Maximum nonhamiltonian tough graphs

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Received 15 December 1988
Revised 17 October 1989

Abstract


Tough nonhamiltonian n-vertex graphs with $n \geqslant 3$ are known to exist only for $n \geqslant 7$. The maximum size among them is shown to be $6 + (n - 3)(n - 4)/2$. All corresponding maximum graphs are $K_t \ast (3K_1 \rightarrow K_{n-4})$ for $n \leqslant 7$ (unique if $n \neq 9$) and, additionally, $K_t \ast K_2 \ast (3K_1 \rightarrow K_3)$ if $n = 9$, where $\ast$ stands for the non-associative join and $3K_1 \rightarrow K_t$ with $t \geqslant 3$ denotes the complete graph $K_t$ together with three disjoint hanging edges (and vertices). This settles a conjecture by the second author.

1. Introduction

In general we use standard terminology and notation. Only simple graphs are considered. In what follows, $G$ stands for a graph, $G = (V, E)$. Then $|E|$ is called the size of $G$; moreover, $n$, $\delta$, $\Delta$ stand for the order, $|V|$, and the minimum and maximum degrees among vertices of $G$, respectively. If $G$ is noncomplete, $G \neq K_n$, then $\Delta'$ stands for the maximum degree among nontotal vertices $x$ of $G$ (i.e., with $\deg x < n - 1$). Let $k(G)$ denote the number of components of $G$. We call $G$ tough (or 1-tough in Chvátal's terminology) if

$$k(G - S) \leqslant |S|$$

for each $S \subseteq V(G)$ such that $k(G - S) \neq 1$.

Toughness is clearly a necessary condition for a graph $G$ to be hamiltonian. On the other hand, among homogeneously traceable graphs (all of which are tough) there are many nonhamiltonian ones. (Recall that, following the second author, $G$ is called homogeneously traceable if each vertex of $G$ is an end-vertex of a hamiltonian path.) However, maximum nonhamiltonian $n$-vertex graphs different from both $K_n$ and $K_2$ (i.e., with $n > 2$ and of maximum size), described without proof by Ore (1963; see [1–2] for a proof), are all nontough. Our aim is to describe all maximum nonhamiltonian tough graphs $G$.  

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To this end, let $*$ denote the non-associative join on disjoint graphs, i.e., $G * F * H = G * F \cup F * H$ provided that $G$, $F$, $H$ are mutually disjoint graphs. Moreover, $3K_1 * \rightarrow K_t$ where $t \geq 3$, stands for the connected graph obtained from the disjoint union $3K_1 \cup K_t$ by adding three independent $3K_1 - K_t$ edges (cf. [9] where $\leftrightarrow$ is called the injective join).

Consider the following tough nonhamiltonian graph $M_n$, denoted $K_1 * K_{n-4}^+ \rightarrow$ in [6], of order $n \geq 7$.

$$M_n = K_1 * (3K_1 * \rightarrow K_{n-4}) \quad n \geq 7.$$ Notice that $M_n$, found by Chvátal, is the smallest nontrivially nonhamiltonian graph among tough graphs. Let

$$G_0 = K_1 * K_2 * (3K_1 * \rightarrow K_3),$$

which is a maximally nonhamiltonian tough graph of order $n = 9$ exhibited in [4]. We shall prove that $M_n$ and $G_0$ for $n = 9$ are the only maximum nontrivially nonhamiltonian tough graphs of order $n$, $n \geq 3$.

**Proposition 1.1.** Graphs $M_n$ and $G_0$ for $n = 9$ are nonhamiltonian tough graphs of order $n$ and size

$$f(n) := 6 + \binom{n-3}{2} \quad n \geq 7.$$ Recall that a path-system is a graph whose each component is a path. The size of the path-system is called its length. Recall that, given nonnegative integers $n$, $p$ and $q$, an $n$-vertex graph $G$ is called non-strongly-$(p, q)$-hamiltonian if there is $V_1 \subset V$ such that $|V_1| \leq p$ and there is a path-system $S$ of length $\leq q$ in the complete graph with the vertex set $V - V_1$ such that the graph $S \cup G - V_1$ has no Hamiltonian cycle containing $S$. Notice that ‘nonstrongly-$(p, q)$-hamiltonian’ means ‘nonhamiltonian’ if $p = q = 0$, ‘non-Hamilton-connected’ if $p = 0$ and $q = 1$, and ‘non-$p$-hamiltonian’ if $q = 0$ and $p > 0$. Hence if $p = q = 0$ then the following result coincides with the above-mentioned result of Ore.

