Nonhomogeneous biharmonic problem in the half-space, \( L^p \) theory and generalized solutions

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Abstract
In this paper, we study the biharmonic equation in the half-space \( \mathbb{R}^N_+ \), with \( N \geq 2 \). We prove in \( L^p \) theory, with \( 1 < p < \infty \), existence and uniqueness results. We consider data and give solutions which live in weighted Sobolev spaces.

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1. Introduction

The purpose of this paper is the resolution of the biharmonic problem with nonhomogeneous boundary conditions

\[
(P) \quad \begin{cases}
\Delta^2 u = f & \text{in } \mathbb{R}^N_+ , \\
u = g_0 & \text{on } \Gamma = \mathbb{R}^{N-1} , \\
\partial_N u = g_1 & \text{on } \Gamma .
\end{cases}
\]

Since this problem is posed in the half-space, it is important to specify the behaviour at infinity for the data and solutions. We have chosen to impose such conditions by setting our problem in

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weighted Sobolev spaces, where the growth or decay of functions at infinity are expressed by means of weights. These weighted Sobolev spaces provide a correct functional setting for unbounded domains, in particular because the functions in these spaces satisfy an optimal weighted Poincaré-type inequality. The weights chosen here behave at infinity as powers to $|x|$. The reason of this choice is given by the behaviour at infinity of the fundamental solution $E_N$ to the biharmonic operator in $\mathbb{R}^N$. Let us recall for instance that

$$E_3(x) = c_3|x|, \quad E_4(x) = c_4 \ln |x|, \quad E_5(x) = \frac{c_5}{|x|},$$

and in particular if $f \in \mathcal{D}(\mathbb{R}^N)$, the convolution $E_N * f$ behaves at infinity as $E_N$. In this work, we shall consider more general data $f$; and the solutions will have a behaviour at infinity which will naturally depend on the one of data in $\mathbb{R}^N_+$ and on the boundary. We have also tried to give another motivation to this choice, more precisely for the biharmonic problem, in Section 2.4, after the definition of spaces.

Our analysis is based on the isomorphism properties of the biharmonic operator in the whole space and the resolution of the Dirichlet and Neumann problems for the Laplacian in the half-space. This last one is itself based on the isomorphism properties of the Laplace operator in the whole space and also on the reflection principle inherent in the half-space. Note here the double difficulty arising from the unboundedness of the domain in any direction and from the unboundedness of the boundary itself.

This paper is organized as follows. Section 2 is devoted to the notations and fundamental results. In Section 3, we study the biharmonic operator in the whole space and we establish isomorphism properties which we will use in the sequel. At last, Section 4 is devoted to the resolution of problem $(P)$. The main result is Theorem 4.1, where we obtain generalized solutions $u \in W^{2,p}_l(\mathbb{R}^N_+)$ to biharmonic problem, where $l$ indicates the behaviour at infinity of these solutions. In a forthcoming work, we shall examine the case of regular data and the homogeneous problem with singular boundary conditions.

2. Notations, spaces, motivation and known results

2.1. Notations

For any real number $p > 1$, we always take $p'$ to be the Hölder conjugate of $p$, i.e.

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Let $\Omega$ be an open set of $\mathbb{R}^N$, $N \geq 2$. Writing a typical point $x \in \mathbb{R}^N$ as $x = (x', x_N)$, where $x' = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$ and $x_N \in \mathbb{R}$, we will especially look on the upper half-space $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N; x_N > 0\}$. We let $\overline{\mathbb{R}^N_+}$ denote the closure of $\mathbb{R}^N_+$ in $\mathbb{R}^N$ and let $\Gamma = \{x \in \mathbb{R}^N; x_N = 0\} = \mathbb{R}^{N-1}$ denote its boundary. Let $|x| = (x_1^2 + \cdots + x_N^2)^{1/2}$ denote the Euclidean norm of $x$, we will use two basic weights

$$\varrho = (1 + |x|^2)^{1/2} \quad \text{and} \quad \lg \varrho = \ln(2 + |x|^2).$$
We denote by $\partial_i$ the partial derivative $\frac{\partial}{\partial x_i}$, similarly $\partial^2_i = \partial_i \circ \partial_i = \frac{\partial^2}{\partial x_i \partial x_i}$, $\partial^2_{ij} = \partial_i \circ \partial_j = \frac{\partial^2}{\partial x_i \partial x_j}$, \ldots.

More generally, if $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{N}^N$ is a multi-index, then

$$\partial^\lambda = \partial_1^{\lambda_1} \cdots \partial_N^{\lambda_N} = \frac{\partial^{\vert \lambda \vert}}{\partial x_1^{\lambda_1} \cdots \partial x_N^{\lambda_N}},$$

where $\vert \lambda \vert = \lambda_1 + \cdots + \lambda_N$.

In the sequel, for any integer $q$, we shall use the following polynomial spaces:

- $\mathcal{P}_q$ is the space of polynomials of degree smaller than or equal to $q$;
- $\mathcal{P}_q^\Delta$ is the subspace of harmonic polynomials of $\mathcal{P}_q$;
- $\mathcal{P}_q^{\Delta^2}$ is the subspace of biharmonic polynomials of $\mathcal{P}_q$;
- $\mathcal{P}_q^\Delta$ is the subspace of polynomials of $\mathcal{P}_q^\Delta$, odd with respect to $x_N$, or equivalently, which satisfy the condition $\varphi(x', 0) = 0$;
- $\mathcal{N}_q^\Delta$ is the subspace of polynomials of $\mathcal{P}_q^\Delta$, even with respect to $x_N$, or equivalently, which satisfy the condition $\partial_N \varphi(x', 0) = 0$;

with the convention that these spaces are reduced to $\{0\}$ if $q < 0$.

For any real number $s$, we denote by $[s]$ the integer part of $s$.

Given a Banach space $B$, with dual space $B'$ and a closed subspace $X$ of $B$, we denote by $B' \perp X$ the subspace of $B'$ orthogonal to $X$, i.e.

$$B' \perp X = \{ f \in B'; \forall v \in X, \langle f, v \rangle = 0 \} = (B/X)^\prime.$$

Lastly, if $k \in \mathbb{Z}$, we shall constantly use the notation $\{1, \ldots, k\}$ for the set of the first $k$ positive integers, with the convention that this set is empty if $k$ is nonpositive.

### 2.2. Weighted Sobolev spaces

For any nonnegative integer $m$, real numbers $p > 1$, $\alpha$ and $\beta$, we define the following space:

$$W^{m,p}_{\alpha, \beta}(\Omega) = \{ u \in \mathcal{D}'(\Omega); \ 0 \leq \vert \lambda \vert \leq k, \ \varrho^{\alpha - m + \vert \lambda \vert}(\log \varrho)^{\beta - 1} \partial^\lambda u \in L^p(\Omega); \ k + 1 \leq \vert \lambda \vert \leq m, \ \varrho^{\alpha - m + \vert \lambda \vert}(\log \varrho)^{\beta} \partial^\lambda u \in L^p(\Omega) \}, \quad (1)$$

where

$$k = \begin{cases} -1 & \text{if } \frac{N}{p} + \alpha \notin \{1, \ldots, m\}, \\ m - \frac{N}{p} - \alpha & \text{if } \frac{N}{p} + \alpha \in \{1, \ldots, m\}. \end{cases}$$

In the case $\beta = 0$, we simply denote the space by $W^{m,p}_{\alpha}(\Omega)$. Note that $W^{m,p}_{\alpha, \beta}(\Omega)$ is a reflexive Banach space equipped with its natural norm:
\[ \|u\|_{W^{m,p}_{\alpha,\beta}(\Omega)} = \left( \sum_{0 \leq |\lambda| \leq k} \| \varrho^{\alpha-m+|\lambda|} (\log \varrho)^{\beta-1} \partial^\lambda u \|_{L^p(\Omega)}^p \right)^{1/p} + \sum_{k+1 \leq |\lambda| \leq m} \| \varrho^{\alpha-m+|\lambda|} (\log \varrho)^{\beta} \partial^\lambda u \|_{L^p(\Omega)}^p \right)^{1/p}. \]

We also define the semi-norm:

\[ |u|_{W^{m,p}_{\alpha,\beta}(\Omega)} = \left( \sum_{|\lambda| = m} \| \varrho^\alpha (\log \varrho)^\beta \partial^\lambda u \|_{L^p(\Omega)}^p \right)^{1/p}. \]

The weights in the definition (1) are chosen so that the corresponding space satisfies two properties. On the one hand, \( \mathcal{D}(\mathbb{R}^N_+) \) is dense in \( W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \). On the other hand, the following Poincaré-type inequality holds in \( W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \):

**Theorem 2.1.** Let \( \alpha \) and \( \beta \) be two real numbers and \( m \geq 1 \) an integer such that

\[ \frac{N}{p} + \alpha \notin \{1, \ldots, m\} \quad \text{or} \quad (\beta - 1) p \neq -1. \]

Then the semi-norm \( |\cdot|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)} \) defines on \( W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \) a norm which is equivalent to the quotient norm, with \( q^* = \inf(q, m - 1) \), where \( q \) is the highest degree of the polynomials contained in \( W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \).

Now, we define the space

\[ \hat{W}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) = \overline{\mathcal{D}(\mathbb{R}^N_+)} \|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}, \]

which will be characterized in Lemma 2.3 as the subspace of functions with null traces in \( W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \). From that, we can introduce the space \( W^{-m,p}_{\alpha,-\beta}(\mathbb{R}^N_+) \) as the dual space of \( \hat{W}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \). In addition, we have the following Poincaré inequality on \( \hat{W}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \) (cf. [3]):

**Theorem 2.2.** Under the assumptions of Theorem 2.1, \( |\cdot|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)} \) is a norm on \( \hat{W}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \) which is equivalent to the full norm \( \|\cdot\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)} \).

