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The Number of Generators of a Colength N Ideal in a Power Series Ring

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If I is an ideal in $R = k[[x_1,..., x_r]]$, the colength of I is defined to be $\dim_k R/I$. Given N a positive integer, and an ordered system of parameters $x_1,..., x_r$, I will define an ideal I_N, intuitively the ideal of colength N closest to an ideal of the form $m^j = (x_1,..., x_r)^j$ for an appropriate j. If $N = \dim_k R/m^j$ for some j, the ideal $I_N = m^j$ and has generators all of the same degree; otherwise I_N has generators in adjacent degrees. Always there exists j such that $m^j \supset I_N \supset m^{j+1}$.

In the main theorem, Theorem 1, I show that if I is any colength N graded ideal in R, then the number of generators of $I \leqslant$ the number of generators of I_N. The number of generators of I refers to the minimal number of homogeneous generators. Since an ideal of finite colength contains a power of the maximal ideal, this result is true for the polynomial ring as well as for the power series ring.

If I is any colength N ideal in R and I^* is the associated graded ideal of I, then the number of generators of $I \leqslant$ the number of generators of I^* and the colength $I^* = N$. Hence, the theorem implies that the number of generators of any colength N ideal $\leqslant$ the number of generators of I_N.

The proof of the main theorem relies on a result of Macaulay [3] which reduces the theorem to the case where I is an ideal of colength N generated by monomials. In that case, by pulling out and filling in appropriate monomials in I, a new monomial ideal I' is obtained which has the same colength as I and more generators, and is in a sense "nearer" to I_N. The theorem then follows by induction. Figures are included illustrating the proof of Theorem 1 in the case $r = 2$, 3, and 4.

Proposition 2 describes asymptotic bounds for the number of generators of m^s in terms of the colength N of m^s. First, if r is held fixed and $s \to \infty$, then

$$\#\text{gens } m^s \cong (r/\sqrt[r]{r!})\, N^{(1-1/r)}.$$

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Second, if s is held fixed and $r \to \infty$, then

$$\#\text{gens } m^s \cong (s/\sqrt[s]{s!})^{-s/(s-1)} N^{s/(s-1)}.$$

An immediate consequence of Theorem 1 and Proposition 2 is Theorem 3, giving a sharp upper bound on the number of generators of an ideal I of colength N in R; the bound is

$$\#\text{gens } I \leqslant (r/\sqrt[r]{r!})\, N^{(1-1/r)}(1+\varepsilon),$$

when $N \geqslant N(r, \varepsilon)$. Previously, upper bounds of the form $cN^{(1-1/r)}$ were obtained on the number of generators of a colength N ideal in R, by Briançon and Iarrobino [2]. The result of this paper is the first in which the upper bound is sharp. Boratynski *et al.* in [1] have bounded the number of generators of a colength N ideal in a local Cohen–Macaulay ring R. In case $R = K[[x_1, ..., x_r]]$, their result is that $\#\text{gens } I \leqslant (r!/\sqrt[r]{r!})\, N^{(1-1/r)} + (r-1)$. For a general survey of related problems concerning the number of generators of ideals in local rings, see the book by Sally [4].

I now describe the ideal I_N. Order the monomials in R lexicographically: If $m = x_1^{i_1} \cdots x_r^{i_r}$, $m' = x_1^{j_1} \cdots x_r^{j_r}$, then $m < m'$ if

(i) $\deg m < \deg m'$

or

(ii) $\deg m = \deg m'$, $m \neq m'$, and if l is the smallest positive integer such that $i_l \neq j_l$, then $i_l > j_l$.

