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On reduction maps and support problem in K-theory and abelian varieties

Stefan Barańczuk

Department of Mathematics, Adam Mickiewicz University, Poznań, Poland

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Abstract

In this paper we consider orders of images of nontorsion points by reduction maps for abelian varieties defined over number fields and for odd dimensional K-groups of number fields. As an application we obtain the generalization of the support problem for abelian varieties and K-groups.

1. Introduction

By Supp(m) we will denote the set of prime numbers dividing a positive number m. Pál Erdös asked the following question:

Suppose that for some integers x, y, the following condition holds

$$\operatorname{Supp}\left(x^n - 1\right) = \operatorname{Supp}\left(y^n - 1\right)$$

for every natural number n. Is then $x = y$?

Corrales-Rodrigáñez and Schoof answered the question and proved its analogue for number fields and for elliptic curves in [C-RS].

Schinzel proved the support problem for the pair of sets of natural numbers in [S].

E-mail address: stefbar@amu.edu.pl.

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Banaszak, Gajda and Krasoń examined the support problem for abelian varieties for which the images of the l-adic representation is well controlled and for K-theory of number fields in [BGK1,BGK2].

The support problem for abelian varieties over number fields was considered independently by Khare and Prasad in [KP].

Larsen in [Lar] gave a solution of the support problem for all abelian varieties over number fields.

Weston gave in [We] a solution to a question of Gajda which is related to the support problem for abelian varieties. In [BGK3] Banaszak, Gajda and Krasoń considered similar question as in Weston's paper in the framework of Mordell–Weil systems. In the present work I apply this framework.

In this paper we consider the generalization of the support problem for K-theory and abelian varieties; namely, we deal with the pair of sets of points instead of pair of points. Let us state, for example, Theorem 8.2.

Theorem 8.2. *Let A be an abelian variety defined over number field F such that* $\mathrm{End}(A)$ *is an integral domain. Let $P_1, \dots, P_n, Q_1, \dots, Q_n \in A(F)$ be the points of infinite order. Assume that for almost every prime l the following condition holds*:

For every set of positive integers $m_1, \dots, m_n$ and for almost every prime v

$$m_1 r_v(P_1) + \cdots + m_n r_v(P_n) = 0 \quad \textit{implies} \quad m_1 r_v(Q_1) + \cdots + m_n r_v(Q_n) = 0.$$

Then there exist $k_i \in \mathbb{Z} \setminus \{0\}$, $\beta_i \in \mathrm{End}(A) \setminus \{0\}$ such that $k_i P_i + \beta_i Q_i = 0$ for every $i \in \{1, \dots, n\}$.

Intuitively, the condition from the generalized support problem means that for almost every v each linear dependence satisfied by the points $r_v(P_1), \dots, r_v(P_n)$ is also satisfied by the points $r_v(Q_1), \dots, r_v(Q_n)$.

The main technical result of the paper is Theorem 5.1 which lets us control the images of linearly independent points of K-groups and abelian varieties over number fields via reduction maps. These theorems are the refinement of Theorem 3.1 of [BGK3] and are proven using similar methods. I have recently found out that Pink has proven by a different method a result similar to Theorem 5.1 in the abelian variety case, cf. [Pink, Corollary 4.3].

2. Groups of the Mordell–Weil type

The following axiomatic setup of Mordell–Weil systems was developed in [BGK3].

Notation.

$\mathbb{N}$	the set of positive integers
l	a prime number
F	a number field, $\mathcal{O}_F$ its ring of integers
$\bar{F}$	fixed algebraic closure of F
G_F	$= G(\bar{F}/F)$
v	a finite prime of $\mathcal{O}_F$, $\kappa_v = \mathcal{O}_F/v$ the residue field at v
g_v	$= G(\bar{\kappa_v}/\kappa_v)$
T_l	a free $\mathbb{Z}_l$ -module of finite rank d

V_l	$= T_l \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$
A_l	$= V_l / T_l$
S_l	a fixed finite set of primes of $\mathcal{O}_F$ containing all primes above l
$\rho_l : G_F \to GL(T_l)$	a Galois representation unramified outside the set S_l
$\bar{\rho_{l^k}} : G_F \to GL(T_l / l^k)$	the residual representation induced by ρ_l
F_{l^k}	the number field $\bar{F}^{\operatorname{Ker} \bar{\rho_{l^k}}}$
F_{l^∞}	$= \bigcup_k F_{l^k}$
G_l	$= G(F_l / F)$
G_{l^k}	$= G(F_{l^k} / F)$
G_{l^∞}	$= G(F_{l^\infty} / F)$
$C[l^k]$	the subgroup of l^k-torsion elements of an abelian group C
C_l	$= \bigcup_k C[l^k]$, the l-torsion subgroup of C.

Let L/F be a finite extension contained in $\bar{F}$ and w a finite prime in L. We write $w \notin S_l$ to indicate that w is not over any prime in S_l.

Let $\mathcal{O}$ be a ring with unity, free as $\mathbb{Z}$-module, which acts on T_l in such a way that the action commutes with the G_F action. All modules over the ring $\mathcal{O}$ considered in this paper are left $\mathcal{O}$-modules.

Let $\{B(L)\}_L$ be a direct system of finitely generated $\mathcal{O}$-modules indexed by all finite field extensions L/F. We assume that for every embedding $L \hookrightarrow L'$ of extensions of F the induced structure map $B(L) \to B(L')$ is a homomorphism of $\mathcal{O}$-modules.

Similarly, for every prime v of F we define a direct system $\{B_v(\kappa_w)\}_{\kappa_w}$ of $\mathcal{O}$ -modules where κ_w is a residue field for a prime w over v in a finite extension L/F. We suppose that the system $\{B_v(\kappa_w)\}_{\kappa_w}$ is compatible with G_F action. Namely, if $\kappa_{w'}$ is a residue field for a prime w' over w in a finite extension L'/L then a natural map $B_v(\kappa_w) \to B_v(\kappa_{w'})$ assumes G_F action in the following way: if $\sigma \in G_F$ then the map $B_v(\kappa_{\sigma(w)}) \to B_v(\kappa_{\sigma(w')})$ is the image of the map $B_v(\kappa_w) \to B_v(\kappa_{w'})$ under σ.

