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# On the finitistic dimension conjecture II: Related to finite global dimension

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Dedicated to Claus Michael Ringel on the occasion of his 60th birthday

**Abstract**

In this paper, we study the finitistic dimensions of artin algebras by establishing a relationship between the global dimensions of the given algebras, on the one hand, and the finitistic dimensions of their subalgebras, on the other hand. This is a continuation of the project in [J. Pure Appl. Algebra 193 (2004) 287–305]. For an artin algebra $A$ we denote by $\mathrm{gl.dim}(A)$, $\mathrm{fin.dim}(A)$ and $\mathrm{rep.dim}(A)$ the global dimension, finitistic dimension and representation dimension of $A$, respectively. The Jacobson radical of $A$ is denoted by $\mathrm{rad}(A)$. The main results in the paper are as follows: Let $B$ be a subalgebra of an artin algebra $A$ such that $\mathrm{rad}(B)$ is a left ideal in $A$. Then (1) if $\mathrm{gl.dim}(A) \leqslant 4$ and $\mathrm{rad}(A) = \mathrm{rad}(B)A$, then $\mathrm{fin.dim}(B) < \infty$. (2) If $\mathrm{rep.dim}(A) \leqslant 3$, then $\mathrm{fin.dim}(B) < \infty$. The results are applied to pullbacks of algebras over semi-simple algebras. Moreover, we have also the following dual statement: (3) Let $\varphi : B \longrightarrow A$ be a surjective homomorphism between two algebras $B$ and $A$. Suppose that the kernel of $\varphi$ is contained in the socle of the right $B$-module $B_B$. If $\mathrm{gl.dim}(A) \leqslant 4$, or $\mathrm{rep.dim}(A) \leqslant 3$, then $\mathrm{fin.dim}(B) < \infty$. Finally, we provide a class of algebras with representation dimension at most three: (4) If $A$ is stably hereditary and $\mathrm{rad}(B)$ is an ideal in $A$, then $\mathrm{rep.dim}(B) \leqslant 3$.


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## 1. Introduction

Given an artin algebra $A$, the famous finitistic dimension conjecture says that there exists a uniform bound for the finite projective dimensions of all finitely generated (left) $A$-modules of finite projective dimension. This conjecture implies the Nakayama conjecture. There are a few cases for which this conjecture is verified to be true (see [9,10,13,17,18,8]). In general, this conjecture seems to be far from being solved. Recently, we start with [20] to study the finitistic dimension conjecture by comparing the finitistic dimensions of a pair of algebras instead of focusing only on one single algebra, namely, we consider the following question: suppose two artin algebras $A$ and $B$ are related to each other in a certain manner, for example, $B$ is a subalgebra or factor algebra of $A$. If one of them has finite finitistic dimension, what could we say about the finitistic dimension of the other? In [20] we investigated the case where one of them is representation finite, and got the finiteness of the finitistic dimension of the other. Moreover, under a mild assumption on the ground field, it was proved in [20] that the finitistic dimension conjecture is equivalent to the following statement: if $B$ is a subalgebra of $A$ such that $\mathrm{rad}(B)$ is a left ideal in $A$ and if $A$ has finite finitistic dimension, then $B$ has finite finitistic dimension. Thus, in order to understand the finititic dimension conjecture, it is helpful to study the homological or representation-theoretical behaviors of subalgebras through extension algebras.

In the present paper, we continue to study the above question. Here, we consider the case where one of the given algebras has finite global dimension instead of finite representation-type, and want to approach the finiteness of the finitistic dimension of the other on which we do not impose any homological conditions. The main hypothesis for our question is the so-called radical-full homomorphism (see 3.5 below), which relates the two artin algebras considered together. This radical condition seems to be a right way to study subalgebras via extension algebras. Clearly, the notion "radical-full" extends the notion of radical embedding in [8]. Note that even under the strong condition that subalgebras have the same Jacobson radical as a given algebra does, the subalgebras might be very complicated, namely, a subalgebra of a representation-finite algebra might be representation-wild, and a subalgebra of an algebra of finite global dimension might be of infinite global dimension. From this point of view, it seems that the study of the finitistic dimensions of subalgebras via extension algebras would be much more challenging.

The main result in this paper is the following:

**Theorem 1.1.** *Let $B$ be a subalgebra of an artin algebra $A$ such that* $\mathrm{rad}(B)$ *is a left ideal in $A$. Then*:

(1) *If the inclusion map of $B$ into $A$ is radical-full and if* $\mathrm{gl.dim}(A) \leqslant 4$, *then* $\mathrm{fin.dim}(B) < \infty$, *where* $\mathrm{gl.dim}(A)$ *and* $\mathrm{fin.dim}(A)$ *denote the global dimension and the finitistic dimension of $A$, respectively.*
(2) *If* $\mathrm{rep.dim}(A) \leqslant 3$, *then* $\mathrm{fin.dim}(B) < \infty$, *where* $\mathrm{rep.dim}(A)$ *stands for the representation dimension of $A$.*
(3) *If $A$ is stably hereditary and* $\mathrm{rad}(B)$ *is an ideal in $A$, then* $\mathrm{rep.dim}(B) \leqslant 3$. *In particular, the finitistic dimension of $B$ is finite.*

Result (1) extends a result of [8] in different direction, and (2) and (3) generalize some results in [12,22], respectively.

Note that algebras of smaller global dimension were studied by many other authors in the literature, and seem to have special interest in homological algebra and in the representation theory of artin algebras.

The paper is organized as follows: In Section 2 we make a preparation for the proof of the main result. In Sections 3 and 4 we give the proof of the main result and, at the same time, deduce some consequences of the main result. In particular, the main result is applied to pullback algebras and tensor products of algebras. The idea of the proof of the main result is to establish an exact sequence of syzygy modules over $A$ and $B$, which can link the different syzygies together. In the last section, we display some examples to illustrate our main result. Also, some open questions related to the main result are mentioned.

## 2. Preliminaries

In this section we recall some basic definitions and results needed in the paper.

Let $A$ be an artin algebra, that is, $A$ is a finitely generated module over its center which is assumed to be a commutative artin ring. We denote by $A$-mod the category of all finitely generated left $A$-modules and by $\mathrm{rad}(A)$ the Jacobson radical of $A$. Given an $A$-module $M$, we denote by $\mathrm{proj.dim}(M)$ the projective dimension of $M$.

Let $K(A)$ be the quotient of the free abelian group generated by the isomorphism classes $[M]$ of modules $M$ in $A$-mod modulo the relations:

(1) $[Y] = [X] + [Z]$ if $Y \simeq X \oplus Z$; and
(2) $[P] = 0$ if $P$ is projective.

Thus $K(A)$ is a free abelian group with the basis of non-isomorphism classes of non-projective indecomposable $A$-modules in $A$-mod. Igusa and Todorov [12] use the noetherian property of the ring of integers and define a function $\Psi$ on this abelian group, which depends on the algebra $A$ and takes values of non-negative integers.

The following result is due to Igusa and Todorov [12].

**Lemma 2.1.** *For any artin algebra $A$ there is a function $\Psi$ defined on the objects of $A$-mod such that*

(1) $\Psi(M) = \mathrm{proj.dim}(M)$ *if $M$ has finite projective dimension. Moreover, if $M$ is indecomposable and* $\mathrm{proj.dim}(M) = \infty$, *then* $\Psi(M) = 0$.
(2) *For any natural number $n$,* $\Psi(\bigoplus_{j=1}^{n} M) = \Psi(M)$.
(3) *For any $A$-modules $X$ and $Y$,* $\Psi(X) \leqslant \Psi(X \oplus Y)$.
(4) *If* $0 \to X \to Y \to Z \to 0$ *is an exact sequence in $A$-mod with* $\mathrm{proj.dim}(Z) < \infty$, *then* $\mathrm{proj.dim}(Z) \leqslant \Psi(X \oplus Y) + 1$.
(5) *If* $0 \to X \to Y \to Z \to 0$ *is an exact sequence in $A$-mod with $Z$ indecomposable, then* $\Psi(Z) \leqslant \Psi(X \oplus Y) + 1$.

Note that given an exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ in $A$-mod, there are two relevant exact sequences

$$0 \longrightarrow \Omega(Y) \longrightarrow \Omega(Z) \oplus P \longrightarrow X \rightarrow 0$$

and

$$0 \longrightarrow \Omega^2(Z) \longrightarrow \Omega(X) \oplus P' \longrightarrow \Omega(Y) \rightarrow 0,$$

where $\Omega^i$ is the $i$th syzygy operator, and $P$, $P'$ are projective modules. So the following result is a consequence of 2.1.

**Lemma 2.2.** *If* $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ *is an exact sequence in* $A$-mod, *then*

(1) $\text{proj.dim}(Y) \leqslant \Psi(\Omega(X) \oplus \Omega^2(Z)) + 2$ *if* $\text{proj.dim}(Y) < \infty$,
(2) $\text{proj.dim}(X) \leqslant \Psi(\Omega(Y \oplus Z)) + 1$ *if* $\text{proj.dim}(X) < \infty$.

The following two lemmas are standard homological facts, which we need.

**Lemma 2.3.** *Let* $A$ *be an artin algebra and let* $M$ *be an* $A$*-module. If there is an exact sequence*

$$0 \rightarrow X_s \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$$

*of* $A$*-modules with* $\text{proj.dim}(X_i) \leqslant k$ *for all* $i$, *then* $\text{proj.dim}({}_A M) \leqslant s + k$.

**Lemma 2.4** (*Schanuel's lemma*). *If there are two exact sequences*

$$0 \longrightarrow K_1 \longrightarrow P_1 \longrightarrow M \longrightarrow 0,$$

$$0 \longrightarrow K_2 \longrightarrow P_2 \longrightarrow M \longrightarrow 0$$

*in* $A$-mod *with* $P_1$, $P_2$ *projective, then* $K_1 \oplus P_2 \simeq K_2 \oplus P_1$.

Finally, let us recall the definition of finitistic dimension.

**Definition 2.5.** Given an artin algebra $A$, the *finitistic dimension* of $A$, denoted by $\text{fin.dim}(A)$, is defined as

$$\text{fin.dim}(A) = \sup\{\text{proj.dim}({}_A M) \mid M \in A\text{-mod and proj.dim}({}_A M) < \infty\}.$$

Note that $\text{fin.dim}(A)$ may be different from $\text{fin.dim}(A^{op})$, where $A^{op}$ is the opposite algebra of $A$. Concerning this notion there is the following famous conjecture:

*Finitistic dimension conjecture* [5]: $\mathrm{fin.dim}(A) < \infty$ for every artin algebra $A$.