We shall now recall some properties of the weighted Sobolev spaces \( W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \). We have the algebraic and topological inbeddings:

\[ W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \hookrightarrow W^{m-1,p}_{\alpha-1,\beta}(\mathbb{R}^N_+) \hookrightarrow \cdots \hookrightarrow W^{0,p}_{\alpha-m,\beta}(\mathbb{R}^N_+) \quad \text{if} \quad \frac{N}{p} + \alpha \notin \{1, \ldots, m\}. \]

When \( \frac{N}{p} + \alpha = j \in \{1, \ldots, m\} \), then we have:

\[ W^{m,p}_{\alpha,\beta} \hookrightarrow \cdots \hookrightarrow W^{m-j+1,p}_{\alpha-j+1,\beta} \hookrightarrow W^{m-j,p}_{\alpha-j,\beta-1} \hookrightarrow \cdots \hookrightarrow W^{0,p}_{\alpha-m,\beta-1}. \]
Note that in the first case, for any $\gamma \in \mathbb{R}$ such that $\frac{N}{p} + \alpha - \gamma \notin \{1, \ldots, m\}$ and $m \in \mathbb{N}$, the mapping

$$u \in W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \mapsto \varphi^\gamma u \in W^{m,p}_{\alpha-\gamma,\beta}(\mathbb{R}^N_+)$$

is an isomorphism. In both cases and for any multi-index $\lambda \in \mathbb{N}^N$, the mapping

$$u \in W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \mapsto \partial_\lambda u \in W^{m-|\lambda|,p}_{\alpha,\beta}(\mathbb{R}^N_+)$$

is continuous. Finally, it can be readily checked that the highest degree $q$ of the polynomials contained in $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ is given by

$$q = \begin{cases} 
    m - \left(\frac{N}{p} + \alpha\right) - 1, & \text{if } \frac{N}{p} + \alpha \in \{1, \ldots, m\} \text{ and } (\beta - 1)p \geq -1, \text{ or} \\
    [m - \left(\frac{N}{p} + \alpha\right)], & \text{otherwise.} 
  \end{cases} \tag{2}$$

2.3. The spaces of traces

In order to define the traces of functions of $W^{m,p}_{\alpha}(\mathbb{R}^N_+)$ (here we do not consider the case $\beta \neq 0$), for any $\sigma \in ]0, 1[$, we introduce the space:

$$W^{\sigma,p}_{0}(\mathbb{R}^N) = \left\{ u \in \mathcal{D}'(\mathbb{R}^N); \ w^{-\sigma}u \in L^p(\mathbb{R}^N) \text{ and } \forall i = 1, \ldots, N, \\
\int_0^{+\infty} t^{-1-\sigma p} dt \int_{\mathbb{R}^N} |u(x + te_i) - u(x)|^p dx < \infty \right\}, \tag{3}$$

where $w = \varrho$ if $N/p \neq \sigma$ and $w = \varrho (\log \varrho)^{1/\sigma}$ if $N/p = \sigma$, and $e_1, \ldots, e_N$ is the canonical basis of $\mathbb{R}^N$. It is a reflexive Banach space equipped with its natural norm:

$$\|u\|_{W^{\sigma,p}_{0}(\mathbb{R}^N)} = \left( \| u/w^\sigma \|_{L^p(\mathbb{R}^N)}^p + \sum_{i=1}^{+\infty} \int_0^{+\infty} t^{-1-\sigma p} dt \int_{\mathbb{R}^N} |u(x + te_i) - u(x)|^p dx \right)^{1/p}$$

which is equivalent to the norm

$$\left( \| u/w^\sigma \|_{L^p(\mathbb{R}^N)}^p + \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+\sigma p}} dx dy \right)^{1/p}.$$

Similarly, for any real number $\alpha \in \mathbb{R}$, we define the space:
\[ W_{\alpha}^{s,p}(\mathbb{R}^N) = \left\{ u \in \mathcal{D}'(\mathbb{R}^N); \ w^{\alpha-s}u \in L^p(\mathbb{R}^N), \right. \]
\[
\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\varrho^\alpha(x)u(x) - \varrho^\alpha(y)u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy < \infty \left\}, \]

where \( w = \varrho \) if \( N/p + \alpha \neq \sigma \) and \( w = \varrho(\lg \varrho)^{1/(\sigma - \alpha)} \) if \( N/p + \alpha = \sigma \). For any \( s \in \mathbb{R}^+ \), we set

\[ W_{\alpha}^{s,p}(\mathbb{R}^N) = \left\{ u \in \mathcal{D}'(\mathbb{R}^N); \ 0 \leq |\lambda| \leq k, \ \varrho^{\alpha-s+|\lambda|}(\lg \varrho)^{-1}\partial^\lambda u \in L^p(\mathbb{R}^N); \right. \]
\[
k + 1 \leq |\lambda| \leq |s| - 1, \ \varrho^{\alpha-s+|\lambda|}\partial^\lambda u \in L^p(\mathbb{R}^N); \ |\lambda| = |s|, \ \partial^\lambda u \in W_{\alpha}^{s,p}(\mathbb{R}^N) \left\}, \right.

where \( k = s - N/p - \alpha \) if \( N/p + \alpha \in \{\sigma, \ldots, \sigma + [s] - 1\} \), with \( \sigma = s - [s] \) and \( k = -1 \) otherwise. It is a reflexive Banach space equipped with the norm:

\[
\|u\|_{W_{\alpha}^{s,p}(\mathbb{R}^N)} = \left( \sum_{0 \leq |\lambda| \leq k} \|\varrho^{\alpha-s+|\lambda|}(\lg \varrho)^{-1}\partial^\lambda u\|_{L^p(\mathbb{R}^N)}^p \right)^{1/p} + \sum_{k+1 \leq |\lambda| \leq |s| - 1} \|\varrho^{\alpha-s+|\lambda|}\partial^\lambda u\|_{L^p(\mathbb{R}^N)}^p + \|\partial^\lambda u\|_{W_{\alpha}^{s,p}(\mathbb{R}^N)}.
\]

We can similarly define, for any real number \( \beta \), the space:

\[ W_{\alpha,\beta}^{s,p}(\mathbb{R}^N) = \left\{ v \in \mathcal{D}'(\mathbb{R}^N); \ (\lg \varrho)^{\beta}v \in W_{\alpha}^{s,p}(\mathbb{R}^N) \right\}. \]

We can prove some properties of the weighted Sobolev spaces \( W_{\alpha,\beta}^{s,p}(\mathbb{R}^N) \). We have the algebraic and topological imbeddings in the case where \( N/p + \alpha \notin \{\sigma, \ldots, \sigma + [s] - 1\} \):

\[
W_{\alpha,\beta}^{s,p}(\mathbb{R}^N) \hookrightarrow W_{\alpha-1,\beta}^{s-1,p}(\mathbb{R}^N) \hookrightarrow \ldots \hookrightarrow W_{\alpha-[s],\beta}^{s-[s],p}(\mathbb{R}^N),
\]
\[
W_{\alpha,\beta}^{s,p}(\mathbb{R}^N) \hookrightarrow W_{\alpha+[s]-s,\beta}^{s-[s]+p}(\mathbb{R}^N) \hookrightarrow \ldots \hookrightarrow W_{\alpha-s,\beta}^{0,p}(\mathbb{R}^N).
\]

When \( N/p + \alpha = j \in \{\sigma, \ldots, \sigma + [s] - 1\} \), then we have:

\[
W_{\alpha,\beta}^{s,p}(\mathbb{R}^N) \hookrightarrow \ldots \hookrightarrow W_{\alpha-j+1,\beta}^{s-j+1,p}(\mathbb{R}^N) \hookrightarrow W_{\alpha-j,\beta-1}^{s-j,p}(\mathbb{R}^N) \hookrightarrow \ldots \hookrightarrow W_{\alpha-[s],\beta-1}^{0,p}(\mathbb{R}^N),
\]
\[
W_{\alpha,\beta}^{s,p}(\mathbb{R}^N) \hookrightarrow \ldots \hookrightarrow W_{\alpha-[s]+s,\beta}^{s-[s]-1,p}(\mathbb{R}^N) \hookrightarrow W_{\alpha-s-j-1,\beta}^{s-j+1,p}(\mathbb{R}^N) \hookrightarrow \ldots \hookrightarrow W_{\alpha-s,\beta-1}^{0,p}(\mathbb{R}^N).
\]

If \( u \) is a function on \( \mathbb{R}^N_+ \), we denote its trace of order \( j \) on the hyperplane \( \Gamma \) by

\[ \forall j \in \mathbb{N}, \ \gamma_j u : x' \in \mathbb{R}^{N-1} \mapsto \partial^j_N u(x', 0). \]

Let us recall the following trace lemma due to Hanouzet (cf. [9]) and extended by Amrouche–Nečasová (cf. [3]) to this class of weighted Sobolev spaces:
Lemma 2.3. For any integer $m \geq 1$ and real number $\alpha$, the mapping

$$\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{m-1}) : \mathcal{D}(\mathbb{R}_+^N) \to \prod_{j=0}^{m-1} \mathcal{D}(\mathbb{R}^{N-1}),$$

can be extended to a linear continuous mapping, still denoted by $\gamma$,

$$\gamma : W^{m,p}_\alpha(\mathbb{R}_+^N) \to \prod_{j=0}^{m-1} W^{m-j-1/p,p}_\alpha(\mathbb{R}^{N-1}).$$

Moreover $\gamma$ is surjective and $\ker \gamma = \dot{W}^{m,p}_\alpha(\mathbb{R}_+^N)$.

2.4. Motivation

Problem $(P)$ has been investigated by Boulmezaoud (cf. [6]) in weighted Sobolev spaces in $L^2$ theory for $N \geq 3$ and without the critical cases corresponding to logarithmic factors. The aim of this work is to give results in $L^p$ theory, with $1 < p < \infty$, to reduce critical values and especially to reach weaker solutions from more singular data.

Let us throw light on this functional framework in the $L^2$ case. If we consider problem $(P)$ with homogeneous boundary conditions, i.e. $g_0 = g_1 = 0$, we can give the following variational formulation: For any given $f \in V'$, find $u \in V$ such that

$$\forall v \in V, \quad \int_{\mathbb{R}^N} \Delta u \Delta v \, dx = (f, v)_{V' \times V}.$$

Which is the appropriate space $V$ to use the Lax–Milgram’s lemma? We must have firstly, for any $v \in V$, $\Delta v \in L^2(\mathbb{R}_+^N)$ and secondly, the coercivity condition for the bilinear form: $(u, v) \mapsto \int_{\mathbb{R}_+^N} \Delta u \Delta v \, dx$.

By Theorem 2.2, we have:

$$\forall v \in \dot{W}^{2,2}_0(\mathbb{R}_+^N), \quad \|v\|_{\dot{W}^{2,2}_0(\mathbb{R}_+^N)} \leq C \|\nabla^2 v\|_{L^2(\mathbb{R}_+^N)^{N^2}}.$$

Moreover,

$$\forall v \in \dot{W}^{2,2}_0(\mathbb{R}_+^N), \quad \|\nabla^2 v\|_{L^2(\mathbb{R}_+^N)^{N^2}} = \|\Delta v\|_{L^2(\mathbb{R}_+^N)},$$

hence the coercivity of the form. Consequently, problem $(P)$ with $g_0 = g_1 = 0$ is well posed on $V = \dot{W}^{2,2}_0(\mathbb{R}_+^N)$. Which are the appropriate spaces of traces for the complete problem? Thanks to Lemma 2.3,

$$u \in W^{2,2}_0(\mathbb{R}_+^N) \quad \Rightarrow \quad (\gamma_0 u, \gamma_1 u) \in W^{3/2,2}_0(\mathbb{R}^{N-1}) \times W^{1/2,2}_0(\mathbb{R}^{N-1}),$$

consequently we must take $g_0, g_1 \in W^{3/2,2}_0(\mathbb{R}^{N-1}) \times W^{1/2,2}_0(\mathbb{R}^{N-1})$ in the problem with non-homogeneous boundary conditions.
Remark 2.4. If we consider the problem for the operator $I + \Delta^2$:

$$\begin{align*}
(Q) \quad \begin{cases}
  u + \Delta^2 u = f & \text{in } \mathbb{R}^N_+,
  u = g_0 & \text{on } \Gamma,
  \partial_N u = g_1 & \text{on } \Gamma,
\end{cases}
\end{align*}$$

we have the following variational formulation with $g_0 = g_1 = 0$: For any given $f \in V'$, find $u \in V$ such that

$$\int_{\mathbb{R}^N_+} uv \, dx + \int_{\mathbb{R}^N_+} \Delta u \Delta v \, dx = \langle f, v \rangle_{V' \times V}.$$ 

This form satisfies naturally the coercivity condition on $V = H^2_0(\mathbb{R}^N_+)$, where $H^2_0(\mathbb{R}^N_+)$ denotes here the classical Sobolev space of functions $v \in H^2(\mathbb{R}^N_+)$ such that $v = \partial_N v = 0$ on $\Gamma$. For the nonhomogeneous problem, we must take $(g_0, g_1) \in H^{3/2}(\mathbb{R}^{N-1}) \times H^{1/2}(\mathbb{R}^{N-1})$.