We make the following assumptions on the action of the G_F and $\mathcal{O}$:

(A1) for each l, each finite extension L/F and any prime w of L, such that $w \notin S_l$ we have $T_l^{Fr_w} = 0$, where $Fr_w \in g_w$ denotes the arithmetic Frobenius at w;

(A2) for every L and $w \notin S_l$ there are natural maps $\psi_{l,L}$, $\psi_{l,w}$ and r_w respecting G_F and $\mathcal{O}$ actions such that the diagram commutes:

$$\begin{array}{ccc} B(L) \otimes \mathbb{Z}_l & \xrightarrow{\ r_w\ } & B_v(\kappa_w)_l \\ \psi_{l,L} \big\downarrow & & \cong \big\downarrow \psi_{l,w} \\ H^1_{f,S_l}(G_L, T_l) & \xrightarrow{\ r_w\ } & H^1(g_w, T_l) \end{array} \qquad (1)$$

where $H^1_{f,S_l}(G_L, T_l)$ is the group defined by Bloch and Kato [BK]. The left (respectively the right) vertical arrow in the diagram (1) is an embedding (respectively an isomorphism) for every L (respectively for every $w \notin S_l$);

(A3) either for every L the map $\psi_{l,L}$ is an isomorphism for almost all l or $B(\bar{F})$ is a discrete G_F-module divisible by l for almost l;

(A4) for every L we have: $B(\bar{F})^{G_L} \cong B(L)$ and there is a Galois equivariant and $\mathcal{O}$-equivariant map $j_{L,l}$ such that the following diagram commutes:

$$\begin{array}{ccc} & & B(L) \\ & \overset{j_{L,l}}{\nearrow} & \downarrow \\ & & B(L) \otimes \mathbb{Z}_l \\ & & \big\downarrow \\ H^0(G_L, A_l) & \hookrightarrow & H^1_{f,S_l}(G_L, T_l) \end{array}$$

and by abuse of notation we will consider $H^0(G_L, A_l)$ as a subgroup of $B(L)$.

As in [Ri] we impose the following four axioms on the representations which we consider:

(B1) $\mathrm{End}_{G_l}(A_l[l]) \cong \mathcal{O}/l\mathcal{O}$, for almost all l and $\mathrm{End}_{G_{l^\infty}}(T_l) \cong \mathcal{O} \otimes \mathbb{Z}_l$, for all l;
(B2) $A_l[l]$ is a semisimple $\mathbb{F}_l[G_l]$-module for almost all l and V_l is a semisimple $\mathbb{Q}_l[G_{l^\infty}]$-module for all l;
(B3) $H^1(G_l, A_l[l]) = 0$ for almost all l and $H^1(G_{l^\infty}, T_l)$ is a finite group for all l;
(B4) for each finitely generated subgroup $\Gamma \subset B(F)$ the group

$$\Gamma' = \{P \in B(F)\colon\ mP \in \Gamma \text{ for some } m \in \mathbb{N}\}$$

is such that Γ'/Γ has a finite exponent.

For a point $R \in B(L)$ (respectively a subgroup $\Gamma \subset B(F)$) we denote $\hat{R} = \psi_{l,L}(R)$ (respectively $\hat{\Gamma} = \psi_{l,L}(\Gamma)$).

Definition 2.1. The system of modules $\{B(L)\}_L$ fulfilling the above axioms is called a *Mordell–Weil system.*

3. Examples of Mordell–Weil systems

In all cases below the axioms (A_1)–(A_4) and (B_1)–(B_4) are satisfied by [BGK3], proofs of Theorems 4.1 and 4.2 loc. cit. In particular, in the case of abelian varieties, the assumptions are fulfilled due to results of Faltings [Fa], Zarhin [Za], Serre [Serre], and Mordell and Weil.

The cyclotomic character. Let $T_l = \mathbb{Z}_l(1)$, $V_l = \mathbb{Q}_l(1)$, $A_l = \mathbb{Q}_l/\mathbb{Z}_l(1)$. Let S be an arbitrary finite set of primes in $\mathcal{O}_F$. We put

$$B(L) = \mathcal{O}_{L,S}^{\times}$$

for any finite extension L/F and we have $\mathcal{O} = \mathbb{Z}$.

Algebraic K-theory of number fields. Let n be a natural number. For every finite extension L/F consider the Dwyer–Friedlander maps [DF]:

$$K_{2n+1}(L) \to K_{2n+1}(L) \otimes \mathbb{Z}_l \to H^1\big(G_L; \mathbb{Z}_l(n+1)\big),$$

where the action of G_L on $\mathbb{Z}_l(n+1)$ is given by the $(n+1)$th tensor power of the cyclotomic character.

Let C_L be the subgroup of $K_{2n+1}(L)$ generated by the l-parts of kernels of the maps $K_{2n+1}(L) \to H^1(G_L; \mathbb{Z}_l(n+1))$ for all primes l. By [DF] C_L is finite and according to Quillen–Lichtenbaum conjecture should be trivial. We put

$$B(L) = K_{2n+1}(L)/C_L$$

and we have $\mathcal{O} = \mathbb{Z}$. The map $\psi_{L,l}$ is induced by the Dwyer–Friedlander map.

Abelian varieties over number fields. Let A be an abelian variety over number field F and let

$$\rho_l : G_F \to GL\big(T_l(A)\big)$$

be the l-adic representation given by the action of absolute Galois group on the Tate module $T_l(A)$ of A. In this case we put $B(L) = A(L)$ for every finite field extension L/F and $\mathcal{O} = \mathrm{End}(A)$.

4. Kummer theory for l-adic representations

We introduce the Kummer theory for l-adic representations, following [Ri,BGK3]. Let Λ be a finitely generated free $\mathcal{O}$-submodule of $B(F)$ with basis $P_1, \ldots, P_r$.

For natural numbers k we have the Kummer maps:

$$\phi^k_{P_i} : G(\bar{F}/F_{l^k}) \to A_l\big[l^k\big],$$
$$\phi^k_{P_i}(\sigma) = \sigma\left(\frac{1}{l^k}\hat{P}_i\right) - \frac{1}{l^k}\hat{P}_i.$$

These maps are well defined by the definition of maps $j_{L,l}$ in axiom (A4).

We define

$$\Phi^k : G(\bar{F}/F_{l^k}) \to \bigoplus_{i=1}^{r} A_l\big[l^k\big], \quad \Phi^k = \big(\phi^k_{P_1}, \ldots, \phi^k_{P_r}\big).$$

We define the field:

$$F_{l^k}\left(\frac{1}{l^k}\hat{\Lambda}\right) := \bar{F}^{\,\mathrm{Ker}\,\Phi^k}.$$

Taking the inverse limit in the following commutative diagram

$$\begin{array}{ccc} G(\bar{F}/F_{l^k}) & \xrightarrow{\phi_{P_i}^k} & A_l[l^k], \\ \downarrow & & \downarrow \times l \\ G(\bar{F}/F_{l^{k-1}}) & \xrightarrow{\phi_{P_i}^{k-1}} & A_l[l^{k-1}] \end{array}$$

we obtain a map:

$$\phi_{P_i}^{\infty}: G(\bar{F}/F_{l^\infty}) \to T_l.$$

We define:

$$\Phi^{\infty}: G(\bar{F}/F_{l^\infty}) \to \bigoplus_{i=1}^{r} T_l, \quad \Phi = (\phi_{P_1}^{\infty}, \dots, \phi_{P_r}^{\infty}).$$

Lemma 4.1. *For k big enough*

$$F_{l^k}\left(\frac{1}{l^k}\hat{\Lambda}\right) \cap F_{l^{k+1}} = F_{l^k}.$$

Proof. The proof follows the lines of step 1 of the proof of Proposition 2.2 in [BGK4] that partly repeats the argument in the proof of Lemma 5 in [KP].

