8\omega_r} \{ S_{rr}, q_r, (p_r^2 + q_r^2) \} + \frac{f_r}{2\omega_r}.
\]
For the second case, we can determine the equilibrium solutions by using the same method in Section 4. Fig. 6 illustrates the computed amplitudes for the first and third symmetric modes as functions of the detuning parameter $\sigma_2$ in the neighborhood of the primary resonance of the third symmetric mode with $f_3 = 0.50$ and $\sigma_1 = -1.5$. The single-mode response curves are, as expected, of the hardening type, and a saddle-node bifurcation of the single-mode solution occurs at SN2 ($\sigma_2 = 2.512$). Referring to Fig. 6, for a relatively high excitation frequency ($\sigma_2 > 0.516$), only the third symmetric in-plane mode is excited directly. Therefore, the nonlinear response of the suspended cable can be determined by the single-mode solution, although three-to-one internal resonances between the third and first symmetric modes may be activated due to the fact that $S_i \neq 0$ ($i = 1, 3$).

For the two-mode equilibrium solutions, there exist two solution branches with one branch being always unstable. Following the stable solution branch, as the detuning parameter $\sigma_2$ increases from a very small value, the amplitude of the first symmetric mode decreases while the amplitude of the third symmetric mode grows all the way until a Hopf bifurcation occurs at HB1 ($\sigma_2 = -3.48$), where the stable solution loses its stability.

![Fig. 6. Frequency–response curves of the cable C1 with $\Omega \approx \omega_3$ and $f_3 = 0.50$.](image1)

![Fig. 7. Frequency–response curves of the cable C2 with $\Omega \approx \omega_4$ and $f_4 = 0.50$.](image2)
Then, the unstable solution regains its stability via a reverse Hopf bifurcation at HB2 ($r_2 = 0.462$). At last, this stable solution undergoes a saddle-node bifurcation, which occurs at SN1 ($r_2 = 0.513$). Increasing $r_2$ beyond SN1, the response jumps to a single-mode equilibrium solution.

Fig. 7 shows the frequency–response curves of the cable C2 for the case of primary resonance of fourth symmetric mode with $f_4 = 0.50$. Clearly, the two-mode equilibrium solution branches move closer to each other in this case. And the unstable region due to two Hopf bifurcations is still observed. Compared with Fig. 6, the amplitudes of the first symmetric mode decreases as the elasto-geometric parameter $k$ increases, which may be due to the fact that the alternation of mode shape function leads to the weakening of the nonlinear interaction’s strength.

Fig. 8 shows the time history of horizontal tensions of the cable C1 (Fig. 8a) and the cable C2 (Fig. 8b) for the case of primary resonance of the third/fourth symmetric mode. It is noted that the maximum tensions of the cable C1 are greater than those of the cable C2. Moreover, the first symmetric mode due to the internal resonances play a dominant role in the horizontal tensions in this case. Similar to the case of primary resonance of the first symmetric mode, no negative horizontal tensions is observed.

5. Dynamic solutions and stability

Because the Cartesian form of the modulation equations has the standard form as

$$\dot{x} = F(x)$$

it is used to examine the periodic solution and chaotic solution, and can be solved by the shooting method (Nayfeh and Balachandran, 1995). After the periodic solution $x_0$ is constructed, the pseudo-arclength path-following algorithm can be used to trace the periodic solution branch, and the Floquet theory (Nayfeh and Balachandran, 1995) can be used to determine its stability. Because the frequency/force–response curves of the cable C1 exhibit more unstable regions, the following analysis is only focused on the cable C1. In the following figures, the open circles denote the unstable periodic solutions, and the filled circles denote the stable periodic solutions.

5.1. Primary resonance of the first symmetric mode

On continuing the periodic solutions emerging from HB1 and HB2 in Fig. 2, Fig. 9 shows the amplitude of the periodic solutions of the modulation equations as functions of the detuning parameter $\sigma_2$ with $f_1 = 0.50$ and $\sigma_1 = -1.5$, where PD and CF denote the period-doubling bifurcation point and cyclic-fold bifurcation point respectively. In Fig. 9, the amplitudes indicate the maximum and minimum values of $p_1$, which occur on the limit cycles. The figure exhibits rich qualitative nonlinear dynamic behavior, including Hopf bifurcations, period-doubling bifurcations, and the cyclic-fold bifurcation. It is observed from Fig. 9 that the Hopf bifurcation (HB1) is supercritical due to the fact that the periodic solution starting from HB1 is stable, and the other Hopf bifurcation is subcritical. Starting from $\sigma_2 = -1.97$, the limit cycle grows from zero amplitude to
some finite size as $\sigma_2$ increases. The period-1 (P-1) solution is stable over the detuning interval $\sigma_2 \in (-1.97, -1.685)$, and loses its stability via a period-double bifurcation indicated by PD1. As $\sigma_2$ further increases from PD1, the amplitude of the unstable periodic solution increases. When $\sigma_2$ is increased beyond $-0.852$, the periodic solution regains its stability via another period-doubling bifurcation (PD3). On continuing the period-2 (P-2) solution branches emerging from PD1, we can obtain a cascade of period-doubling bifurcations, which eventually leads to chaos, and then reverses to a P-1 solution. Around $\sigma_2 = -0.784$, the P-1 solution seems to jump to a chaotic solution encircling an unstable two-mode equilibrium solution.