Let R_j denote the vector space spanned by all degree j forms in R, and $W(j, d)$ denote the vector space spanned by the first d degree j monomials in the order. If N is a positive integer then there exists a positive integer j such that

$$\sum_{l=0}^{j-1} \#R_l \leqslant N < \sum_{l=0}^{j} \#R_l.$$

Let

$$t = \left(\sum_{l=0}^{j} \#R_l\right) - N.$$

Define

$$I_N = (W(j, t), m^{j+1}).$$

Then

$$\text{colength } I_N = \dim R/I_N = \sum_{l=0}^{j-1} \#R_l + (\#R_j - t)$$

$$= \left(\sum_{l=0}^{j} \#R_l\right) - t = N.$$

THEOREM 1. *If I is any colength N graded ideal, the number of generators of $I \leqslant$ the number of generators of I_N. (Special case: If $N = \dim_k R/m^j$, then $I_N = m^j$ and has $\dim_k R_j$ generators.)*

Proof. Step 1: Reduction to ideals generated by monomials.

Consider graded ideals I with fixed dimension type $T = t_0, \ldots, t_j, \ldots,$ and colength N, where $t_j = \dim_k I_j$, with $I_j = I \cap R_j$. Let $s_I(j) =$ number of deg j generators of I in a minimal generating set. Then $t_{j+1} = \dim R_1 I_j + s_I(j+1)$ by linear algebra. Macaulay shows in [3] that for vector spaces of forms, V, of degree j with fixed dimension t_j, the dimension of $R_1 V$ is minimal if $V = W(j, t_j)$. It follows that $t_{j+1} \geqslant \dim R_1 W(j, t_j)$ and that the total number of generators of I, $\sum_{j=0}^{\infty} s_I(j)$, is maximal among I of dimension type T, for $I_T =^{\text{def}} \bigoplus_{j=0}^{\infty} W(j, t_j)$. Observe that I_T is an ideal since $t_{j+1} \geqslant \dim R_1 W(j, t_j)$ implies $R_1 W(j, t_j) = W(j+1, \ \dim R_1 W(j, t_j)) \subseteq W(j+1, t_{j+1})$. Thus, to prove the theorem, I need only show that the number of generators of $I_T \leqslant$ the number of generators of I_N, for all dimension types T of colength N.

Step 2: Induction over the partially ordered set of types.

Partially order the dimension types so that $T < S$ if

(i) initial degree $I_T <$ initial degree I_S,

or

(ii) initial degree $I_T =$ initial degree I_S, and the number of generators of I_T of initial degree $>$ the number of generators of I_S of initial degree.

Observe that if $T_N =$ dimension type of I_N and $T =$ dimension type of any colength N ideal, then $T \leqslant T_N$; also there are only finitely many possible dimension types T for colength N ideals.

The proof of Theorem 1 is by descending induction, starting at T_N, on the partially ordered set $\{T \mid T$ is the dimension type of a colength N ideal$\}$. Suppose, by way of induction, that $T < T_N$ is a fixed dimension type and that for all T' with $T < T' \leqslant T_N$, the number of generators of $I_{T'} \leqslant$ the number of generators of I_N.

Let

$a =$ last monomial of initial degree in I_T;

$b =$ first monomial among those of largest degree not in I_T.

Since $T < T_N$, $\deg a < \deg b$. Suppose

$$a = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}, \qquad i_n \neq 0$$

$$b = x_1^{i'_1} x_2^{i'_2} \cdots x_m^{i'_m}, \qquad i'_m \neq 0.$$

Case 1. $n \leqslant m$.

In this case, modify the ideal I_T by pulling out the monomial a and filling in the monomial b. The result is an ideal $I_{T'}$ of type T' with $T < T'$ and colength N. As a result of pulling out a, $r - n + 1$ new generators are created for $I_{T'}$:

$$x_1^{i_1} \cdots x_{n-1}^{i_{n-1}} x_n^{i_n+1}, x_1^{i_1} \cdots x_n^{i_n} x_{n+1}, ..., x_1^{i_1} \cdots x_n^{i_n} x_r.$$

As a result of filling in b, $r - m + 1$ old generators are lost for $I_{T'}$:

$$x_1^{i'_1} \cdots x_{m-1}^{i'_{m-1}} x_m^{i'_m+1}, x_1^{i'_1} \cdots x_m^{i'_m} x_{m+1}, ..., x_1^{i'_1} \cdots x_m^{i'_m} x_r.$$