Related to this conjecture there are several other homological conjectures (see the last 6 conjectures of the total 13 conjectures in the book [4]):

*Strong Nakayama conjecture* [7]: If $M$ is a non-zero module over an artin algebra $A$, then there is an integer $n \geqslant 0$ such that $\mathrm{Ext}^n_A(M, A) \neq 0$.

*Generalized Nakayama conjecture* [3]: If $0 \to {}_AA \to I_0 \to I_1 \to \cdots$ is a minimal injective resolution of an artin algebra $A$, then any indecomposable injective is a direct summand of some $I_j$. Equivalently, if $M$ is a finitely generated $A$-module such that $\mathrm{add}(A) \subseteq \mathrm{add}(M)$ and $\mathrm{Ext}^i_A(M, M) = 0$ for all $i \geqslant 1$, then $M$ is projective.

*Nakayama conjecture* [16]: If all $I_j$ in a minimal injective resolution of an artin algebra $A$, say $0 \to {}_AA \to I_0 \to I_1 \to \ldots$ , are projective, then $A$ is self-injective.

*Gorenstein symmetry conjecture*: Let $A$ be an artin algebra. If the injective dimension of ${}_AA$ is finite, then the injective dimension of $A_A$ is finite.

In general, all the five conjectures are still open. They have the following well-known relationship, for a proof we refer to [3,23].

**Proposition 2.6.** (1) *The finitistic dimension conjecture implies the strong Nakayama conjecture*.

(2) *The strong Nakayama conjecture implies the generalized Nakayama conjecture.*

(3) *The generalized Nakayama conjecture implies the Nakayama conjecture.*

(4) *The finitistic dimension conjecture implies the Gorenstein symmetry conjecture.*

Thus, the finitistic dimension possesses a strong homological property and can be far more revealing measures of homological complexity of an algebra at hand, while infinite global dimension often does not reveal much about that complexity.

## 3. Radical-full homomorphisms and finitistic dimensions

In this section, we shall study a given pair $B \subseteq A$ of algebras, or more generally, an increasing chain of subalgebras in $A$, and want to approach the finiteness of the finitistic dimension of $B$ by assuming the finiteness of global dimension of $A$. In general, there is no expected good relationship between the finitistic dimensions of a pair $B \subseteq A$ of algebras. This can be seen from a matrix algebra and its upper triangular matrix subalgebra. Thus the finitistic dimension, as a homological complexity, of the module category over a subalgebra seems more complicated than that over an extension algebra. So, our philosophy is to control a complicated object (that is, a subalgebra) by using a relatively simple object (that is, an extension algebra). To this end, we shall introduce the so-called radical-full homomorphism of algebras, this enables us to compare the complexities of module categories between an algebra and its subalgebras.

Let us start with the following result:

**Theorem 3.1.** *Let*

$$B = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_{s-1} \subseteq A_s = A$$

*be a chain of subalgebras of* $A$ *such that* $\mathrm{rad}(A_{i-1})$ *is a left ideal in* $A_i$ *and* $\mathrm{proj.dim}({}_{A_{i-1}}A_i) < \infty$ *for all* $1 \leqslant i \leqslant s$. *If* $\mathrm{gl.dim}(A)$ *is finite*, *then* $\mathrm{fin.dim}(B)$ *is finite*.

Before we start the proof of Theorem 3.1, we cite the following lemma proved in [20, Erratum, Lemma 0.2].

**Lemma 3.2.** *Suppose* $B$ *is a subalgebra of* $A$ *such that* $\mathrm{rad}(B)$ *is a left ideal in* $A$. *For any* $B$*-module* $X$ *and integer* $i \geqslant 2$, *there is a projective* $A$*-module* $Q$ *and an* $A$*-module* $Z$ *such that* $\Omega^i_B(X) \simeq \Omega_A(Z) \oplus Q$ *as* $A$*-modules.*

**Proof of Theorem 3.1.** First, we show that $\mathrm{proj.dim}({}_BA_j) < \infty$ for all $j$ by induction on $j$. If $j = 0$ then the statement is obvious. Assume that $\mathrm{proj.dim}({}_BA_i) < \infty$ for all $i \leqslant j-1$. Since the projective dimension of the $A_{j-1}$-module $A_j$ is finite, there is a finite projective resolution for the $A_{j-1}$-module ${}_{A_{j-1}}A_j$:

$$0 \to P_m \to \cdots \to P_1 \to P_0 \to A_j \to 0$$

with all $P_r$ being projective $A_{j-1}$-modules. By the induction hypothesis, each $P_r$ as a $B$-module has finite projective dimension. Thus this exact sequence together with Lemma 2.3 yields the desired conclusion.

Now suppose $M$ is a $B$-module with finite projective dimension. Let us denote by $\Omega_i$ the first syzygy operator of $A_i$-modules. Since $\mathrm{rad}(B)$ is a left ideal in $A_1$, we know that $\Omega^2_0(M)$ is an $A_1$-module by Lemma 3.2. Similarly, $\Omega^2_j \cdots \Omega^2_1\Omega^2_0(M)$ is an $A_{j+1}$-module. Let $P_j(1) \longrightarrow P_j(0) \longrightarrow \Omega^2_{j-1} \cdots \Omega^2_1\Omega^2_0(M) \longrightarrow 0$ be an $A_j$-projective presentation. Then we have an exact sequence

$$0 \to \Omega^2_{s-1} \cdots \Omega^2_1\Omega^2_0(M) \to P_{s-1}(1) \to P_{s-1}(0) \to \cdots \to P_1(1) \to P_1(0) \to \Omega^2_0(M) \to 0.$$

Since $\mathrm{gl.dim}(A) < \infty$, we have a projective resolution of the $A_s$-module $\Omega^2_{s-1} \cdots \Omega^2_1\Omega^2_0(M)$:

$$0 \to Q_t \to \cdots \to Q_1 \to Q_0 \to \Omega^2_{s-1} \cdots \Omega^2_1\Omega^2_0(M) \to 0,$$

where $t = \mathrm{gl.dim}(A)$, and all $Q_j$ are projective $A$-modules. By putting two exact sequences together, we get a new exact sequence

$$0 \to Q_t \to \cdots \to Q_0 \to P_{s-1}(1) \to P_{s-1}(0) \to \cdots \to P_1(1) \to P_1(0) \to \Omega^2_0(M) \to 0.$$

Since $\mathrm{proj.dim}({}_BA_j) < \infty$ for all $0 \leqslant j \leqslant s$, we see that $\mathrm{proj.dim}({}_BQ_j)$ and $\mathrm{proj.dim}({}_BP_j)$ are finite. Let $m = \max\{\mathrm{proj.dim}({}_BA_j) \mid j = 1, 2, \ldots, s\}$. Then, $\mathrm{proj.dim}({}_BM) \leqslant 2s + t + m + 2$ by Lemma 2.3. This shows that the finitistic dimension of $B$ is finite. $\square$

The next result is a variation of Theorem 3.1.

**Proposition 3.3.** *Let*

$$A_0 = B \subseteq A_1 \subseteq \cdots \subseteq A_{s-1} \subseteq A_s = A$$

*be a chain of subalgebras of $A$ such that* $\operatorname{rad}(A_{i-1})$ *is a left ideal in $A_i$ for all $i$ and that* $\operatorname{proj.dim}({}_{A_{i-1}}A_i) < \infty$ *for all* $1 \leqslant i \leqslant s-1$. *If* $\operatorname{gl.dim}(A) \leqslant 1$, *then* $\operatorname{fin.dim}(B)$ *is finite.*

**Proof.** Suppose $M$ is a $B$-module with finite projective dimension. As in the above proof of 3.1, we have an exact sequence

$$0 \to \Omega^2_{s-1} \cdots \Omega^2_1 \Omega^2_0(M) \to P_{s-1}(1) \to P_{s-1}(0) \to \cdots \to P_1(1) \to P_1(0) \to P_1 \to P_0 \to M \to 0$$

of $B$-modules. Note that $\Omega^2_{s-1} \cdots \Omega^2_1 \Omega^2_0(M)$ is an $A_s$-module. Since we do not know that the $A_0$-module $A_s$ has a finite projective dimension, the argument in the proof of Theorem 3.1 does not work. However, since $A = A_s$ is a hereditary algebra, there is an exact sequence $0 \to P_{s+1} \to P_s \to \Omega^2_{s-1} \cdots \Omega^2_1 \Omega^2_0(M) \to 0$, where $P_s$ and $P_{s+1}$ are projective $A$-modules. Note that the $B$-module $\Omega^2_{s-1} \cdots \Omega^2_1 \Omega^2_0(M)$ has finite projective dimension. Hence, by Lemmas 2.3 and 2.1, we have

$$\begin{aligned}
\operatorname{proj.dim}({}_BM) &\leqslant 2s + 1 + \max\{\operatorname{proj.dim}({}_B\Omega^2_{s-1} \cdots \Omega^2_1 \Omega^2_0(M)),\\
&\qquad \operatorname{proj.dim}({}_BP_j(i)),\ j = 1, \ldots, s-1, i = 1, 2\}\\
&\leqslant 2s + 1 + \max\{\operatorname{proj.dim}({}_B\Omega^2_{s-1} \cdots \Omega^2_1 \Omega^2_0(M)),\\
&\qquad \operatorname{proj.dim}({}_BA_j),\ j = 1, \ldots, s-1\}\\
&\leqslant 2s + 1 + \max\{\Psi(P_{s+1} \oplus P_s) + 1, \operatorname{proj.dim}({}_BA_j),\ j = 1, \ldots, s-1\}\\
&\leqslant 2s + 1 + \max\{\Psi({}_BA_s) + 1,\ \operatorname{proj.dim}({}_BA_j),\ j = 1, \ldots, s-1\}.
\end{aligned}$$

This shows that the finitistic dimension of $B$ is finite. $\square$

**Corollary 3.4.** *If $B$ is a subalgebra of a hereditary artin algebra $A$ such that the radical of $B$ is a left ideal in $A$, then the finitistic dimension of $B$ is finite.*

If we impose one more condition on the radical of the subalgebra $B$, then we may relax the restriction on $A$. Recall that in [8] an injective morphism $f : B \to A$ is called a radical embedding if $f(\operatorname{rad}(B)) = \operatorname{rad}(A)$. Before we start with our discussion, it is convenient to introduce the following notion which is a proper generalization of "radical embedding".