When $\sigma_2$ decreases from 0.91 (HB2), a small decrease in $\sigma_2$ leads to a limit cycle due to a Hopf bifurcation. As $\sigma_2$ increases, the limit cycle shrinks in size, and loses stability via a cyclic-fold bifurcation at 0.9105 (CF), resulting in a jump to the stable equilibrium solution branch. Decreasing $\sigma_2$ from 0.909, the limit cycle undergoes another cascade of period-doubling bifurcations leading to chaos. Next, we use the numerical simulation to examine the periodic solutions of the modulation equations.

Fig. 10 shows the two-dimensional projections of the phase portraits onto the $p_1 - q_3$ plane as the detuning parameter $\sigma_2$ slowly varies. When $\sigma_2$ increases past the Hopf bifurcation point (HB1), a small limit cycle is born and grows in size, deforms, then undergoes a sequence of period-doubling bifurcations leading to chaos, as shown in Fig. 10a–c. As $\sigma_2$ further increases, the chaotic attractor deforms and encounters an explosive bifurcation. This type of interior crisis, due to the collisions of the chaotic attractor with the chaotic saddle (Tyrkiel, 2005) on its basin boundary, gives rise to a sudden change of another chaotic attractor into one larger chaotic attractor, as shown in Fig. 10d. The simultaneous contacts between the fractal basin boundary and the attractors lead to a destruction of the basin boundary and an abrupt widening of the attractor. Then, a sequence of reverse period-doubling bifurcations occurs, leading to a limit cycle, as shown in Fig. 10e–f.

When $\sigma_2$ is decreased beyond CF, another period-doubling route to chaos and crisis are also observed. In brief, it seems to assign the following sequence to these global bifurcations:

Hopf bifurcation → period – doubling bifurcations → chaos → crisis → chaos
→ reverse period – doubling bifurcations → jumping → chaos
→ reverse period – doubling bifurcations → cyclic – fold bifurcation
→ Hopf bifurcation.

Next, the periodic solutions past the Hopf bifurcation at HB1 in Fig. 4 are discussed in detail. Fig. 11 shows the amplitude of the periodic solutions as functions of the excitation amplitude $f_1$ with $\sigma_2 = -2.0$. Referring to Fig. 11, the newly born limit cycle (P-1 solution) is stable until PD1 where it loses its stability via a period-
doubling bifurcation. The P-1 solution regains its stability by a reverse period-doubling bifurcation at PD9. Along the P-2 solution branch emerging from PD1, many period-doubling bifurcations, indicated by PD3-8, are found. On continuing the periodic solutions emerging from these period-doubling bifurcation points, cascades of period-doubling bifurcations can be detected.

To illustrate the nonlinear dynamic behavior of the modulation equations in the excitation amplitude range $f_1 \in (0.528, 1.227)$, Fig. 12 show the two-dimensional projections of the phase portraits onto the $p_1-q_3$ plane as the parameter $f_1$ slowly varies. Fig. 12a–d shows one period-doubling route to chaos. A representative chaotic
The attractor is shown in Fig. 12d. Then, as $f_1$ increases, a cascade of reverse period-doubling bifurcations occurs leading to P-2 solution (see Fig. 12e–f). The same process occurs in the excitation amplitude range $f_1 \in (1.262, 1.412)$. In addition, it is worth noting the presence of the interior crisis in this range. Another cascade of period-doubling bifurcations starts at PD7 ($f_1 = 1.498$). This sequence of period-doubling values converges quickly to a value around 1.510. As $f_1$ increases to 1.545 approximately, the chaotic attractor suffers an explosive bifurcation, which is similar to that shown in Fig. 10d. As $f_1$ increases further more, the chaotic attractor undergoes a cascade of reverse period-doubling bifurcation, leading to a stable P-1 solution.

![Fig. 12. Two-dimensional projections of the phase portraits onto the $p_1-q_3$ phase space. (a) P-1 $f_1 = 0.675$, (b) P-2 $f_1 = 0.687$, (c) P-4 $f_1 = 0.690$, (d) chaos $f_1 = 0.693$, (e) P-4 $f_1 = 1.105$, (f) P-2 $f_1 = 1.141$.](image1)

Fig. 13. The periodic solutions when $\Omega \approx \omega_3$ and $f_3 = 0.50$. 