Therefore, if $n \leqslant m$,

$$\begin{aligned} \#\text{gens } I_{T'} &= \#\text{gens } I_T + \text{change in } \#\text{gens} \\ &\quad \text{from pulling } a + \text{change in } \#\text{gens from filling } b \\ &= \#\text{gens } I_T + ((r - n + 1) - 1) + (1 - (r - m + 1)) \\ &= \#\text{gens } I_T + (m - n) \geqslant \#\text{gens } I_T. \end{aligned}$$

Now by the induction hypothesis for T', $\#\text{gens } I_T \leqslant \#\text{gens } I_N$.

Note. If $R = k[[x, y]]$, then always $n \leqslant m = 2$. The possible ideals I_T, where T is the type of a colength 10 ideal in $k[x, y]$ are displayed in Fig. 1.

Case 2. $n > m$ (*cf. Figs.* 2, 3).

It is always true that $\deg a \leqslant \deg b$, and $\deg a = \deg b \Rightarrow I_T = I_N$. Suppose $\deg a < \deg b$. Because of the choice of I_T, all monomials of degree equal degree of a, but which are less than a, are in I_T. Thus all monomials of degree equal degree of b which are less than $ax_r^{\deg b - \deg a}$ are in I_T. Therefore, $b > ax_r^{\deg b - \deg a}$; and since the degrees of b and $ax_r^{\deg b - \deg a}$ are equal, it must be true that their x_1 exponents satisfy $i'_1 \leqslant i_1$. Therefore $\deg a < \deg b$ and $i'_1 \leqslant i_1 \Rightarrow i_2 + i_3 + \cdots + i_n \leqslant i'_2 + i'_3 + \cdots + i'_m$. Now let l denote the largest integer less than or equal to m satisfying

$$i_l + i_{l+1} + \cdots + i_n \leqslant i'_l + i'_{l+1} + \cdots + i'_m. \tag{1}$$

It follows from the choice of l, that

$$i_{l+1} + \cdots + i_n > i'_{l+1} + \cdots + i'_m \qquad \text{if} \quad l < m, \tag{2}$$

or

$$i_{l+1} + \cdots + i_n > 0 \qquad \text{if} \quad l = m. \tag{2'}$$

I treat first the case $l < m$ and later the case $l = m$.