2.5. The Laplace equation in $\mathbb{R}^N_+$

We shall now recall the fundamental results of the Laplace equation in the half-space, with nonhomogeneous Dirichlet or Neumann boundary conditions. These results have been proved by Boulmezaoud (cf. [5]) in the particular case where $p = 2$ for $N \geq 3$, then generalized by Amrouche–Nečasová (cf. [3]) and Amrouche (cf. [4]) in $L^p$ theory for $N \geq 2$, with solutions of some critical cases by means of logarithmic factors in the weight. Let us also quote the partial results of Maz'ya–Plamenevskiǐ–Stupyalis (cf. [8]) for the Stokes system in $\mathbb{R}^3_+$ with the velocity obtained in $W^{1,2}_0(\mathbb{R}^3_+)$ or $W^{2,2}_1(\mathbb{R}^3_+)$, and those of Tanaka (cf. [10]) for the same problem and the velocity vector field in $W^{m+2,2}_0(\mathbb{R}^3_+)$ with $m > 0$.

Let us first recall the main result of the Dirichlet problem

$$\begin{align*}
(P_D) \quad \begin{cases}
  \Delta u = f & \text{in } \mathbb{R}^N_+,
  u = g & \text{on } \Gamma,
\end{cases}
\end{align*}$$

with a different behaviour at infinity according to $l$.

Theorem 2.5 (Amrouche–Nečasová). Let $l \in \mathbb{Z}$ such that

$$\frac{N}{p'} \notin \{1, \ldots, l\} \quad \text{and} \quad \frac{N}{p} \notin \{1, \ldots, -l\}. \quad (4)$$

For any $f \in W^{-1,p}_i(\mathbb{R}^N_+)$ and $g \in W^{-1,1/p-p}_i(\Gamma)$ satisfying the compatibility condition

$$\forall \varphi \in \mathcal{A}^{\Delta}_{[1+l-N/p']}, \quad \langle f, \varphi \rangle_{W^{-1,p}_i(\mathbb{R}^N_+) \times W^{1,p'}_i(\mathbb{R}^N_+)} = \langle g, \partial_N \varphi \rangle_{\Gamma}, \quad (5)$$

where $\langle \cdot, \cdot \rangle_{\Gamma}$ denotes the duality between $W^{1/p',p}_i(\Gamma)$ and $W^{-1/p',p'}_i(\Gamma)$, problem $(P_D)$ has a solution $u \in W^{1,p}_i(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{A}^{\Delta}_{[1-l-N/p]}$. 
The second recall deals with the problem with more regular data.

**Theorem 2.6 (Amrouche–Nečasová).** Let \( l \in \mathbb{Z} \) and \( m \geq 1 \) be two integers such that

\[
\frac{N}{p'} \notin \{1, \ldots, l + 1\} \quad \text{and} \quad \frac{N}{p} \notin \{1, \ldots, -l - m\}.
\]

For any \( f \in W_{m+l}^{m-1,p}(\mathbb{R}^N_+) \) and \( g \in W_{m+l}^{m+1-1/p,p}(\Gamma) \), satisfying the compatibility condition (5), problem \((P_D)\) has a solution \( u \in W_{m+l}^{m+1,p}(\mathbb{R}^N_+) \), unique up to an element of \( \mathcal{A}_{1-l-N/p}^\Delta \).

Concerning the Neumann problem

\[
(P_N) \quad \begin{cases}
\Delta u = f & \text{in } \mathbb{R}^N_+,
\partial_N u = g & \text{on } \Gamma,
\end{cases}
\]

let us first recall the existence and unicity result with the weakest hypotheses.

**Theorem 2.7 (Amrouche).** Let \( l \in \mathbb{Z} \) such that

\[
\frac{N}{p'} \notin \{1, \ldots, l\} \quad \text{and} \quad \frac{N}{p} \notin \{1, \ldots, -l + 1\}.
\]

For any \( f \in W_{l-1}^{0,p}(\mathbb{R}^N_+) \) and \( g \in W_{l-1}^{1-1/p,p}(\Gamma) \) satisfying the compatibility condition

\[
\forall \varphi \in \mathcal{A}_{1-l-N/p}^\Delta, \quad \langle f, \varphi \rangle_{W_{l-1}^{0,p}(\mathbb{R}^N_+) \times W_{l-1}^{0,p'}(\mathbb{R}^N_+)} + (g, \varphi)_{\Gamma} = 0,
\]

where \( \langle \cdot, \cdot \rangle_{\Gamma} \) denotes the duality between \( W_{l-1}^{1-1/p,p}(\Gamma) \) and \( W_{l+1}^{1-1/p',p'}(\Gamma) \), problem \((P_N)\) has a solution \( u \in W_{l-1}^{1,p}(\mathbb{R}^N_+) \), unique up to an element of \( \mathcal{A}_{2-l-N/p}^\Delta \).

As for the Dirichlet problem, we can prove the following result:

**Theorem 2.8.** Let \( l \in \mathbb{Z} \) and \( m \geq 0 \) be two integers such that

\[
\frac{N}{p'} \notin \{1, \ldots, l\} \quad \text{and} \quad \frac{N}{p} \notin \{1, \ldots, -l - m\}.
\]

For any \( f \in W_{m+l}^{m,p}(\mathbb{R}^N_+) \) and \( g \in W_{m+l}^{m+1-1/p,p}(\Gamma) \) satisfying the compatibility condition (8), problem \((P_N)\) has a solution \( u \in W_{m+l}^{m+2,p}(\mathbb{R}^N_+) \), unique up to an element of \( \mathcal{A}_{2-l-N/p}^\Delta \).

**Remark 2.9.** Note that for these four theorems, the solutions continuously depend on the data with respect to the quotient norm.
3. Biharmonic operator in $\mathbb{R}^N$

In this section, we shall give some isomorphism results relative to the biharmonic operator in the whole space. We shall rest on these for our investigation in the half-space. At first, we characterize the kernel

$$K = \{ v \in W^{2,p}_l(\mathbb{R}^N); \, \Delta^2 v = 0 \text{ in } \mathbb{R}^N \}.$$

**Lemma 3.1.** Let $l \in \mathbb{Z}$.

(i) If $\frac{N}{p} \notin \{1, \ldots, -l\}$, then $K = \mathcal{P}^2_{[2-l-N/p]}$.

(ii) If $\frac{N}{p} \in \{1, \ldots, -l\}$, then $K = \mathcal{P}^2_{1-l-N/p}$.

**Proof.** Let $u \in K$. As we know that $\Delta^2 u = 0$ and moreover $u \in W^{2,p}_l(\mathbb{R}^N) \subset \mathcal{S}'(\mathbb{R}^N)$, the space of tempered distributions, we can deduce that $u$ is a polynomial on $\mathbb{R}^N$. But according to (2), we know that the highest degree $q$ of the polynomials contained in $W^{2,p}_l(\mathbb{R}^N)$ is given by

$$q = \begin{cases} 1 - l - N/p & \text{if } \frac{N}{p} + l \in \{ j \in \mathbb{Z}; \, j \leq 0 \}, \\ [2 - l - N/p] & \text{otherwise}. \end{cases}$$

We can thus see the conditions of the statement appear precisely.

More generally, for any integer $m \in \mathbb{N}$, we define the kernel

$$K^m = \{ v \in W^{m+2,p}_{m+l}(\mathbb{R}^N); \, \Delta^2 v = 0 \text{ in } \mathbb{R}^N \}.$$

The same arguments lead us to a result which includes the precedent, corresponding then to case $m = 0$.

**Lemma 3.2.** Let $l \in \mathbb{Z}$ and $m \in \mathbb{N}$ such that

(i) $\frac{N}{p} \notin \{1, \ldots, -l - m\}$, then $K^m = \mathcal{P}^2_{[2-l-N/p]}$.

(ii) $\frac{N}{p} \in \{1, \ldots, -l - m\}$, then $K^m = \mathcal{P}^2_{1-l-N/p}$.

We can now formulate the first result of isomorphism in $\mathbb{R}^N$:

**Theorem 3.3.** Let $l \in \mathbb{Z}$. Under hypothesis (4), the following operator is an isomorphism:

$$\Delta^2 : W^{2,p}_l(\mathbb{R}^N) / \mathcal{P}^2_{[2-l-N/p]} \to W^{-2,p}_l(\mathbb{R}^N) \perp \mathcal{P}^2_{[2+l-N/p]}.$$

**Proof.** Let us recall (cf. [1]) that under assumption (4), the operator

$$\Delta : W^{2,p}_l(\mathbb{R}^N) / \mathcal{P}^2_{[2-l-N/p]} \to W^{0,p}_l(\mathbb{R}^N) \perp \mathcal{P}^2_{[l-N/p]}$$

(10)
is an isomorphism. By duality, we can deduce that it is the same for the operator
\[ \Delta : W^{0,p}_{l}(\mathbb{R}^{N}) / \mathcal{D}_{l-N/p}^{\Delta} \rightarrow W^{-2,p}_{l}(\mathbb{R}^{N}) \perp \mathcal{D}_{l-N/p}^{\Delta}. \] (11)

If we suppose now that \( l - N/p' < 0 \), we can compose isomorphisms (10) and (11) to deduce that the operator
\[ \Delta^{2} : W^{2,p}_{l}(\mathbb{R}^{N}) / \mathcal{D}_{l-N/p}^{\Delta^{2}} \rightarrow W^{-2,p}_{l}(\mathbb{R}^{N}) \perp \mathcal{D}_{l-N/p'}^{\Delta^{2}} \] (12)
is an isomorphism. By duality, we can deduce that the operator
\[ \Delta^{2} : W^{2,p}_{l}(\mathbb{R}^{N}) / \mathcal{D}_{l-N/p}^{\Delta^{2}} \rightarrow W^{-2,p}_{l}(\mathbb{R}^{N}) \perp \mathcal{D}_{l-N/p'}^{\Delta^{2}} \] (13)
is an isomorphism provided that we have \( -l - N/p < 0 \).

