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An Easy Proof of the Askey–Wilson Integral and Applications of the Method

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1. INTRODUCTION

The first explicit statement of the Askey–Wilson integral,

$$\int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta})_\infty \, d\theta}{(\lambda_1 e^{i\theta}, \lambda_1 e^{-i\theta}, \lambda_2 e^{i\theta}, \lambda_2 e^{-i\theta}, \lambda_3 e^{i\theta}, \lambda_3 e^{-i\theta}, \lambda_4 e^{i\theta}, \lambda_4 e^{-i\theta})_\infty} = \frac{2\pi}{(q)_\infty} \frac{(\lambda_1\lambda_2\lambda_3\lambda_4)_\infty}{(\lambda_1\lambda_2, \lambda_1\lambda_3, \lambda_1\lambda_4, \lambda_2\lambda_3, \lambda_2\lambda_4, \lambda_3\lambda_4)_\infty}, \tag{1}$$

is in Askey and Wilson [5]. However, in [8] the author showed that Rogers [14] gave a formula equivalent to (1) modulo the orthogonality of the continuous q-Hermite polynomials in 1893. [8] also contains extensions of Rogers' ideas and gives several new generalizations and variants of (1). Equation (1) uses the standard notation

$$(a_1, a_2, \ldots, a_k)_\infty = \prod_{i=1}^{k} (a_i)_\infty,$$

where

$$(x)_\infty = \prod_{n\geq 0} (1 - xq^{n-1}).$$

$(x)_\infty$ is called an infinite q-factorial.

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Throughout this paper q is taken to be a fixed complex number satisfying $|q| < 1$. We will also find occasion to use finite q-factorials:

$$(x)_l = \prod_{n=1}^{l} (1 - xq^{n-1}).$$

In this paper we simplify the ideas of [8] and in so doing give an easy proof of (1). Our method has wide applicability and gives new results as well as all of our previous Askey–Wilson type integrals.

2. ORTHOGONAL POLYNOMIALS

Let a function $W(x)$ be given which is non-negative and integrable on an interval (a, b) and satisfies

$$\int_a^b W(x)\, dx > 0.$$

If there exists a sequence of polynomials $\{P_n\}_{n \geq 0}$, P_n of degree n, such that

$$\int_a^b P_m(x) P_n(x) W(x)\, dx = \mu_m \delta_{m,n} \qquad m, n \geq 0, \tag{2}$$

then the polynomials $\{P_n\}_{n \geq 0}$, are called an orthogonal system with respect to the weight function $W(x)$ on (a, b).

One of the principle uses of an orthogonal system $\{P_n\}_{n \geq 0}$ is that it enables one to compute expansions of suitable functions in series of these polynomials. Let $f(x)$ and $g(x)$ be such functions with the following expansions.

$$\begin{aligned} f(x) &= \sum_{m \geq 0} a_m P_m(x), \\ g(x) &= \sum_{n \geq 0} b_n P_n(x). \end{aligned} \tag{3}$$

Of course to find these expansions, the sequences of constants $\{a_m\}_{m \geq 0}$ and $\{b_n\}_{n \geq 0}$ need to be determined. To this end, note that (3) implies that

$$f(x) P_n(x) W(x) = \sum_{m \geq 0} a_m P_m(x) P_n(x) W(x).$$

$$g(x) P_m(x) W(x) = \sum_{n \geq 0} b_n P_n(x) P_m(x) W(x).$$

Integrating on (a, b) and using (2) then gives

$$a_m = \frac{1}{\mu_m} \int_a^b f(x) P_m(x) W(x)\, dx,$$
$$b_n = \frac{1}{\mu_n} \int_a^b g(x) P_n(x) W(x)\, dx. \tag{4}$$