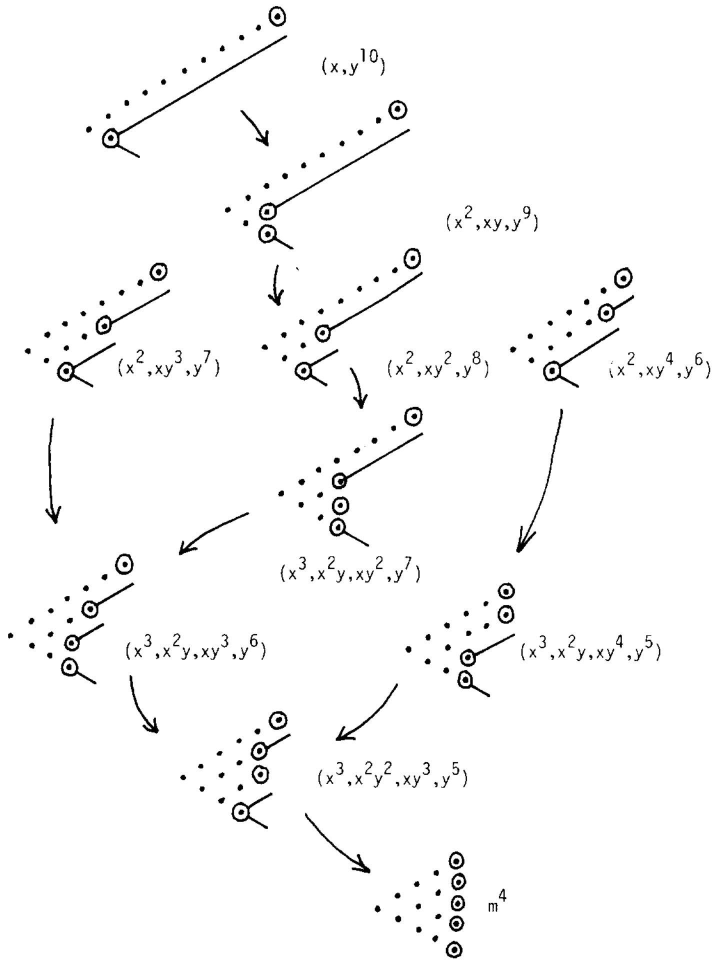

FIG. 1. The ideals I_T, where T Is the type of a colength 10 ideal in $k[[x,y]], x < y$. Explanation: All types in one horizontal level are less than all types in the level directly below. An arrow $A \to B$ indicates that B was obtained from A by pulling out one monomial and filling in another. Generators are circled.

Case 2a; $l < m$. In the first case modify the ideal I_T as follows to get an ideal $I_{T'}$, $T < T' \leqslant T_N$. Pull out all monomials M which satisfy

$$x_1^{i_1} \cdots x_{l-1}^{i_{l-1}} x_l^{(i_l + \cdots + i_n) - (i'_{l+1} + \cdots + i'_m)} x_{l+1}^{i'_{l+1}} \cdots x_m^{i'_m} \leqslant M \leqslant x_1^{i_1} \cdots x_n^{i_n} = a. \quad (3)$$

By (1) and (2) $i_l < (i_l + \cdots + i_n) - (i'_{l+1} + \cdots + i'_m) \leqslant i'_l$ so that the left-hand monomial in (3) is well defined and is less than a. Fill in all monomials M' which satisfy

$$b = x_1^{i'_1} \cdots x_m^{i'_m} \leqslant M' \leqslant x_1^{i'_1} \cdots x_{l-1}^{i'_{l-1}} x_l^{(i'_l + \cdots + i'_m) - (i_{l+1} + \cdots + i_n)} x_{l+1}^{i_{l+1}} \cdots x_n^{i_n}. \quad (4)$$

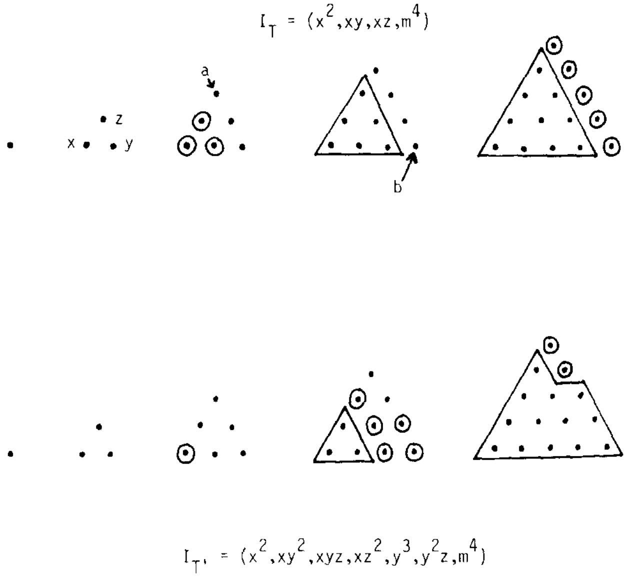

FIG. 2. The critical case $n > m$ in $k[[x, y, z]]$, $x < y < z$. Explanation: In $k[[x, y, z]]$, $n > m$ forces $m = 2$, $n = 3$, thus $a = x^{i_1}y^{i_2}z^{i_3}$, $b = x^{i'_1}y^{i'_2}$; $\bar{M}$ is the set $y^{i_3} \leqslant \bar{M} \leqslant z^{i_3}$. In the diagram $a = xz$, $b = y^3$. To get $I_{T'}$ from I_T we pull out the constant x-power row of monomials in I_T including a and fill in part of the constant x-power row of monomials in I_T starting with b. $I_{T'}$ has the property that #gens $I_{T'} \geqslant$ #gens I_T (in the diagram both have eight generators). Generators are circled and other monomials in I are triangled off.