Consider the following commutative diagram:

$$\begin{array}{ccc} G\big(F_{l^\infty}\big(\frac{1}{l^\infty}\hat{\Lambda}\big)/F_{l^\infty}\big) & \longrightarrow & T_l^r/l^m T_l^r \\ \downarrow & & \downarrow \cong \\ G\big(F_{l^{k+1}}\big(\frac{1}{l^{k+1}}\hat{\Lambda}\big)/F_{l^{k+1}}\big) & \longrightarrow & \big(A_l[l^{k+1}]\big)^r/l^m\big(A_l[l^{k+1}]\big)^r \\ \downarrow & & \downarrow \cong \\ G\big(F_{l^k}\big(\frac{1}{l^k}\hat{\Lambda}\big)/F_{l^k}\big) & \longrightarrow & \big(A_l[l^k]\big)^r/l^m\big(A_l[l^k]\big)^r \end{array}$$

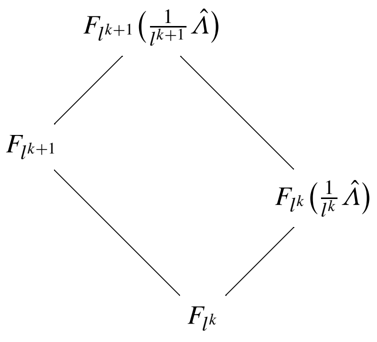
where the horizontal arrows are the Kummer maps and $m \in \mathbb{N}$ is big enough so that $l^m T_l^r \subset \operatorname{Im}(G(F_{l^\infty}(\frac{1}{l^\infty}\hat{\Lambda})/F_{l^\infty}) \to T_l^r)$. Such m exists by [BGK3] Lemma 2.13.

Now we see that for k big enough the images of the maps

$$G\left(F_{l^k}\left(\frac{1}{l^k}\hat{\Lambda}\right)/F_{l^k}\right) \to \big(A_l[l^k]\big)^r/l^m\big(A_l[l^k]\big)^r$$

must be all isomorphic. Hence the maps $G(F_{l^{k+1}}(\frac{1}{l^{k+1}}\hat{\Lambda})/F_{l^{k+1}}) \to G(F_{l^k}(\frac{1}{l^k}\hat{\Lambda})/F_{l^k})$ are surjective.

Now the diagram



shows that

$$F_{l^k}\left(\frac{1}{l^k}\hat{\Lambda}\right) \cap F_{l^{k+1}} = F_{l^k}. \qquad \square$$

5. Main technical result

Theorem 5.1. *Assume that $\rho(G_F)$ contains an open subgroup of the group of homotheties. Let*

$$P_1, \ldots, P_s \in B(F)$$

be points of infinite order, which are linearly independent over $\mathcal{O}$.

Then for any prime l, and for any set $\{k_1, \ldots, k_s\} \subset \mathbb{N} \cup \{0\}$, there are infinitely many primes v, such that the image of the point P_t via the map

$$r_v : B(F) \to B_v(\kappa_v)_l$$

has order equal to l^{k_t} for every $t \in \{1, \ldots, s\}$.

Proof. Let us rename the points $P_1, \ldots, P_s \in B(F)$ in the following way:

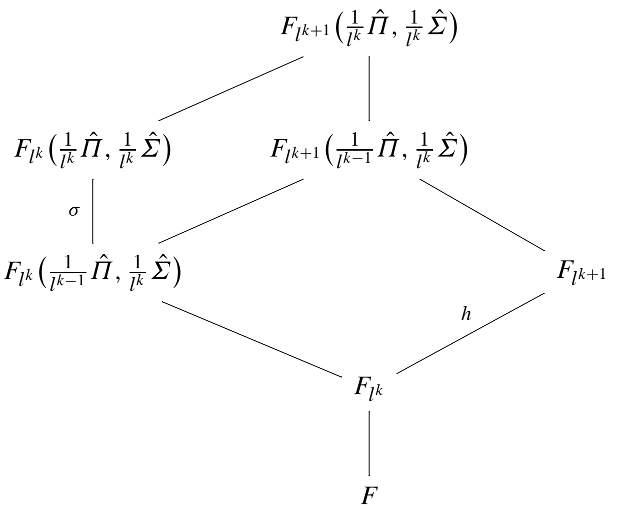
$$P_1, \ldots, P_i, Q_1, \ldots, Q_j \in B(F),$$

and we are going to show that for any prime l, and for any set $\{k_1, \ldots, k_i\} \subset \mathbb{N}$, there are infinitely many primes v, such that the image of the point P_t via the map

$$r_v : B(F) \to B_v(\kappa_v)_l$$

has order equal to l^{k_t} for every $t \in \{1, \ldots, i\}$ and the images of the points $Q_1, \ldots, Q_j$ are trivial. It is enough to prove the theorem in a case when $k_t = 1$ for every $t \in \{1, \ldots, i\}$, since if $r_v(l^{k_t-1}P_t)$ has order equal to l then $r_v(P_t)$ has order equal to l^{k_t}.

We will make use of the following diagram:



where Π (respectively Σ) is the $\mathcal{O}$-submodule of $B(F)$ generated by $P_1, \dots, P_i$ (respectively by $Q_1, \dots, Q_j$).