Recall the derivation of Parseval theorem's for a bilinear inner product $\langle \cdot , \cdot \rangle$ with associated orthogonal functions $\{P_m\}_{m \geq 0}$, $\langle P_m, P_n \rangle = \mu_m \delta_{m,n}$:

$$\begin{aligned} \langle f, g \rangle &= \left\langle \sum_m f_m P_m, \sum_n g_n P_n \right\rangle \\ &= \sum_{m,n} f_m g_n \langle P_m, P_n \rangle \\ &= \sum_{m,n} f_m g_n \mu_m \delta_{m,n} = \sum_m f_m g_m \mu_m. \end{aligned}$$

In the case of interest here $\langle f, g \rangle = \int_a^b f(x) g(x) W(x)\, dx$. Parseval's theorem then gives

$$\int_a^b f(x) g(x) w(x)\, dx = \sum_{m \geq 0} a_m b_m \mu_m. \tag{5}$$

In all our applications below the interval of orthogonality is finite, so the polynomials are complete in L^2, and Parseval's theorem holds.

In the next section we give applications of (5) beginning with an easy proof of the Askey–Wilson integral.

3. THE ASKEY–WILSON INTEGRAL AND OTHER APPLICATIONS

The Askey–Wilson integral (1) is important because it is the key to proving the orthogonality of the Askey–Wilson polynomials which are the most general known set of hypergeometric orthogonal polynomials [12]. A much less general set of orthogonal polynomials, the continuous q-Hermite (also called Rogers–Szègö) polynomials, were first discussed by Rogers in 1893 [14]. In fact (1) follows easily from (5) combined with an identity involving the continuous q-Hermite polynomials first given by Rogers in

1893 [14]. This identity is

$$\frac{(\lambda_1\lambda_2\lambda_3\lambda_4)_\infty}{(\lambda_1\lambda_3, \lambda_1\lambda_4, \lambda_2\lambda_3, \lambda_2\lambda_4)_\infty} = \sum_{n\geq 0} \frac{h_n(\lambda_1, \lambda_2)h_n(\lambda_3, \lambda_4)}{(q)_n}, \tag{6}$$

where

$$h_n(x, y) = \sum_{m=0}^{n} \frac{(q)_n x^m y^{n-m}}{(q)_m (q)_{n-m}},$$

and is known as the q-Mehler formula. The proof of (6) is quite easy; it is given as an exercise in Andrews [4]. For Rogers' proof see Rogers [14] or Bowman [8]. Carlitz rediscovered (6) in [9]. He later in [10] gave two more proofs, the second of which is almost identical to the original proof of Rogers [14].

The continuous q-Hermite polynomials $\{A_n\}_{n\geq 0}$ are defined by $A_n = A_n(e^{i\theta}) = h_n(e^{i\theta}, e^{-i\theta})$ with θ real. Today these polynomials are denoted by $H_n(x; q)$ where $x = \cos\theta$. It is not hard to see that the polynomial $H_n(x)$ is of degree n in x. The orthogonality of $\{A_n\}_{n\geq 0}$ on $(0, \pi)$ is given by

$$\int_0^\pi A_m A_n (e^{2i\theta}, e^{-2i\theta})_\infty \, d\theta = \frac{2\pi}{(q)_\infty}(q)_n \delta_{m,n}. \tag{7}$$

The deepest result used in the proof of this relation is Jacobi's triple product identity. For details see [1–3]. (The first proof was apparently given by Allaway [2]). For brevity we follow Rogers [14] and put $P(z) = (ze^{i\theta}, ze^{-i\theta})\infty$.