By (1) and (2) $i_l \leqslant (i'_l + \cdots + i'_m) - (i_{l+1} + \cdots + i_n) < i'_l$ so that the right-hand monomial in (4) is well defined and is greater than b.

Dividing each term in (3) by $x_1^{i_1} \cdots x_l^{i_l}$ shows that we pulled all monomials $M = x_1^{i_1} \cdots x_l^{i_l}\bar{M}$ with

$$x_l^{(i_{l+1} + \cdots + i_n) - (i'_{l+1} + \cdots + i'_m)}\, x_{l+1}^{i'_{l+1}} \cdots x_m^{i'_m} \leqslant \bar{M} \leqslant x_{l+1}^{i_{l+1}} \cdots x_n^{i_n}. \qquad (5)$$

Dividing each term in (4) by $x_1^{i'_1} \cdots x_{l-1}^{i'_{l-1}} x_l^{(i'_l + \cdots + i'_m) - (i_{l+1} + \cdots + i_n)} = c$ shows that we filled in all monomials $M' = c\bar{M}$ with $\bar{M}$ satisfying (5). Hence the number of monomials lost equals the number of monomials added, so colength $I_{T'} =$ colength $I_T = N$.

I now show that the number of generators of $I_{T'}$ equals the number of generators of I_T. Let a' denote the left-hand monomial in (3) and b' denote the right-hand monomial in (4). Since all monomials of degree equal degree of a which are less than a are in I_T, the new generators of degree, deg $a + 1$, in $I_{T'}$ will be all monomials M with

