Convolution with measures on flat curves in low dimensions

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Abstract

We prove $L^p \to L^q$ convolution estimates for the affine arclength measure on certain flat curves in $\mathbb{R}^d$ when $d \in \{2, 3, 4\}$. For $d = 2, 3$, we also establish certain related Lorentz space estimates.
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1. Introduction

Let γ be a curve in $\mathbb{R}^d$ given by

$$\gamma(t) = \left(t, \frac{t^2}{2}, \dots, \frac{t^{d-1}}{(d-1)!}, \phi(t)\right) \tag{1.1}$$

where $\phi \in C^{(d)}(a, b)$, where $\phi^{(j)}(t) > 0$ for $t \in (a, b)$ and $j = 0, 1, 2, \dots, d$, and where $\phi^{(d)}$ is nondecreasing. Such curves are termed *simple* in [10]. We are interested in the possibility of proving $L^p \to L^q$ convolution estimates for the affine arclength measure λ on (1.1), given by $d\lambda = \phi^{(d)}(t)^{2/(d^2+d)} \, dt$. We begin by recalling a theorem from [13]. (In this note, $|E|$ will stand for the Lebesgue measure of E.)

Theorem 1.1. *Suppose $d = 2$. The inequality*

$$\|\lambda * \chi_E\|_3 \leqslant (12)^{1/3}|E|^{2/3}$$

holds for all measurable $E \subset \mathbb{R}^2$.

Theorem 1.1 is equivalent to a weak-type $(3/2, 3)$ estimate for the operator given by convolution with λ, an estimate which is uniform over the class of measures λ described above. Here are two questions which are raised by Theorem 1.1:

 (i) is there an analogous strong-type estimate, and
(ii) are there analogs of Theorem 1.1 if $d > 2$?

Having no idea how to attack these interesting questions in their natural generality, we follow the usual practice of asking what can be said along such lines by imposing additional hypotheses on ϕ. The requirement

$$\left(\prod_{j=1}^{n} \phi^{(d)}(s_j)\right)^{1/n} \leqslant A\phi^{(d)}\left(\frac{s_1 + \cdots + s_n}{n}\right), \tag{1.2}$$

to hold for $s_j \in (a, b)$, was used with $n = d$ in [3] to obtain Fourier restriction estimates for curves (1.1). It is obvious that if $\beta \geqslant d$ then condition (1.2) holds with $A = 1$ for $\phi(t) = t^\beta$ on the interval $(0, \infty)$. Moreover, as was observed in [3], if we define $\phi_0(t) = t^\beta$ for some $\beta > d$ and then define

$$\phi_j(t) = \int\limits_0^t (t - u)^{d-1} \exp\left(-\frac{1}{\phi_{j-1}^{(d)}(u)}\right) du$$

for $j \geqslant 1$, each of the functions ϕ_j satisfies (1.2) with $A = 1$ on $(0, \infty)$. This yields a sequence of functions which are progressively flatter at the origin. (See Section 4 of [3] for other examples of flat functions satisfying (1.2).) In this note we will assume the $n = 2$ version of (1.2) which, with $\omega \doteq (\phi^{(d)})^{2/(d^2+d)}$, we write as

$$\left(\omega(s_1)\omega(s_2)\right)^{1/2} \leqslant A\omega\left(\frac{s_1 + s_2}{2}\right). \tag{1.3}$$

We will obtain convolution estimates in only the dimensions $d = 2, 3$ and 4:

Theorem 1.2. *Suppose $d = 2$ and assume (1.3). Then there is the Lorentz space estimate*

$$\|\lambda * f\|_{L^3} \leqslant C(A)\|f\|_{L^{3/2,3}}.$$

Theorem 1.3. *Suppose $d = 3$ and assume (1.3). Then, for any $\epsilon > 0$, there is the Lorentz space estimate*

$$\|\lambda * f\|_{L^2} \leqslant C(A)\|f\|_{L^{3/2,2-\epsilon}}.$$

Theorem 1.4. *Suppose $d = 4$ and assume* (1.3). *If*

$$\frac{1}{p} - \frac{1}{q} = \frac{1}{10} \quad and \quad \frac{4}{10} < \frac{1}{p} < \frac{7}{10} \tag{1.4}$$

then there is the Lebesgue space estimate

$$\|\lambda * f\|_{L^q} \leqslant C(A, p)\|f\|_{L^p}.$$

Here are some comments:

(a) Theorem 1.2 is the best possible Lorentz space estimate, even in the nondegenerate case $\phi(t) = t^2/2$ (see Section 3 in [15]). It implies the sharp $L^p \to L^q$ mapping property, an $L^{3/2} \to L^3$ estimate.

(b) Theorem 1.3 is analogous to a result from [8] for polynomial curves (whose proof we will follow). Theorem 1.3 implies the sharp $L^p \to L^q$ estimates, which hold for

$$\frac{1}{p} - \frac{1}{q} = \frac{1}{6} \quad and \quad \frac{1}{2} \leqslant \frac{1}{p} \leqslant \frac{2}{3}. \tag{1.5}$$

But there are sharper Lorentz space estimates for the nondegenerate case $\phi(t) = t^3/6$ in [1] and for polynomial curves (for all dimensions d) in [16].