**Definition 3.5.** A homomorphism $f : B \longrightarrow A$ between two algebras $A$ and $B$ is said to be (left) radical-full if $\mathrm{rad}({}_BA) = \mathrm{rad}({}_AA)$, that is, $\mathrm{rad}(B)A = \mathrm{rad}(A)$. This is equivalent to saying that the radical of $A$ is generated as a right ideal in $A$ by the image of the radical of $B$ under $f$.

Here we require that the homomorphism $f$ between algebras preserves the identity. Clearly, the composition of two radical-full homomorphisms is again radical-full, and a surjective homomorphism $f$ is radical-full. Note also that, for algebras over an algebraically closed field, the tensor product of two radical-full maps is again radical-full, but not a radical embedding in general.

Given a homomorphism $\varphi : B \longrightarrow A$, we denote by $F_\varphi$ or simply by $F$ the restriction functor from $A$-mod to $B$-mod if there is no confusion.

The following is a generalization of [20, Proposition 4.2 (6)].

**Lemma 3.6.** *An algebra homomorphism $\varphi : B \rightarrow A$ between two algebras $B$ and $A$ is radical-full if and only if* $\mathrm{rad}({}_BFX) = F\mathrm{rad}({}_AX)$ *for all $A$-module $X$, and if and only if* $F\mathrm{top}_A(X) = \mathrm{top}_B(FX)$ *for all $A$-module $X$, where* $\mathrm{top}_A(X)$ *stands for the top of the $A$-module $X$.*

**Proof.** Suppose that the homomorphism $\varphi$ is radical-full, that is, $\mathrm{rad}(A) = \mathrm{rad}({}_BA) = \mathrm{rad}(B)A = \varphi(\mathrm{rad}(B))A$. If ${}_AX$ is an $A$-module, then $\mathrm{rad}({}_AX) = \mathrm{rad}(A)X = \varphi(\mathrm{rad}(B))AX = \varphi(\mathrm{rad}(B))X = \mathrm{rad}(B) \cdot X = \mathrm{rad}({}_BX) = \mathrm{rad}(F_AX)$. Thus $\mathrm{rad}({}_BFX) = F\,\mathrm{rad}({}_AX)$. The converse statement is trivially true.

The last statement follows from the first one. $\square$

Now let us prove one of our main results in this paper.

**Theorem 3.7.** *Let $B$ be a subalgebra of an artin algebra $A$ such that* $\mathrm{rad}(B)$ *is a left ideal in $A$. Suppose that the inclusion map of $B$ into $A$ is radical-full. If* $\mathrm{gl.dim}(A) \leqslant 4$, *then* $\mathrm{fin.dim}(B) < \infty$.

As a direct consequence of Theorem 3.7, we have

**Corollary 3.8.** *Let*

$$A_0 = B \subseteq A_1 \subseteq \cdots \subseteq A_{s-1} \subseteq A_s = A$$

*be a chain of subalgebras of $A$ such that* $\mathrm{rad}(A_{i-1}) = \mathrm{rad}(A_i)$ *for all $i$. If* $\mathrm{gl.dim}(A) \leqslant 4$, *then* $\mathrm{fin.dim}(B)$ *is finite.*

**Proof of Theorem 3.7.** Suppose that $B$ is a subalgebra of an algebra $A$ such that $\mathrm{rad}(B)$ is a left ideal in $A$ and $\mathrm{rad}(A) = \mathrm{rad}(B)A$. Then we have the following exact sequence of $A$-modules:

$$0 \longrightarrow \Omega_B\big(P_A(\Omega_A^j\Omega_B^2(X))\big) \longrightarrow \Omega_B(\Omega_A^j\Omega_B^2(X)) \longrightarrow \Omega_A^{j+1}(\Omega_B^2(X)) \longrightarrow 0$$

for all $j \geqslant 0$ and all $X \in B$-mod, where $P_A(M)$ stands for the projective cover of an $A$-module $M$.

Indeed, we know that $\Omega_B^2(X)$ is an $A$-module for all $B$-module $X$, so $\Omega_A^j(\Omega_B^2(X))$ is well defined for $j \geqslant 0$. Let $\pi_1 : P_1 \longrightarrow {}_B\Omega_A^j(\Omega_B^2(X))$ be a projective cover of the $B$-module $\Omega_A^j(\Omega_B^2(X))$. The inclusion of $\Omega_B(\Omega_A^j(\Omega_B^2(X)))$ into $P_1$ is denoted by $v$. Then we have the following commutative diagram:

$$\begin{tikzcd}
 & & & {}_B\Omega_A^{j+1}(\Omega_B^2(X)) \arrow[d] & \\
0 \arrow[r] & \Omega_B({}_BP_A(\Omega_A^j(\Omega_B^2(X)))) \arrow[r, "\mu"] \arrow[d, "\pi_4"'] & P_1 \arrow[r, "\pi_3"] \arrow[d, equal] & {}_BP_A(\Omega_A^j(\Omega_B^2(X))) \arrow[r] \arrow[d, "\pi_2"'] & 0 \\
0 \arrow[r] & \Omega_B(\Omega_A^j(\Omega_B^2(X)) \arrow[r, "v"] & P_1 \arrow[r, "\pi_1"] & {}_B\Omega_A^j(\Omega_B^2(X)) \arrow[r] & 0,
\end{tikzcd}$$

where $\pi_3$ exists since $P_1$ is projective and $\pi_2$ is surjective. The condition $\mathrm{rad}(A) = \mathrm{rad}(B)A$ implies that the top of $P_1$ is isomorphic to the top of ${}_BP_A(\Omega_A^j(\Omega_B^2(X)))$ by 3.6. Thus $\pi_3$ is surjective and the kernel of $\pi_3$ is $\Omega_B({}_BP_A(\Omega_A^j(\Omega_B^2(X))))$. We denote by $\mu$ the inclusion of $\Omega_B({}_BP_A(\Omega_A^j(\Omega_B^2(X))))$ into $P_1$. Now $\pi_4$ is just the restriction of the identity map $id_{P_1}$ to the submodule $\Omega_B({}_BP_A(\Omega_A^j(\Omega_B^2(X))))$. Note that if $f : {}_AP \longrightarrow {}_AM$ is a projective cover of $M$, then the syzygy of $M$ can be described as the kernel of the induced map $\mathrm{rad}({}_AP) \longrightarrow \mathrm{rad}({}_AM)$. Also, note that for any $A$-module $M$ we have $\mathrm{rad}({}_AM) = \mathrm{rad}(B)AM = \mathrm{rad}(B)M = \mathrm{rad}({}_BM)$. Hence the above diagram gives the following commutative diagram:

$$\begin{tikzcd}
 & & \Omega_A^{j+1}(\Omega_B^2(X)) \arrow[d] \\
0 \longrightarrow \Omega_B({}_BP_A(\Omega_A^j(\Omega_B^2(X)))) \arrow[r, "\mu"] \arrow[d, "\pi_4"'] & \mathrm{rad}({}_BP_1) \arrow[r, "\pi_3"] \arrow[d, equal] & \mathrm{rad}({}_BP_A(\Omega_A^j(\Omega_B^2(X)))) \longrightarrow 0 \arrow[d, "\pi_2"'] \\
0 \longrightarrow \Omega_B(\Omega_A^j(\Omega_B^2(X)) \arrow[r, "v"] & \mathrm{rad}({}_BP_1) \arrow[r, "\pi_1"] & \mathrm{rad}({}_B\Omega_A^j(\Omega_B^2(X))) \longrightarrow 0.
\end{tikzcd}$$

Note that $\pi_3 : \mathrm{rad}({}_BP_1) \longrightarrow \mathrm{rad}({}_BP_A(\Omega_A^j(\Omega_B^2(X))))$ is an $A$-homomorphism between $A$-modules. Since all homomorphisms in the diagram are $A$-homomorphisms, the snake lemma yields the following exact sequence of $A$-modules:

$$(*) \qquad 0 \longrightarrow \Omega_B\big({}_BP_A(\Omega_A^j(\Omega_B^2(X)))\big) \xrightarrow{\pi_4} \Omega_B(\Omega_A^j(\Omega_B^2(X)) \longrightarrow \Omega_A^{j+1}(\Omega_B^2(X)) \longrightarrow 0.$$

This is what we want to prove. Now we put $j = 0$ in $(*)$ and get an exact sequence

$$0 \longrightarrow \Omega_B\big({}_B P_A(\Omega_B^2(X))\big) \longrightarrow \Omega_B^3(X) \longrightarrow \Omega_A(\Omega_B^2(X)) \longrightarrow 0.$$

From this sequence we obtain the following commutative diagram in $A$-mod:

$$\begin{tikzcd}
 & & \Omega_A^2(\Omega_B^2(X)) \arrow[r, equal] \arrow[d] & \Omega_A^2(\Omega_B^2(X)) \arrow[d] & \\
0 \arrow[r] & \Omega_B({}_B P_A((\Omega_B^2(X)))) \arrow[r] \arrow[d, equal] & \Omega_B(P_A(\Omega_B^2(X))) \oplus P_A(\Omega_A(\Omega_B^2(X))) \arrow[r] \arrow[d] & P_A(\Omega_A(\Omega_B^2(X))) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega_B({}_B P_A((\Omega_B^2(X)))) \arrow[r] & \Omega_B^3(X) \arrow[r] & \Omega_A(\Omega_B^2(X)) \arrow[r] & 0.
\end{tikzcd}$$

This provides us the following exact sequence in $A$-mod:

$$(**) \quad 0 \longrightarrow \Omega_A^2(\Omega_B^2(X)) \longrightarrow \Omega_B(P_A(\Omega_B^2(X))) \oplus P_A(\Omega_A(\Omega_B^2(X))) \longrightarrow \Omega_B^3(X) \longrightarrow 0.$$

From this exact sequence we get the following exact sequence by applying the syzygy operator:

$$0 \longrightarrow \Omega_A \Omega_B(P_A(\Omega_B^2(X))) \longrightarrow \Omega_A \Omega_B^3(X) \oplus Q' \longrightarrow \Omega_A^2(\Omega_B^2(X)) \longrightarrow 0,$$

where $Q'$ is a projective $A$-module. This further yields the following exact sequence:

$$0 \longrightarrow \Omega_A^2 \Omega_B^3(X) \longrightarrow \Omega_A^3(\Omega_B^2(X)) \oplus Q \longrightarrow \Omega_A \Omega_B(P_A(\Omega_B^2(X))) \longrightarrow 0,$$

where $Q$ is a projective $A$-module. By Lemma 3.2, there is an $A$-module $Y$ and a projective $A$-module $Q''$ such that $\Omega_B^2(X) \simeq \Omega_A(Y) \oplus Q''$. So the above exact sequence can be rewritten as

$$0 \longrightarrow \Omega_A^2 \Omega_B^3(X) \longrightarrow \Omega_A^4(Y) \oplus Q \longrightarrow \Omega_A \Omega_B(P_A(\Omega_B^2(X))) \longrightarrow 0.$$