$$x_r \cdot (\text{first monomial before } a') < M \leqslant x_r \cdot a,$$

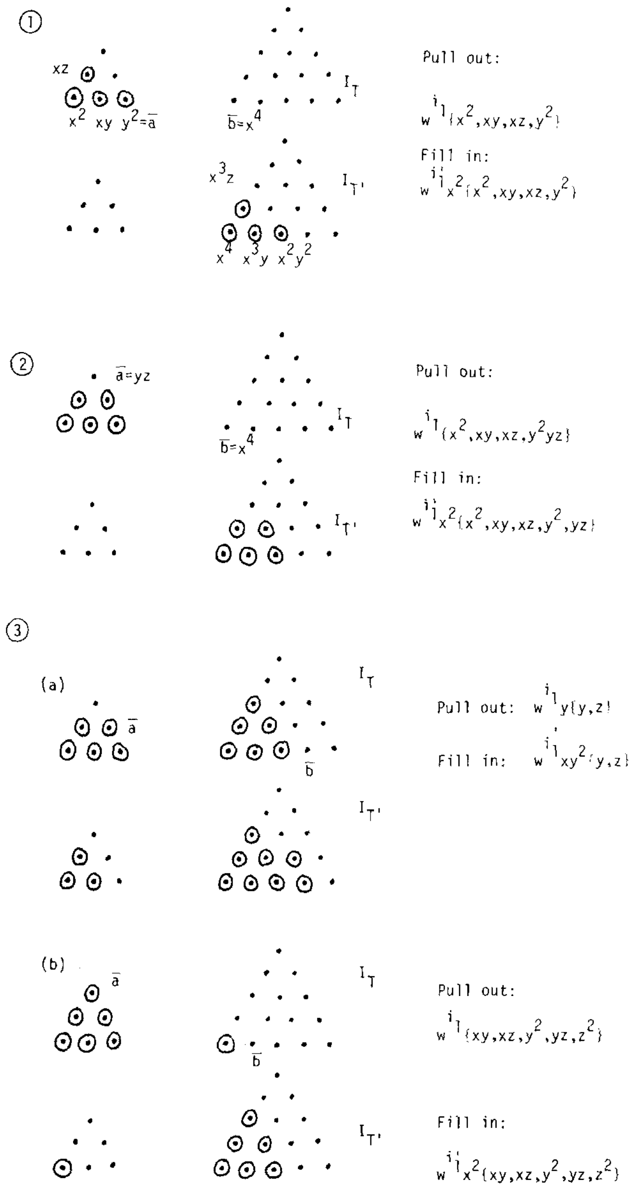

FIG. 3. The critical case $n > m$ in $k[[w, x, y, z]]$, $w < x < y < z$. There are three subcases: (1) $m = 2$, $n = 3$; $a = w^{i_1}x^{i_2}y^{i_3}$; $b = w^{i'_1}x^{i'_2}$. (2) $m = 2$, $n = 4$; $a = w^{i_1}x^{i_2}y^{i_3}z^{i_4}$; $b = w^{i'_1}x^{i'_2}$. (3) $m = 3$, $n = 4$; $a = w^{i_1}x^{i_2}y^{i_3}$; $b = w^{i'_1}x^{i'_2}y^{i'_3}$. Only the monomials having the same degree as a and b and the same constant w power respectively are diagramed for each ideal. These are the only monomials that change from I_T to $I_{T'}$. Set $a = w^{i_1}\bar{a}$, $b = w^{i'_1}\bar{b}$. Always it is true that $\deg \bar{a} < \deg \bar{b}$. Subcase (3) splits into two subsubcases. (a) There is enough room to switch a single constant w, constant x power row. (b) There is not enough room.

or

$$x_1^{i_1} \cdots x_{l-1}^{i_{l-1}} x_l^{(i_l + \cdots + i_n) - (i'_{l+1} + \cdots + i'_m)} x_{l+1}^{i'_{l+1}} \cdots \times x_{m-2}^{i'_{m-2}} x_{m-1}^{i'_{m-1}+1} x_m^{i'_m - 1} x_r < M \leqslant x_1^{i_1} \cdots x_n^{i_n} x_r. \tag{6}$$

Since all monomials of degree equal degree of b which are less than b are in I_T and since all monomials of degree equal degree $b+1$ are in I_T, the generators lost from those of I_T will be all monomials M' of degree, deg $b+1$, with

$$x_r \cdot (\text{first monomial before } b) < M' \leqslant x_r \cdot b',$$

or

$$x_1^{i'_1} \cdots x_{m-2}^{i'_{m-2}} x_{m-1}^{i'_{m-1}+1} x_m^{i'_m - 1} x_r < M' \leqslant x_1^{i'_1} \cdots \times x_{l-1}^{i'_{l-1}} x_l^{(i'_l + \cdots + i'_m) - (i_{l+1} + \cdots + i_n)} x_{l+1}^{i_{l+1}} \cdots x_n^{i_n} x_r. \tag{7}$$

Dividing each term in (7) by $x_1^{i'_1} \cdots x_{l-1}^{i'_{l-1}} x_l^{(i'_l + \cdots + i'_m) - (i_{l+1} + \cdots + i_n)} = c'$ shows that the generators lost will be all monomials $M' = c'\bar{M}'$ with

$$x_l^{(i_{l+1} + \cdots + i_n) - (i'_{l+1} + \cdots + i'_m)} x_{l+1}^{i'_{l+1}} \cdots x_{m-2}^{i'_{m-2}} x_{m-1}^{i'_{m-1}+1} x_m^{i'_m - 1} x_r < \bar{M}' \leqslant x_{l+1}^{i_{l+1}} \cdots x_n^{i_n} x_r \tag{8}$$

Dividing each term in (6) by $x_1^{i_1} \cdots x_l^{i_l}$ shows that the new generators are all monomials $M = x_1^{i_1} \cdots x_l^{i_l} \bar{M}'$ with $\bar{M}'$ satisfying (8). Therefore the number of new generators equals the number of generators lost. Hence the number of generators of $I_{T'}$ equals the number of generators of I_T. By the induction hypothesis applied to $I_{T'}$, the number of generators of $I_T \leqslant$ the number of generators of I_N.