(c) Theorem 1.4 is much less satisfactory. One would like, for example, at least the sharp $L^p \to L^q$ mapping properties, which correspond to the endpoints in (1.4).

(d) An analog of Theorem 1.4 for all dimensions d, as well as analogs of the endpoint results of [15] and [16], might follow from an analog of the band structure construction of [6] for the curves and measures considered in this note. But, in view of the complicated nature of a Jacobian determinant associated with our operators, it is not clear how to obtain such a band structure.

(e) The papers [9,4,5] contain some earlier results for convolution with affine arclength measures in dimensions 2 and 3.

Section 2 contains the proofs of Theorems 1.2–1.4 and Section 3 contains the proofs for the lemmas required in Section 2.

2. Proofs of theorems

Proof of Theorem 1.2. According to the proof of Theorem 5 in [14], which abstracts an argument from [2], it is enough to establish the estimate

$$\int_a^b \left(\int_a^b \chi_E\big(\gamma(t_2) - \gamma(t_1)\big)\omega(t_2)\, dt_2 \right)^2 \omega(t_1)\, dt_1 \leqslant C(A)|E|$$

for measurable $E \subset \mathbb{R}^2$. The inequality

$$\int_a^b \left(\int_{t_1}^b \chi_E\big(\gamma(t_2) - \gamma(t_1)\big)\omega(t_2)\, dt_2 \right)^2 \omega(t_1)\, dt_1 \leqslant 4|E|$$

from [13], which is true without any additional hypothesis like (1.3), shows that it suffices to establish the estimate

$$\int_a^b \left(\int_a^{t_1} \chi_E\big(\gamma(t_2) - \gamma(t_1)\big)\omega(t_2)\,dt_2 \right)^2 \omega(t_1)\,dt_1 \leqslant C(A)|E|. \tag{2.1}$$

The mapping

$$(t_1, t_2) \mapsto \gamma(t_2) - \gamma(t_1)$$

is one-to-one by the convexity of the curve γ. If $J(t_1, t_2)$ is the absolute value of the Jacobian determinant of this mapping, then (2.1) is equivalent to

$$\int \left(\int \chi_\Omega(t_1, t_2)\omega(t_2)\,dt_2 \right)^2 \omega(t_1)\,dt_1 \leqslant C(A) \int \int \chi_\Omega(t_1, t_2) J(t_1, t_2)\,dt_2\,dt_1 \tag{2.2}$$

if $\Omega \subset \{(t_1, t_2): a < t_2 < t_1 < b\}$. We will need the following estimate, a consequence of Lemma 2.1 below,

$$J(t_1, t_2) \geqslant c(A)|t_1 - t_2|\omega(t_2)\omega(t_1)^2. \tag{2.3}$$

Lemma 2.1. *Suppose γ is as in (1.1) and let $J(t_1, \ldots, t_d)$ be the absolute value of the Jacobian determinant of one of the mappings*

$$(t_1, \ldots, t_d) \mapsto \gamma(t_1) \pm \gamma(t_2) \pm \cdots \pm \gamma(t_d).$$

Suppose that (1.3) holds and that $n_1 \leqslant \cdots \leqslant n_d$ are positive numbers satisfying $n_1 + \cdots + n_d = d(d+1)/2$. Suppose that $\{i_1, \ldots, i_d\} = \{1, \ldots, d\}$ and that $a < t_{i_1} < \cdots < t_{i_d} < b$. Then

$$J(t_1, \ldots, t_d) \geqslant c \left(\prod_{j=1}^d \omega(t_{i_j})^{n_j} \right) V(t_1, \ldots, t_d), \tag{2.4}$$

where

$$V(t_1, \ldots, t_d) = \left| \prod_{1 \leqslant i < j \leqslant d} (t_j - t_i) \right|$$

and where c depends only on A from (1.3) and on $n_1, \ldots, n_d$.

Given (2.3), inequality (2.2) will follow from

$$\int \left(\int \chi_\Omega(t_1, t_2)\omega(t_2)\,dt_2 \right)^2 \omega(t_1)\,dt_1$$

$$\leqslant C \int \left(\int \chi_\Omega(t_1, t_2)\omega(t_1)|t_1 - t_2|\omega(t_2)\,dt_2 \right)\omega(t_1)\,dt_1. \tag{2.5}$$

To see (2.5), we will use the following lemma.