Note that for an algebra $\Lambda$, $\text{gl.dim}(\Lambda) \leqslant n$ if and only if $\Omega_\Lambda^n(M)$ is projective for all $\Lambda$-modules $M$. Since $\text{gl.dim}(A) \leqslant 4$ by assumption, we know that the middle term of the last exact sequence is a projective $A$-module. Note also that there is another canonical exact sequence

$$0 \longrightarrow \Omega_A^2 \Omega_B(P_A(\Omega_B^2(X))) \longrightarrow P_A(\Omega_A \Omega_B(P_A(\Omega_B^2(X)))) \longrightarrow \Omega_A \Omega_B(P_A(\Omega_B^2(X))) \longrightarrow 0$$

with a middle term projective. Thus, by Schanuel's lemma (see 2.4), we have

$$\Omega_A^2 \Omega_B^3(X) \oplus P_A(\Omega_A \Omega_B(P_A(\Omega_B^2(X)))) \simeq \Omega_A^2 \Omega_B(P_A(\Omega_B^2(X))) \oplus \Omega_A^4(Y)) \oplus Q.$$

This implies that $\Omega_A^2 \Omega_B^3(X)$ is a direct summand of $\Omega_A^2 \Omega_B(P_A(\Omega_B^2(X))) \oplus \Omega_A^4(Y) \oplus Q$.

Now it follows from $(**)$ that there is an exact sequence

$$0 \longrightarrow \Omega_A^2\Omega_B^3(X) \longrightarrow \Omega_B(P_A(\Omega_B^3(X))) \oplus P_A(\Omega_A(\Omega_B^3(X))) \longrightarrow \Omega_B^4(X) \longrightarrow 0.$$

If ${}_BX$ is a $B$-module with finite projective dimension, then, by Lemmas 2.2 and 2.1, we have the following estimation:

$$\begin{aligned}
\text{proj.dim}({}_BX) &\leqslant \text{proj.dim}(\Omega_B^4(X)) + 4\\
&= \Psi(\Omega_B^4(X)) + 4\\
&\leqslant \Psi\big(\Omega_A^2\Omega_B^3(X) \bigoplus \Omega_B(P_A(\Omega_B^3(X))) \oplus P_A(\Omega_A(\Omega_B^3(X)))\big) + 1 + 4\\
&\leqslant \Psi\big(\Omega_A^2\Omega_B(P_A(\Omega_B^2(X))) \oplus \Omega_A^4(Y) \oplus Q \bigoplus \Omega_B(P_A(\Omega_B^3(X)))\\
&\qquad \oplus P_A(\Omega_A(\Omega_B^3(X)))\big) + 5\\
&\leqslant \Psi\big(\Omega_A^2\Omega_B({}_BA) \oplus {}_BA \oplus \Omega_B({}_BA)\big) + 5.
\end{aligned}$$

This shows that the finitistic dimension of $B$ is bounded above by $\Psi\big(\Omega_A^2\Omega_B({}_BA) \oplus {}_BA \oplus \Omega_B({}_BA)\big) + 5$. $\square$

Recall that an $A$-module $K$ is called a $d$th syzygy module ($d \geqslant 1$) if there is an exact sequence of $A$-modules: $0 \to K \to P_{d-1} \to \cdots \to P_0 \to M \to 0$ with the $P_i$ projective. By $\Omega_A^d(A\text{-mod})$ we denote the full subcategory of $A$-mod consisting of all $d$th syzygy modules.

From the proof of 3.7 we have the following statement:

**Proposition 3.9.** *Let $B$ be a subalgebra of an algebra $A$ such that* $\text{rad}(B)$ *is a left ideal in $A$. Suppose that the inclusion map is radical-full. If add$(\Omega_A^3(A\text{-mod}))$ is of finite type, that is, there are only finitely many non-isomorphic indecomposable modules in add$(\Omega_A^3(A\text{-mod}))$, then* $\text{fin.dim}(B) < \infty$.

**Proof.** This is a consequence of the exact sequence $(**)$, Lemmas 3.2 and 2.1. $\square$

Now let us deduce some consequences of Theorem 3.7.

**Corollary 3.10.** *Let $B$ be a subalgebra of an algebra $A$ such that* $\text{rad}(B)$ *is a left ideal in $A$ and that the canonical inclusion is radical-full. If* $\text{gl.dim}(A) \leqslant 4$, *then, for any $A$-$B$-bimodule ${}_AM_B$, the triangular matrix algebra* $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ *has finite finitistic dimension.*

**Proof.** Under the assumption we know from Theorem 3.7 that fin.dim$(B)$ is finite. By a well-known result, we have

$$\text{fin.dim}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix} \leqslant \text{fin.dim}(A) + \text{fin.dim}(B) + 1 \leqslant \text{fin.dim}(B) + 5.$$

Thus the corollary follows. □

As another corollary of Theorem 3.7, we consider the pullback of two algebras of global dimension at most four.

**Corollary 3.11.** *Let $\bar{A}$, $A_1$ and $A_2$ be three algebras with $\bar{A}$ semi-simple. Given surjective homomorphisms $f_i : A_i \longrightarrow \bar{A}$ of algebras for $i = 1, 2$, we denote by $A$ the pullback of $f_1$ and $f_2$ over $\bar{A}$. If* gl.dim$(A_i) \leqslant 4$ *for $i = 1, 2$, then the finitistic dimension of $A$ is finite.*

**Proof.** By definition, $A = \{(x_1, x_2) \in A_1 \oplus A_2 \mid f_1(x_1) = f_2(x_2)\}$. The radical of $A_1 \oplus A_2$ is rad$(A_1) \oplus$ rad$(A_2)$. Since $\bar{A}$ is semi-simple, rad$(A_i)$ is mapped to zero under $f_i$. This implies that rad$(A_1) \oplus$ rad$(A_2) \subseteq$ rad$(A)$. The pullback diagram

$$\begin{array}{ccc} A & \xrightarrow{p_1} & A_1 \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f_1} \\ A_2 & \xrightarrow{f_2} & \bar{A} \end{array}$$

shows that the projection $p_i$ is surjective since each $f_i$ is surjective. Thus rad$(A)$ is mapped to rad$(A_i)$ under $p_i$. This yields that rad$(A)$ is included in rad$(A_1) \oplus$ rad$(A_2)$, and thus rad$(A)$ = rad$(A_1) \oplus$ rad$(A_2)$. Now the corollary follows from Theorem 3.7. □

The following is a typical case of 3.11.

**Corollary 3.12.** *Let $A$ be an algebra and $I, J$ two ideals in $A$ such that rad$(A) \subseteq I + J$. If $A/I$ and $A/J$ both have global dimension at most* 4, *then the finitistic dimension of $A/(I \cap J)$ is finite.*

**Proof.** We have the following pullback diagram:

$$\begin{array}{ccc} A/(I \cap J) & \xrightarrow{p_1} & A/I \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f_1} \\ A/J & \xrightarrow{f_2} & A/(I + J). \end{array}$$

Note that the algebra $A/(I + J)$ is semi-simple by the condition rad$(A) \subseteq I + J$. Thus the corollary follows immediately from 3.11. □

As a concrete example of 3.12, we have the following corollary.

**Corollary 3.13.** *Let $A_1$ and $A_2$ be two finite-dimensional $k$-algebras over a field $k$ such that $A_i/\mathrm{rad}(A_i)$ is splitting semi-simple for $i = 1, 2$. If $\mathrm{gl.dim}(A_i) \leqslant 4$ for all $i$, then the finitistic dimension of $(B \otimes_k C)/(\mathrm{rad}(B) \otimes_k \mathrm{rad}(C))$ is finite.*

**Proof.** Let $A$ denote the tensor algebra of $A_1$ and $A_2$. We define $I = \mathrm{rad}(A_1) \otimes_k A_2$ and $J = A_1 \otimes_k \mathrm{rad}(A_2)$. Then $A/I$ is isomorphic to a product of full matrix algebras over $A_2$ and $A/J$ is isomorphic to a product of full matrix algebras over $A_1$. Since $\mathrm{rad}(A) = I + J$ and $I \cap J = \mathrm{rad}(A_1) \otimes_k \mathrm{rad}(A_2)$, the corollary follows immediately from 3.12. $\square$

Concerning tensor algebras, let us also point out the following result:

**Corollary 3.14.** *Let $k$ be an algebraically closed field. Let $A_i$ be a finite-dimensional $k$-algebra with $\mathrm{gl.dim}(A_i) \leqslant 2$ for $i = 1, 2$. If $B$ is a subalgebra of the tensor algebra of $A_1 \otimes_k A_2$ such that $\mathrm{rad}(B) = \mathrm{rad}(A_1) \otimes_k A_2 + A_1 \otimes_k \mathrm{rad}(A_2)$, then $\mathrm{fin.dim}(B)$ is finite. In particular, if $A$ is an Auslander algebra, then any subalgebra $B$ of the enveloping algebra of $A$ with the same radical has finite finitistic dimension.*

Recall that an artin algebra is called an Auslander algebra if its global dimension is at most 2 and its dominant dimension is at least 2.

**Proof of Corollary 3.14.** Note that under the assumption on the field $k$ we have that $\mathrm{rad}(A_1 \otimes_k A_2) = \mathrm{rad}(A_1) \otimes_k A_2 + A_1 \otimes_k \mathrm{rad}(A_2)$ and $\mathrm{gl.dim}(A_1 \otimes_k A_2) = \mathrm{gl.dim}(A_1) + \mathrm{gl.dim}(A_2)$. Thus the corollary is a consequence of 3.7. $\square$

Concerning the finitistic dimensions of certain subalgebras of the form $eAe$, we have the following result:

**Corollary 3.15.** *Let $A$ be an artin algebra of global dimension at most 4. Suppose that $B$ is a subalgebra of $A$ such that $\mathrm{rad}(B) = \mathrm{rad}(A)$.*

(1) *If $e$ is an idempotent element in $B$ such that $eA$ is projective as a left $eAe$-module, then $\mathrm{fin.dim}(eBe) < \infty$.*
(2) *If $I$ is an idempotent ideal in $A$ such that ${}_AI$ is projective, then $\mathrm{fin.dim}\big(B/(B \cap I)\big) < \infty$.*

**Proof.** (1) Since ${}_{eAe}eA$ is projective and since $\mathrm{gl.dim}(A) \leqslant 4$, we have $\mathrm{gl.dim}(eAe) \leqslant 4$. It follows from $\mathrm{rad}(eAe) = e\,\mathrm{rad}(A)e$ that $\mathrm{rad}(eBe) = \mathrm{rad}(eAe)$. Now the corollary follows from Theorem 3.7 for the pair $eBe \subseteq eAe$.