To combine (12) and (13), it remains to be noted that if \( l - N/p' < 0 \), then we have \( \mathcal{D}_{l-N/p}^{\Delta} = \mathcal{D}_{l-N/p}^{\Delta^{2}} = \mathcal{D}_{l-N/p'}^{\Delta} \); and symmetrically, if \( -l - N/p < 0 \), we have \( \mathcal{D}_{l-N/p}^{\Delta} = \mathcal{D}_{l-N/p}^{\Delta^{2}} = \mathcal{D}_{l-N/p'}^{\Delta} \). Moreover, if we note that the union of those two cases covers all integers \( l \in \mathbb{Z} \), we can deduce that for any \( l \in \mathbb{Z} \) satisfying (4), the operator
\[ \Delta^{2} : W^{2,p}_{l}(\mathbb{R}^{N}) / \mathcal{D}_{l-N/p}^{\Delta^{2}} \rightarrow W^{-2,p}_{l}(\mathbb{R}^{N}) \perp \mathcal{D}_{l-N/p'}^{\Delta^{2}} \] (14)
is an isomorphism. \( \square \)

We can establish now a result for more regular data, with two preliminary lemmas.

**Lemma 3.4.** Let \( m \geq 1 \) and \( l \leq -2 \) be two integers such that
\[ \frac{N}{p} \notin \{1, \ldots, -l - m\}, \] (15)
then the following operator is an isomorphism:
\[ \Delta^{2} : W^{m+2,p}_{m+l}(\mathbb{R}^{N}) / \mathcal{D}_{l-N/p}^{\Delta^{2}} \rightarrow W^{m-2,p}_{m+l}(\mathbb{R}^{N}). \]

**Proof.** We use here another isomorphism result (cf. [2]). Let \( m \geq 1 \) and \( l \leq -1 \) be two integers. Under hypothesis (15), the Laplace operator
\[ \Delta : W^{m+1,p}_{m+l}(\mathbb{R}^{N}) / \mathcal{D}_{l-N/p}^{\Delta} \rightarrow W^{m-1,p}_{m+l}(\mathbb{R}^{N}), \] (16)
is an isomorphism. Then, replacing \( m \) by \( m - 1 \) and \( l \) by \( l + 1 \), we can obtain that for \( m \geq 2 \), \( l \leq -2 \), under hypothesis (15), the operator
\[ \Delta : W^{m,p}_{m+l}(\mathbb{R}^{N}) / \mathcal{D}_{l-N/p}^{\Delta} \rightarrow W^{m-2,p}_{m+l}(\mathbb{R}^{N}), \] (17)
is an isomorphism. Moreover (cf. [1]), for \( l \leq -2 \), the operator
\[ \Delta : W^{1,p}_{1+l}(\mathbb{R}^{N}) / \mathcal{D}_{l-N/p}^{\Delta} \rightarrow W^{-1,p}_{1+l}(\mathbb{R}^{N}) \quad \text{if} \ N/p \notin \{1, \ldots, -l - 1\}, \] (18)
is an isomorphism. Then, combining (17) and (18), we can deduce that for \( m \geq 1 \) and \( l \leq -2 \), under hypothesis (15), the operator
\[
\Delta : W^{m,p}_{m+l}(\mathbb{R}^N) / \mathcal{P}_{[-l-N/p]} \rightarrow W^{m-2,p}_{m+l}(\mathbb{R}^N),
\]
is an isomorphism. Replacing now \( m \) by \( m+1 \) and \( l \) by \( l-1 \) in (16), we obtain that for \( m \geq 0 \) and \( l \leq 0 \), under hypothesis (15), the operator
\[
\Delta : W^{m+2,p}_{m+l}(\mathbb{R}^N) / \mathcal{P}_{[2-l-N/p]} \rightarrow W^{m,p}_{m+l}(\mathbb{R}^N),
\]
is an isomorphism. The lemma follows from the composition of isomorphisms (19) and (20). □

**Lemma 3.5.** Let \( m \geq 1 \) be an integer such that
\[
\frac{N}{p'} \neq 1 \quad \text{or} \quad m = 1,
\]
then the biharmonic operator
\[
\Delta^2 : W^{m+2,p}_{m-1}(\mathbb{R}^N) / \mathcal{P}^{\Delta^2}_{[3-N/p]} \rightarrow W^{m-2,p}_{m-1}(\mathbb{R}^N) \perp \mathcal{P}_{[1-N/p']}
\]
is an isomorphism.

**Proof.** Let us note that it suffices to prove that the operator is surjective. Here again, we compose two Laplace operators. We have the following isomorphism (cf. [1]): for \( m \in \mathbb{N} \),
\[
\Delta : W^{1+m,p}_{m}(\mathbb{R}^N) / \mathcal{P}^{\Delta}_{[1-N/p]} \rightarrow W^{1+m,p}_{m}(\mathbb{R}^N) \perp \mathcal{P}_{[1-N/p']} \quad \text{if} \quad \frac{N}{p'} \neq 1 \quad \text{or} \quad m = 0.
\]
Replacing \( m \) by \( m-1 \), we obtain that for \( m \geq 1 \), the operator
\[
\Delta : W^{m,p}_{m-1}(\mathbb{R}^N) / \mathcal{P}^{\Delta}_{[1-N/p]} \rightarrow W^{m-2,p}_{m-1}(\mathbb{R}^N) \perp \mathcal{P}_{[1-N/p']} \quad \text{if} \quad \frac{N}{p'} \neq 1 \quad \text{or} \quad m = 1,
\]
is an isomorphism. Composing with (20), for \( l = -1 \), we obtain the result. □

We can now give a global result for the biharmonic operator.

**Theorem 3.6.**

(i) Let \( l \in \mathbb{Z} \) such that
\[
\frac{N}{p'} \notin \{1, \ldots, l+1\} \quad \text{and} \quad \frac{N}{p} \notin \{1, \ldots, -l-1\},
\]
then the biharmonic operator
\[
\Delta^2 : W^{3,p}_{l+1}(\mathbb{R}^N) / \mathcal{P}^{\Delta^2}_{[2-l-N/p]} \rightarrow W^{-1,p}_{l+1}(\mathbb{R}^N) \perp \mathcal{P}^{\Delta^2}_{[2+l-N/p']}
\]
is an isomorphism.
(ii) Let \( l \in \mathbb{Z} \) and \( m \geq 2 \) be two integers such that
\[
\frac{N}{p} \notin \{1, \ldots, l+2\} \quad \text{and} \quad \frac{N}{p} \notin \{1, \ldots, -l-m\},
\]
then the biharmonic operator
\[
\Delta^2 : W^{m+2,p}_{m+l}(\mathbb{R}^N) / \mathcal{P}_{2-l-N/p}^2 \to W^{m-2,p}_{m+l}(\mathbb{R}^N) \perp \mathcal{P}^2_{2+l-N/p'}
\]
is an isomorphism.

**Proof.** For \( l \leq -1 \), it is clear that Lemmas 3.4 and 3.5 exactly cover points (i) and (ii). It remains to establish the theorem for \( l \geq 0 \).

According to [1], for \( l \geq 0 \), the following operator is an isomorphism:
\[
\Delta : W^{1,p}_{l+1}(\mathbb{R}^N) \to W^{-1,p}_{l+1}(\mathbb{R}^N) \perp \mathcal{P}^\Delta_{2-l-N/p'} \quad \text{if} \quad N/p' \notin \{1, \ldots, l+1\}. \tag{23}
\]
For \( m \geq 1 \) and \( l \geq 1 \), we also have the isomorphism:
\[
\Delta : W^{m+1,p}_{m+l}(\mathbb{R}^N) \to W^{m-1,p}_{m+l}(\mathbb{R}^N) \perp \mathcal{P}^\Delta_{2+l-N/p'} \quad \text{if} \quad N/p' \notin \{1, \ldots, l+1\}. \tag{24}
\]
Replacing \( m \) by \( m-1 \) and \( l \) by \( l+1 \), we deduce for \( m \geq 2 \) and \( l \geq 0 \), the isomorphism:
\[
\Delta : W^{m,p}_{m+l}(\mathbb{R}^N) \to W^{m-2,p}_{m+l}(\mathbb{R}^N) \perp \mathcal{P}^\Delta_{2+l-N/p'} \quad \text{if} \quad N/p' \notin \{1, \ldots, l+2\}. \tag{25}
\]
Replacing \( m \) by \( m+1 \) and \( l \) by \( l-1 \) in (24), we obtain for \( m \geq 1 \) and \( l \geq 2 \), the isomorphism:
\[
\Delta : W^{m+2,p}_{m+l}(\mathbb{R}^N) \to W^{m,p}_{m+l}(\mathbb{R}^N) \perp \mathcal{P}^\Delta_{l-N/p'} \quad \text{if} \quad N/p' \notin \{1, \ldots, l\}. \tag{26}
\]
And now replacing \( m \) by \( m+1 \) in (21), we obtain for \( m \geq 1 \), the isomorphism:
\[
\Delta : W^{m+2,p}_{m+1}(\mathbb{R}^N) / \mathcal{P}^\Delta_{1-N/p} \to W^{m,p}_{m+1}(\mathbb{R}^N) \perp \mathcal{P}_{1-N/p'} \quad \text{if} \quad N/p' \neq 1. \tag{27}
\]
Finally, if we return to (16) with \( l = -1 \) and \( m+1 \) instead of \( m \), we have for \( m \geq 1 \), the isomorphism:
\[
\Delta : W^{m+2,p}_{m}(\mathbb{R}^N) / \mathcal{P}_{2-N/p} \to W^{m,p}_{m}(\mathbb{R}^N). \tag{28}
\]
Then, combining (26), (27) and (28), we obtain for \( m \geq 1 \) and \( l \geq 0 \), the isomorphism:
\[
\Delta : W^{m+2,p}_{m+l}(\mathbb{R}^N) / \mathcal{P}_{2-l-N/p} \to W^{m,p}_{m+l}(\mathbb{R}^N) \perp \mathcal{P}^\Delta_{2+l-N/p'} \quad \text{if} \quad N/p' \notin \{1, \ldots, l\}. \tag{29}
\]
It remains to justify orthogonality conditions to compose (29) with (23) and (25), which will give us respectively the isomorphisms of points (i) and (ii).

Let \( f \in W^{m-2,p}_{m+l}(\mathbb{R}^N) \perp \mathcal{P}^\Delta_{2+l-N/p'} \) with \( m \geq 1 \), then we have \( f \perp \mathcal{P}^\Delta_{2+l-N/p'} \) and according to (23) or (25), there exists \( u \in W^{m,p}_{m+l}(\mathbb{R}^N) \) such that \( \Delta u = f \). We will show that
4. Generalized solutions of $\Delta^2$ in $\mathbb{R}^N_+$

In this section, we shall deal with problem $(P)$ in the half-space.