Lemma 2.2. *Suppose ω is nonnegative and nondecreasing on some interval $(c, d]$. Suppose that $c_1, \ldots, c_K \in \mathbb{R}$. For $\rho > 0$ let*

$$E_\rho = \left\{ t \in (c, d]: \ \omega(t)^{K-1} \omega(d) \prod_{\ell=1}^{K} |t - c_l| \leqslant \rho^K \right\}.$$

Then

$$\int_{E_\rho} \omega(t) \, dt \leqslant C(K) \rho.$$

Indeed, fix t_1 and define ρ by

$$\rho = \frac{1}{2C(1)} \int \chi_\Omega(t_1, t_2) \omega(t_2) \, dt_2,$$

where $C(1)$ is the constant in Lemma 2.2 corresponding to $K = 1$. It follows from Lemma 2.2 (with $d = t_1$) that

$$\int_{\{\omega(t_1)|t_2 - t_1| \geqslant \rho\}} \chi_\Omega(t_1, t_2) \omega(t_2) \, dt_2 \geqslant \frac{1}{2} \int \chi_\Omega(t_1, t_2) \omega(t_2) \, dt_2$$

and so

$$\int \chi_\Omega(t_1, t_2) \omega(t_1) |t_1 - t_2| \omega(t_2) \, dt_2 \geqslant \frac{1}{4C(1)} \left(\int \chi_\Omega(t_1, t_2) \omega(t_2) \, dt_2 \right)^2.$$

Now integrating with respect to t_1 gives (2.5). $\quad\square$

Proof of Theorem 1.4. We will apply the iterated TT^* method introduced by Christ in [6] and (see, e.g., the discussion and references in [16]) employed by many others since then. Thus, assuming some familiarity with Christ's method, Theorem 1.4 will follow if we establish the inequality (2.9) below, where E, α, and β are as follows: let $\Omega \subset (a, b)^4$ be a set of the form

$$\Omega = \big\{ (t_1, t_2, t_3, t_4): \ t_1 \in \Omega_0, \ t_2 \in \Omega(t_1), \ t_3 \in \Omega(t_1, t_2), \ t_4 \in \Omega(t_1, t_2, t_3) \big\}$$

where

$$\lambda(\Omega_0) \geqslant \beta > 0, \qquad \lambda\big(\Omega(t_1)\big) \geqslant \alpha > 0 \quad \text{for each } t_1 \in \Omega_0, \tag{2.6}$$

$$\lambda\big(\Omega(t_1, t_2)\big) \geqslant \beta \quad \text{whenever } t_1 \in \Omega_0, \ t_2 \in \Omega(t_1), \quad \text{and} \tag{2.7}$$

$$\lambda\big(\Omega(t_1, t_2, t_3)\big) \geqslant \alpha \quad \text{whenever } t_1 \in \Omega_0, \ t_2 \in \Omega(t_1), \ t_3 \in \Omega(t_1, t_2). \tag{2.8}$$

(Here we are writing λ for the measure

$$d\lambda(t) = \omega(t) \, dt = \phi^{(4)}(t)^{1/10} \, dt$$

on (a, b) as well as for its image on γ.) The set E is defined by

$$E = \big\{ \gamma(t_1) - \gamma(t_2) + \gamma(t_3) - \gamma(t_4)\colon (t_1, t_2, t_3, t_4) \in \Omega \big\},$$

and the desired inequality is

$$|E| \geqslant c(A)\alpha^7\beta^3. \tag{2.9}$$

By passing to a subset of Ω and replacing α and β by $\alpha/24$ and $\beta/24$, we can assume that there is some permutation $\{i_1, i_2, i_3, i_4\}$ of $\{1, 2, 3, 4\}$ such that if $(t_1, t_2, t_3, t_4) \in \Omega$ then

$$t_{i_1} < t_{i_2} < t_{i_3} < t_{i_4}.$$

If $J(t_1, t_2, t_3, t_4)$ is the absolute value of the Jacobian determinant of the mapping

$$(t_1, t_2, t_3, t_4) \mapsto \gamma(t_1) - \gamma(t_2) + \gamma(t_3) - \gamma(t_4),$$

we will use the following inequality, a consequence of Lemma 2.1,

$$J(t_1, t_2, t_3, t_4) \geqslant c(A)\omega(t_{i_4})^9 \left(\prod_{j=1}^{3} \omega(t_{i_j}) \right)^{1/3} V(t_1, t_2, t_3, t_4). \tag{2.10}$$

We will also need the following lemma.

Lemma 2.3. *Suppose ω is nonnegative and nondecreasing on an interval $[c, d)$. Suppose $\eta > 0$ and $r > 1$ satisfy*

$$\eta < \frac{1}{r'} \doteq 1 - \frac{1}{r}.$$

Suppose $E \subset [c, d)$ and let

$$\rho = \int_E \omega(t)\,dt.$$

Then, for $t_0 \in \mathbb{R}$,

$$\rho^{1+r\eta}\, \omega(c)^{r-(1+r\eta)} \leqslant C(\eta, r) \int_E \omega(t)^r |t - t_0|^{r\eta}\, dt.$$

If $t_1, t_2 \in \mathbb{R}$ then also

$$\rho^{1+r\eta}|t_2 - t_1|^{r\eta}\omega(c)^{r-(1+r\eta)} \leqslant C(\eta, r) \int_E \omega(t)^r \big(|t - t_1| \cdot |t - t_2| \big)^{r\eta}\, dt.$$

Now define I by

$$I = \int\limits_{\Omega_0} \int\limits_{\Omega(t_1)} \int\limits_{\Omega(t_1,t_2)} \int\limits_{\Omega(t_1,t_2,t_3)} \omega(t_{i_4})^9 \left(\prod_{j=1}^{3} \omega(t_{i_j}) \right)^{1/3} V(t_1, t_2, t_3, t_4) \, dt_4 \, dt_3 \, dt_2 \, dt_1$$

so that, because of (2.10), we have

$$|E| \geqslant c(A)I. \tag{2.11}$$

(The change of variables needed for the estimate (2.11) is justified as in [11, p. 549].) In view of (2.11), (2.9) will follow if we show that

$$I \gtrsim \alpha^7 \beta^3. \tag{2.12}$$

(The constants implied by $\lesssim$ and $\gtrsim$ will not depend on any parameters.)