(2) Since ${}_AI$ is projective, it follows that $\mathrm{gl.dim}(A/I) \leqslant \mathrm{gl.dim}(A) \leqslant 4$. Now we consider the pair $(B+I)/I \subseteq A/I$. By our assumption, we know that $\mathrm{rad}\big((B+I)/I\big) = (\mathrm{rad}(B) + I)/I = \mathrm{rad}(A/I)$. Statement (2) follows now from Theorem 3.7 since we have $(B + I)/I \simeq B/(B \cap I)$. $\square$

Now let us mention the following construction of a pair $B \subseteq A$ with $\mathrm{rad}(B) = \mathrm{rad}(A)$.

Suppose we have two pairs $B \subseteq A$ and $C \subseteq D$ of algebras such that $\mathrm{rad}(B) = \mathrm{rad}(A)$ and $\mathrm{rad}(C) = \mathrm{rad}(D)$. We assume further that $B = S \oplus \mathrm{rad}(B)$, $C = S \oplus \mathrm{rad}(C)$, $A = T \oplus \mathrm{rad}(A)$ and $D = T \oplus \mathrm{rad}(D)$, where $S \subseteq T$ are commutative maximal semi-simple subalgebras of $B$ and $A$, respectively. Now we may form the trivially twisted extension of $B$ and $C$, and the trivially twisted extension of $A$ and $D$ (for details we refer to [21]). Let $\mathcal{A}(B, C)$ denote the trivially twisted extension of $B$ and $C$. If $C = B^{\mathrm{op}}$, we call $\mathcal{A}(B, B^{\mathrm{op}})$ the dual extension of $B$, which is denoted by $\mathcal{A}(B)$. Then we have a pair $\mathcal{A}(B, C) \subseteq \mathcal{A}(A, D)$ with the equal radicals. In particular, we have the following result:

**Proposition 3.16.** *Let $B$ be a subalgebra of an algebra $A$ with the same radical. If* $\mathrm{gl.dim}(A) \leqslant 2$, *then the dual extension of $B$ has finite finitistic dimension.*

**Proof.** Since the global dimension of the dual extension of $A$ is the double of the global dimension of $A$, we know that $\mathrm{gl.dim}(\mathcal{A}(A)) \leqslant 4$ by our assumption on $A$. Now the result follows from Theorem 3.7. $\square$

Let us state the dual statement of Theorem 3.7.

**Proposition 3.17.** *Let $\varphi : B \longrightarrow A$ be a surjective homomorphism between two algebras $B$ and $A$. Suppose that the kernel of $\varphi$ is contained in* $\mathrm{soc}(B_B)$. *If* $\mathrm{gl.dim}(A) \leqslant 4$, *then the finitistic dimension of $B$ is finite.*

**Proof.** The proof is similar to that of 3.7. We sketch the points which are different from the ones in 3.7.

(1) We may assume that $A = B/I$ with $I = \ker(\varphi)$. So a $B$-module $Y$ is an $A$-module if and only if $IY = 0$. Since the socle of $B_B$ is $\{x \in B | x\,\mathrm{rad}(B) = 0\}$, we see that $\mathrm{rad}({}_BX)$ is also an $A$-module for all ${}_BX$. This implies that $\Omega_B(X)$ is an $A$-module for all ${}_BX$ since it is the kernel of the map $\mathrm{rad}({}_BP) \longrightarrow \mathrm{rad}({}_BX)$ induced by a projective cover ${}_BP \longrightarrow {}_BX$.
(2) The map $\varphi$ is radical-full: since $\mathrm{rad}(A) = \varphi(\mathrm{rad}(B))$, we have $\mathrm{rad}({}_BA) = \mathrm{rad}(B) \cdot A = \varphi(\mathrm{rad}(B))A = \mathrm{rad}(A)A = \mathrm{rad}(A)$. Thus Lemma 3.6 can be applied to our case.
Now the rest of the arguments in 3.7 works in our case. We omit it. $\square$

Finally, let us consider the case of $\mathrm{gl.dim}(A) \leqslant 5$. In this case we have the following result:

**Proposition 3.18.** *Let $B$ be a subalgebra of an algebra $A$ such that* $\mathrm{rad}(B) = \mathrm{rad}(A)$. *Furthermore, we assume that there are two natural numbers $t$ and $s$ with $s \geqslant 2$ such that* $\mathrm{add}\{\Omega_B^s(X) \mid \mathrm{proj.dim}({}_BX) < \infty\} \subseteq \mathrm{add}(\Omega_A \Omega_B^{t+1}(B\text{-mod}))$. *If* $\mathrm{gl.dim}(A) \leqslant 5$, *then* $\mathrm{fin.dim}(B) < \infty$.

**Proof.** It follows from the proof of 3.7 that we have the following two exact sequences for any $B$-module ${}_BX$:

$$(*) \qquad 0\longrightarrow \Omega_A^2\Omega_B^q(X)\longrightarrow \Omega_B(P_A(\Omega_B^q(X)))\oplus P_A(\Omega_A(\Omega_B^q(X)))\longrightarrow \Omega_B^{q+1}(X)\longrightarrow 0$$

with $q\geqslant 2$. Applying the syzygy operator $\Omega_A^2$, we get the following exact sequence:

$$0\longrightarrow \Omega_A^4\Omega_B^q(X)\longrightarrow \Omega_A^2\Omega_B(P_A(\Omega_B^q(X)))\oplus P'\longrightarrow \Omega_A^2(\Omega_B^{q+1}(X))\longrightarrow 0,$$

where $P'$ is a projective $A$-module. Now we apply $\Omega_A$ to get the following exact sequence:

$$0\longrightarrow \Omega_A^3\Omega_B(P_A(\Omega_B^q(X)))\longrightarrow \Omega_A^3\Omega_B^{q+1}(X)\oplus P\longrightarrow \Omega_A^4\Omega_B^q(X)\longrightarrow 0.$$

By Lemma 3.2, we may substitute $\Omega_A^4\Omega_B^q(X)$ by $\Omega_A^5(Y)$ for some $A$-module $Y$.

Since gl.dim$(A)\leqslant 5$, the module $\Omega_A^5(Y)$ is projective. Thus the last exact sequence splits. This implies that $\Omega_A^3\Omega_B^{q+1}(X)$ is a direct summand of $\Omega_A^5(Y)\oplus\Omega_A^3\Omega_B(P_A(\Omega_B^q(X)))$. Now we assume that the projective dimension of ${}_BX$ is finite. Since add$\Omega_B^{s+1}$($B$-mod) is contained in add$\Omega_B^s$($B$-mod), we have a $B$-module $X'$ such that $\Omega_B^{s+1}(X)$ is a direct summand of $\Omega_A\Omega_B^{t+1}(X')$ by assumption, and therefore $\Omega_A^2\Omega_B^{s+1}(X)$ is a direct summand of $\Omega_A^3\Omega_B^{t+1}(X')$. Hence we have proved that $\Omega_A^2\Omega_B^{s+1}(X)$ is a direct summand of $\Omega_A^5(Y)\oplus\Omega_A^3\Omega_B(P_A(\Omega_B^t(X')))$.

Now it follows from $(*)$ by putting $q=s+1$ that the following estimation can be made:

$$\begin{aligned}
\text{proj.dim}({}_BX) &\leqslant \text{proj.dim}(\Omega_B^{s+2}(X))+s+2\\
&\leqslant \Psi\big(\Omega_A^2\Omega_B^{s+1}(X)\oplus\Omega_B(P_A(\Omega_B^{s+1}(X)))\\
&\oplus\ P_A(\Omega_A\Omega_B^{s+1}(X))\big)+1+s+2\\
&\leqslant \Psi\big(\Omega_A^5(Y)\oplus\Omega_A^3\Omega_B(P_A(\Omega_B^t(X')))\oplus\Omega_B(P_A(\Omega_B^{s+1}(X)))\\
&\qquad\oplus P_A(\Omega_A\Omega_B^{s+1}(X))\big)+s+3\\
&\leqslant \Psi\big({}_BA\oplus\Omega_A^3\Omega_B({}_BA)\oplus\Omega_B({}_BA)+s+3.
\end{aligned}$$

This shows that the finitistic dimension of $B$ is bounded above by $\leqslant\Psi\big({}_BA\oplus\Omega_A^3\Omega_B({}_BA)\oplus\Omega_B({}_BA)\big)+s+3$. $\square$

Finally, we remark that for a pair $B\subseteq A$ of algebras the “radical-full” condition in Theorem 3.7 does not imply that rad$(B)$ is a left ideal in $A$. This can be seen by considering the tensor algebras of the pair $k[x]/(x^2)\subseteq k(\circ\longrightarrow\circ)$. We leave the verification to the reader.

## 4. Finitistic dimensions and representation dimensions

It is known by a result in [12] that the representation dimension of an artin algebra being bounded above by 3 implies the finiteness of the finitistic dimension. In this section, we shall point out that for a given pair $B \subseteq A$ of algebras with $\mathrm{rad}(B)$ a left ideal in $A$ one can get the finite finitistic dimension for $B$ by bounding the representation dimension of $A$.

Recall that given an algebra $A$, the *representation dimension* of $A$ is defined by Auslander [2] as follows:

$$\begin{aligned}\mathrm{rep.dim}(A) = \inf\{\mathrm{gl.dim}(\Lambda) \mid \Lambda \text{ is an artin algebra with } \mathrm{dom.dim}(\Lambda) \geqslant 2 \text{ and } \mathrm{End}({}_\Lambda T)\\ \text{is Morita equivalent to } A, \text{where } T \text{ is the injective envelope of } \Lambda\}.\end{aligned}$$

Auslander also proved in [2] that the above definition is equivalent to the following definition:

$$\mathrm{rep.dim}(A) = \inf\{\mathrm{gl.dim}(\mathrm{End}_A(M)) \mid M \text{ is a generator–cogenerator for } A\text{-mod}\},$$

where $M$ is called a generator if every indecomposable projective $A$-module is isomorphic to a direct summand of $M$; and a cogenerator if every indecomposable injective $A$-module is isomorphic to a direct summand of $M$.