For any $q \in \mathbb{Z}$, we introduce the space $\mathcal{B}_q$ as a subspace of $\mathcal{D}^2_q$:

$$\mathcal{B}_q = \{ u \in \mathcal{D}^2_q : u = \partial_N u = 0 \text{ on } \Gamma \}.$$ 

We shall establish the main theorem:

**Theorem 4.1.** Let $l \in \mathbb{Z}$ such that

$$\frac{N}{p'} \notin \{1, \ldots, l\} \quad \text{and} \quad \frac{N}{p} \notin \{1, \ldots, -l\}. $$

For any $f \in W_{l-1}^{2,p}(\mathbb{R}^N_+)$, $g_0 \in W_{l-1}^{2,1/p,p}(\Gamma)$ and $g_1 \in W_{l-1}^{1,1/p,p}(\Gamma)$ satisfying the compatibility condition

$$\forall \varphi \in \mathcal{D}_{[2+l-N/p']} \cup \mathcal{D}_{[2+l-N/p]}, \quad \langle f, \varphi \rangle_{W_l^{2,p}(\mathbb{R}^N_+) \times W_{l-1}^{p',p}(\mathbb{R}^N_+)} + \langle g_1, \Delta \varphi \rangle_{\Gamma} - \langle g_0, \partial_N \Delta \varphi \rangle_{\Gamma} = 0, \quad (30)$$

we shall establish the main theorem:

Theorem 4.1. Let $f \in W_{l-1}^{1,p}(\mathbb{R}^N_+)$, $g_0 \in W_{l-1}^{2,p}(\mathbb{R}^N_+)$ such that $f = \Delta \varphi$, and $g_0 = \partial_N \Delta \varphi$ on $\Gamma$, then the operator

$$\Delta^2 : W_{m+l}^{m+2,p}(\mathbb{R}^N_+) / \mathcal{D}_{[2+l-N/p']} \to W_{m+l}^{m-2,p}(\mathbb{R}^N_+)$$

is an isomorphism. \(\square\)
problem (P) admits a solution \( u \in W^{2,p}_l(\mathbb{R}^N_+) \), unique up to an element of \( \mathcal{B}_{[2-l-N/p]} \), and there exists a constant \( C \) such that

\[
\inf_{q \in \mathcal{B}_{[2-l-N/p]}} \| u + q \|_{W^{2,p}_l(\mathbb{R}^N_+)} \leq C \left( \| f \|_{W^{2,p}_l(\mathbb{R}^N_+)} + \| g_0 \|_{W^{1/p,p}_l(\Gamma)} + \| g_1 \|_{W^{1-1/p,p}_l(\Gamma)} \right).
\]

NB: \( \langle g_1, \Delta \varphi \rangle _\Gamma \) denotes the duality bracket \( \langle g_1, \Delta \varphi \rangle _{W^{2-1/p,p}_l(\mathbb{R}^N_+) \times W^{-1/p,p'}_l(\Gamma)} \), and \( \langle g_0, \partial N \Delta \varphi \rangle _\Gamma \) the duality bracket \( \langle g_0, \partial N \Delta \varphi \rangle _{W^{2-1/p,p}_l(\mathbb{R}^N_+) \times W^{-1/p,p'}_l(\Gamma)} \).

4.1. Characterization of the kernel

Let us denote by \( \mathcal{K} \) the kernel of the operator \( (\Delta^2, \gamma_0, \gamma_1) : W^{2,p}_l(\mathbb{R}^N_+) \rightarrow W^{-2,p}_l(\mathbb{R}^N_+) \times W^{-1/p,p}_l(\Gamma) \times W^{1-1/p,p}_l(\Gamma) \), i.e.

\[
\mathcal{K} = \{ u \in W^{2,p}_l(\mathbb{R}^N_+); \Delta^2 u = 0 \text{ in } \mathbb{R}^N_+, u = \partial_N u = 0 \text{ on } \Gamma \}.
\]

The following characterization uses the reflection principle (cf. Farwig [7]).

**Lemma 4.2.** Let \( l \in \mathbb{Z} \).

(i) If \( \frac{N}{p} \notin \{1, \ldots, -l\} \), then \( \mathcal{K} = \mathcal{B}_{[2-l-N/p]} \).

(ii) If \( \frac{N}{p} \in \{1, \ldots, -l\} \), then \( \mathcal{K} = \mathcal{B}_{1-l-N/p} \).

**Proof.** Given \( u \in \mathcal{K} \), we set

\[
\tilde{u}(x', x_N) = \begin{cases} u(x', x_N) & \text{if } x_N \geq 0, \\
(-u - 2x_N \partial_N u - x_N^2 \Delta u)(x', -x_N) & \text{if } x_N < 0. \end{cases}
\]

Then we have \( \tilde{u} \in \mathcal{K}'(\mathbb{R}^N) \) and we show that \( \Delta^2 \tilde{u} = 0 \) in \( \mathbb{R}^N \). We can deduce that \( \tilde{u} \), and consequently \( u \), is a polynomial. Furthermore, \( u \in W^{2,p}_l(\mathbb{R}^N_+) \) implies that its maximum degree is the same as in Lemma 3.1. \( \square \)

More generally, for any \( m \in \mathbb{N} \), we denote by \( \mathcal{K}^m \) the kernel of the operator

\[
(\Delta^2, \gamma_0, \gamma_1) : W^{m+2,p}_{m+l}(\mathbb{R}^N_+) \rightarrow W^{m-2,p}_{m+l}(\mathbb{R}^N_+) \times W^{m+2-1/p,p}_l(\Gamma) \times W^{m+1-1/p,p}_l(\Gamma),
\]

i.e.

\[
\mathcal{K}^m = \{ u \in W^{m+2,p}_{m+l}(\mathbb{R}^N_+); \Delta^2 u = 0 \text{ in } \mathbb{R}^N_+, u = \partial_N u = 0 \text{ on } \Gamma \}.
\]

Identical arguments lead us to the following result:

**Lemma 4.3.** Let \( l \in \mathbb{Z} \) and \( m \in \mathbb{N} \).
(i) If $\frac{N}{P} \notin \{1, \ldots, -l - m\}$, then $K^m = B_{2-l-N/P}$.
(ii) If $\frac{N}{P} \in \{1, \ldots, -l - m\}$, then $K^m = B_{1-l-N/P}$.

We now introduce the two operators $\Pi_D$ and $\Pi_N$, defined by

$$\forall r \in \mathcal{A}_k^\Delta, \quad \Pi_D r = \frac{1}{2} \int_0^{x_N} t r(x', t) \, dt,$$

$$\forall s \in \mathcal{N}_k^\Delta, \quad \Pi_N s = \frac{1}{2} x_N \int_0^{x_N} s(x', t) \, dt.$$

So we obtain the second characterization of $K^m$:

**Lemma 4.4.** Let $l \in \mathbb{Z}$ and $m \in \mathbb{N}$. Under hypothesis (4), we have

$$K^m = B_{2-l-N/P} = \Pi_D \mathcal{A}_{[-l-N/P]}^\Delta \oplus \Pi_N \mathcal{N}_{[-l-N/P]}^\Delta. \quad (31)$$

**Proof.** A direct calculation with these operators yields the following formulas:

$$\forall r \in \mathcal{A}_k^\Delta, \quad \begin{cases} \Delta \Pi_D r = r & \text{in } \mathbb{R}_+^N, \\ \partial_N \Pi_D r = \frac{1}{2} x_N r & \text{in } \mathbb{R}_+^N, \\ \Pi_D r = \partial_N \Pi_D r = 0 & \text{on } \Gamma, \end{cases} \quad (32)$$

and

$$\forall s \in \mathcal{N}_k^\Delta, \quad \begin{cases} \Delta \Pi_N s = s & \text{in } \mathbb{R}_+^N, \\ \partial_N \Pi_N s = \frac{1}{2} \left( x_N s + \int_0^{x_N} s(x', t) \, dt \right) & \text{in } \mathbb{R}_+^N, \\ \Pi_N s = \partial_N \Pi_N s = 0 & \text{on } \Gamma. \end{cases} \quad (33)$$

Moreover, for any $r \in \mathcal{A}_k^\Delta$ and $s \in \mathcal{N}_k^\Delta$, $\Pi_D r \in \mathcal{P}_{k+2}$ and $\Pi_N s \in \mathcal{P}_{k+2}$. Thus, if $r \in \mathcal{A}_{[-l-N/P]}^\Delta$ and $s \in \mathcal{N}_{[-l-N/P]}^\Delta$, we can deduce that $\Pi_D r \in \mathcal{P}_{2-l-N/P}$ and $\Pi_N s \in \mathcal{P}_{2-l-N/P}$.

Conversely, if we consider $u \in B_{2-l-N/P}$, then we have $\Delta u \in \mathcal{P}_{[-l-N/P]}^\Delta$. Since $\mathcal{P}_{[-l-N/P]}^\Delta = \mathcal{A}_{[-l-N/P]}^\Delta \oplus \mathcal{N}_{[-l-N/P]}^\Delta$, there exists $(r, s) \in \mathcal{A}_{[-l-N/P]}^\Delta \times \mathcal{N}_{[-l-N/P]}^\Delta$ such that $\Delta u = r + s$ in $\mathbb{R}_+^N$. According to formulas (32) and (33), the function $z = u - \Pi_D r - \Pi_N s$ satisfies: $\Delta = 0$ in $\mathbb{R}_+^N$ and $z = \partial_N z = 0$ on $\Gamma$. The function $z$ belonging to $\mathcal{A}_{[-l-N/P]}^\Delta \cap \mathcal{N}_{[-l-N/P]}^\Delta = \{0\}$, then $u = \Pi_D r + \Pi_N s$. Furthermore, the sum (31) is direct, because if $(r, s) \in \mathcal{A}_{[-l-N/P]}^\Delta \times \mathcal{N}_{[-l-N/P]}^\Delta$ such that $\Pi_D r = \Pi_N s = u$, then $\Delta u = r + s$. That implies $\Delta u = 0$ in $\mathbb{R}_+^N$ with $u = \partial_N u = 0$ on $\Gamma$, hence $u = 0$ in $\mathbb{R}_+^N$. \qed

The following proposition clarifies the kernel $B_{2-l-N/P}$ in the simplest cases.
Proposition 4.5. Let \( l \in \mathbb{Z} \) such that \( \frac{N}{p} \notin \{1, \ldots, -l \} \).

(i) If \( -l - N/p < 0 \), then \( B_{[2-l-N/p]} = \{0\} \).

(ii) If \( 0 < -l - N/p < 1 \), then \( B_{[2-l-N/p]} = \mathbb{R}x_N^2 \).

Proof. If \( -l - N/p < 0 \), then we have \( B_{[2-l-N/p]} \subset P_1 \). Now, if \( \varphi \in P_1 \) with \( \varphi = \partial_N \varphi = 0 \) on \( \Gamma \), we necessarily have \( \varphi = 0 \). If \( 0 < -l - N/p < 1 \), then \( B_{[2-l-N/p]} = B_2 = \{ \varphi \in P_2 \mid \varphi = \partial_N \varphi = 0 \text{ on } \Gamma \} \). Now, if \( \varphi \in P_2 \) with \( \varphi = \partial_N \varphi = 0 \) on \( \Gamma \), a direct calculation shows that \( \varphi(x) = cx_N^2 \), where \( c \in \mathbb{R} \). □

Remark 4.6. This proposition yields an answer to important particular cases:

(i) If \( l \geq 0 \) or \( (l = -1 \text{ and } N/p > 1) \), then \( B_{[2-l-N/p]} = \{0\} \).

(ii) If \( l = -1 \) and \( N/p < 1 \), then \( B_{[3-N/p]} = B_2 = \mathbb{R}x_N^2 \).

4.2. The compatibility condition

We shall now show the necessity of condition (30) in Theorem 4.1.