**Theorem 1.2** (Corollary 2 in [10]). If $n$, $p$ and $q$ are integers such that $n \geq 3$, $p \geq 0$, $q \geq 0$ and $s := p + q \leq n - 3$ and $G$ is a non-strongly-$(p, q)$-hamiltonian graph of order $n$ and the largest possible size then

$$G = K_1 * K_{1+s} * K_{n-s-2}$$

or additionally,

$$G = 3K_1 * K_{2+s} \quad \text{if } n - s = 5.$$

If $s \geq 1$ then graphs exhibited in Theorem 1.2 are tough.
2. Preliminaries

The following result is well known.

**Proposition 2.1.** If \([v_1, v_2, \ldots, v_n]\) is a hamiltonian path of a nonhamiltonian graph of order \(n \geq 3\) then \(\deg v_1 + \deg v_n \leq n - 1\).

It is easily seen that if \(G\) is a tough (or homogeneously traceable) graph then \(\delta \geq 2\) provided that \(n \geq 3\). We shall use the following results.

**Theorem 2.2** (Skupień [7]). For every vertex \(x\) of a homogeneously traceable nonhamiltonian graph of order \(n \geq 3\), there exists a vertex \(y\) connected to \(x\) by a hamiltonian path and such that \(\deg x + \deg y \leq n - 2\).

**Corollary 2.3** (Skupień [7]). If \(G\) is a homogeneously traceable nonhamiltonian graph of order \(n \geq 3\) then \(\Delta + \delta \leq n - 2\) (whence \(\Delta \leq n - 4\)).

Notice, however, that a tough nonhamiltonian graph \(G\) can have \(\Delta = n - 1\). Therefore the following result is more general than the theorem above. Its proof, however, follows the lines of Skupień’s proof.

**Theorem 2.4.** Let \(G\) be a tough nonhamiltonian graph with \(n \geq 3\) vertices such that each vertex of \(G\) has at most one neighbour of degree 2 and let \(x\) be an end-vertex of a hamiltonian path in \(G\). Then there exists a vertex \(y\) and a hamiltonian \(x-y\) path of \(G\) such that \(\deg x + \deg y \leq n - 2\).

**Corollary 2.5.** If \(n \geq 3\) and \(G\) is a tough maximally nonhamiltonian \(n\)-vertex graph in which each vertex has at most one neighbour of degree 2 then \(\Delta' \leq n - 2 - \delta\) (\(\leq n - 4\)).

**Theorem 2.6** (Jung [5]). Let \(G\) be a tough graph on \(n \geq 11\) vertices. If \(G\) satisfies

\[ \forall x, y \in V: xy \notin E \Rightarrow \deg x + \deg y \geq n - 4, \]

then \(G\) is hamiltonian.

**Corollary 2.7.** If \(G\) is a tough graph, \(n \geq 11\) and \(\delta \geq (n - 4)/2\) then \(G\) is hamiltonian.

The next result involves the following notions. Given a graph \(G\), let \(p(G)\) denote the length of the longest simple path in \(G\) and let \(s(G)\) be the scattering number of \(G\), i.e.,

\[ s(G) = \max\{k(G - S) - |S|: S \subset V(G), k(G - S) \neq 1\}. \]

**Proposition 2.8.** Assume that \(G\) is a tough graph and \(G = H \ast K_r \ast K_a\) where \(K_i\) stands for a complete graph of order \(i\) and \(H\) is a graph of order \(r + b\) with \(r \geq 2\).
\[ a \geq 1 \text{ and } b \geq 1 \text{ and such that, for each component } F \text{ of } H, \]
\[ p(F) = |V(F)| - 1 \text{ or } p(F) + s(F) \geq |V(F)|. \]

Then \( G \) is hamiltonian.

**Proof.** Since \( G \) is tough, \( k(H) < r - 1 \), whence \( H \) has nontrivial components. Let \( k(H) = r - t \) where \( t \geq 1 \). Assume that \( H \) has \( r - t - c \) isolated vertices and \( c \) nontrivial components, which are denoted \( F_1, \ldots, F_c \). Then
\[
\sum_{i=1}^c |V(F_i)| = (r + b) - (r - t - c) = b + t + c,
\]
\[
\sum_{i=1}^c |E(F_i)| \geq (r + b) - (r - t) = b + t \quad \text{(because the cyclomatic number of } H \text{ is nonnegative). Observe that if } H \text{ has a path-system of length greater than } b \text{ then } G \text{ is hamiltonian. Thus, it suffices to show that the number } w := \sum_{i=1}^c p(F_i) \geq b + 1.
\]