In [2] it was shown that $A$ is representation-finite if and only if $\mathrm{rep.dim}(A) \leqslant 2$. For some new results on the representation dimension we refer to [19,22,14,8,6].

One can also define the so-called weak representation dimension of $A$, denoted by $\mathrm{wrep.dim}(A)$, as follows:

$$\mathrm{wrep.dim}(A) = \inf\{\mathrm{gl.dim}(\mathrm{End}_A(M)) \mid M \text{ is a generator for } A\text{-mod}\}.$$

Clearly, for any algebra $A$, $\mathrm{wrep.dim}(A) \leqslant \mathrm{rep.dim}(A)$. The following lemma is well known in [2].

**Lemma 4.1.** *Let $A$ be an artin algebra and let $M$ be a generator–cogenerator for* $A$-mod. *Suppose $m$ is a non-negative integer. Then* $\mathrm{gl.dim}(\mathrm{End}({}_A M)) \leqslant m$ *if and only if for each $A$-module $Y$ there is an exact sequence*

$$0 \longrightarrow M_{m-2} \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow Y \longrightarrow 0,$$

*with* $M_j \in \mathrm{add}({}_A M)$ *for* $j = 0, \ldots, m-2$, *such that*

$$0 \longrightarrow \mathrm{Hom}_A(X, M_{m-2}) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(X, M_1) \longrightarrow \mathrm{Hom}_A(X, M_0) \longrightarrow \mathrm{Hom}_A(X, Y) \longrightarrow 0$$

*is exact for all* $X \in \mathrm{add}({}_A M)$.

We have the following result, which generalizes a result in [12] which says that an algebra has finite finitistic dimension if its representation dimension is at most 3.

**Theorem 4.2.** *Let $B$ be a subalgebra of an artin algebra $A$ such that $rad(B)$ is a left ideal in $A$. If* $\text{rep.dim}(A) \leqslant 3$, *then* $\text{fin.dim}(B) < \infty$.

**Proof.** Suppose that $A$ has the representation dimension at most 3. Then there is an $A$-module $M$ which is a generator–cogenerator for $A$-mod such that $\text{gl.dim}(\text{End}(_AM)) = \text{rep.dim}(A) \leqslant 3$. In particular, Lemma 4.1 holds true for this module $M$. Now we take an arbitrary $B$-module $X$ with finite projective dimension. Then $\Omega_B^2(X)$ is an $A$-module thanks to Lemma 3.2. By Lemma 4.1, there is an exact sequence of $A$-modules

$$0 \longrightarrow M_1 \longrightarrow M_0 \longrightarrow \Omega_B^2(X) \longrightarrow 0,$$

with $M_1, M_0$ in $\text{add}(_AM)$. Now we have the following estimation by Lemma 2.1:

$$\begin{aligned} \text{proj.dim}(_BX) &\leqslant \text{proj.dim}(\Omega_B^2(X)) + 2 \\ &= \Psi(\Omega_B^2(X)) + 2 \\ &\leqslant \Psi(_BM_1 \oplus {}_BM_0) + 1 + 2 \\ &\leqslant \Psi(_BM) + 3. \end{aligned}$$

Since $M$ is a fixed $A$-module, the restriction $_BM$ is a fixed $B$-module, and thus the projective dimension of $_BX$ is bounded above by $\Psi(_BM) + 3$. This completes the proof. $\square$

Similarly, we have

**Proposition 4.3.** *Let $\varphi : B \longrightarrow A$ be a surjective homomorphism between two algebras $B$ and $A$. Suppose that the kernel of $\varphi$ is contained in* $\text{soc}(B_B)$. *If* $\text{rep.dim}(A) \leqslant 3$, *then the finitistic dimension of $B$ is finite.*

**Proof.** Let $A = B/I$ with $I = \ker(\varphi)$. Given an $B$-module $X$, the first syzygy $\Omega_B(X)$ of $X$ is an $B/I$-module since it is contained in the radical of a projective $B$-module and $\text{soc}(B_B)\,\text{rad}(B) = 0$. If $\text{rep.dim}(B/I) \leqslant 2$, then the proposition is true by a result in [20]. Thus we may assume that $\text{rep.dim}(B/I) = 3$. By definition of the representation dimension, there is a $B/I$-module $M$ such that $\text{rep.dim}(B/I) = \text{gl.dim}(\text{End}(_{B/I}M)) = 3$. Thus, by Lemma 4.1, there is an exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_0 \longrightarrow \Omega_B(X) \longrightarrow 0$$

in $B/I$-mod with $M_1$, $M_0 \in \mathrm{add}({}_{B/I}M)$. By Lemma 2.1, we have

$$\begin{aligned} \mathrm{proj.dim}({}_BX) &\leqslant \mathrm{proj.dim}(\Omega_B(X)) + 1 \\ &= \Psi(\Omega_B(X)) + 1 \\ &\leqslant \Psi({}_BM_1 \oplus {}_BM_0) + 1 + 1 \\ &\leqslant \Psi({}_BM) + 2. \end{aligned}$$

Thus $\mathrm{fin.dim}(B) < \infty$. $\square$

Let us mention a few consequences of Theorem 4.2. Recall that an artin algebra is said to be *stably hereditary* in [22] if (1) each indecomposable submodule of an indecomposable projective module is either projective or simple, and (2) each indecomposable factor module of an indecomposable injective module is either injective or simple. Here we require that an indecomposable projective–injective module satisfies either (1) or (2), but not necessarily both the conditions.

Note that stably hereditary algebra is a proper generalization of the stably equivalent to hereditary algebra, namely, algebras which are stably equivalent to hereditary algebras are stably hereditary, but the converse is not true, in general.

**Corollary 4.4.** *If $B$ is a subalgebra of one of the following algebras $A$ such that $rad(B)$ is a left ideal in $A$, then the finitistic dimension of $B$ is finite.*

(1) *$A$ is a stably hereditary algebra*;
(2) *$A$ is a special biserial algebra*;
(3) *$A$ is the trivial extension of an iterated tilted algebra*;
(4) *$A$ is an algebra such that $Hom_A(-, A)$ or $Hom_A(D(A), -)$ has finite length.*

**Proof.** All algebras displayed in the corollary have representation dimension at most 3. This was proved for (1) in [22], for (2) in [8], and for (3) and (4) in [6]. Thus the corollary follows immediately from Theorem 4.2. $\square$

**Remark.** Since stable equivalences of Morita-type preserve representation dimension, we may replace the algebra $A$ in 4.4 by any algebra $C$ such that there is a stable equivalence of Morita type between $C$ and $A$. For more information on stable equivalences of Morita type for general finite-dimensional algebras we refer to a recent paper [15] and the references therein.

**Corollary 4.5.** *Let $\bar{A}$, $A_1$ and $A_2$ be three algebras with $\bar{A}$ semi-simple. Given surjective homomorphisms $f_i : A_i \longrightarrow \bar{A}$ of algebras for $i = 1, 2$, we denote by $A$ the pullback of $f_1$ and $f_2$ over $\bar{A}$. If* $\mathrm{rep.dim}(A_i) \leqslant 3$ *for $i = 1, 2$, then the finitistic dimension of $A$ is finite.*

As a special case of 4.3, we have the following result:

**Corollary 4.6.** (1) *Let $A$ be an algebra with an ideal $I$ such that* $I\,\mathrm{rad}(A) = 0$. *If* $\mathrm{rep.dim}(A/I) \leqslant 3$, *then the finitistic dimension of $A$ is finite.*

(2) *Let $A$ be an algebra and $N$ its Jacobson radical with the nilpotency index $n$. If* $\mathrm{rep.dim}(A/N^{n-1}) \leqslant 3$, *then the finitistic dimension of $A$ is finite.*

**Proof.** (1) The condition $I\,\mathrm{rad}(A) = 0$ implies that $I \subseteq \mathrm{soc}(A_A)$. Thus the corollary follows from 4.3.

(2) follows from (1). $\square$

In the following we shall provide a class of algebras with representation dimension at most 3.

**Theorem 4.7.** *Let $B$ be a subalgebra of an Artin algera $A$ with the same identity such that* $\mathrm{rad}(B)$ *is an ideal in $A$. If $A$ is a stably hereditary algebra, then* $\mathrm{rep.dim}(B) \leqslant 3$.

Before we start the proof of Theorem 4.7, we first show the following lemma:

**Lemma 4.8.** *The following two statements are equivalent for an Artin algebra*:

(1) *Every indecomposable submodule of an indecomposable projective module is either projective or simple.*
(2) *Every indecomposable submodule of a projective module is either projective or simple.*
*Dually, the following statements are equivalent*:
(1′) *Every indecomposable factor module of an indecomposable injective module is either injective or simple.*
(2′) *Every indecomposable factor module of an injective module is either injective or simple.*

**Proof.** We prove that (1) implies (2). Suppose $X$ is an indecomposable submodule of a projective module $P$. We decompose $P$ into a direct sum of indecomposable modules, say $P = \oplus_{i=1}^{n} P_i$ with $P_i$ indecomposable, and have a homomorphism $\alpha_i : X \longrightarrow P_i$ for each $i$. We may assume that all $\alpha_i \neq 0$. Since the image of $\alpha_i$ is a submodule of the indecomposable projective module $P_i$, we know that it must be a direct sum of a projective module $P'$ and a semi-simple module by (1). If $P'$ is not zero, then there is a surjective map from $X$ to $P'$, thus $X$ is projective. So we assume that the image of $\alpha_i$ is semi-simple for all $i$. In this case, $X$ is a submodule of the semi-simple module $\mathrm{Im}(\alpha_1) \oplus \mathrm{Im}(\alpha_2) \oplus \cdots \oplus \mathrm{Im}(\alpha_n)$. Thus $X$ must be simple. Altogether, we have proved (2). $\square$

**Proof of Theorem 4.7.** We cite the following properties from [20] where it is assumed that $\mathrm{rad}(B)$ is a left ideal in $A$:

(1) The restriction functor $F : A\text{-mod} \longrightarrow B\text{-mod}$ is an exact faithful functor, and has a right adjoint $G = \mathrm{Hom}_B({}_BA_A, -) : B\text{-mod} \longrightarrow A\text{-mod}$ and a left adjoint $E =: A \otimes_B - : B\text{-mod} \longrightarrow A\text{-mod}$. In particular, $E$ preserves projective modules and $G$ preserves injective modules.