It follows by Lemma 4.1 applied to the $\mathcal{O}$-submodule of $B(F)$ generated by $lP_1, \dots, lP_i, Q_1, \dots, Q_j$ that for k big enough

$$F_{l^k}\left(\frac{1}{l^{k-1}}\hat{\Pi}, \frac{1}{l^k}\hat{\Sigma}\right) \cap F_{l^{k+1}} = F_{l^k}.$$

Step 1. Consider the following commutative diagram:

$$\begin{array}{ccc}
G\left(F_{l^\infty}\left(\frac{1}{l^\infty}\hat{\Pi}, \frac{1}{l^\infty}\hat{\Sigma}\right)/F_{l^\infty}\left(\frac{1}{l^\infty}\hat{\Sigma}\right)\right) & \longrightarrow & T_l^i \\
\downarrow & & \downarrow \\
G\left(F_{l^\infty}\left(\frac{1}{l^k}\hat{\Pi}, \frac{1}{l^\infty}\hat{\Sigma}\right)/F_{l^\infty}\left(\frac{1}{l^\infty}\hat{\Sigma}\right)\right) & \hookrightarrow & \left(A_l\left[l^k\right]\right)^i \\
\downarrow & & \downarrow l \\
G\left(F_{l^\infty}\left(\frac{1}{l^{k-1}}\hat{\Pi}, \frac{1}{l^\infty}\hat{\Sigma}\right)/F_{l^\infty}\left(\frac{1}{l^\infty}\hat{\Sigma}\right)\right) & \hookrightarrow & \left(A_l\left[l^{k-1}\right]\right)^i
\end{array} \tag{2}$$

The horizontal arrows in the diagram (2) are the Kummer maps. The upper horizontal arrow has finite cokernel by [BGK3, Lemma 2.13], so for k big enough the horizontal arrows have cokernels bounded independently of k. Hence for k big enough there exists $\sigma \in G(F_{l^\infty}(\frac{1}{l^k}\hat{\Pi}, \frac{1}{l^\infty}\hat{\Sigma})/F_{l^\infty}(\frac{1}{l^{k-1}}\hat{\Pi}, \frac{1}{l^\infty}\hat{\Sigma}))$ such that σ maps via the Kummer map

$$G\left(F_{l^\infty}\left(\frac{1}{l^k}\hat{\Pi}, \frac{1}{l^\infty}\hat{\Sigma}\right)\Big/ F_{l^\infty}\left(\frac{1}{l^{k-1}}\hat{\Pi}, \frac{1}{l^\infty}\hat{\Sigma}\right)\right) \to \left(A_l[l]\right)^i \tag{3}$$

to an element whose all i projections on the direct summands $(A_l[l])^i$ are nontrivial. Then the following tower of fields

$$\begin{array}{ccc} & F_{l^\infty}\left(\frac{1}{l^k}\hat{\Pi}, \frac{1}{l^\infty}\hat{\Sigma}\right) & \\ \diagup & & \diagdown \\ F_{l^\infty}\left(\frac{1}{l^{k-1}}\hat{\Pi}, \frac{1}{l^\infty}\hat{\Sigma}\right) & & F_{l^k}\left(\frac{1}{l^k}\hat{\Pi}, \frac{1}{l^k}\hat{\Sigma}\right) \\ \diagdown & & \diagup \\ & F_{l^k}\left(\frac{1}{l^{k-1}}\hat{\Pi}, \frac{1}{l^k}\hat{\Sigma}\right) & \end{array} \tag{4}$$

shows that there exist $\sigma \in G(F_{l^k}(\frac{1}{l^k}\hat{\Pi}, \frac{1}{l^k}\hat{\Sigma})/F_{l^k}(\frac{1}{l^{k-1}}\hat{\Pi}, \frac{1}{l^k}\hat{\Sigma}))$ such that σ maps via the Kummer map

$$G\left(F_{l^k}\left(\frac{1}{l^k}\hat{\Pi}, \frac{1}{l^k}\hat{\Sigma}\right) \Big/ F_{l^k}\left(\frac{1}{l^{k-1}}\hat{\Pi}, \frac{1}{l^k}\hat{\Sigma}\right)\right) \to \left(A_l[l]\right)^i \tag{5}$$

to an element whose all i projections on the direct summands $(A_l[l])^i$ are nontrivial.

Step 2. Let k be big enough that there is an element σ constructed in a previous step and such that there exists a nontrivial homothety $h \in G(F_{l^{k+1}}/F_{l^k})$ acting on the module T_l as a multiplication by $1 + l^k u_0$, for some $u_0 \in \mathbb{Z}_l^\times$.

We choose an automorphism

$$\gamma \in G\left(F_{l^{k+1}}\left(\frac{1}{l^k}\hat{\Pi}, \frac{1}{l^k}\hat{\Sigma}\right)/F\right)$$

such that

$$\gamma|_{F_{l^k}(\frac{1}{l^k}\hat{\Pi}, \frac{1}{l^k}\hat{\Sigma})} = \sigma,$$
$$\gamma\,|_{F_{l^{k+1}}} = h.$$

By the Tchebotarev Density Theorem there exist infinitely many prime ideals v in $\mathcal{O}_F$ such that γ is equal to the Frobenius element for the prime v in the extension $F_{l^{k+1}}(\frac{1}{l^k}\hat{\Pi}, \frac{1}{l^k}\hat{\Sigma})/F$. In the remainder of the proof we work with prime ideals v we have just selected.

Step 3. Using the same argument as in [BGK3], step 4, we show that $lr_v(P_1), \ldots, lr_v(P_i)$, $r_v(Q_1), \ldots, r_v(Q_j)$ are trivial in $B_v(\kappa_v)_l$:

Let P denote any of the points $lP_1, \ldots, lP_i, Q_1, \ldots, Q_j$. Let l^c be an order of the point $r_v(P)$ in the group $B_v(\kappa_v)_l$ (see the axiom (A2)). Let w_1 be a prime ideal of $F_{l^{k+1}}(\frac{1}{l^{k-1}}\hat{\Pi}, \frac{1}{l^k}\hat{\Sigma})$ below w. A point $\frac{1}{l^k}P \in B(F_{l^{k+1}}(\frac{1}{l^{k-1}}\hat{\Pi}, \frac{1}{l^k}\hat{\Sigma}))$ and its image $r_{w_1}(\frac{1}{l^k}P) \in B_v(\kappa_{w_1})_l$ has order equal l^{k+c}. By the choice v the point $r_{w_1}(\frac{1}{l^k}P)$ comes from an element of $B_v(\kappa_v)_l$. The right vertical arrow in the diagram (1) is an isomorphism, hence by the choice of v,

$$h\left(r_{w_1}\left(\frac{1}{l^k}P\right)\right) = \left(1 + l^k u_0\right) r_{w_1}\left(\frac{1}{l^k}P\right).$$

But $r_{w_1}(\frac{1}{l^k}P) \in B_v(\kappa_v)_l$, hence $h(r_{w_1}(\frac{1}{l^k}P)) = r_{w_1}(\frac{1}{l^k}P)$, again by the choice of v. Thus $l^k r_{w_1}(\frac{1}{l^k}P) = 0$ and $c = 0$.