Suppose on the contrary that \( w < b + 1 \). Let \( P_i \) be a longest path in \( F_i \), let \( B_i = V(F_i) - V(P_i) \quad t = 1, \ldots, c \), and let \( B = \bigcup_{i=1}^c B_i \). Then
\[
|B| = (b + t + c) - (w + c) = b + t - w > b + t - (b + 1) = t \geq 0.
\]
Therefore the set \( B \) is non-empty. Let \( I = \{ i : B_i \neq \emptyset \} \) and let \( J = \{ 1, \ldots, c \} - I \). Hence if \( i \in I \) then \( p(F_i) \neq |V(F_i)| - 1 \) and therefore, by the assumption, \( 1 + p(F_i) + |B_i| = |V(F_i)| \leq p(F_i) + s(F_i) \), whence \( 1 + |B_i| \leq s(F_i) \). Therefore \( |I| + |B| \leq \sum_{i \in I} s(F_i) \). Assume that \( s(F_i) = k(F_i - A_i) - |A_i| \) for some \( A_i \subset V(F_i), \ i \in I \). Let \( A = \bigcup_{i \in I} A_i \cup V(K) \). Hence
\[
k(G - A) - |A| = (r - t - c) + \sum_{i \in I} k(F_i - A_i) + |J| + 1 - \sum_{i \in I} |A_i| - r
= |J| + \sum_{i \in I} s(F_i) + 1 - t - c \geq |J| + |I| + |B| + 1 - t - c
= |B| + 1 - t > 0,
\]
contrary to the toughness of \( G \). \( \square \)

**Remark.** Under the assumptions of Proposition 2.8, if each component \( F \) of \( H \) is of order at most five then \( G \) is hamiltonian. In fact, then \( p(F) = |V(F)| - 1 \) or \( p(F) + s(F) \geq |V(F)| \) can easily be seen. The smallest connected graph \( F \) which violates the last condition is the 6-vertex graph of the letter \( H \).

### 3. Main result

Now we are going to prove Conjecture (3.3) of [9].

**Theorem 3.1.** The maximum size of a tough nontrivially nonhamiltonian \( n \)-vertex graph \( G \) is \( f(n) := 6 + \binom{n}{2} \cdot 3 \), \( n \geq 7 \). The corresponding maximum graphs are \( M_n \) for each \( n \geq 7 \) and, additionally, \( G_9 \) for \( n = 9 \).
Proof. Let $g(n)$ be the maximum size to be determined and let $G$ stand for the corresponding maximum tough nonhamiltonian graph of order $n$ (and size $g(n)$). Assume $n \geq 7$ because there is no tough nonhamiltonian graph of order $n$ for $3 \leq n \leq 6$. Hence, by Proposition 1.1, $g(n) \geq f(n)$. We can see, by inspecting the list [4] of all maximally nonhamiltonian graphs of order $n \leq 10$, that our Theorem is true for $n \leq 10$. Hence, assume $n > 11$.

In what follows we shall use the fact that toughness is an increasing property, i.e., the addition of a new edge to a tough graph does not spoil the toughness. Notice, moreover, that $G$ is maximally nonhamiltonian, whence every two non-adjacent vertices of $G$ are connected by a hamiltonian path.

In what follows $d$ is the number of total (i.e., degree-$(n-1)$) vertices in $G$.

Case 1: $\Delta < n-1$ (and $d = 0$).

Then $G$ is homogeneously traceable whence, by Corollary 2.3, $\Delta \leq n-4$. Suppose that $\Delta = n-4 - \rho$, where $0 \leq \rho \leq n-6$. However, if $\rho \geq 3$ then the sum of vertex degrees $2g(n) \leq n(n-7) < n^2 - 7n + 24 = 2f(n)$, a contradiction. Hence $\rho \in \{0, 1, 2\}$. Let $x$ be a vertex of $G$ of degree $\Delta$. Then $x$ has $\rho + 3$ nonneighbours, say $y_1, y_2, \ldots, y_{\rho+3}$, where $x \neq y_j$. By Proposition 2.1, $\deg y_j \leq \rho + 3$ for each $j$ and, by Theorem 2.2, $\deg y_j \leq \rho + 2$ for some subscript $t$. Hence

$$2g(n) \leq (\rho + 2)(\rho + 3) + \rho + 2 + (n-\rho-3)(n-\rho-4) = n^2 - 7n - 2\rho n + 2\rho^2 + 13\rho + 20 < 2f(n),$$

a contradiction.