Lemma 4.7. Let \( l \in \mathbb{Z} \) such that

\[ \frac{N}{p} \notin \{1, \ldots, l\}. \] (34)

Let \( f \in W_{l}^{-2,p}(\mathbb{R}_+^N) \), \( g_0 \in W_{l}^{2-1/p,p}(\Gamma) \) and \( g_1 \in W_{l}^{1-1/p,p}(\Gamma) \). If problem (P) admits a solution in \( W_{l}^{2,p}(\mathbb{R}_+^N) \), then we have the compatibility condition:

\[ \forall \varphi \in B_{[2+l-N/p]}, \quad \{f, \varphi\}_{W_{l}^{-2,p}(\mathbb{R}_+^N) \times W_{l}^{-2,p}(\mathbb{R}_+^N)} + \{g_1, \Delta \varphi\}_{\Gamma} - \{g_0, \partial_N \Delta \varphi\}_{\Gamma} = 0, \]

where \( \{g_1, \Delta \varphi\}_{\Gamma} \) denotes the duality bracket \( \{g_1, \Delta \varphi\}_{W_{l}^{1-1/p,p}(\Gamma) \times W_{l}^{-1/p',p'}(\Gamma)} \) and \( \{g_0, \partial_N \Delta \varphi\}_{\Gamma} \) denotes the duality bracket \( \{g_0, \partial_N \Delta \varphi\}_{W_{l}^{2-1/p,p}(\Gamma) \times W_{l}^{-1-1/p',p'}(\Gamma)} \).

Remark 4.8. By Proposition 4.5, if \( l - N/p < 0 \) and particularly if \( l \leq 0 \), we have \( B_{[2+l-N/p']} = \{0\} \). Thus there is no compatibility condition in these cases.

Proof. So we assume that \( l \geq 1 \). The first point is to justify the dualities in the spaces of traces. Noting that under hypothesis (34), for any \( \varphi \in B_{[2+l-N/p']} \), we have \( \varphi \in W_{l+1}^{3,p'_{l+1}}(\mathbb{R}_+^N) \) and also \( \varphi \in W_{l+2}^{A',p'_{l+2}}(\mathbb{R}_+^N) \), we can deduce that \( \Delta \varphi|_{\Gamma} \in W_{l+1}^{1-1/p',p'}(\Gamma) \) and \( \partial_N \Delta \varphi|_{\Gamma} \in W_{l+2}^{1-1/p',p'}(\Gamma) \). It remains to verify the imbeddings

\[ W_{l+1}^{1-1/p',p'}(\Gamma) \hookrightarrow W_{l}^{-1/p',p'}(\Gamma), \] (35)

\[ W_{l+2}^{1-1/p',p'}(\Gamma) \hookrightarrow W_{l}^{-1-1/p',p'}(\Gamma). \] (36)

(i) To show (35), we break down this imbedding into
where (38) is equivalent by duality to

\[ W_{l}^{1/p',p'}(\mathbb{R}^{N-1}) \hookrightarrow W_{l-1/p'}^{0,p}(\mathbb{R}^{N-1}). \]  

(39)

Observe that (37) holds if and only if \( \frac{N-1}{p'} - l + 1 \neq 1 - \frac{1}{p'} \), i.e. \( \frac{N}{p} \neq l \), which is included in (34). Likewise (39) is satisfied if and only if \( \frac{N-1}{p'} + l \neq 1 \), i.e. \( \frac{N}{p} \neq -l + 1 \), which cannot happen for \( l \geq 1 \).

(ii) Similarly, the imbedding (36) is equivalent to

\[ W_{l+1}^{1/p',p'}(\mathbb{R}^{N-1}) \hookrightarrow W_{l-1/p'}^{0,p}(\mathbb{R}^{N-1}). \]  

(40)

\[ W_{l}^{1+1/p',p'}(\mathbb{R}^{N-1}) \hookrightarrow W_{l-1-1/p'}^{0,p}(\mathbb{R}^{N-1}). \]  

(41)

The imbedding (40) holds if and only if \( \frac{N}{p} \neq l - 1 \), which is included in (34). The imbedding (41) is satisfied if and only if \( \frac{N}{p} \notin \{-l + 1, -l + 2\} \). Since \( l \geq 1 \), it suffices that \( \frac{N}{p} \neq 1 \) for \( l = 1 \). Assume that \( l = 1 \) and \( \frac{N}{p} = 1 \), then we have \( \frac{N}{p} = N - 1 \) and thus \( \mathcal{D}_{[2+l-N/p']}^{1} = \mathcal{D}_{[4-N]} \). If \( N \geq 3 \), there is no compatibility condition because \( \mathcal{D}_{[4-N]} = \{0\} \). If \( N = 2 \), then we have \( p = p' = 2 \) and \( \frac{N}{p'} = 1 \), but that is excluded by (34).

Now it is clear that for any \( u \in \mathcal{D}(\mathbb{R}^{N}) \) we have

\[ \forall \varphi \in \mathcal{D}_{[2+l-N/p]}^{1}, \quad \int_{\Gamma} \varphi \Delta^{2} u \, dx = \int_{\Gamma} u \Delta \partial_{N} \varphi \, dx' - \int_{\Gamma} \partial_{N} u \Delta \varphi \, dx'. \]

Let \( u \in W_{l}^{2,p}(:,\mathbb{R}^{N}) \) and \( \varphi \in \mathcal{D}_{[2+l-N/p']}^{1} \). Thanks to the density of \( \mathcal{D}(\mathbb{R}^{N}) \) in \( W_{l}^{2,p}(\mathbb{R}^{N}) \), there exists a sequence \( (u_{k})_{k \in \mathbb{N}} \subset \mathcal{D}(\mathbb{R}^{N}) \) such that \( u_{k} \to u \) in \( W_{l}^{2,p}(\mathbb{R}^{N}) \). Therefore \( \Delta^{2} u_{k} \to \Delta^{2} u \) in \( W_{l}^{-2,p}(\mathbb{R}^{N}) \), \( u_{k} \to u \) in \( W_{l}^{2-1/p',p'}(\Gamma) \) and \( \partial_{N} u_{k} \to \partial_{N} u \) in \( W_{l}^{1-1/p,p}(\Gamma) \). Writing the previous formula for any \( u_{k} \), we obtain by passing to the limit as \( k \to \infty \)

\[ \forall \varphi \in \mathcal{D}_{[2+l-N/p']}^{1}, \quad \langle \Delta^{2} u, \varphi \rangle_{W_{l}^{-2,p}(\mathbb{R}^{N}) \times W_{l}^{2-1/p',p'}(\mathbb{R}^{N})} = \langle u, \partial_{N} \Delta \varphi \rangle_{\Gamma} - \langle \partial_{N} u, \Delta \varphi \rangle_{\Gamma}. \]

This proves the necessity of condition (30). \( \square \)

4.3. The homogeneous problem

Here we consider the homogeneous problem in \( \mathbb{R}^{N} \), i.e. \( f = 0 \), with standard boundary conditions. Let the problem

\[
(P^{0}) \quad \begin{cases}
\Delta^{2} u = 0 & \text{in } \mathbb{R}^{N}, \\
u = g_{0} & \text{on } \Gamma, \\
\partial_{N} u = g_{1} & \text{on } \Gamma,
\end{cases}
\]
with \( g_0 \in W^{2-1/p}_l (\Gamma) \) and \( g_1 \in W^{1-1/p}_l (\Gamma) \).

**Lemma 4.9.** Let \( l \in \mathbb{Z} \). Under hypothesis (4), for any \( g_0 \in W^{2-1/p}_l (\Gamma) \) and \( g_1 \in W^{1-1/p}_l (\Gamma) \) satisfying the compatibility condition

\[
\forall \varphi \in B_{[2+l-N/p]}, \quad (g_1, \Delta \varphi)_\Gamma - (g_0, \partial_N \varphi)_\Gamma = 0, \tag{42}
\]

problem \((P^0)\) admits a solution \( u \in W^{2,p}_l (\mathbb{R}^N_+) \), unique up to an element of \( B_{[2+l-N/p]} \), with the estimate

\[
\inf_{q \in B_{[2+l-N/p]}} \|u + q\|_{W^{2,p}_l (\mathbb{R}^N_+)} \leq C (\|g_0\|_{W^{2-1/p}_l (\Gamma)} + \|g_1\|_{W^{1-1/p}_l (\Gamma)}).
\]

**Proof.** Firstly, thanks to Lemma 4.4, note that condition (42) is equivalent to both conditions

\[
\forall r \in A_{[l-N/p]}, \quad (g_0, \partial_N r)_\Gamma = 0, \tag{43}
\]

\[
\forall s \in N_{[l-N/p]}, \quad (g_1, s)_\Gamma = 0. \tag{44}
\]

Consider the Dirichlet problem:

\[
(\mathcal{R}^0) \quad \begin{cases} 
\Delta \vartheta = 0 & \text{in } \mathbb{R}^N_+ \\
\vartheta = g_0 & \text{on } \Gamma.
\end{cases}
\]

Since \( g_0 \in W^{2-1/p}_l (\Gamma) = W^{1+1-l/p}_l (\Gamma) \), Theorem 2.6 holds with \( m = 1 \) and \( l - 1 \) instead of \( l \). Then hypothesis (6) becomes \( \frac{N}{p} \notin \{1, \ldots, l\} \) and \( \frac{N}{p} \notin \{1, \ldots, -l\} \). Moreover compatibility condition (5) corresponds precisely to (43). We can deduce that problem \((\mathcal{R}^0)\) admits a solution \( \vartheta \in W^{2,p}_l (\mathbb{R}^N_+) \).