![Fig. 13. The periodic solutions when $\Omega \approx \omega_3$ and $f_3 = 0.50$.](image2)
5.2. Primary resonance of the third symmetric mode

Choosing the HB1 and HB2 points as the initial points (Fig. 6), we obtain the periodic solution branches with $f_3 = 0.50$ and $\sigma_1 = -1.50$, as shown in Fig. 13. As can be seen in Fig. 13, the periodic solution branches emerging from two Hopf bifurcations are overlapped each other. And Fig. 14 shows the time-period ($T$) of the periodic solutions as functions of $\sigma_2$. Referring to Fig. 14, it is observed that the periodic solutions double their period at period-doubling bifurcation points as the detuning parameter $\sigma_2$ increases or decreases.

Fig. 15 shows two-dimensional projections of the phase portraits onto the $p_1-q_3$ plane as the detuning parameter $\sigma_2$ slowly varies. As $\sigma_2$ increases from $-3.480$, the period-1 limit cycle is born, as shown in

![Fig. 14. The time-period of the periodic solutions when $\Omega \approx \omega_3$ and $f_3 = 0.50$.](image)

![Fig. 15. Two-dimensional projections of the phase portraits onto the $p_1-q_3$ phase space. (a) P-1 $\sigma_2 = -2.450$, (b) P-2 $\sigma_2 = -0.075$, (c) chaos $\sigma_2 = -1.940$, (d) crisis $\sigma_2 = -1.935$, (e) chaos $\sigma_2 = 0.445$, (f) P-2 $\sigma_2 = 0.450$.](image)
Fig. 15a. Then the stable P-1 solution loses its stability via a period-doubling bifurcation at PD1 (σ_2 = -2.103), and the period-2 is shown in Fig. 15b. As σ_2 increases further, the P-2 solution undergoes a cascade of period-doubling bifurcations at σ_2 = -1.986 (P-4), σ_2 = -1.948 (P-8),... At last, this cascade of period-doubling bifurcations leads to chaos. A representative chaotic attractor is shown in Fig. 15c. As σ_2 increases further, a boundary crisis occurs. The chaotic attractor continues a possible long time in the vicinity of the chaotic saddle until it collides with an unstable orbit within its basin of attraction, and this collision results in a sudden destruction of the chaotic attractor and its basin of attraction and its disappearance from the phase portrait. Then the attractor settles down to the stable single-mode equilibrium solution, as shown in Fig. 15d. Similar phenomena was observed by Chin and Nayfeh (1997) in the hinged-clamped beam with three-to-one internal resonances. However, when σ_2 increases from 0.441, another type of chaotic attractor arises (Fig. 15e). Then the chaotic attractor undergoes a sequence of reverse period-doubling bifurcations, resulting in one stable limit cycle. And Fig. 15f shows one representative period-2 limit cycle.

6. Numerical simulations of the equations of motion

It is well known that solutions of the modulation equations provide an approximation to the various types of solutions of the suspended cable’s Galerkin-discretized models. The equilibrium solution of the modulation equations relates to the periodic motion of the suspended cable. The periodic solution of the modulation equations corresponds to the amplitude-modulated motion of the suspended cable. And the chaotic solution of the modulation equations is expected to relate to the amplitude-modulated chaotic motion of the suspended cable (Bajaj and Tousi, 1990).

The finite-amplitude dynamics of the suspended cable may exhibit very rich deterministic nonlinear phenomena. And a very interesting review on this issue is given by Rega (2004b). In this study, to demonstrate the relationship between the solutions of the modulation equations and those of two DOF model, two-dimensional Galerkin-discretized models are numerically integrated by employing the fourth-order Runge–Kutta scheme in this section, and the initial condition for the numerical integrations are set to ones obtained by the modulation equations. In this section, our discussion is limited to the parameter range where the modulation equations exhibit periodic solutions.

6.1. Primary resonance of the first symmetric mode

Fig. 16 shows the steady state motion of the suspended cable for σ_2 = -1.75. Fig. 16a shows the time history of the motion. And a Poincaré section of the motion for the surface of section at t = 0 (mod 2π) is shown in Fig. 16b. The closed curve of the Poincaré section implies that the motion of the cable is quasi-periodic with the modulation frequency incommensurate with the fast frequency. It is observed that the motion is on a two-
torus (Fig. 16b). It is also noted that the modulation period is 45–60 times the excitation period. By varying the detuning parameter $\sigma_2$, the phase-locked motions on the two-torus can be obtained. However, integration of the original equations of motion does not capture some phenomena (like torus-doubling, chaotic motion) predicted by the modulation equations. Some explanation for this may be appreciable effects of the higher order small terms ($O(\varepsilon^4)$) (Bajaj and Tousi, 1990), and the value of $\varepsilon$ (where $\varepsilon$ is set to one).