Proof of the Askey–Wilson integral. Put $P_m = A_m$, $f(\theta) = 1/P(\lambda_1)P(\lambda_2)$, and $g(\theta) = 1/P(\lambda_3)P(\lambda_4)$, which by (6) have the expansions

$$f(\theta) = \frac{1}{P(\lambda_1)P(\lambda_2)} = \sum_{m\geq 0} \frac{h_m(\lambda_1, \lambda_2)}{(\lambda_1\lambda_2)_\infty (q)_m} A_m,$$

and

$$g(\theta) = \frac{1}{P(\lambda_3)P(\lambda_4)} = \sum_{n\geq 0} \frac{h_n(\lambda_3, \lambda_4)}{(\lambda_3\lambda_4)_\infty (q)_n} A_n.$$

Hence by (5) and (7),

$$\int_0^{\pi} \frac{(e^{-2i\theta}, e^{2i\theta})_\infty \, d\theta}{P(\lambda_1)P(\lambda_2)P(\lambda_3)P(\lambda_4)}$$

$$= \frac{2\pi}{(q)_\infty} \sum_{m\geq 0} \frac{h_m(\lambda_1, \lambda_2)h_m(\lambda_3, \lambda_4)}{(\lambda_1\lambda_2)_\infty(\lambda_3\lambda_4)_\infty(q)_m}$$

$$= \frac{2\pi}{(q)_\infty} \frac{(\lambda_1\lambda_2\lambda_3\lambda_4)_\infty}{(\lambda_1\lambda_2, \lambda_1\lambda_3, \lambda_1\lambda_4, \lambda_2\lambda_3, \lambda_2\lambda_4, \lambda_3\lambda_4)_\infty},$$

where the last equality followed again from (6). ∎

A proof very similar to this one was given by Ismail and Stanton [13]. Their proof, however, makes use of the linearization formula for the polynomials A_m (equivalent to the q-Mehler formula) and involves a four-fold summation which is reduced using the q-binomial theorem, reindexing and rearranging terms, instead of using the Parseval identity as we do here. These differences result in our proof being substantially shorter. The integral evaluations in Bowman [8] can also be proved more quickly by this method.

It is interesting to note that the technique above can be stated compactly in terms of the Poisson kernel. Let the Poisson kernel associated with the arbitrary orthogonal system P_n above be defined by

$$K(x, y, s) = \sum_{n\geq 0} P_n(x)P_n(y)s^n/\mu_n.$$

Then we have the following formula:

$$\int_a^b K(x, y, s)K(x, z, t)W(x)\,dx = K(y, z, st). \tag{8}$$

The proof again is just an application of the Parseval theorem. To obtain the Askey–Wilson integral note that the Poisson kernel for the continuous q-Hermite polynomials is the q-Mehler formula.

We now give several new applications of the proof technique.

Define the polynomials $J_k(\lambda_1; \lambda_4, \lambda_5, \dots, \lambda_{m+3})$ by

$$J_k(\lambda_1; \lambda_4, \lambda_5, \dots, \lambda_{m+3})$$

$$= \sum_{\substack{n_1+\cdots+n_m=k \\ n_i\geq 0}} \begin{bmatrix} k \\ n_1, \dots, n_m \end{bmatrix} (\lambda_1\lambda_5)_{n_1}(\lambda_1\lambda_6)_{n_1+n_2}$$

$$\times \cdots (\lambda_1\lambda_{m+3})_{n_1+\cdots+n_{m-1}} \lambda_4^{n_1}\lambda_5^{n_2} \cdots \lambda_{m+3}^{n_m},$$

where the q-multinomial brackets are defined by

$$\begin{bmatrix} k \\ n_1, \dots, n_m \end{bmatrix} = \frac{(q)_k}{(q)_{n_1}(q)_{n_2} \cdots (q)_{n_m}}.$$

Our first theorem is a vector extension of the Askey–Wilson integral.

THEOREM 1. *For* $|\lambda_k^{(l)}| < 1$, $2 \le k \le m+3$, $l = 1, 2$,

$$\int_0^\pi \Bigg(\prod_{j=1}^{2} \frac{1}{\left(\lambda_2^{(j)} e^{-i\theta}, \lambda_3^{(j)} e^{-i\theta}\right)_\infty} \prod_{l=2}^{m+3} \frac{1}{\left(\lambda_l^{(j)} e^{i\theta}\right)_\infty} \sum_{k \ge 0} \frac{\left(\lambda_2^{(j)} e^{i\theta}, \lambda_3^{(j)} e^{i\theta}\right)_k}{\left(\lambda_2^{(j)} \lambda_3^{(j)}\right)_k (q)_k}$$