Case 2b; $l = m$. The remaining case is when $l = m$; i.e., $i_m + \cdots + i_n \leqslant i'_m$. Then pull out all monomials M which satisfy

$$x_1^{i_1} \cdots x_{m-1}^{i_{m-1}} x_m^{i_m + \cdots + i_n} \leqslant M \leqslant x_1^{i_1} \cdots x_n^{i_n} = a.$$

Fill in all monomials M' that satisfy

$$b = x_1^{i'_1} \cdots x_m^{i'_m} \leqslant M' \leqslant x_1^{i'_1} \cdots x_{m-1}^{i'_{m-1}} x_m^{i'_m - (i_{m+1} + \cdots + i_n)} x_{m+1}^{i_{m+1}} \cdots x_n^{i_n}.$$

Proceeding as before we find colength $I_{T'} =$ colength $I_T = N$ and number of generators of $I_{T'}$ equals the number of generators of I_T. Now use the induction hypothesis again to get the number of generators of $I_T \leqslant$ the number of generators of I_N. ∎

I now give asymptotic bounds for the number of generators of m^s in terms of the colength N of m^s. An idea of the constants involved is given by $r/\sqrt[r]{r!} < e$ and $\lim_{r\to\infty} r/\sqrt[r]{r!} = e$.

PROPOSITION 2. *Given* $m^s \subset k[[x_1,\ldots,x_r]]$, *colength* $m^s = N$.

(A) *If r is held fixed and $s \to \infty$, then*

$$\#\text{gens } m^s \cong (r/\sqrt[r]{r!})\, N^{(1-1/r)}.$$

In fact $(r/\sqrt[r]{r!})\, N^{(1-1/r)} \leqslant \#\text{gens } m^s \leqslant (r/\sqrt[r]{r!})\, N^{(1-1/r)}[1 + (r-1)/2s]$.

(B) *If s is held fixed and $r \to \infty$, then*

$$\#\text{gens } m^s \cong (s/\sqrt[s]{s!})^{-s/(s-1)} N^{s/(s-1)}.$$

In fact

$$(s/\sqrt[s]{s!})^{-s/(s-1)} N^{s/(s-1)} \leqslant \#\text{gens } m^s \leqslant (s/\sqrt[s]{s!})^{-s/(s-1)} N^{s/(s-1)}[1 + (s-1)/2r].$$

Proof of A.

$$\left.\begin{aligned} N &= \binom{r+s-1}{s-1} = \frac{(r+s-1)!}{r!(s-1)!} \\ \#\text{gens } m^s &= \binom{r+s-1}{s} = \frac{(r+s-1)!}{(r-1)!\,s!} \end{aligned}\right\} \Rightarrow \frac{rN}{s} = \#\text{gens } m^s$$

$$\#\text{gens } m^s/(r/\sqrt[s]{r!})\, N^{(1-1/r)} = (rN/s)/(r/\sqrt[r]{r!})\, N^{(1-1/r)} = \frac{(Nr!)^{1/r}}{s}$$

$$= \frac{1}{s}\left(\frac{(r+s-1)!}{(s-1)!}\right)^{1/r} = \frac{1}{s}((r+s-1)\cdot(r+s-2)\cdot \cdots \cdot (r+s-r))^{1/r}.$$

Now if $l_1,\ldots, l_r$ are positive numbers such that $\sum_{i=1}^r l_i = K$, K fixed, then $\prod_{i=1}^r l_i$ is a maximum when $l_i = K/r$, $\forall\, i = 1,\ldots, r$. (This is easily seen if $r = 2$, and the general case follows from the $r = 2$ case.) Thus

$$\begin{aligned} 1 &\leqslant \frac{1}{s}((r+s-1)\cdot(r+s-2)\cdot \ldots \cdot (r+s-r))^{1/r} \\ &\leqslant \frac{1}{s}\left(\left(\frac{(r+s-1)+(r+s-2)+\cdots+s}{r}\right)^r\right)^{1/r} \\ &= \frac{1}{s}\left(\frac{r^2+rs-(r(r+1)/2)}{r}\right) = 1 + \left(\frac{r-1}{2s}\right). \end{aligned}$$