We will, unfortunately, need to consider several cases. To begin, if $4 = i_4$, we will use Lemma 2.3 with $r = 5$ and $\eta = 3/5$ to estimate

$$\int\limits_{\Omega(t_1,t_2,t_3)} \omega(t_4)^5 \prod_{j=1}^{3} |t_4 - t_j| \, dt_4 \geqslant \int\limits_{\Omega(t_1,t_2,t_3)} \omega(t_4)^5 |t_4 - t_{i_3}|^3 \, dt_4$$

$$\gtrsim \left(\int\limits_{\Omega(t_1,t_2,t_3)} \omega(t_4) \, dt_4 \right)^4 \omega(t_{i_3}).$$

With the inequality $\omega(t_{i_4}) \geqslant \omega(t_{i_3})$ this gives

$$I \gtrsim \int\limits_{\Omega_0} \int\limits_{\Omega(t_1)} \int\limits_{\Omega(t_1,t_2)} \left(\int\limits_{\Omega(t_1,t_2,t_3)} \omega(t_4) \, dt_4 \right)^4$$

$$\cdot \omega(t_{i_3})^{16/3} \omega(t_{i_2})^{1/3} \omega(t_{i_1})^{1/3} V(t_1, t_2, t_3) \, dt_3 \, dt_2 \, dt_1. \tag{2.13}$$

If $4 = i_{k_0}$ for some $k_0 = 1, 2, 3$, then

$$\omega(t_4)^{1/3} \omega(t_{i_4})^{11/3} \geqslant \omega(t_4)^3 \omega(t_{i_{k_0+1}})$$

by the monotonicity of ω. Thus

$$\int\limits_{\Omega(t_1,t_2,t_3)} \omega(t_4)^{1/3} \omega(t_{i_4})^{11/3} \prod_{j=1}^{3} |t_4 - t_j| \, dt_4 \geqslant \int\limits_{\Omega(t_1,t_2,t_3)} \omega(t_4)^3 \omega(t_{i_{k_0+1}}) \prod_{j=1}^{3} |t_4 - t_j| \, dt_4$$

$$\gtrsim \left(\int\limits_{\Omega(t_1,t_2,t_3)} \omega(t_4) \, dt_4 \right)^4,$$

where the last inequality follows from an application of Lemma 2.2 as at the end of the proof of Theorem 1.2 but with $K = 3$ instead of $K = 1$. Therefore

$$I \gtrsim \int\limits_{\Omega_0} \int\limits_{\Omega(t_1)} \int\limits_{\Omega(t_1,t_2)} \left(\int\limits_{\Omega(t_1,t_2,t_3)} \omega(t_4)\, dt_4 \right)^4$$

$$\times \omega(t_{i_4})^{16/3} \prod_{k=1, k \neq k_0}^{3} \omega(t_{i_k})^{1/3} V(t_1, t_2, t_3)\, dt_3\, dt_2\, dt_1. \qquad (2.14)$$

Now if $\{j_1, j_2, j_3\}$ is the permutation of $\{1, 2, 3\}$ such that $t_{j_1} < t_{j_2} < t_{j_3}$ whenever $t_1 \in \Omega_0$, $t_2 \in \Omega(t_1)$, $t_3 \in \Omega(t_1, t_2)$, then (2.13), (2.14), and (2.8) imply that

$$I \gtrsim \alpha^4 \int\limits_{\Omega_0} \int\limits_{\Omega(t_1)} \int\limits_{\Omega(t_1,t_2)} \omega(t_{j_3})^{16/3} \omega(t_{j_2})^{1/3} \omega(t_{j_1})^{1/3} V(t_1, t_2, t_3)\, dt_3\, dt_2\, dt_1. \qquad (2.15)$$

If $3 = j_3$,

$$\int\limits_{\Omega(t_1,t_2)} \omega(t_3)^3 |t_3 - t_2| \cdot |t_3 - t_1|\, dt_3 \gtrsim |t_1 - t_2| \int\limits_{\Omega(t_1,t_2)} \omega(t_3)^3 |t_3 - t_{j_2}|\, dt_3$$

$$\gtrsim |t_1 - t_2| \left(\int\limits_{\Omega(t_1,t_2)} \omega(t_3)\, dt_3 \right)^2 \omega(t_{j_2}),$$

where the $\gtrsim$ results from an application of Lemma 2.3 with $r = 3$ and $\eta = 1/3$. With (2.15), (2.7), and the monotonicity of ω this gives

$$I \gtrsim \alpha^4 \beta^2 \int\limits_{\Omega_0} \int\limits_{\Omega(t_1)} \omega(t_{j_2})^{7/2} \omega(t_{j_1})^{1/2} |t_1 - t_2|^2\, dt_2\, dt_1. \qquad (2.16)$$