Step 4. Using similar argument, as in [BGK3] step 5, we show that $r_v(P_1), \dots, r_v(P_i)$ have order divisible by l in $B_v(\kappa_v)_l$:

Let w_2 denote the prime ideal in $F_{l^k}(\frac{1}{l^k}\hat{\Pi}, \frac{1}{l^k}\hat{\Sigma})$ below w and let u_2 denote the prime in F_{l^k} below w_2. Consider the following commutative diagram:

$$\begin{array}{ccc} B(F)/l^k B(F) & \longrightarrow & B_v(\kappa_v)/l^k B_v(\kappa_v) \\ \downarrow & & \downarrow \\ B(F_{l^k})/l^k B(F_{l^k}) & \longrightarrow & B_v(\kappa_{u_2})/l^k B_v(\kappa_{u_2}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}\big(G(\bar{F}/F_{l^k})^{ab}; A_l\big[l^k\big]\big) & \longrightarrow & \mathrm{Hom}\,\big(g_{u_2}; A_l[l^k]\big). \end{array} \qquad (6)$$

Every point P_j maps via the left vertical arrow in the diagram (6) to the Kummer map $\phi^k_{P_j}$. The homomorphism $\phi^k_{P_j}$ factors through the group $G(F_{l^k}(\frac{1}{l^k}\hat{\Pi}, \frac{1}{l^k}\hat{\Sigma})/F_{l^k})$. We denote this factorization with the same symbol $\phi^k_{P_j}$. By the choice of the automorphism γ the element $\phi^k_{P_j}(\gamma|_{F_{l^k}((1/l^k)\hat{\Pi},(1/l^k)\hat{\Sigma})}) \in A_l[l^k]$ is nontrivial. Hence the element $\phi^k_{P_j} \in \mathrm{Hom}(G(\bar{F}/F_{l^k})^{ab}; A_l[l^k])$ is nontrivial. Thus by the choice of v image of the element $\phi^k_{P_j}$ via map $\mathrm{Hom}(H^{ab}_{l^k}; A_l[l^k]) \to \mathrm{Hom}(g_{u_2}; A_l[l^k])$ from diagram (6) is nontrivial. Hence the image of P_j via the bottom horizontal arrow in the diagram (6) is nontrivial.

Thus every point $r_v(P_1), \dots, r_v(P_i)$ has the order divisible by l in $B_v(\kappa_v)_l$. But step 3 shows that every point $lr_v(P_1), \dots, lr_v(P_i)$ is trivial. Hence elements $r_v(P_1), \dots, r_v(P_i)$ have orders equal l. □

6. Support problem for *K*-theory and abelian varieties

Let $\mathcal{O}$ be an integral domain.

Theorem 6.1. *Let $P_1, \dots, P_n, P_0, Q_1, \dots, Q_n, Q_0 \in B(F)$ be the points nontorsion over $\mathcal{O}$. Assume that for almost every prime l the following condition holds in the group $B_v(\kappa_v)_l$:*

For every set of nonnegative integers $m_1, \dots, m_n$ and for almost every prime v

$$m_1 r_v(P_1) + \cdots + m_n r_v(P_n) = r_v(P_0) \quad \textit{implies} \quad m_1 r_v(Q_1) + \cdots + m_n r_v(Q_n) = r_v(Q_0).$$

Then there exist $\alpha_i, \beta_i \in \mathcal{O} \setminus \{0\}$ such that $\alpha_i P_i + \beta_i Q_i = 0$ in $B(F)$ for every $i \in \{0, \dots, n\}$.

Proof. Set $m_i = 0$ for every $i \in \{1, \dots, n\}$. We get

$$r_v(P_0) = 0 \quad \text{implies} \quad r_v(Q_0) = 0 \qquad (7)$$

for almost every prime v. Assume that P_0 and Q_0 are linearly independent in $B(F)$ over $\mathcal{O}$. By Theorem 5.1 there are infinitely many primes v such that $r_v(P_0)=0$ and $r_v(Q_0)$ has order l. This contradicts (7). Hence there exist $\alpha_0, \beta_0 \in \mathcal{O}\setminus\{0\}$ such that $\alpha_0 P_0 + \beta_0 Q_0 = 0$ in $B(F)$.

Now fix $m_1 = \cdots = m_{j-1} = m_{j+1} = \cdots = m_n = 0$. Let m_j be a natural number such that $m_j P_j + P_0$, $(m_j+1)P_j + P_0$, $(m_j+2)P_j + P_0$, $m_j Q_j + Q_0$, $(m_j+1)Q_j + Q_0$, $(m_j+2)Q_j + Q_0$ be nontorsion points.

As above we show that there exist $x_0, y_0, x_1, y_1, x_2, y_2 \in \mathcal{O}\setminus\{0\}$ such that

$$\begin{cases} x_0(m_j P_j + P_0) + y_0(m_j Q_j + Q_0) = 0, \\ x_1((m_j+1)P_j + P_0) + y_1((m_j+1)Q_j + Q_0) = 0, \\ x_2((m_j+2)P_j + P_0) + y_2((m_j+2)Q_j + Q_0) = 0. \end{cases}$$

Hence

$$\begin{cases} x_1 y_0 P_j + y_1 y_0 Q_j = (x_0 y_1 - x_1 y_0)(m_j P_j + P_0), \\ 2(x_2 y_0 P_j + y_2 y_0 Q_j) = (x_0 y_2 - x_2 y_0)(m_j P_j + P_0). \end{cases} \tag{8}$$

If $(x_0 y_1 - x_1 y_0) = 0$ or $(x_0 y_2 - x_2 y_0) = 0$ we are done. So assume that

$$(x_0 y_1 - x_1 y_0)(x_0 y_2 - x_2 y_0) \neq 0. \tag{9}$$

Hence from (8) we get

$$y_0\big(x_0(x_2 y_1 - x_1 y_2) + x_2(x_0 y_1 - x_1 y_0)\big)P_j = y_0\big(y_0(x_1 y_2 - x_2 y_1) + y_2(x_1 y_0 - x_0 y_1)\big)Q_j.$$

If $y_0(x_0(x_2 y_1 - x_1 y_2) + x_2(x_0 y_1 - x_1 y_0)) \neq 0$ or $y_0(y_0(x_1 y_2 - x_2 y_1) + y_2(x_1 y_0 - x_0 y_1)) \neq 0$ we are done. So assume that

$$\begin{cases} y_0(x_0(x_2 y_1 - x_1 y_2) + x_2(x_0 y_1 - x_1 y_0)) = 0, \\ y_0(y_0(x_1 y_2 - x_2 y_1) + y_2(x_1 y_0 - x_0 y_1)) = 0. \end{cases}$$

Then

$$\begin{cases} x_0(x_2 y_1 - x_1 y_2) + x_2(x_0 y_1 - x_1 y_0) = 0, \\ y_0(x_1 y_2 - x_2 y_1) + y_2(x_1 y_0 - x_0 y_1) = 0. \end{cases}$$