Case 2: $\Delta = n-1$ and $d \geq 2$.

Let $X = \{x_1, x_2, \ldots, x_d\}$, denote the set of total vertices of $G$. Hence $\delta = d$, but if $\delta = d = 2$ then $G$, being tough, has exactly one vertex of degree 2. Owing to Corollary 2.7,

$$d \leq \delta \leq (n-5)/2.$$  

Let $z_0, z_0 \in V(G) - X$, be a vertex of degree $\Delta'$. Then, by Corollary 2.5, $\Delta' \leq n-2-s$. Let $\Delta' = n-2-s$. Then $s \geq d$ and there is the set $Y$ of $s+1$ vertices, say $y_1, y_2, \ldots, y_s+1$, each of which is different from and non-adjacent to $z_0$. Let $Z = V - (X \cup Y)$. By Proposition 2.1, $\deg y_j \leq s + 1$ for each $y_j \in Y$ and, by Theorem 2.4, $\deg y_j \leq s$ for some $t$. Hence

$$2g(n) = \sum_{x \in X} \deg x + \sum_{y \in Y} \deg y + \sum_{z \in Z} \deg z = d(n-1) + s + s(s + 1) + (n-d-s-1)(n-2-s) = n^2 - n(2s + 3) + 2s^2 + 3s + d + 2.$$  

Suppose that $s \geq d + 2$. Then $2g(n) \leq n^2 - n(2s + 3) + 3s^2 + 4s$. Because $3s^2 + 4s - 24 \leq (2s + 2)(2s - 4)$ for each $s$, $3s^2 + 4s - 24 < n(2s - 4)$ if $2s + 2 < n$, which implies $g(n) < f(n)$, a contradiction. Hence $2s + 2 \geq n$, i.e., $|Y| \geq n/2$. Therefore $\Delta' \leq n - |Y| - 1 \leq n/2 - 1$, whence (because $d \leq (n-5)/2$)

$$2g(n) \leq d(n-1) + (n-d)(n/2 - 1) = n^2/2 - n + dn/2 \leq 3n^2/4 - 9n/4 < 2f(n),$$

a contradiction.
Thus $s \in \{d, d + 1\}$. Hence the subgraph $\langle Y \rangle$ induced by $Y$ is a union of cycles, paths and isolated vertices.

Suppose that the subgraph $\langle Z \rangle$ induced by $Z$ is complete, $\langle Z \rangle = K_{|Z|}$. Then, for each $z \in Z$, $\deg z = \deg z_0 = \Delta'$, whence there is no $Z - Y$ edge in $G$. Therefore $G = \langle Y \rangle \ast K_{d} \ast K_{|Z|}$, whence, by Proposition 2.8, $G$ is hamiltonian, a contradiction. Hence $|Z| \geq 3$ and, for some two vertices $u, v \in Z$, $uv \notin E(G)$.

Moreover, $d + s + 4 \leq n$. Consequently,

$$\sum_{z \in Z} \deg z = \deg u + \deg v + \sum_{z \in Z - \{u, v\}} \deg z,$$

whence

$$2g(n) \leq d(n - 1) + s + s(s + 1) + n - 1 + (n - (d + s + 3))(n - 2 - s)$$

$$= n^2 - n(2s + 4) + w(d, s) =: \varphi(n, d)$$

where $i := s - d \in \{0, 1\}$ and

$$w(d, s) := 2s^2 + ds + 7s + d + 5.$$

Suppose that $s = d$, i.e., $i = 0$. Then $w(d, s) = w(d, d) = 3d^2 + 8d + 5$. Furthermore,

$$2f(n) - \varphi_0(n, d) = n(2d - 3) - 3d^2 - 8d + 19$$

$$= n - 9 \text{ if } d = 2, \text{ a contradiction if } d = 2;$$

$$\geq (d - 2)^2 \text{ (because } n \geq 2d + 5),$$

$$\text{ a contradiction for } d > 2.$$

Hence $s = d + 1$, i.e., $i = 1$. Then

$$w(d, s) = w(d, d + 1) = 3d^2 + 13d + 14$$

and

$$2f(n) - \varphi_1(n, d) = n(2d - 1) - 3d^2 - 13d + 10$$

$$= 3n - 18 \text{ if } d = 2, \text{ a contradiction if } d = 2,$$

$$\geq (d - 4)(d - 1) + 1, \text{ a contradiction if } d \geq 4;$$

$$= 5n - 56 \text{ if } d = 3, \text{ a contradiction if } n \geq 12 \text{ and } d = 3.$$

A possible counterexample has $n = 11$, $d = 3 = \delta$, $\Delta' = 5$ and $2g(n) = 2f(n) = 68$, whence it has the degree sequence $(10^3, 5^7, 3)$ and therefore does not exist (cannot be nonhamiltonian).