If $3 = j_2$, the second conclusion of Lemma 2.3, with $r = 13/6$ and $\eta = 6/13$, gives

$$\int\limits_{\Omega(t_1,t_2)} \omega(t_3)^{13/6} |t_3 - t_2| \cdot |t_3 - t_1|\, dt_3 \gtrsim |t_1 - t_2| \left(\int\limits_{\Omega(t_1,t_2)} \omega(t_3)\, dt_3 \right)^2 \omega(t_{j_1})^{1/6}.$$

From (2.15) it then follows that

$$I \gtrsim \alpha^4 \beta^2 \int\limits_{\Omega_0} \int\limits_{\Omega(t_1)} \omega(t_{j_3})^{7/2} \omega(t_{j_1})^{1/2} |t_1 - t_2|^2\, dt_2\, dt_1. \qquad (2.17)$$

And if $3 = j_1$ then

$$\int\limits_{\Omega(t_1,t_2)} \omega(t_3)\omega(t_{j_2})|t_3-t_2|\cdot|t_3-t_1|\,dt_3 \geqslant |t_1-t_2| \int\limits_{\Omega(t_1,t_2)} \omega(t_3)\omega(t_{j_2})|t_3-t_{j_2}|\,dt_3$$

$$\gtrsim |t_1-t_2|\left(\int\limits_{\Omega(t_1,t_2)} \omega(t_3)\,dt_3\right)^2$$

by Lemma 2.2 with $K=1$, and so (2.15) gives

$$I \gtrsim \alpha^4\beta^2 \int\limits_{\Omega_0}\int\limits_{\Omega(t_1)} \omega(t_{j_3})^{7/2}\omega(t_{j_2})^{1/2}|t_1-t_2|^2\,dt_2\,dt_1. \tag{2.18}$$

Thus if $\{k_1, k_2\}$ is the permutation of $\{1, 2\}$ such that $t_{k_1} < t_{k_2}$ whenever $t_1 \in \Omega_0$, $t_2 \in \Omega(t_1)$, then (2.12) will follow from (2.16), (2.17), and (2.18) if we establish that

$$\int\limits_{\Omega_0}\int\limits_{\Omega(t_1)} \omega(t_{k_2})^{7/2}\omega(t_{k_1})^{1/2}|t_1-t_2|^2\,dt_2\,dt_1 \gtrsim \alpha^3\beta. \tag{2.19}$$

If $2 = k_2$, then

$$\int\limits_{\Omega(t_1)} \omega(t_2)^{7/2}|t_1-t_2|^2\,dt_2 \gtrsim \left(\int\limits_{\Omega(t_1)} \omega(t_2)\,dt_2\right)^3 \omega(t_1)^{1/2}$$

by Lemma 2.3 with $r = 7/2$, $\eta = 4/7$, and (2.19) follows from (2.6). If $2 = k_1$, then

$$\omega(t_{k_2})^{7/2}\omega(t_{k_1})^{1/2} \geqslant \omega(t_1)^2\omega(t_2)^2$$

and

$$\int\limits_{\Omega(t_1)} \omega(t_2)^2\omega(t_1)|t_1-t_2|^2\,dt_2 \gtrsim \left(\int\limits_{\Omega(t_1)} \omega(t_2)\,dt_2\right)^3$$

by Lemma 2.2 with $K=2$. Again, (2.19) follows from (2.6), and the proof of Theorem 1.4 is complete. $\quad\square$

Proof of Theorem 1.3. The sharp $L^p \to L^q$ estimates (for the indices in (1.5)) can be obtained by the method of [12]. But to obtain the Lorentz space estimates in Theorem 1.3, we will follow the proof of the $d = 3$ case in [8], again using the method of Christ. Thus we will begin by establishing the following claim (which, by itself, implies the almost sharp Lebesgue space estimates corresponding to strict inequality in (1.5)): suppose that $\Omega \subset (a, b)^3$ is a set of the form

$$\Omega = \big\{(t_1, t_2, t_3):\ t_1 \in \Omega_0,\ t_2 \in \Omega(t_1),\ t_3 \in \Omega(t_1, t_2)\big\}$$

where

$$\lambda(\Omega_0) \geqslant \alpha > 0, \qquad \lambda\big(\Omega(t_1)\big) \geqslant \beta > 0 \quad \text{for each } t_1 \in \Omega_0 \quad \text{and}$$

$$\lambda\big(\Omega(t_1, t_2)\big) \geqslant \alpha \quad \text{whenever } t_1 \in \Omega_0, \ t_2 \in \Omega(t_1). \tag{2.20}$$