Hence we get

$$(x_0 y_1 - x_1 y_0)(x_0 y_2 - x_2 y_0) = 0$$

that contradicts (9). □

Theorem 6.2. *Let $P_1, \ldots, P_n, Q_1, \ldots, Q_n \in B(F)$ be the points nontorsion over $\mathcal{O}$. Assume that for almost every prime l the following condition holds in the group $B_v(\kappa_v)_l$:*

For every set of natural numbers $m_1, \ldots, m_n$ and for almost every prime v

$$m_1 r_v(P_1) + \cdots + m_n r_v(P_n) = 0 \quad \textit{implies} \quad m_1 r_v(Q_1) + \cdots + m_n r_v(Q_n) = 0.$$

Then there exist $\alpha_i, \beta_i \in \mathcal{O}\setminus\{0\}$ such that $\alpha_i P_i + \beta_i Q_i = 0$ in $B(F)$ for every $i \in \{1, \ldots, n\}$.

Proof. The proof of the theorem is analogous to the proof of Theorem 6.1:

Let m_j be a natural number such that

$$m_1P_1+\cdots+m_{j-1}P_{j-1}+m_jP_j+m_{j+1}P_{j+1}+\cdots+m_nP_n,$$
$$m_1P_1+\cdots+m_{j-1}P_{j-1}+(m_j+1)P_j+m_{j+1}P_{j+1}+\cdots+m_nP_n,$$
$$m_1P_1+\cdots+m_{j-1}P_{j-1}+(m_j+2)P_j+m_{j+1}P_{j+1}+\cdots+m_nP_n,$$
$$m_1Q_1+\cdots+m_{j-1}Q_{j-1}+m_jQ_j+m_{j+1}Q_{j+1}+\cdots+m_nQ_n,$$
$$m_1Q_1+\cdots+m_{j-1}Q_{j-1}+(m_j+1)Q_j+m_{j+1}Q_{j+1}+\cdots+m_nQ_n,$$
$$m_1Q_1+\cdots+m_{j-1}Q_{j-1}+(m_j+2)Q_j+m_{j+1}Q_{j+1}+\cdots+m_nQ_n$$

be nontorsion points. There exist $x_0, y_0, x_1, y_1, x_2, y_2 \in \mathcal{O}\setminus\{0\}$ such that

$$\begin{cases} x_0(m_1P_1+\cdots+m_{j-1}P_{j-1}+m_jP_j+m_{j+1}P_{j+1}+\cdots+m_nP_n) \\ \quad + y_0(m_1Q_1+\cdots+m_{j-1}Q_{j-1}+m_jQ_j+m_{j+1}Q_{j+1}+\cdots+m_nQ_n)=0, \\ x_1(m_1P_1+\cdots+m_{j-1}P_{j-1}+(m_j+1)P_j+m_{j+1}P_{j+1}+\cdots+m_nP_n) \\ \quad + y_1(m_1Q_1+\cdots+m_{j-1}Q_{j-1}+(m_j+1)Q_j+m_{j+1}Q_{j+1}+\cdots+m_nQ_n)=0, \\ x_2(m_1P_1+\cdots+m_{j-1}P_{j-1}+(m_j+2)P_j+m_{j+1}P_{j+1}+\cdots+m_nP_n) \\ \quad + y_2(m_1Q_1+\cdots+m_{j-1}Q_{j-1}+(m_j+2)Q_j+m_{j+1}Q_{j+1}+\cdots+m_nQ_n)=0. \end{cases}$$

The rest of the proof follows the lines of the proof of Theorem 6.1. □

Remark 6.2.1. Assume that $\mathcal{O}=\mathcal{O}_E$ for some number field E. Assume that there exist $\alpha, \beta \in \mathcal{O}_E\setminus\{0\}$ such that $\alpha P+\beta Q=0$ in $B(F)$. Then there exist $z\in\mathbb{Z}\setminus\{0\}$ such that $z\frac{\beta}{\alpha}\in\mathcal{O}_E$ (see [Mol, p. 46]). Hence $zP+z\frac{\beta}{\alpha}Q=0$ in $B(F)$. We can then replace the expression "$\alpha_i, \beta_i \in \mathcal{O}\setminus\{0\}$" in Theorem 6.1 by "$\alpha_i\in\mathbb{Z}\setminus\{0\}$, $\beta_i\in\mathcal{O}\setminus\{0\}$."

7. The case $\mathcal{O}=\mathbb{Z}$

We consider the special case $\mathcal{O}=\mathbb{Z}$. The following lemma was proved in the abelian varieties case using different method by Larsen in [Lar]:

Lemma 7.1. *Let $P, Q\in B(F)$ be points of infinite order. Assume that for every prime number l the following condition holds in the group $B_v(\kappa_v)_l$:*

For every natural number n and for almost every prime v:

$$nr_v(P)=0 \quad \textit{implies} \quad nr_v(Q)=0. \tag{10}$$

Then there is an integer e such that $Q=eP$.

Proof. By Theorem 6.2 there are $\alpha, \beta\in\mathbb{Z}\setminus 0$ such that $\alpha P=\beta Q$. Let l^k be the largest power of prime number l that divides β, $\beta=bl^k$. By (10) we have

$$\alpha r_v(P)=0 \quad \text{implies} \quad \alpha r_v(Q)=0,$$

hence

$$\beta r_v(Q) = 0 \quad \text{implies} \quad \alpha r_v(Q) = 0$$

and

$$bl^k r_v(Q) = 0 \quad \text{implies} \quad \alpha r_v(Q) = 0.$$

But obviously $\alpha r_v(Q) = 0$ implies $b\alpha r_v(Q) = 0$. Hence we get

$$l^k r_v(bQ) = 0 \quad \text{implies} \quad \alpha r_v(bQ) = 0. \tag{11}$$

By Theorem 5.1 there are infinitely many primes v such that the order of $r_v(bQ)$ is l^k. So by (11) we get $\alpha r_v(bQ) = 0$ and l^k divides α.