The motions of the suspended cable with the excitation amplitude varies are shown in Fig. 17. When the excitation amplitude increases from 0.720, the periodic solution branch for the modulation equations exhibits only stable limit cycle. Therefore, the motion on the torus is quasi-periodic. As the excitation amplitude increases, a chaotic attractor arises due to the destruction of the torus motion via torus breakdown. As the excitation amplitude increases further, the cable’s response exhibits quasi-periodic motion again. This phenomenon is further illustrated by Fig. 17a, which shows the Poincaré section of the quasi-periodic motion. When the excitation amplitude is increased beyond 3.560, a chaotic attractor arises again, as shown in Fig. 17b. Referring to Fig. 17b, the Poincaré section in the $u_1-u_3$ plane has a distinct butterfly shape.

6.2. Primary resonance of the third symmetric mode

At last, we investigate the nonperiodic motion of the suspended cable for the case of the primary resonance of the third symmetric in-plane mode. The periodic motion of the suspended cable takes place before the Hopf

![Fig. 17](image1.png)

**Fig. 17.** The Poincaré section of the nonperiodic motion when $\sigma_2 = -2.0$. (a) $f_1 = 2.150$; (b) $f_1 = 3.850$.

![Fig. 18](image2.png)

**Fig. 18.** The Poincaré section of the nonperiodic motion when $f_3 = 0.5$. (a) $\sigma_2 = -2.75$; (b) $\sigma_2 = -2.05$. 
bifurcation that is predicted to occur at $\sigma_2 = -3.48$. Numerical integration verifies the existence of this Hopf bifurcation. The response depicted by Fig. 18a corresponds to $\sigma_2 = -2.75$, which implies that the motion of the suspended cable is quasi-periodic and corresponds to a closed curve in the Poincaré section. As the detuning parameter $\sigma_2$ increases from $-2.56$, the quasi-periodic motion gives rise to the chaotic motion. A representative Poincaré section of the chaotic motion is shown in Fig. 18b. Compared with Fig. 18a, we can guess that the torus seems to undergo a very short and incomplete cascade of torus-doubling and the destruction of the torus, resulting in a chaotic attractor. As $\sigma_2$ increases further, the chaotic motion goes back to the quasi-periodic motion.

7. Conclusions

We investigate the nonlinear interactions and chaotic dynamics of the suspended cables subjected to primary resonance excitations. The Galerkin method is applied to discretize the integral–partial-differential equation of motion. Then the method of multiple scales is used to determine the second-order uniform expansions of the solution and to derive four first-order nonlinear ordinary-differential equations governing the modulation of the amplitudes and phases of the two interacting modes.

The Newton–Raphson method and the pseudo-arclength path-following method are used to calculate the equilibrium solutions of the modulation equations, and their stability is determined by the classical method of linearization. The equilibrium solutions of the modulation equations undergo the saddle-node, supercritical and subcritical Hopf bifurcations.

The dynamic solutions of the modulation equations are obtained by the shooting method and the pseudo-arclength path-following method. The Floquet theory is applied to determine the stability of the dynamic solutions. The complex dynamic solutions, including the period-doubling bifurcations, cyclic-fold bifurcation, crisis, are found.

The fourth-order Runge–Kutta scheme is employed to illustrate the relationship between the solutions of the modulation equations and those of Galerkin-discretized models. It is shown that some phenomena predicted by the modulation equations cannot reflect all the details of the dynamics of the discretized models.

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Appendix A

The symmetric in-plane eigenmodes are given by (Irvine, 1981)

$$
\phi_i(x) = c_i \left[ 1 - \tan \left( \frac{1}{2} \omega_i \right) \sin \omega_i x - \cos \omega_i x \right],
$$

where $c_i$ are chosen so that the modes satisfy the orthonormality condition. The eigenfrequencies are determined by

$$
\frac{1}{2} \omega_i - \tan \left( \frac{\omega_i}{2} \right) = \frac{1}{2 \lambda^2} \omega_i^3 = 0,
$$

where $\lambda^2 = EA/mg(8b/l)^3$.

The coefficients of Eq. (4) are given by

$$
A_{kij} = \int_0^1 \left( x \phi_i'' \int_0^1 \phi_j' \phi_i' \, dx + \frac{1}{2} x y'' \int_0^1 \phi_i' \phi_j' \, dx \right) \phi_k \, dx, \quad \Gamma_{kijh} = \int_0^1 \frac{1}{2} x \phi_i'' \phi_k \, dx \int_0^1 \phi_j' \phi_h' \, dx,
$$

$$
f_k = \int_0^1 F(x) \phi_k(x) \, dx, \quad \mu_k = \int_0^1 c(x) \phi_k(x) \, dx.
$$

(A.3)
References