If

$$E = \big\{\gamma(t_1) - \gamma(t_2) + \gamma(t_3): \ (t_1, t_2, t_3) \in \Omega\big\},$$

then we have

$$|E| \geqslant c(A)\alpha^4 \beta^2. \tag{2.21}$$

As before, we can assume that there is some permutation $\{i_1, i_2, i_3\}$ of $\{1, 2, 3\}$ such that if $(t_1, t_2, t_3) \in \Omega$ then

$$t_{i_1} < t_{i_2} < t_{i_3}.$$

With $J(t_1, t_2, t_3)$ the absolute value of the Jacobian determinant of the mapping

$$(t_1, t_2, t_3) \mapsto \gamma(t_1) - \gamma(t_2) + \gamma(t_3),$$

we will need the following consequence of Lemma 2.1:

$$J(t_1, t_2, t_3) \geqslant c(A)\omega(t_{i_3})^5 \left(\prod_{j=1}^{2} \omega(t_{i_j})\right)^{1/2} V(t_1, t_2, t_3). \tag{2.22}$$

Define I by

$$I = \int_{\Omega_0} \int_{\Omega(t_1)} \int_{\Omega(t_1, t_2)} \omega(t_{i_3})^5 \left(\prod_{j=1}^{2} \omega(t_{i_j})\right)^{1/2} V(t_1, t_2, t_3)\, dt_3\, dt_2\, dt_1$$

so that, because of (2.22), we have

$$|E| \geqslant c(A) I.$$

(Again, the change of variables here is justified as in [11].) Then (2.21) will follow from

$$I \gtrsim \alpha^4 \beta^2. \tag{2.23}$$

Since the proof of (2.23) is very similar to the proof of (2.12), we will only sketch the argument. The first step is to obtain the inequality

$$I \gtrsim \alpha^3 \int_{\Omega_0} \int_{\Omega(t_1)} \omega(t_{j_2})^{5/2} \omega(t_{j_1})^{1/2} |t_1 - t_2|\, dt_2\, dt_1,$$

where $\{t_1, t_2\} = \{t_{j_1}, t_{j_2}\}$ and $t_{j_1} < t_{j_2}$. Recalling (2.20), this is done by using Lemma 2.3, with $r = 7/2$ and $\eta = 4/7$, if $3 = i_3$ and by using Lemma 2.2, with $K = 2$, if $3 = i_2$ or $3 = i_1$. The proof of (2.23) is then concluded by showing that

$$\int\limits_{\Omega_0} \int\limits_{\Omega(t_1)} \omega(t_{j_2})^{5/2} \omega(t_{j_1})^{1/2} |t_1 - t_2| \, dt_2 \, dt_1 \gtrsim \beta^2 \alpha$$

by using Lemma 2.3 with $r = 5/2$, $\eta = 2/5$ if $t_1 < t_2$ and Lemma 2.2 with $K = 1$ if $t_2 < t_1$. This proves (2.23) and thus, as mentioned above, establishes the almost sharp Lebesgue space bounds by the method of [6].

To obtain the Lorentz space bounds claimed in Theorem 1.3, we follow the proof of the analogous result in [8] (itself based on a further argument of Christ [7]). Thus it is enough to establish an analogue of Lemma 1 in [8] for our curves γ. The crux of the matter is to show the following: if $\Omega \subset (a, b)^3$ is a set of the form

$$\Omega = \big\{(t_1, t_2, t_3): t_1 \in \Omega_0, \ t_2 \in \Omega(t_1), \ t_3 \in \Omega(t_1, t_2)\big\},$$

where

$$\lambda(\Omega_0) \geqslant \beta > 0, \qquad \lambda\big(\Omega(t_1)\big) \geqslant \beta |G|/|E| > 0 \quad \text{for each } t_1 \in \Omega_0 \quad \text{and}$$

$$\lambda\big(\Omega(t_1, t_2)\big) \geqslant \delta > 0 \quad \text{whenever } t_1 \in \Omega_0, \ t_2 \in \Omega(t_1),$$

and if

$$E' = \big\{\gamma(t_1) - \gamma(t_2) + \gamma(t_3): \ (t_1, t_2, t_3) \in \Omega\big\},$$

then we have

$$\big|E'\big| \geqslant c(A)\delta^3 \left(\frac{\beta |G|}{|E|}\right)^2 \beta.$$

This can be established by exactly the argument given above for (2.21). $\quad\square$

3. Proofs of lemmas

Proof of Lemma 2.1. Assume without loss of generality that

$$a < t_1 < \cdots < t_d < b. \tag{3.1}$$

It is enough to prove the lemma in the special case when each n_j can be written as

$$n_j = \frac{d(d+1)}{2} \cdot \frac{l_j}{2^n} \tag{3.2}$$

for some large integer n and positive integers l_j. To see this, find n and $l_1 \leqslant \cdots \leqslant l_d$ such that

$$\sum_{j=1}^{d} l_j = 2^n$$

and

$$n_j \leqslant \frac{d(d+1)}{2} \cdot \frac{l_j}{2^n}, \quad j = 2, \ldots, d. \tag{3.3}$$

Then note that, since

$$\sum_{j=1}^{d} n_j = \sum_{j=1}^{d} \frac{d(d+1)l_j}{2^{n+1}},$$

it follows from (3.1), the monotonicity of ω, and (3.3), that

$$\prod_{j=1}^{d} \omega(t_j)^{n_j} \leqslant \prod_{j=1}^{d} \omega(t_j)^{\frac{d(d+1)l_j}{2^{n+1}}}.$$