Now repeating an argument from the proof of the Theorem 3.12 of [BGK3] we show that $Q = \frac{\alpha}{\beta} P$ with $\frac{\alpha}{\beta} \in \mathbb{Z}$:

We have $\frac{\alpha}{l^k} P = \frac{\beta}{l^k} Q + R$ where $R \in B(F)[l^k]$. By Theorem 5.1 and by Assumption 10 there are infinitely many primes v such that $r_v(P) = r_v(Q) = 0$. Hence we get $r_v(R) = 0$ for infinitely many primes v. But the map

$$r_v : B(F)_{\text{tor}} \to B_v(\kappa_v)$$

is an embedding for any prime $v \notin S_l$ by Lemma 3.11 of [BGK3]. Thus $R = 0$. □

Lemma 7.2. *Let $P_1, P_2, Q_1, Q_2 \in B(F)$ be points of infinite order. Assume that for every prime number l the following condition holds in the group $B_v(\kappa_v)_l$:*

For every set of natural numbers m_1, m_2 and for almost every prime v:

$$m_1 r_v(P_1) + m_2 r_v(P_2) = 0 \quad \textit{implies} \quad m_1 r_v(Q_1) + m_2 r_v(Q_2) = 0. \tag{12}$$

Then there is an integer e such that $Q_1 = eP_1$ and $Q_2 = eP_2$.

Proof. By Theorem 6.2 there are integers $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that $\alpha_1 P_1 = \beta_1 Q_1$, $\alpha_2 P_2 = \beta_2 Q_2$. We can assume that $\alpha_1, \alpha_2 > 0$.

Now we have to consider two cases.

First, assume that P_1 and P_2 are linearly independent over $\mathbb{Z}$. Hence P_1 and $|b|Q_2$ are also linearly independent, where $\beta_2 = bl^k$ and l^k is the largest power of prime number l that divides β_2.

By Theorem 5.1 there are infinitely many primes v such that $r_v(P_1) = 0$ and $r_v(|b|\,Q_2)$ has order l^k.

By (12), for $m_1 = |\beta_1|$ and $m_2 = \alpha_2$, and by the choice of v we have:

$$|\beta_2| r_v(Q_2) = 0 \quad \text{implies} \quad \alpha_2 r_v(Q_2) = 0,$$
$$l^k r_v\big(|b|Q_2\big) = 0 \quad \text{implies} \quad \alpha_2 r_v\big(|b|Q_2\big) = 0.$$

Again by the choice of v

$$\alpha_2 r_v\big(|b|Q_2\big) = 0.$$

Hence l^k divides α_2. Now we repeat again the argument from the proof of Theorem 3.12 of [BGK3] showing that $Q_2 = e_2 P_2$ for some nonzero integer e_2 and analogously $Q_1 = e_1 P_1$ for some nonzero integer e_1.

Now by (12)

$$r_v(P_1) + r_v(P_2) = 0 \quad \text{implies} \quad r_v(Q_1) + r_v(Q_2) = 0.$$

Hence

$$r_v(P_1) + r_v(P_2) = 0 \quad \text{implies} \quad (e_1 - e_2) r_v(P_2) = 0. \tag{13}$$

Let now k be arbitrary natural number and l be arbitrary prime number. By Theorem 5.1 there are infinitely many primes v such that $r_v(P_1 + P_2) = 0$ and $r_v(P_2)$ has order l^k. Hence by (13), l^k divides $e_1 - e_2$. So $e_1 - e_2 = 0$.

Now we assume that P_1 and P_2 are linearly dependent over $\mathbb{Z}$, i.e. there are numbers $x \in \mathbb{N}$ and $y \in \mathbb{Z} \setminus \{0\}$ such that $xP_1 = yP_2$. Hence $\alpha_2\beta_1 x Q_1 = \alpha_1\beta_2 y Q_2$. Put $m_2 = m_1\alpha_1\beta_2 y \operatorname{sgn}(\beta_2 y)$ in (12):

$$\begin{aligned} &m_1 r_v(P_1) + m_1\alpha_1\beta_2 y \operatorname{sgn}(\beta_2 y) r_v(P_2) = 0 \quad \text{implies} \\ &m_1 r_v(Q_1) + m_1\alpha_1\beta_2 y \operatorname{sgn}(\beta_2 y) r_v(Q_2) = 0, \end{aligned}$$

hence

$$m_1\big[1 + \alpha_1\beta_2 x \operatorname{sgn}(\beta_2 y)\big] r_v(P_1) = 0 \quad \text{implies} \quad m_1\big[1 + \alpha_2\beta_1 x \operatorname{sgn}(\beta_2 y)\big] r_v(Q_1) = 0.$$

Putting $P := [1 + \alpha_1\beta_2 x \operatorname{sgn}(\beta_2 y)]P_1$, $Q := [1 + \alpha_2\beta_1 x \operatorname{sgn}(\beta_2 y)]Q_1$ we get

$$m_1 r_v(P) = 0 \quad \text{implies} \quad m_1 r_v(Q) = 0.$$

By Lemma 7.1 there is an integer s such that $Q = sP$. So

$$\big[1 + \alpha_2\beta_1 x \operatorname{sgn}(\beta_2 y)\big] Q_1 = s\big[1 + \alpha_1\beta_2 x \operatorname{sgn}(\beta_2 y)\big] P_1.$$

Hence

$$Q_1 = \big[s\big(1 + \alpha_1\beta_2 x \operatorname{sgn}(\beta_2 y)\big) - \alpha_1\alpha_2 x \operatorname{sgn}(\beta_2 y)\big] P_1,$$

i.e. there is an integer e_1 such that $Q_1 = e_1 P_1$. Analogously there is an integer e_2 such that $Q_2 = e_2 P_2$.

Now, by (12), for $m_1 = x$, $m_2 = l^k - y$ where k is an arbitrary natural number and l is an arbitrary prime number such that $l^k - y > 0$, we get

$$l^k r_v(P_2) = 0 \quad \text{implies} \quad y(e_1 - e_2) r_v(P_2) = 0.$$

By Theorem 5.1 there are infinitely many primes v such that $r_v(P_2)$ has order l^k, so l^k divides $y(e_1 - e_2)$. But k was arbitrary, so $e_1 - e_2 = 0$. □

Theorem 7.3. *Let $P_1, \dots, P_n, Q_1, \dots, Q_n \in B(F)$ be the points of infinite order. Assume that for every prime number l the following condition holds in the group $B_v(\kappa_v)_l$:*

For every set of natural numbers $m_1, \dots, m_n$ and for almost every prime v

$$m_1 r_v(P_1) + \dots + m_n r_v(P_n) = 0 \quad \textit{implies} \quad m_1 r_v(Q_1) + \dots + m_n r_v(Q_n) = 0.$$

Then there exists an integer e such that $Q_i = eP_i$ in $B(F)$ for every $i \in \{1, \dots, n\}$.

Proof. There is $s \in \mathbb{N}$ such that $P := sP_2 + P_3 + \dots + P_n$, $\bar{P} := (s+1)P_2 + P_3 + \dots + P_n$, $Q := sP_2 + P_3 + \dots + P_n$, $\bar{Q} := (s+1)Q_2 + Q_3 + \dots + Q_n$ are nontorsion points.