Consider now the Neumann problem:

\[
(\mathcal{S}^0) \quad \begin{cases} 
\Delta \zeta = 0 & \text{in } \mathbb{R}^N_+ \\
\partial_N \zeta = g_1 & \text{on } \Gamma.
\end{cases}
\]

Theorem 2.8 holds with \( m = 0 \). Moreover compatibility condition (8) corresponds precisely to (44). We can deduce that problem \((\mathcal{S}^0)\) admits a solution \( \zeta \in W^{2,p}_l (\mathbb{R}^N_+) \). So we can readily verify that the function defined by

\[
u = x_N \partial_N (\zeta - \vartheta) + \vartheta
\]

is a solution to \((P^0)\). However we must show that \( u \in W^{2,p}_l (\mathbb{R}^N_+) \). For this, we remark that \( u \) satisfies

\[
(\mathcal{I}) \quad \begin{cases} 
\Delta u = 2\partial_N^2 (\zeta - \vartheta) & \text{in } \mathbb{R}^N_+ \\
u = g_0 & \text{on } \Gamma,
\end{cases}
\]

with \( 2\partial_N^2 (\zeta - \vartheta) \in W^{0,p}_l (\mathbb{R}^N_+) \) and \( g_0 \in W^{2-1/p}_l (\Gamma) \).
(i) If \( \frac{N}{p} \neq -l + 1 \), then we have the imbedding \( W^{2,p}_l(\mathbb{R}^N_+) \hookrightarrow W^{1,p}_l(\mathbb{R}^N_+) \). By (45), we deduce that \( u \in W^{1,p}_l(\mathbb{R}^N_+) \). Furthermore we have the following Green formula:

\[
\forall r \in \mathcal{A}_{[l-N/p]}, \quad \{ \nabla u, r \}_{W^{-1,p}_l(\mathbb{R}^N_+) \times W^{1,p}_l(\mathbb{R}^N_+)} = (u, \partial_N r)_{W^{-1/p,p}_l(\Gamma) \times W^{-1/p,p}_l(\Gamma)},
\]

i.e.

\[
\forall r \in \mathcal{A}_{[l-N/p]}, \quad \{ 2\partial^2_N (\zeta - \vartheta), r \}_{W^{-1,p}_l(\mathbb{R}^N_+) \times W^{1,p}_l(\mathbb{R}^N_+)} = (g_0, \partial_N r). \]

Thus the compatibility condition of problem \((\mathcal{T})\) is satisfied and thanks to Theorem 2.8, it admits a solution \( y \in W^{2,p}_l(\mathbb{R}^N_+) \), unique up to an element of \( \mathcal{A}_{[2-l-N/p]} \). So the function \( z = u - y \in W^{1,p}_l(\mathbb{R}^N_+) \) and satisfies

\[
\begin{align*}
(K) \quad & \Delta z = 0 \quad \text{in} \ \mathbb{R}^N_+, \\
& z = 0 \quad \text{on} \ \Gamma.
\end{align*}
\]

We can deduce that \( z \in \mathcal{A}_{[2-l-N/p]} \), i.e. \( u = y + r \) with \( r \in \mathcal{A}_{[2-l-N/p]} \subset W^{2,p}_l(\mathbb{R}^N_+) \), which shows that \( u \in W^{2,p}_l(\mathbb{R}^N_+) \).

(ii) If \( \frac{N}{p} = -l + 1 \), the previous imbedding does not hold. Then we only have \( W^{2,p}_l(\mathbb{R}^N_+) \hookrightarrow W^{1,p}_{l-1,-1}(\mathbb{R}^N_+) \), with the introduction of a logarithmic weight in the second space. By (45), we can deduce that \( u \in W^{1,p}_{l-1,-1}(\mathbb{R}^N_+) \). Furthermore we have \( l - \frac{N}{p} < 0 \), thus there is no compatibility condition for \((\mathcal{T})\) which admits consequently a solution \( y \in W^{2,p}_l(\mathbb{R}^N_+) \), unique up to an element of \( \mathcal{A}_1 \). The end of the proof is similar to the previous case. \( \square \)

We can now extend this result to more regular data.

Lemma 4.10. Let \( l \in \mathbb{Z} \) and \( m \geq 1 \). Under hypothesis (9), for any \( g_0 \in W^{m+2-1/p,p}_m(\Gamma) \) and \( g_1 \in W^{m+1-1/p,p}_m(\Gamma) \), satisfying the compatibility condition (42), problem \((\mathcal{P}^0)\) admits a solution \( u \in W^{m+2,p}_m(\mathbb{R}^N_+) \), unique up to an element of \( \mathcal{A}_{[2-l-N/p]} \) with the estimate

\[
\inf_{q \in \mathcal{A}_{[2-l-N/p]}} \| u + q \|_{W^{m+2,p}_m(\mathbb{R}^N_+)} \leq C(\| g_0 \|_{W^{m+2-1/p,p}_m(\Gamma)} + \| g_1 \|_{W^{m+1-1/p,p}_m(\Gamma)}).
\]

Proof. We strictly resume the proof of Lemma 4.9. In this case, we note that \( g_0 \in W^{m+1+1/p-1/p,p}(\mathbb{R}^N_+) \) and \( g_1 \in W^{m+1-1/p,p}_m(\Gamma) \), then we use Theorems 2.6 and 2.8. To show that \( u \in W^{m+2,p}_m(\mathbb{R}^N_+) \), we must distinguish two cases. If \( \frac{N}{p} \neq -l - m + 1 \), then we have the imbedding \( W^{m+2,p}_m(\mathbb{R}^N_+) \hookrightarrow W^{m+1,p}_{m+1-1}(\mathbb{R}^N_+) \). If \( \frac{N}{p} = -l - m + 1 \), then we have \( W^{m+2,p}_m(\mathbb{R}^N_+) \hookrightarrow W^{m+1,p}_{m+1-1}(\mathbb{R}^N_+) \). In the second case, we must remark that \( l - \frac{N}{p} < 0 \), so there is again no compatibility condition for \((\mathcal{T})\). \( \square \)
Note that we have the chain of imbeddings $W^{m+2,p}_{m+l}(\mathbb{R}^N_+) \hookrightarrow W^{m+1,p}_{m+l-1}(\mathbb{R}^N_+) \hookrightarrow \cdots \hookrightarrow W^{2,p}_l(\mathbb{R}^N_+)$ if and only if $\frac{N}{p} \notin \{-l - m + 1, \ldots, -l\}$, and then Lemma 4.10 is a regularity result with respect to Lemma 4.9.

4.4. Existence of a solution to problem $(\mathcal{P})$

We come back to the general problem $(\mathcal{P})$ and Theorem 4.1. By Lemma 2.3, there exists a lifting function $u_g \in W^{2,p}_l(\mathbb{R}^N_+)$ of $(g_0, g_1)$, i.e. $u_g = g_0$ on $\Gamma$ and $\partial_N u_g = g_1$ on $\Gamma$, such that

$$\|u_g\|_{W^{2,p}_l(\mathbb{R}^N_+)} \leq C (\|g_0\|_{W^{2-1/p,p}_{l}(\Gamma)} + \|g_1\|_{W^{1-1/p,p}_{l}(\Gamma)}).$$

Set $h = f - \Delta^2 u_g \in W^{-2,p}_l(\mathbb{R}^N_+)$ and $v = u - u_g$, then problem $(\mathcal{P})$ is equivalent to the following with homogeneous boundary conditions:

$$\Delta^2 v = h \quad \text{in} \quad \mathbb{R}^N_+, \quad v = \partial_N v = 0 \quad \text{on} \quad \Gamma.$$

Then, the compatibility condition (30) for problem $(\mathcal{P})$ becomes:

$$\forall \phi \in \mathcal{B}_{2+l-N/p'}, \quad \langle h, \phi \rangle_{W^{-2,p}_{l}(\mathbb{R}^N_+) \times \hat{W}^{-2,p'}_{l} (\mathbb{R}^N_+)} = 0. \quad (46)$$

So, we can consider now the lifted problem

$$(\mathcal{P}^*) \quad \begin{cases} 
\Delta^2 u = f & \text{in} \quad \mathbb{R}^N_+, \\
u = 0 & \text{on} \quad \Gamma, \\
\partial_N u = 0 & \text{on} \quad \Gamma,
\end{cases}$$

where $f \in W^{-2,p}_{l}(\mathbb{R}^N_+)$ and $f \perp \mathcal{B}_{2+l-N/p'}$.

Give at first a characterization of $W^{-2,p}_{l}(\mathbb{R}^N_+)$:

**Lemma 4.11.** For any $f \in W^{-2,p}_{l}(\mathbb{R}^N_+)$, there exists $F = (F_{ij})_{1 \leq i, j \leq N} \in W^{0,p}_{l}(\mathbb{R}^N_+)^{N^2}$ such that

$$f = \text{div div } F = \sum_{i,j=1}^N \partial_{ij}^2 F_{ij},$$

with the estimate

$$\sum_{i,j=1}^N \|F_{ij}\|_{W^{0,p}_{l}(\mathbb{R}^N_+)} \leq C \|f\|_{W^{-2,p}_{l}(\mathbb{R}^N_+)}.$$

**Proof.** We know by Hardy’s inequality that the norm and the semi-norm in $\hat{W}^{-2,p'}_{l} (\mathbb{R}^N_+)$ are equivalent, i.e. there exists a constant $C$ such that

$$\forall u \in \hat{W}^{-2,p'}_{l} (\mathbb{R}^N_+), \quad \|\nabla^2 u\|_{\hat{W}^{0,p'}_{l}(\mathbb{R}^N_+)^{N^2}} \leq \|u\|_{\hat{W}^{-2,p'}_{l} (\mathbb{R}^N_+)} \leq C \|\nabla^2 u\|_{\hat{W}^{0,p'}_{l}(\mathbb{R}^N_+)^{N^2}}.$$
Let
\[ T : \dot{W}_{-l}^{2,p'}(\mathbb{R}^N_+) \to W_{-l}^{0,p'}(\mathbb{R}^N_+)^N \]
\[ u \mapsto \nabla^2 u. \]

By the previous inequalities, \( T \) is a linear continuous injective mapping. We set
\[ G = T(\dot{W}_{-l}^{2,p'}(\mathbb{R}^N_+)), \]
equipped with the norm of \( W_{-l}^{0,p'}(\mathbb{R}^N_+)^N \), and \( S = T^{-1} : G \to \dot{W}_{-l}^{2,p'}(\mathbb{R}^N_+) \). The mapping \( h \in G \mapsto \langle f, Sh \rangle_{W_{-l}^{-2,p'}(\mathbb{R}^N_+)^N} \) is a linear functional on \( G \). Thanks to Hahn–Banach theorem, we can extend it to a linear functional on \( W_{-l}^{0,p'}(\mathbb{R}^N_+)^N \) denoted by \( \Phi \). Thanks to Riesz representation theorem, we know that there exists \( F = (F_{ij}) \in W_{l}^{0,p}(\mathbb{R}^N_+)^N \) such that
\[ \forall h = (h_{ij}) \in W_{-l}^{0,p'}(\mathbb{R}^N_+)^N, \quad \langle \Phi, h \rangle = \int_{\mathbb{R}^N_+} F_{ij} h_{ij} \, dx, \]
with Einstein convention of summation on repeated indices. Particularly, if \( h \in G \), we have
\[ \langle f, Sh \rangle = \int_{\mathbb{R}^N_+} F_{ij} h_{ij} \, dx, \]
i.e.
\[ \forall u \in \dot{W}_{-l}^{2,p'}(\mathbb{R}^N_+), \quad \langle f, u \rangle = \int_{\mathbb{R}^N_+} F_{ij} \partial_{ij}^2 u \, dx. \]

We can deduce that
\[ \forall u \in \mathcal{D}(\mathbb{R}^N_+), \quad \langle f, u \rangle = \{ \partial_{ij}^2 F_{ij}, u \}, \]
i.e. \( f = \text{div div } F = \partial_{ij}^2 F_{ij} \).

Now we can establish a first isomorphism result in the half-space:

**Proposition 4.12.** Let \( l \in \mathbb{Z} \). Under hypothesis (4), with \( 2 + l - N - p' < 0 \) or \( 2 - l - N / p < 0 \), the biharmonic operator
\[ \Delta^2 : \dot{W}_{l}^{2,p}(\mathbb{R}^N_+) / \mathcal{B}_{[2-l-N/p]} \to W_{l}^{-2,p}(\mathbb{R}^N_+) \perp \mathcal{B}_{[2+l-N/p']} \]
is an isomorphism.