There exists a nonnegative function $\psi = \psi_d(u; t_1, \ldots, t_d)$ supported in $[t_1, t_d]$ such that

$$J(t_1, \ldots, t_d) = \int_{t_1}^{t_d} \omega(u)^{d(d+1)/2} \psi(u) \, du. \tag{3.4}$$

(It is easy to check that one can take $\psi_2(u; t_2, t_2) = \chi_{[t_1,t_2]}(u)$. For $d = 3, 4, \ldots$, ψ_d is defined recursively by a procedure described in detail in Section 2 of [3].) The choice $\phi(t) = t^d/d!$ in (1.1) shows that

$$\int_{t_1}^{t_d} \psi(u) \, du = c(d) V(t_1, \ldots, t_d).$$

For sufficiently small $\delta > 0$, define

$$t_\delta = \bigl(1 - (d-1)\delta\bigr)t_d + \delta(t_1 + t_2 + \cdots + t_{d-1}).$$

The inequality (2.4) will follow from (3.4), the monotonicity of ω, the inequality

$$\int_{t_\delta}^{t_d} \psi(u; t_1, \ldots, t_d) \, du \geqslant c(\delta) V(t_1, \ldots, t_d), \tag{3.5}$$

and the fact that there is a $\delta = \delta(n_1, \ldots, n_d) > 0$ such that

$$\omega(t_\delta)^{d(d+1)/2} \geqslant c(A; n_1, \ldots, n_d) \prod_{j=1}^{d} \omega(t_j)^{n_j}. \tag{3.6}$$

The proof of (3.5) is by induction on d. Since

$$\psi(u; t_1, t_2) = \chi_{[t_1, t_2]}(u),$$

the case $d = 2$ is clear. The inductive step requires an identity from [3]:

$$\psi(u; t_1, \ldots, t_d) = \int_{t_1}^{t_2} \cdots \int_{t_{d-1}}^{t_d} \psi(u; s_1, \ldots, s_{d-1}) \, ds_1 \cdots ds_{d-1}.$$

Thus

$$\int_{t_\delta}^{t_d} \psi(u) \, du = \int_{t_1}^{t_2} \cdots \int_{t_{d-1}}^{t_d} \int_{\{u \geq t_\delta\}} \psi(u; s_1, \ldots, s_{d-1}) \, du \, ds_1 \cdots ds_{d-1}. \qquad (3.7)$$

We need the following additional fact from [3]: suppose $\lambda_j \in (0, 1)$ for $j = 1, \ldots, d-1$ and let

$$t_j' = \lambda_j t_j + (1 - \lambda_j) t_{j+1}$$

for $j = 1, \ldots, d-1$. Then

$$\int_{t_1'}^{t_2} \cdots \int_{t_{d-1}'}^{t_d} V(s_1, \ldots, s_{d-1}) \, ds_1 \cdots ds_{d-1} \geq c(\lambda_1, \ldots, \lambda_{d-1}) V(t_1, \ldots, t_d). \qquad (3.8)$$

Now choose $\lambda_1, \ldots, \lambda_{d-1} \in (0, 1)$ and $\delta' > 0$ such that if $t_j' \leq s_j \leq t_{j+1}$ for $j = 1, \ldots, d-1$ then

$$s_{\delta'} \doteq \left(1 - (d-2)\delta'\right) s_{d-1} + \delta'(s_1 + s_2 + \cdots + s_{d-2})$$

$$\geq t_\delta = \left(1 - (d-1)\delta\right) t_d + \delta(t_1 + t_2 + \cdots + t_{d-1}). \qquad (3.9)$$

(Here is how to make this choice: we can assume that $t_d = 1$. If

$$1 - (d-2)\delta' > 0,$$

then (3.9) holds for all $s_j \in [t_j', t_{j+1}]$ if and only if it holds for $s_j = t_j'$. So fix $s_j = t_j'$ for $j = 1, \ldots, d-1$. Then, with $\lambda = (\lambda_1, \ldots, \lambda_{d-1})$,

$$s_{\delta'} = (1 - \lambda_{d-1})\left(1 - (d-2)\delta'\right) + \sum_{j=1}^{d-1} c_j(\delta', \lambda) t_j$$

where

$$\left| c_j(\delta', \lambda) \right| = O\left(\delta' + \|\lambda\|\right).$$

Assume that δ' and λ are chosen so that

$$(1 - \lambda_{d-1})(1 - (d-2)\delta') \geqslant (1 - (d-1)\delta) \quad \text{and} \quad |c_j(\delta', \lambda)| \leqslant \delta.$$

Since

$$t_\delta = (1 - (d-1)\delta) + \delta(t_1 + \cdots + t_{d-1}),$$

it then follows from the fact that $s_{\delta'} = t_\delta = 1$ when $t_1 = \cdots = t_{d-1} = 1$ that $s_{\delta'} \geqslant t_\delta$ if $0 \leqslant t_j \leqslant 1$.)