By the assumption of the theorem the following condition holds for every set of natural numbers m_1, m_2 and for almost every prime v:

$$m_1 r_v(P_1) + m_2 r_v(P) = 0 \quad \text{implies} \quad m_1 r_v(Q_1) + m_2 r_v(Q) = 0,$$
$$m_1 r_v(P_1) + m_2 r_v(\bar{P}) = 0 \quad \text{implies} \quad m_1 r_v(Q_1) + m_2 r_v(\bar{Q}) = 0.$$

By Lemma 7.2 there is an integer e such that $Q_1 = eP_1$, $Q = eP$, $\bar{Q} = e\bar{P}$, i.e.

$$\begin{cases} e[sP_2 + P_3 + \dots + P_n] = sQ_2 + Q_3 + \dots + Q_n, \\ e[(s+1)P_2 + P_3 + \dots + P_n] = (s+1)Q_2 + Q_3 + \dots + Q_n, \end{cases}$$

hence $Q_2 = eP_2$. Analogously $Q_i = eP_i$ for every $i \in \{3, \dots, n\}$. □

8. Corollaries of Theorems 6.1 and 6.2

We obtain the specializations of Theorems 6.1 and 6.2 for cyclotomic character, K-theory and abelian varieties (see Section 3). Let us state these results in abelian variety case.

Theorem 8.1. *Let A be an abelian variety defined over number field F such that* $\mathrm{End}(A)$ *is an integral domain. Let $P_1, \dots, P_n, P_0, Q_1, \dots, Q_n, Q_0 \in A(F)$ be the points of infinite order. Assume that for almost every prime l the following condition holds*:

For every set of nonnegative integers $m_1, \dots, m_n$ and for almost every prime v

$$m_1 r_v(P_1) + \dots + m_n r_v(P_n) = r_v(P_0) \quad \textit{implies} \quad m_1 r_v(Q_1) + \dots + m_n r_v(Q_n) = r_v(Q_0).$$

Then there exist $k_i \in \mathbb{Z} \setminus \{0\}$, $\beta_i \in \mathrm{End}(A) \setminus \{0\}$ such that $k_i P_i + \beta_i Q_i = 0$ for every $i \in \{0, \dots, n\}$.

Proof. By Theorem 6.1 there exist $\alpha_i, \beta_i \in \mathrm{End}(A) \setminus \{0\}$ such that $\alpha_i P_i + \beta_i Q_i = 0$ for every $i \in \{0, \dots, n\}$. But $\mathrm{End}(A)$ is an integral domain, hence A is simple and $\mathrm{End}(A) \otimes \mathbb{Q}$ is division algebra. So there exists an endomorphism $\gamma_i \in \mathrm{End}(A)$ such that $\gamma_i \alpha_i = [k_i]$ for some $k_i \in \mathbb{Z} \setminus \{0\}$. Hence $k_i P_i + \gamma_i \beta_i Q_i = 0$. □

Theorem 8.2. *Let A be an abelian variety defined over number field F such that* $\mathrm{End}(A)$ *is an integral domain. Let $P_1, \dots, P_n, Q_1, \dots, Q_n \in A(F)$ be the points of infinite order. Assume that for almost every prime l the following condition holds*:

For every set of positive integers $m_1, \dots, m_n$ and for almost every prime v

$$m_1 r_v(P_1) + \cdots + m_n r_v(P_n) = 0 \quad \textit{implies} \quad m_1 r_v(Q_1) + \cdots + m_n r_v(Q_n) = 0.$$

Then there exist $k_i \in \mathbb{Z} \setminus \{0\}$, $\beta_i \in \operatorname{End}(A) \setminus \{0\}$ such that $k_i P_i + \beta_i Q_i = 0$ for every $i \in \{1, \dots, n\}$.

9. Corollaries of Theorem 7.3

Theorem 9.1. *Let $p_1, \dots, p_s, q_1, \dots, q_s \in F^*$ and suppose that for almost every prime ideal $\wp$ in $\mathcal{O}_F$ and for every set of natural numbers $m_1, \dots, m_s$ the following condition holds*:

$$\prod_{i=1}^{s} p_i^{m_i} = 1 \ (\text{mod } \wp) \quad \textit{implies} \quad \prod_{i=1}^{s} q_i^{m_i} = 1 \ (\text{mod } \wp).$$

Then there exists $e \in \mathbb{Z} \setminus \{0\}$ such that $q_i = p_i^e$ for every $i \in \{1, \dots, s\}$.

Remark 9.1.2. Schinzel [S, Theorem 1] proved by a different method a similar result. Theorem 9.1 is a bit more general since it assumes only positive coefficients m_i.

Theorem 9.2. *Let $P_1, \dots, P_s, Q_1, \dots, Q_s \in K_{2n+1}(F)/C_F$ be the points of infinite order, where $n \geqslant 1$. Assume that for every prime l the following condition holds*:

For every set of natural numbers $m_1, \dots, m_s$ and for almost every prime v

$$m_1 r_v(P_1) + \cdots + m_s r_v(P_s) = 0 \quad \textit{implies} \quad m_1 r_v(Q_1) + \cdots + m_s r_v(Q_s) = 0.$$

Then there exists $e \in \mathbb{Z} \setminus \{0\}$ such that $Q_i = eP_i$ for every $i \in \{1, \dots, s\}$.

Theorem 9.3. *Let A be an abelian variety defined over number field F such that $\operatorname{End}(A) = \mathbb{Z}$. Let $P_1, \dots, P_n, Q_1, \dots, Q_n \in A(F)$ be the points of infinite order. Assume that for every prime l the following condition holds*:

For every set of natural numbers $m_1, \dots, m_n$ and for almost every prime v

$$m_1 r_v(P_1) + \cdots + m_n r_v(P_n) = 0 \quad \textit{implies} \quad m_1 r_v(Q_1) + \cdots + m_n r_v(Q_n) = 0.$$

Then there exists $e \in \mathbb{Z} \setminus \{0\}$ such that $Q_i = eP_i$ for every $i \in \{1, \dots, n\}$.

Corollary 9.4. *Let E be an elliptic curve without complex multiplication defined over number field F. Let $P_1, \dots, P_n, Q_1, \dots, Q_n \in A(F)$ be the points of infinite order. Assume that for every prime l the following condition holds*:

For every set of natural numbers $m_1, \dots, m_n$ and for almost every prime v

$$m_1 r_v(P_1) + \cdots + m_n r_v(P_n) = 0 \quad \textit{implies} \quad m_1 r_v(Q_1) + \cdots + m_n r_v(Q_n) = 0.$$

Then there exists $e \in \mathbb{Z} \setminus \{0\}$ such that $Q_i = eP_i$ for every $i \in \{1, \dots, n\}$.

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