**Proof.** Let us first assume that \( 2 + l - N / p' < 0 \). Let \( f \in W_{l}^{-2,p}(\mathbb{R}^N_+) \). Then by Lemma 4.11, we can write \( f = \partial_{ij}^2 F_{ij} \) with \( (F_{ij})_{1 \leq i,j \leq N} \in W_{l}^{0,p}(\mathbb{R}^N_+)^N \). If we extend \( F_{ij} \) to \( \mathbb{R}^N \) by 0, we obtain \( \tilde{F}_{ij} \in W_{l}^{0,p}(\mathbb{R}^N_+)^N \), and thus \( \tilde{f} = \partial_{ij}^2 \tilde{F}_{ij} \in W_{l}^{-2,p}(\mathbb{R}^N_+) \) as extension of \( f \) such that
The function $u = z - v$ answers to problem $(P^*)$ in this case. So we have shown that if $2 + l - N/p' < 0$, the operator

$$\Delta^2 : \tilde{W}_l^2, p(\mathbb{R}^N) / \mathscr{B}_{[2 - l - N/p]} \rightarrow \tilde{W}_l^{-2, p}(\mathbb{R}^N)$$

is an isomorphism. Thus by duality we obtain the isomorphism

$$\Delta^2 : \tilde{W}_l^2, p(\mathbb{R}^N) \rightarrow W_l^{-2, p}(\mathbb{R}^N) \perp \mathscr{B}_{[2 + l - N/p]}$$

if $2 - l - N/p < 0$. \(\square\)

It remains to solve $(P^*)$ if

$$2 + l - N/p' \geq 0 \quad \text{and} \quad 2 - l - N/p \geq 0. \quad (50)$$

It suffices to check the cases $l \in \{-1, 0, 1\}$, outside which condition (50) does not hold. For that, we establish a preliminary proposition:

**Proposition 4.13.** Let $l \in \{-1, 0\}$ such that $N/p \neq 1$ if $l = -1$. For any $f \in W_l^{0, p}(\mathbb{R}^N)$, there exists $z \in W_l^{4, p}(\mathbb{R}^N)$ such that $\Delta^2 z = f$.

**Proof.** Under these hypotheses, consider the extension $\tilde{f}$ of $f$ to $\mathbb{R}^N$ by $0$, so $\tilde{f} \in W_l^{0, p}(\mathbb{R}^N)$. Show at first that there exists $\tilde{z} \in W_l^{4, p}(\mathbb{R}^N)$ such that $\Delta^2 \tilde{z} = \tilde{f}$.

(a) If $l = -1$, then $\tilde{f} \in W_{-1}^{0, p}(\mathbb{R}^N)$ and we have $N/p \neq 1$. Thus Lemma 3.4 of isomorphism in $\mathbb{R}^N$ holds with $m = 2$ and $l = -3$, hence the existence of $\tilde{z} \in W_{-1}^{4, p}(\mathbb{R}^N)$ such that $\Delta^2 \tilde{z} = \tilde{f}$.

(b) If $l = 0$, then $\tilde{f} \in L^p(\mathbb{R}^N)$. Here again Lemma 3.4 holds with $m = 2$ and $l = -2$, hence the existence of $\tilde{z} \in W_0^{4, p}(\mathbb{R}^N)$ such that $\Delta^2 \tilde{z} = \tilde{f}$.

Then we come back to the restriction $z = \tilde{z}\vert_{\mathbb{R}^N}$ for which we have naturally $\Delta^2 z = f$ in $\mathbb{R}^N$. \(\square\)

Now we can fill the gap of Proposition 4.12:

**Proposition 4.14.** Let $l \in \{-1, 0, 1\}$ such that

$$\frac{N}{p'} \neq 1 \quad \text{if} \quad l = 1 \quad \text{and} \quad \frac{N}{p} \neq 1 \quad \text{if} \quad l = -1.$$
Then the biharmonic operator
\[ \Delta^2 : \hat{W}_l^{-2,p}(\mathbb{R}^N_+) / \mathcal{B}_{[2-l-N/p]} \rightarrow \hat{W}_l^{-2,p}(\mathbb{R}^N_+) \perp \mathcal{B}_{[2+l-N/p']} \]
is an isomorphism.

**Proof.** At first we will use Lemma 4.11 and Proposition 4.13 to solve \((P^*)\) for \(l \in \{-1, 0\} \).

Let \( f \in \hat{W}_l^{-2,p}(\mathbb{R}^N_+) \) with \( l \in \{-1, 0\} \) verifying (4). By Lemma 4.11, there exists \( F = (F_{ij})_{1 \leq i, j \leq N} \in \hat{W}_{l}^{0,p}(\mathbb{R}^N_+)^{N^2} \) such that \( f = \text{div} \text{div} F \). It suffices to apply Proposition 4.13 to all the components \( F_{ij} \) of \( F \) to find \( Z = (Z_{ij})_{1 \leq i, j \leq N} \in \hat{W}_{l}^{4,p}(\mathbb{R}^N_+)^{N^2} \) such that \( \Delta^2 Z = F \) in \( \mathbb{R}^N \).

Setting \( \tilde{z} = \text{div} \text{div} Z \), we obtain \( z \in \hat{W}_l^{2,p}(\mathbb{R}^N_+) \) such that \( \Delta^2 \tilde{z} = f \) in \( \mathbb{R}^N_+ \) because the operators \( \text{div} \text{div} \) and \( \Delta \) commute. Thus we have \( z|_{\Gamma} \in \hat{W}_l^{2-1/p,p}(\Gamma) \) and \( \partial_N z|_{\Gamma} \in \hat{W}_l^{1-1/p,p}(\Gamma) \), and Lemma 4.9 asserts the existence of a solution \( v \in \hat{W}_l^{2,p}(\mathbb{R}^N_+) \) to problem (47), since we have still \( \mathcal{B}_{[2+l-N/p']} = \{ 0 \} \) (cf. Remark 4.8). Then the function \( u = \tilde{z} - v \) answer again to problem \((P^*)\) for \( l \in \{-1, 0\} \).

Finally to solve the case \( l = 1 \), we proceed by duality from the case \( l = -1 \). We have the isomorphism
\[ \Delta^2 : \hat{W}_{-1}^{-2,p}(\mathbb{R}^N_+) / \mathcal{B}_{[3,-N/p]} \rightarrow \hat{W}_{-1}^{-2,p}(\mathbb{R}^N_+) \quad \text{if} \quad \frac{N}{p} \neq 1, \quad (51) \]
therefore by duality, the isomorphism
\[ \Delta^2 : \hat{W}_{1}^{-2,p}(\mathbb{R}^N_+) \rightarrow \hat{W}_{1}^{-2,p}(\mathbb{R}^N_+) \perp \mathcal{B}_{[3,-N/p']} \quad \text{if} \quad \frac{N}{p} \neq 1. \quad (52) \]

**Remark 4.15.** It is also possible to solve directly the case \( l = 1 \). The first step is to extend Proposition 4.13 to \( l = 1 \) with \( N/p' \neq 1 \). Here we consider the extension \( \tilde{f} \in \hat{W}_1^{0,p}(\mathbb{R}^N) \) of \( f \in \hat{W}_1^{0,p}(\mathbb{R}^N_+) \) defined by
\[ \tilde{f}(x',x_N) = \begin{cases} 
 f(x',x_N) & \text{if } x_N > 0, \\
 0 & \text{if } x_N = 0, \\
 -f(x',-x_N) & \text{if } x_N < 0. 
\end{cases} \]
Then we use Lemma 3.5 with \( m = 2 \), which asserts the existence of a function \( \tilde{z} \in \hat{W}_1^{4,p}(\mathbb{R}^N) \) such that \( \Delta^2 \tilde{z} = \tilde{f} \) in \( \mathbb{R}^N \), if \( N/p' \neq 1 \) and \( \tilde{f} \perp \mathcal{P}_{[1-N/p']}^\Delta \). There are two cases: either \( N/p' > 1 \), then \( \mathcal{P}_{[1-N/p']}^\Delta = \{ 0 \} \) and there is no condition on \( \tilde{f} \); or \( N/p' < 1 \), then \( \mathcal{P}_{[1-N/p']}^\Delta \) and we must have \( \tilde{f} \perp \mathcal{P}_0 \). But \( N/p' < 1 \) implies that \( \hat{W}_1^{0,p}(\mathbb{R}^N) \leftrightarrow L^1(\mathbb{R}^N) \) and we have
\[ \int_{\mathbb{R}^N} \tilde{f} \, dx = 0, \]
as a straightforward consequence of this extension of \( f \). That exactly means that \( \tilde{f} \perp \mathcal{P}_0 \). Thus \( z = \tilde{z}|_{\mathbb{R}^N_+} \in \hat{W}_1^{4,p}(\mathbb{R}^N_+) \) satisfies \( \Delta^2 z = f \) in \( \mathbb{R}^N_+ \).
The second step is to resume the proof of Proposition 4.14 for \( l = 1 \) with \( N/p' \neq 1 \). If \( N/p' > 1 \), we have still \( \mathcal{B}_{[3-N/p']} = \{0\} \), so the same reasoning holds; if \( N/p' < 1 \), we know that \( \mathcal{B}_{[3-N/p']} = \mathbb{R}x_N^2 \) and Lemma 4.9 requires the following compatibility condition for problem (47):

\[
\forall \varphi \in \mathbb{R}x_N^2, \quad \langle \partial_N z, \Delta \varphi \rangle_\Gamma - \langle z, \partial_N \Delta \varphi \rangle_\Gamma = 0,
\]

which boils down to

\[
\langle \partial_N z, 1 \rangle_\Gamma = 0. \tag{53}
\]

But remember that \( f \) must satisfy the orthogonality condition for \((P^*)\), i.e.

\[
\langle f, x_N^2 \rangle_{W^{-2,p}(\mathbb{R}^N_+) \times W^{-2,p}(\mathbb{R}^N_+)} = 0
\]

and moreover we have \( f = \Delta^2 z \) in \( \mathbb{R}^N_+ \); thus

\[
\langle \Delta^2 z, x_N^2 \rangle_{W^{2-p}(\mathbb{R}^N_+) \times W^{2-p}(\mathbb{R}^N_+)} = 0.
\]

It suffices to write the Green formula

\[
\langle \Delta^2 z, x_N^2 \rangle_{W^{-2,p}(\mathbb{R}^N_+) \times W^{-2,p}(\mathbb{R}^N_+)} = -\langle \partial_N z, \Delta x_N^2 \rangle_\Gamma = -2 \langle \partial_N z, 1 \rangle_\Gamma,
\]

to see that (53) holds.

To finish the proof of Theorem 4.1, it remains to combine Propositions 4.12 and 4.14, which provides the isomorphism

\[
\Delta^2 : \tilde{W}^{2,p}_l(\mathbb{R}^N_+) / \mathcal{B}_{[2-l-N/p]} \rightarrow W^{-2,p}(\mathbb{R}^N_+) \perp \mathcal{B}_{[2+l-N/p]}, \tag{54}
\]

for any \( l \in \mathbb{Z} \) verifying (4). This answers globally to problem \((P^*)\) and thus to general problem \((P)\) by means of the lifting function mentioned above.

References