Now

$$(3.7) \geqslant \int_{t_1'}^{t_2} \cdots \int_{t_{d-1}'}^{t_d} \int_{\{u \geqslant t_\delta\}} \psi(u; s_1, \ldots, s_{d-1}) \, du \, ds_1 \cdots ds_{d-1}$$

$$\geqslant \int_{t_1'}^{t_2} \cdots \int_{t_{d-1}'}^{t_d} \int_{\{u \geqslant s_{\delta'}\}} \psi(u; s_1, \ldots, s_{d-1}) \, du \, ds_1 \cdots ds_{d-1}$$

$$\geqslant c(\delta') \int_{t_1'}^{t_2} \cdots \int_{t_{d-1}'}^{t_d} V(s_1, \ldots, s_{d-1}) \, ds_1 \cdots ds_{d-1}$$

$$\geqslant c(\delta'; \lambda_1, \ldots, \lambda_{d-1}) V(t_1, \ldots, t_d),$$

where the second inequality is due to (3.9) and the fact that $\psi(u; s_1, \ldots, s_d)$ is nonnegative, the third to the induction hypothesis, and the fourth to (3.8). This completes the proof by induction on d of (3.5).

To see (3.6), recall from (3.2) that

$$n_j = \frac{d(d+1)}{2} \cdot \frac{l_j}{2^n}$$

for some large integer n and positive integers l_j satisfying

$$\sum_{j=1}^{d} \frac{l_j}{2^n} = 1.$$

Choose $\delta > 0$ so small that

$$\delta < \frac{l_j}{2^n}$$

for $j = 1, \ldots, d-1$. Note that, since $t_j < t_d$ if $j < d$,

$$t_\delta = (1 - (d-1)\delta)t_d + \delta(t_2 + \cdots + t_{d-1}) \geqslant \sum_{j=1}^{d} \frac{l_j}{2^n} t_j.$$

Now the inequality

$$A^n \omega\left(\frac{s_1 + \cdots + s_{2^n}}{2^n}\right) \geqslant \left(\prod_{j=1}^{2^n} \omega(s_j)\right)^{1/2^n}$$

(which follows from iterating (1.3)) and the monotonicity of ω imply that

$$\omega(t_\delta) \geqslant \omega\left(\frac{1}{2^n}\sum_{j=1}^d l_j t_j\right) \geqslant A^{-n} \prod_{j=1}^d \omega(t_j)^{\frac{l_j}{2^n}}.$$

This gives (3.6).   □

Proof of Lemma 2.2. By scaling we can assume that $\rho = 1$. Partition $(c, d]$ into disjoint intervals $I_j = (a_j, a_{j+1}]$ such that $2^j \leqslant \omega \leqslant 2^{j+1}$ on I_j. Assume $d \in I_{j_0}$. We will need the inequality

$$\left|\left\{t \in \mathbb{R}: \prod_{l=1}^K |t - c_l| \leqslant \tau\right\}\right| \leqslant C(K)\tau^{1/K}, \quad \tau > 0. \tag{3.10}$$

(To see (3.10), observe that $\mathbb{R}$ can be partitioned into at most $2K$ intervals J_p with the property that

$$\prod_{l=1}^K |t - c_l| \geqslant |t - c_{l(p)}|^K, \quad t \in J_p.)$$

From (3.10) it follows that if

$$E_j = \left\{t \in I_j: \omega(t)^{K-1}\omega(d) \prod_{l=1}^K |t - c_l| \leqslant 1\right\},$$

then

$$|E_j| \leqslant \frac{C(K)}{2^{(j(K-1)+j_0)/K}}.$$

Thus

$$\int_{E_j} \omega(t)\,dt \leqslant C(K)2^{(j-j_0)/K},$$

and the conclusion of Lemma 2.2 follows by summing a geometric series.   □

Proof of Lemma 2.3. We begin by observing that

$$\rho = \int_E \omega(t)|t - t_0|^\eta |t - t_0|^{-\eta}\, dt$$

$$\leqslant \left(\int_E \omega(t)^r |t - t_0|^{r\eta}\, dt \right)^{1/r} \left(\int_E |t - t_0|^{-r'\eta}\, dt \right)^{1/r'}$$

$$\leqslant C(r, \eta) \left(\int_E \omega(t)^r |t - t_0|^{r\eta}\, dt \right)^{1/r} |E|^{1-\eta-1/r}.$$

Thus, by the monotonicity of ω,

$$\rho^{1+r\eta} \big(\omega(c)|E| \big)^{r-1-r\eta} \leqslant \rho^{1+r\eta} \rho^{r-1-r\eta} = \rho^r$$

$$\leqslant C(r, \eta) \int_E \omega(t)^r |t - t_0|^{r\eta}\, dt \cdot |E|^{r-1-r\eta}.$$

This gives the first conclusion of Lemma 2.3. Using the estimate

$$\int_E \big(|t - t_1| \cdot |t - t_2| \big)^{-r'\eta}\, dt \leqslant C(r, \eta)|E|^{1-r'\eta} |t_1 - t_2|^{-r'\eta},$$

the second conclusion follows similarly. $\quad\square$

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