Case 3: $\Delta = n - 1$ and $d = 1$.

Let $n_2$ be the number of degree-2 vertices in $G$. Clearly, no two of them can be adjacent. Suppose $n_2 \leq 1$. Then, by Corollary 2.5, $\Delta' = n - 4 - t$ where $t \geq 0$.

If $t \geq 3$ then $2g(n) \leq n - 1 + (n - 1)(n - 7) < 2f(n)$, a contradiction. Hence $t \in \{0, 1, 2\}$. Let $\deg z_0 = \Delta'$ for some $z_0 \in V(G)$. Then there are $t + 3$ vertices different from and non-adjacent to $z_0$, of which one is of degree $\leq t + 2$ and the
remaining ones are of degree \( \leq t + 3 \) (by Theorem 2.4 and Proposition 2.1). Hence

\[
2g(n) \leq n - 1 + (t + 3)^2 - 1 + (n - t - 4)^2 < 2f(n),
\]

a contradiction. Hence \( n_2 \geq 2 \). Because \( G \) is tough, no nontotal vertex in \( G \) can be a common neighbour of any two degree-2 vertices of \( G \). Suppose \( u_1 \) and \( u_2 \) are the only two degree-2 vertices of \( G \). Hence there are two nontotal vertices, \( z_1 \) and \( z_2 \), in \( G \) such that \( u_iz_i \in E(G) \) \((i = 1, 2)\) and \( z_1 \neq z_2 \). Assume that notation is chosen so that \( \text{deg} z_1 \geq \text{deg} z_2 \). Suppose \( z_1z_2 \notin E(G) \). Then \( \text{deg} z_1 + \text{deg} z_2 \leq n - 1 \) by Proposition 2.1, whence

\[
g(n) \leq n - 1 + 2 + \binom{n - 4}{2} < f(n),
\]
a contradiction. Therefore \( z_1z_2 \in E(G) \). By Proposition 2.1, \( \text{deg} z_1 \leq n - 4 \). Hence there are two vertices, say \( y_1 \) and \( y_2 \), both different from \( u_2 \) and from each \( z_i \) and both non-adjacent to \( z_1 \). Hence, by Proposition 2.1,

\[
\text{deg} z_1 + \text{deg} y_1 + \text{deg} y_2 + \text{deg} z_2 \leq 2(n - 1),
\]
whence

\[
g(n) \geq 2(n - 1) - 1 + \text{deg} y_1 + \text{deg} u_2 - 2 + \binom{n - 6}{2} \leq f(n),
\]
a contradiction. Thus \( n_2 \geq 3 \). Then \( G = M_n \) is the only possibility.

All cases have been examined. \( \square \)

4. Concluding remarks

**Conjecture.** For integers \( d \) and \( h \) such that \( 0 \leq h < d \), if \( G \) is a maximum nonhamiltonian \( n \)-vertex graph with exactly \( d \) total vertices and the scattering number \( s(G) \leq -h \), then \( n \geq 5 + 2d + 3h \) and

\[
G = (d - h - 1)K_1 \ast K_d \ast ((2h + 3)K_1 \ast K_{n - 2d - h - 2}),
\]
whence

\[
\delta \geq \max\{h + 2, d\} \quad \text{and} \quad |E| = d^2 + (h + 2)d + 2h + 3 + \binom{n - d - h - 2}{2}.
\]

This conjecture presents an analogue of a special case \((s = 0)\) in Theorem 3 of [10].

Our main result presents the complete solution of the case \( i = 1 \) and \( h = 0 \) of the conjecture 1.2 in the recent paper [3] by Hendry. Our method does not work in remaining cases, namely, for \( i = 0 \) (\( G \) is nontraceable) and \( i = 2 \) (\( G \) is non-Hamilton-connected). Our Conjecture is influenced by that of Hendry because originally we considered only \( h = 0 \), that is, a tough graph \( G \) (without involving \( s(G) \)).
It remains an open problem to find maximum homogeneously traceable nonhamiltonian graphs \((d = 0)\) on \(n\) vertices for \(n \geq 11\), see [8] for \(n = 9, 10\) and for related problems.

References