Therefore,

$$1 \leqslant \#\text{gens}\, m^s/(r/\sqrt[r]{r!})\, N^{(1-1/r)} \leqslant 1 + [(r-1)/2s]. \tag{9}$$

It follows that if r is held fixed and $s \to \infty$

$$\#\text{gens}\, m^s \cong (r/\sqrt{r!})\, N^{(1-1/r)}.$$

Proof of B. Since $N = (r+s-1)!/r!(s-1)!$ and $\#\text{gens}\, m^s = (r+s-1)!/(r-1)!\,s!$, interchanging r and s has the effect of interchanging N and $\#\text{gens}\, m^s$. Thus by (9),

$$1 \leqslant N/((s/\sqrt[s]{s!})(\#\text{gens}\, m^s)^{(1-1/s)} \leqslant 1 + [(s-1)/2r].$$

Therefore,

$$1 \leqslant \#\text{gens}\, m^s/((s/\sqrt[s]{s!})^{-s/(s-1)} N^{s/(s-1)} \leqslant \{1 + [(s-1)/2r]\}^{-s/(s-1)}.$$

This means that if s is held fixed and $r \to \infty$,

$$\#\text{gens}\, m^s \cong (s/\sqrt[s]{s!})^{-s/(s-1)} N^{s/(s-1)}. \quad \blacksquare$$

An immediate consequence of Theorem 1 and Proposition 2 is a sharp upper bound on the number of generators of a graded ideal I of colength N in $R = k[[x_1,\ldots,x_r]]$.

THEOREM 3. *Let I be a graded ideal of colength N. Then*

$$\#\text{gens}\, I \leqslant (r/\sqrt[r]{r!})\, N^{(1-1/r)}(1 + \varepsilon(N,r)),$$

where $\varepsilon(N,r)$ is a function of N and r, and for fixed r, $\lim_{N\to\infty}\varepsilon(N,r) = 0$.

Proof. Let $s(N)$ be the unique positive integer such that $m^{s(N)-1} \not\supseteq I_N \supset m^{s(N)}$. From the definition of I_N, it is clear that $\#\text{gens}\, m^{s(N)-1} \leqslant \#\text{gens}\, I_N \leqslant \#\text{gens}\, m^{s(N)}$, and also that colength $m^{s(N)-1} < N \leqslant \text{colength } m^{s(N)}$. Thus if we set $N' = \text{colength } m^{s(N)}$, then $N' - N < \#\text{gens}\, m^{s(N)-1}$. Therefore, by Theorem 1 and Proposition 2,

$$\#\text{gens}\, I \leqslant \#\text{gens}\, I_N \leqslant \#\text{gens}\, m^{s(N)} \leqslant (r/\sqrt[r]{r!})\, N'^{(1-1/r)} \left(1 + \frac{r-1}{2s(N)}\right)$$

$$= (r/\sqrt[r]{r!})(N + (N'-N))^{(1-1/r)} \left(1 + \frac{r-1}{2s(N)}\right)$$

$$\leqslant (r/\sqrt[r]{r!}) \left(N + (r/\sqrt[r]{r!})\, N^{(1-1/r)} \left(1 + \frac{r-1}{2s(N)-2}\right)\right)^{(1-1/r)}$$

$$\times \left(1 + \frac{r-1}{2s(N)}\right)$$

$$= (r/\sqrt[r]{r!})\, N^{(1-1/r)} \left(1 + (r/\sqrt[r]{r!})\, N^{-1/r} \left(1 + \frac{r-1}{2s(N)-2}\right)\right)^{(1-1/r)}$$

$$\times \left(1 + \frac{r-1}{2s(N)}\right).$$

By the binomial theorem, the last line equals

$$(r/\sqrt[r]{r!})\, N^{(1-1/r)}(1 + \varepsilon(N, r)),$$

where $\varepsilon(N, r)$ is a function of N and r, and for fixed r, $\lim_{N \to \infty} \varepsilon(N, r) = 0$. ∎

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