(2) For any $B$-module $Y$ there is a $B$-homomorphism $\mu_Y : GY \to Y$ such that the induced map $\mathrm{Hom}_A(X, GY) \longrightarrow \mathrm{Hom}_B(X, Y)$ is an isomorphism for all $A$-module $X$.
(3) The kernel and the cokernel of $\mu_Y$ are semi-simple $B$-modules.
(4) Each simple $A$-module is a semi-simple $B$-module via restriction. (In general, $F$ does not preserve simples.)
(5) $\mathrm{add}(B/\mathrm{rad}(B)) = \mathrm{add}(F(A/\mathrm{rad}(A)))$.

Let $V = A \oplus D(A) \oplus A/\mathrm{rad}(A)$. Then we know that $\mathrm{gl.dim}(\mathrm{End}(_A V)) \leqslant 3$ by the proof of [22, Theorem 3.5]. This implies that for any $A$-module $X$, there is an exact sequence

$$0 \longrightarrow V_1 \longrightarrow V_0 \longrightarrow X \longrightarrow 0$$

with $V_i \in \mathrm{add}(V)$ such that $0 \longrightarrow (V, V_1) \longrightarrow (V, V_0) \longrightarrow (V, X) \longrightarrow 0$ is exact. (In the following we shall write $(V, X)$ for the set of all homomorphisms from the module $V$ to the module $X$ if there is no confusion.)

Now we define $M = B \oplus D(B) \oplus V$. In the following we show that for each $B$-module $Y$, there is an exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_0 \longrightarrow Y \longrightarrow 0$$

with $M_i \in \mathrm{add}(M)$ such that $0 \longrightarrow (M', M_1) \longrightarrow (M', M_0) \longrightarrow (M', Y) \longrightarrow 0$ is exact for any $M'$ in $\mathrm{add}(M)$.

If $Y \in \mathrm{add}(M)$, then we define $M_0 = Y$, and the identity map $M_0 \longrightarrow Y$ gives a desired exact sequence.

Now let $Y$ be an indecomposable $B$-module not in $\mathrm{add}(M)$. We denote by $C$ the cokernel of the map $\mu := \mu_Y$, which is a semi-simple $B$-module by (3). The canonical surjective map from $Y$ to $C$ will be denoted by $v$, and the canonical map from the kernel of $\mu$ into $Y$ is denoted by $i'$. Let $\pi' : P \longrightarrow C$ be a projective cover of the $B$-module $C$ and $\Omega_B(C)$ the first syzygy of $C$. Then there is a homomorphism $\pi : P \longrightarrow Y$ such that $\pi v = \pi'$. Now we may form the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
0 \arrow[r] & \ker(\mu) \arrow[r] \arrow[d, "i'"'] & K' \arrow[r] \arrow[d] & \Omega_B(C) \arrow[r] \arrow[d, "i"] & 0 \\
0 \arrow[r] & FGY \arrow[r, "{(1,0)}"] \arrow[d] & FGY \oplus P \arrow[r, "{(0,1)^T}"] \arrow[d, "{(\mu,\pi)^T}"] & P \arrow[r] \arrow[d, "\pi'"] & 0 \\
0 \arrow[r] & \mathrm{Im}(\mu) \arrow[r] & Y \arrow[r, "v"] & C \arrow[r] & 0.
\end{tikzcd}$$

Since $\mathrm{rad}(B)$ is a left ideal in $A$ and $\Omega_B(C) = \mathrm{rad}(P)$, the module $\Omega_B(C)$ is an $A$-module. By (2), the map $i\pi$ factors through $\mu$, that is, there exists a homomorphism $h$ :

$\Omega_B(C)\longrightarrow FGY$ such that $i\pi = h\mu$. Clearly, the map $\Omega_B(C)\longrightarrow K' = \ker(\mu,\pi)^T$ defined by $x \mapsto (-(x)h,(x)i)$ makes the exact sequence $0\longrightarrow\ker(\mu)\longrightarrow K'\longrightarrow\Omega_B(C)\longrightarrow 0$ splitting, thus $K' = \ker(\mu)\oplus\Omega_B(C)$.

Since $GY$ is an $A$-module and $\mathrm{gl.dim}(\mathrm{End}(_AV))\leqslant 3$, there is an exact sequence

$$0\longrightarrow V_1 \xrightarrow{\beta} V_0 \xrightarrow{\alpha} GY\longrightarrow 0$$

of $A$-modules, with $V_i\in\mathrm{add}(V)$, such that $0\longrightarrow(V',V_1)\longrightarrow(V',V_0)\longrightarrow(V',X)\longrightarrow 0$ is exact for all $V'$ in $\mathrm{add}(V)$.

Since we have an adjoint pair $(F,G)$ of functors, this provides us an adjunction isomorphism

$$\xi := \xi_{Z,W} : \mathrm{Hom}_B(FZ,W)\longrightarrow\mathrm{Hom}_A(Z,GW)$$

for $Z$ in $A$-mod and $W$ in $B$-mod, and two natural transformations: the unit $\varepsilon_Z = \xi_{FZ}$ : $Z\longrightarrow GFZ$; and the counit: $\delta_Y : FGY\longrightarrow Y$. Moreover, the composition of the homomorphisms: $GY \xrightarrow{\varepsilon_{GY}} GFGY \xrightarrow{G\delta_Y} GY$, is the identity map of $GY$ (see [11, Proposition 7.2, p. 65]). Using this fact, we shall show that $GFGY$ is isomorphic to $GY\oplus Y'$, with $Y'$ a $A$-module and $FY'$ is semi-simple. In fact, the map $\delta_Y$ is equal to $\mu_Y$, hence we have an exact sequence of $B$-modules:

$$0\longrightarrow\ker(\mu) = \mathrm{Hom}_B(_BA/B,Y)\longrightarrow FGY \xrightarrow{\delta_Y} Y.$$

From this sequence we get the following exact sequence of $A$-modules:

$$0\longrightarrow\mathrm{Hom}_B(_BA,\ker(\mu))\longrightarrow\mathrm{Hom}_B(_BA,FGY)\longrightarrow\mathrm{Hom}_B(_BA,Y).$$

This shows that the kernel of $G\delta_Y$ is isomorphic to $\mathrm{Hom}_B(_BA,\ker(\mu))$. We claim that this is a semi-simple $B$-module. Let $x$ be in $\mathrm{rad}(B)$ and $f\in\mathrm{Hom}_B(_BA,\ker(\mu))$. Then for any element $a\in A$ we have $(a)[x\cdot f] = (ax)f = [(ax)1]f = (ax)[(1)f]$ since $f$ is a $B$-module homomorphism and $ax\in\mathrm{rad}(B)$. Note that $\ker(\mu)$ is a semi-simple $B$-module, this implies that $(ax)[(1)f] = 0$ and $x\cdot f = 0$, and therefore $\mathrm{Hom}_B(_BA_B,\ker(\mu))$ is a semi-simple $B$-module. Since the composition of the homomorphisms: $GY \xrightarrow{\varepsilon_{GY}} GFGY \xrightarrow{G\delta_Y} GY$, is the identity map of $GY$, we see that $GFGY = GY\oplus Y'$ with $Y' = \mathrm{Hom}_B(_BA,\ker(\mu))$.

To obtain a desired exact sequence for $Y$, we consider the following commutative exact diagram:

$$
\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & & \\
 & V_1 \arrow[d] \arrow[rr, equal] & & V_1 \arrow[d, "{(\beta,0,0)}"'] & & \\
0 \arrow[r] & K \arrow[rr, "\gamma"] \arrow[d] & & V_0 \oplus Y' \oplus P \arrow[r, "{\begin{pmatrix} \alpha\mu \\ 0 \\ \pi \end{pmatrix}}"] \arrow[d, "{\begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}"] & Y \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ker(\mu) \oplus Y' \oplus \Omega_B(C) \arrow[rr, "{\begin{pmatrix} i' & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix}}"'] \arrow[d] & & FGY \oplus Y' \oplus P \arrow[r, "{\begin{pmatrix} \mu \\ 0 \\ \pi \end{pmatrix}}"'] \arrow[d] & Y \arrow[r] & 0 \\
 & 0 & & 0 & &
\end{tikzcd}
$$

In the following we show that $K \simeq \ker(\mu) \oplus Y' \oplus \Omega_B(C) \oplus V_1$. For this we only need to show that the map $i' : \ker(\mu) \longrightarrow FGY$ factors through the homomorphism $\alpha' := \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} : V_0 \oplus Y' \longrightarrow FGY \oplus Y'$, considered as a homomorphism of $B$-modules.

Since $\ker(\mu)$ is a semi-simple $B$-module by (3), we know that there is a semi-simple $A$-module $Z$ such that $FZ = \ker(\mu) \oplus {}_B Z'$. Let $f$ be the following composition of maps:

$$FZ \longrightarrow \ker(\mu) \xrightarrow{i'} FGY.$$

Then we have a homomorphism $g := \xi_{Z,FGY}(f) : {}_A Z \longrightarrow GFGY = GY \oplus Y'$. If we write $g = (g_1, g_2)$ with $g_1 : Z \longrightarrow GY$ and $g_2 : Z \longrightarrow Y'$, then $g_1$ factors through $\alpha$ since $Z \in \mathrm{add}(V)$, that is, there is a homomorphism $f_1' : Z \longrightarrow V_0$ of $A$-modules such that $g_1 = f_1'\alpha$, Let $f' = (f_1', g_2) : Z \longrightarrow V_0 \oplus Y'$. Then $g = f'\alpha'$. It follows that $Fg = (Ff')(F\alpha')$ and $f = \xi^{-1}_{Z,FGY}(g) = (Fg)\delta_{FGY} = (Ff')(F\alpha')\delta_{FGY}$. This means that $f$ factors through the restriction of the map $\alpha'$. If $j : \ker(\mu) \longrightarrow FZ$ is the canonical inclusion of $\ker(\mu)$ into $FZ$, then $i' = jf$ and factors through $\alpha'$. This is what we wanted.

Thus we have shown that $K \simeq V_1 \oplus \ker(\mu) \oplus Y' \oplus \Omega_B(C)$. Moreover, we shall show that the $B$-module $K$ lies in $\mathrm{add}(M)$.

Clearly, the kernel of $\mu$ and the module $Y'$, as a semi-simple $B$-module, lie in $\mathrm{add}(M)$ by (3). To prove that $K$ lies in $\mathrm{add}(M)$, it is sufficient to prove that $\Omega_B(C)$ lies in $\mathrm{add}(M)$. To see this, we note that $\Omega_B(C) = \mathrm{rad}(P)$ since $C$ is a semi-simple $B$-module. Assume that $P = \oplus Be_i$, with $e_i$ primitive idempotent elements in $B$ ( but not necessarily primitive in $A$). Then $\mathrm{rad}(P) = \oplus\,\mathrm{rad}(B)e_i$. By assumption, $\mathrm{rad}(B)$ is an ideal in $A$. So $\mathrm{rad}(B)e_i$ is a submodule of $Ae_i$. Since $A$ is stably hereditary, we know that $\mathrm{rad}(B)e_i$ is a direct sum of projective $A$-module and a semi-simple $A$-module by 4.8, this implies that $\mathrm{rad}(B)e_i$ lies in $\mathrm{add}(V)$. Thus $\mathrm{rad}(P)$ lies in $\mathrm{add}({}_B M)$.