$$\times J_k\left(e^{i\theta}; \lambda_4^{(j)}, \dots, \lambda_{m+3}^{(j)}\right) e^{-ik\theta} \Bigg) \left(e^{2i\theta}, e^{-2i\theta}\right)_\infty d\theta$$

$$= \frac{1}{\left(\lambda_2^{(1)}\lambda_3^{(1)}, \lambda_2^{(2)}\lambda_3^{(2)}\right)_\infty} \frac{2\pi}{(q)_\infty} \sum_{k \ge 0} \frac{h_k\left(\lambda_2^{(1)}, \dots, \lambda_{m+3}^{(1)}\right) h_k\left(\lambda_2^{(2)}, \dots, \lambda_{m+3}^{(2)}\right)}{(q)_k}.$$

Remark on notation. The polynomials $h_k(b_1, \dots, b_m)$ are the $\mathbf{a} = \mathbf{0}$ case of the symmetric polynomials $h_k(\mathbf{a}; \mathbf{b})$ studied in [7]. They are defined by

$$h_k(\mathbf{a}; \mathbf{b}) = \sum_{\substack{n_1 + \cdots + n_m = k \\ n_i \ge 0}} \begin{bmatrix} k \\ n_1, \dots, n_m \end{bmatrix} [b_1, a_1]_{n_1} \cdots [b_m, a_m]_{n_m},$$

where $[x, y]_n = (x - y)(x - yq) \cdots (x - yq^{n-1})$ so that

$$h_k(b_1, \dots, b_m) = h_k(\mathbf{b}) = \sum_{\substack{n_1 + \cdots + n_m = k \\ n_i \ge 0}} \begin{bmatrix} k \\ n_1, \dots, n_m \end{bmatrix} b_1^{n_1} b_2^{n_2} \cdots b_m^{n_m}.$$

In Theorem 2 we will use the $\mathbf{b} = \mathbf{0}$ case of these polynomials. There

$$h_k(\mathbf{a}; \mathbf{0}) = \sum_{\substack{n_1 + \cdots + n_m = k \\ n_i \ge 0}} \begin{bmatrix} k \\ n_1, \dots, n_m \end{bmatrix} (-1)^k q^{\binom{n_1}{2} + \cdots + \binom{n_m}{n}} a_1^{n_1} a_2^{n_2} \cdots a_m^{n_m}.$$

Proof. Theorem 4.2.3 of [8] states that

$$\frac{(\lambda\lambda_1\lambda_2\lambda_3)_\infty}{(\lambda\lambda_2)_\infty(\lambda\lambda_3)_\infty}\prod_{i=2}^{m+3}\frac{1}{(\lambda_1\lambda_i)_\infty}$$

$$\times\sum_{k\geq 0}\frac{(\lambda_1\lambda_2)_k(\lambda_1\lambda_3)_k}{(\lambda\lambda_1\lambda_2\lambda_3)_k(q)_k}J_k(\lambda_1;\lambda_4,\lambda_5,\dots,\lambda_{m+3})\lambda^k$$

$$=\sum_{k\geq 0}\frac{h_k(\lambda,\lambda_1)h_k(\lambda_2,\lambda_3,\dots,\lambda_{m+3})}{(q)_k}.$$

Put $\lambda=e^{-i\theta}$, $\lambda_1=e^{i\theta}$ to obtain

$$\frac{(\lambda_2\lambda_3)_\infty}{\left(\lambda_2e^{-i\theta},\lambda_3e^{-i\theta}\right)_\infty}\prod_{l=2}^{m+3}\frac{1}{\left(\lambda_le^{i\theta}\right)_\infty}$$

$$\times\sum_{k\geq 0}\frac{\left(\lambda_2e^{i\theta}\right)_k\left(\lambda_3e^{i\theta}\right)_k}{(\lambda_2\lambda_3)_k(q)_k}J_k(\lambda_4,\dots,\lambda_{m+3})e^{-i\theta k}$$

$$=\sum_{k\geq 0}\frac{A_kh_k(\lambda_2,\dots,\lambda_{m+3})}{(q)_k},\tag{9}$$

for $|\lambda_l|<1$, $2\leq l\leq m+3$. Now produce two copies of this identity for the sets of variables $\{\lambda_2^{(1)},\dots,\lambda_{m+2}^{(1)}\}$ and $\{\lambda_2^{(2)},\dots,\lambda_{m+3}^{(2)}\}$ and call the respective left-hand sides of $f(\theta)$ and $g(\theta)$. Then (9) becomes the expansions of f and g in the set of polynomials $\{A_n\}_{n\geq 0}$. Applying (5) gives the theorem directly. ∎

For $m\geq 1$ define the sequence of rational functions $N_k(\lambda_1;\lambda_2,\dots,\lambda_{m+1})$ by

$$N_k=N_k(\lambda_1;\lambda_2,\dots,\lambda_{m+1})$$

$$=\sum_{\substack{n_1+\cdots+n_m=k\\ n_i\geq 0}}\begin{bmatrix}k\\ n_1,\dots,n_m\end{bmatrix}\frac{q^{\binom{n_1}{2}+\cdots+\binom{n_m}{2}}\lambda_2^{n_1}\cdots\lambda_{m+1}^{n_m}}{(\lambda_1\lambda_2)_{n_1}(\lambda_1\lambda_3)_{n_1+n_2}\cdots(\lambda_1\lambda_m)_{n_1+\cdots+n_{m-1}}}.$$

Then we have the following theorem.

THEOREM 2.

$$\int_0^\pi\left(\prod_{j=1}^{2}\prod_{l=2}^{m+1}\left(\lambda_l^{(j)}e^{i\theta}\right)_\infty\sum_{k\geq 0}\frac{\left(-e^{-i\theta}\right)^kN_k\left(e^{i\theta};\lambda_2^{(j)},\dots,\lambda_{m+1}^{(j)}\right)}{\left(\lambda_{m+1}^{(j)}e^{i\theta}\right)_k(q)_k}\right)\left(e^{2i\theta},e^{-2i\theta}\right)_\infty d\theta$$

$$=\frac{2\pi}{(q)_\infty}\sum_{k\geq 0}\frac{h_k\left(\lambda_2^{(1)},\dots,\lambda_{m+1}^{(1)};\mathbf{0}\right)h_k\left(\lambda_2^{(2)},\dots,\lambda_{m+1}^{(2)};\mathbf{0}\right)}{(q)_k}.$$

Proof. Theorem 4.2.5 of [8] states that

$$\prod_{i=2}^{m+1}(\lambda_1\lambda_i)_\infty \sum_{k\ge 0}\frac{(-\lambda)^k N_k(\lambda_1;\lambda_2,\ldots,\lambda_{m+1})}{(q)_k(\lambda_1\lambda_{m+1})_k} = \sum_{k\ge 0}\frac{h_k(\lambda,\lambda_1)h_k(\lambda_2,\lambda_3,\ldots,\lambda_{m+1};\mathbf{0})}{(q)_k}.$$

Put $\lambda = e^{-i\theta}$ and $\lambda_1 = e^{i\theta}$ in this equation to get

$$\prod_{l=2}^{m+1}\left(\lambda_l e^{i\theta}\right)_\infty \sum_{k\ge 0}\frac{\left(-e^{-i\theta}\right)^k N_k\left(e^{i\theta};\lambda_2,\ldots,\lambda_{m+1}\right)}{\left(\lambda_{m+1}e^{i\theta}\right)_k(q)_k} = \sum_{k\ge 0}\frac{h_k(\lambda_2,\lambda_3,\ldots,\lambda_{m+1};\mathbf{0})}{(q)_k}A_k.$$