Now we define $M_0 = V_0 \oplus P$ and $M_1 = V_1 \oplus \ker(\mu) \oplus \Omega_B(C)$ and shall prove that the exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_0 \xrightarrow{\begin{pmatrix} \alpha\mu \\ \pi \end{pmatrix}} Y \longrightarrow 0$$

induced from the exact sequence $0 \longrightarrow K \xrightarrow{\gamma} V_0 \oplus Y' \oplus \Omega_B(C) \longrightarrow Y \longrightarrow 0$ has the property that for any $M'$ in $\mathrm{add}(M)$, the induced sequence $0 \longrightarrow (M', M_1) \longrightarrow (M', M_0) \longrightarrow (M', Y) \longrightarrow 0$ is exact. To this end, the following three cases are considered:

(a) If $M'$ is a projective $B$-module, then we are done.
(b) If $M'$ is a restriction of an $A$-module, then any homomorphism from ${}_BM'$ to ${}_BY$ factors through $\mu_Y$ by (2), thus factors through $M_0 \longrightarrow Y$. So the map $\mathrm{Hom}_B(M', M_0) \longrightarrow \mathrm{Hom}_B(M', Y)$ is surjective.
(c) We assume that $M' = D(eB)$ is an indecomposable injective $B$-module which does not lie in $\mathrm{add}(B \oplus V)$, where $e$ is a primitive idempotent in $B$. Note that $M'/\mathrm{soc}(M') \simeq D(e\,\mathrm{rad}(B))$ is an $A$-module. Clearly, $D(e\,\mathrm{rad}(B))$ is a factor $A$-module of the injective $A$-module $D(eA)$. Since $A$ is stably hereditary, the $A$-module $D(e\,\mathrm{rad}(B))$ is a direct sum of an injective $A$-module and a semi-simple $A$-module by 4.8, and lies in $\mathrm{add}(V)$. Thus any homomorphism from $D(e\,\mathrm{rad}(B))$ to $Y$ factors through $(\alpha\mu, \pi)^T$, as was shown in (b). Note that each homomorphism from $M'$ to $Y$ is not injective. Otherwise, we would have $Y \simeq M'$ and $Y \in \mathrm{add}(M)$. Hence any homomorphism from $M'$ to $Y$ factors through $M'/\mathrm{soc}(M')$ and therefore factors through $(\alpha\mu, \pi)^T$. This finishes the proof of Theorem 4.7. $\square$

Finally, let us remark that the algebra $B$ in 4.7 is not stably hereditary in general, even though we assume a strong condition that $\mathrm{rad}(B) = \mathrm{rad}(A)$. For example, let $A$ be the path algebra given by the quiver $\circ \longleftarrow \circ \longleftarrow \circ$. We may take $B$ to be the subalgebra of $A$ defined by the following quiver with one relation:

$$\begin{tikzcd} \underset{1}{\circ} \arrow[loop left, "x"] & \underset{2}{\circ} \arrow[l, "\alpha"'] \end{tikzcd} \qquad x^2 = 0.$$

Clearly, the injective $B$-module corresponding to the vertex 1 has an indecomposable factor module of length 2 which is neither simple nor injective. Thus $B$ is not a stably hereditary algebra, however, we do have $\mathrm{rad}(B) = \mathrm{rad}(A)$.

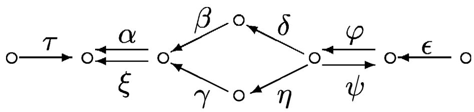
## 5. Examples and problems

We first give some simple examples to show how our methods in this paper can be applied, and then we mention some questions, which are motivated from the results in this paper.

The following is a recipe for getting a pair of algebras $B$ and $A$ satisfying the assumption that $\mathrm{rad}(B) = \mathrm{rad}(A)$. This method might be called "gluing of idempotents".

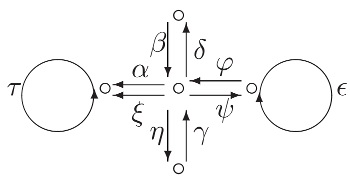
We start with a finite-dimensional basic algebra $A$ with $1_A = \sum_{j=1}^{n} e_i$, where $e_i e_j = \delta_{ij} e_i$ and $\delta_{ij}$ is the Kronecker symbol. Let $I_1, \ldots, I_s$ be a partition of the set $\{1, \ldots, n\}$ and $f_j = \sum_{i \in I_j} e_i$ for $j = 1, \ldots, s$. We define $B$ to be the subalgebra of $A$ generated by $f_1, \ldots, f_s$ and $\mathrm{rad}(A)$. Then $B \subseteq A$ and $\mathrm{rad}(B) = \mathrm{rad}(A)$. From the quiver point of view, this means that we glue idempotents $e_i, i \in I_j$, of $A$ into a new primitive idempotent $f_j$ in $B$. In this way the algebra $B$ become much more complicated.

**Example 1.** Let $A$ be an algebra (over a field) given by the following quiver




with relations: $\delta\beta = \eta\gamma$, $\varphi\delta = \varphi\eta = \varphi\psi = 0$.

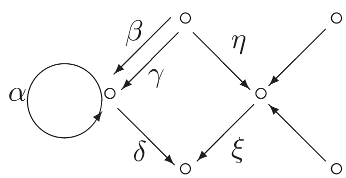
Let us consider the subalgebra $B$ of $A$, which is given by the following quiver with relations:




$$\begin{aligned}&\delta\beta = \eta\gamma, \varphi\delta = \varphi\eta = \varphi\psi = \epsilon^2 = \psi\epsilon = \varphi\alpha = 0,\\ &\varphi\xi = \beta\eta = \beta\psi = \beta\delta = \gamma\delta = \gamma\psi = \gamma\eta = 0,\\ &\tau^2 = \alpha\tau = \xi\tau = 0.\end{aligned}$$

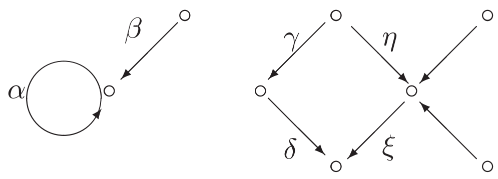
Since algebra $A$ has global dimension equal to 3 and $\mathrm{rad}(B) = \mathrm{rad}(A)$, we have $\mathrm{fin.dim}(B) < \infty$ by Theorem 3.7.

**Example 2.** Let $A$ be an algebra (over a field) given by the following quiver with relations:




$$\eta\xi = \gamma\delta,\ \alpha^4 = \beta\alpha = \gamma\alpha = \beta\delta = \alpha\delta = 0.$$

It is easy to see that $A$ is a subalgebra of the following algebra given by quiver and relations:



$$\eta\xi = \gamma\delta,\ \alpha^4 = \beta\alpha = 0.$$

This algebra has representation dimension 3 by a result in [6, Corollary 2.4]. Thus the algebra $A$ has finite finitistic dimension by 4.2.

Note that, in the two examples above, the algebras which we concerned are neither monomial, nor radical-cube zero, or representation-finite.

**Example 3.** The following example of Igusa–Todorov–Smalø shows that our result, Theorem 3.7, can be used to adjudge the finiteness of finitistic dimensions of certain algebras $B$ even their $\mathcal{P}^{\infty}(B)$ are not contravariantly finite.

Let $B$ be the algebra given by Igusa–Todorov–Smalø, it has the following quiver

$$\begin{tikzcd} 1\circ \arrow[r, "\gamma" description] & \circ 2 \arrow[l, bend right, "\beta"'] \arrow[l, bend left, "\alpha"] \end{tikzcd}$$

with relations $\alpha\gamma = \gamma\alpha = \gamma\beta = 0$. This algebra can be embedded in the following algebra $A$ given by quiver and relations:

$$\begin{tikzcd} \circ & \circ \arrow[l, "\gamma"'] & \circ \arrow[l, shift right, "\beta"'] \arrow[l, shift left, "\alpha"] \end{tikzcd} \qquad \alpha\gamma = 0.$$

Clearly, $\mathrm{rad}(A) = \mathrm{rad}(B)$, and the algebra $A$ has global dimension 2 while algebra $B$ has infinite global dimension.

It follows from Theorem 3.7 that fin.dim$(B)$ is finite. Of course, one may exploit this example to get a more complicated example of a pair $B \subseteq A$ such that $\mathcal{P}^{\infty}(B)$ is not contravariantly finite in $B$-mod while gl.dim$(A) \leqslant 4$. However, we would like to choose this simple example to explain our idea.

Now let us end this section by asking the following questions:

**Question 1.** Let $B$ be a subalgebra of an artin algebra $A$ such that rad$(B)$ is a left ideal in $A$ and that the inclusion map of $B$ into $A$ is radical-full.

(1) Is fin.dim$(B) < \infty$ if gl.dim$(A) \leqslant 5$? (or, more generally, if gl.dim$(A) < \infty$?)
(2) Is fin.dim$(B) < \infty$ if add$(\Omega_A^n(A\text{-mod}))$ is of finite type for a fixed number $n \geqslant 4$?

Note that Theorem 3.7 and (1) would follow from (2) if the answer to (2) is affirmative.

**Question 2.** Let $C \subseteq B \subseteq A$ be a chain of subalgebras of a given artin algebra $A$ such that $\mathrm{rad}(C)$ is a left ideal in $B$ and that $\mathrm{rad}(B)$ is a left ideal in $A$. Is the finitistic dimension conjecture true for $C$ if $\mathrm{rep.dim}(A) \leqslant 3$?

**Question 3.** Let $B$ be a subalgebra of an artin algebra $A$ such that $\mathrm{rad}(B)$ is a left ideal in $A$. Is the finitistic dimension conjecture true for $B$ if $\mathrm{rep.dim}(A) \leqslant 4$?

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Finally, I want to point out that there has been a serious misleading nonsense spreading in the algebra community in China, which said that the finitistic dimension conjecture would be solved by a Japanese in a paper of three pages. This is undoubtedly an evil lie. So far as I know, at the moment when I write this comment, the finitistic dimension conjecture is still open.

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