The rest of the proof proceeds *mutatis mutandis* to the last one. ∎

For the next theorem we use the standard ϕ notation for basic hypergeometric series. The function ${}_k\phi_{k-1}(z)$ for $|z| < 1$ is defined by

$${}_k\phi_{k-1}\left(\begin{matrix} a_1,a_2,\ldots,a_k \\ b_1,b_2,\ldots,b_{k-1}\end{matrix};z\right) = \sum_{n\ge 0}\frac{(a_1)_n(a_2)_n\cdots(a_k)_n}{(b_1)_n(b_2)_n\cdots(b_{k-1})_n(q)_n}z^n.$$

THEOREM 3. *For* $|b_1|, |b_2|, |t_1t_2| < 1$,

$$\int_0^\pi\left(\prod_{j=1}^{2}\frac{\left(t_j\beta e^{i\theta},t_j\beta e^{-i\theta}\right)_\infty}{\left(t_je^{i\theta},t_je^{-i\theta}\right)_\infty}\,{}_3\phi_2\left(\begin{matrix} a_j,t_je^{i\theta},t_je^{-i\theta} \\ t_j\beta e^{i\theta},t_j\beta e^{-i\theta}\end{matrix};b_j\right)\right)\frac{\left(e^{2i\theta},e^{-2i\theta}\right)_\infty}{\left(\beta e^{2i\theta},\beta e^{-2i\theta}\right)_\infty}\,d\theta$$
$$= \frac{2\pi}{(q)_\infty}\frac{(\beta,\beta q,a_1b_1,a_2b_2)_\infty}{\left(\beta^2,b_1,b_2\right)_\infty}\,{}_4\phi_3\left(\begin{matrix} b_1,b_2,\beta^2,\beta \\ a_1b_1,a_2b_2,\beta q\end{matrix};t_1t_2\right).$$

Proof. Put $m = 2$, $d_1 = te^{i\theta}$, $d_2 = te^{-i\theta}$, $c_1 = t\beta e^{i\theta}$, and $c_2 = t\beta e^{-i\theta}$ in [7, Corollary 2.1], to obtain

$${}_3\phi_2\left(\begin{matrix} a,te^{i\theta},te^{-i\theta} \\ t\beta e^{i\theta},t\beta e^{-i\theta}\end{matrix};b\right)\frac{\left(t\beta e^{i\theta},t\beta e^{-i\theta},ab\right)_\infty}{\left(te^{i\theta},te^{-i\theta},b\right)_\infty} = \sum_{k\ge 0}\frac{(b)_k}{(ab)_k}C_k(x;\beta|q)t^k, \tag{10}$$

where $x = \cos\theta$ and $\{C_n\}_{n \geq 0}$ are the continuous q-ultraspherical polynomials. The orthogonality relation for this family is (see [11])

$$\int_0^{\pi} C_m(x; \beta|q) C_n(x; \beta|q) W_\beta(x|q)\, d\theta = \frac{\delta_{m,n}}{y_n(\beta|q)},$$

where $|\beta| < 1$,

$$y_n(\beta|q) = \frac{(q, \beta^2)_\infty (q)_n (1 - \beta q^n)}{2\pi(\beta, \beta q)_\infty (\beta^2)_n (1 - \beta)},$$

and

$$W_\beta(x|q) = \frac{(e^{2i\theta}, e^{-2i\theta})_\infty}{(\beta e^{2i\theta}, \beta e^{-2i\theta})_\infty}.$$

Now produce two copies of (10) for the sets of variables $\{a_1, t_1, b_1\}$ and $\{a_2, t_2, b_2\}$ and call the respective left sides $f(\theta)$ and $g(\theta)$ and apply (5). ▌

Here we have not considered applications of the idea in this paper to polynomials with discrete measures. This will lead to summation theorems instead of integrals. Also, the main idea works for biorthogonal polynomials. We will consider topics like these in future papers.

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