



# SUSY WT identity in a lattice formulation of 2D $\mathcal{N} = (2, 2)$ SYM

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## ABSTRACT

We address some issues relating to a supersymmetric (SUSY) Ward–Takahashi (WT) identity in Sugino's lattice formulation of two-dimensional (2D)  $\mathcal{N} = (2, 2)$   $SU(k)$  supersymmetric Yang–Mills theory (SYM). A perturbative argument shows that the SUSY WT identity in the continuum theory is reproduced in the continuum limit without any operator renormalization/mixing and tuning of lattice parameters. As application of the lattice SUSY WT identity, we show that a prescription for the Hamiltonian density in this lattice formulation, proposed by Kanamori, Sugino and Suzuki, is justified also from a perspective of an operator algebra among correctly-normalized supercurrents. We explicitly confirm the SUSY WT identity in the continuum limit to the first nontrivial order in a semi-perturbative expansion.

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## 1. Introduction and the summary

In the present Letter, we derive an identity in Sugino's lattice formulation of two-dimensional (2D)  $\mathcal{N} = (2, 2)$  supersymmetric  $SU(k)$  Yang–Mills theory (SYM) [1,2] that would become the supersymmetric (SUSY) Ward–Takahashi (WT) identity in the continuum limit. (We term this identity a lattice SUSY WT identity for brevity.) On the basis of formal perturbation theory, we then address the renormalization and mixing of composite operators appearing in the identity. Our consideration is quite parallel in the spirit to the standard analysis of the chiral symmetry on the lattice [3]. Compared with the four-dimensional cousin, four-dimensional  $\mathcal{N} = 1$  SYM [4–9], the situation in 2D  $\mathcal{N} = (2, 2)$  SYM is much simpler or almost trivial, because this 2D model is super-renormalizable. We can in fact argue that, in the continuum limit, the lattice SUSY WT identity reproduces the SUSY WT identity in the continuum target theory without any operator renormalization/mixing and tuning of parameters. This conclusion is consistent with the expected SUSY restoration without fine-tuning in the effective action of elementary fields, which has been discussed within perturbation theory [1]. That consideration on the SUSY restoration in Ref. [1] implies the restoration of the SUSY WT identity in the continuum limit was claimed in Ref. [10] only intuitively. The present analysis remedies this gap.

As an interesting application of the lattice SUSY WT identity, we show that a prescription for the Hamiltonian density in this lattice formulation, advocated in Refs. [11,12] in the context of the spontaneous SUSY breaking, can be justified also from a perspective of a “current” algebra among supercurrents and the Hamiltonian density (our argumentation is analogous to that for the order parameter of the spontaneous chiral symmetry breaking in Ref. [3]). For other numerical application of the present lattice formulation, see Refs. [13–15].

Our argument to this stage is standard but somewhat formal. To partially substantiate our formal argument, we carry out a one-loop calculation that confirms the SUSY WT identity in the first nontrivial order of a semi-perturbative expansion [16] which is justified for small volume lattices.

The present lattice formulation is based on the A model topological twist of 2D  $\mathcal{N} = (2, 2)$  theories [17]. For 2D  $\mathcal{N} = (2, 2)$   $U(k)$  SYM, there exists another type of lattice formulation, proposed initially by Ref. [18] and independently in Ref. [19], that can be understood in terms of the B-model topological twist. For this another type of lattice formulation and related works, see Ref. [20] for a recent review, Ref. [21] and references cited therein.

## 2. Lattice SUSY WT identity

A most salient feature of the lattice formulation of Refs. [1,2] is that it is exactly invariant under a fermionic symmetry  $Q$ , defined

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by<sup>1</sup>

$$\begin{aligned} Q U_\mu(x) &= i\psi_\mu(x)U_\mu(x), \\ Q \psi_\mu(x) &= i\psi_\mu(x)\psi_\mu(x) - i(\phi(x) - U_\mu(x)\phi(x + a\hat{\mu}))U_\mu(x)^{-1}, \\ Q \phi(x) &= 0, \\ Q \bar{\phi}(x) &= \eta(x), \quad Q \eta(x) = [\phi(x), \bar{\phi}(x)], \\ Q \chi(x) &= H(x), \quad Q H(x) = [\phi(x), \chi(x)], \end{aligned} \quad (2.1)$$

where  $U_\mu(x) \in SU(k)$  are conventional gauge link variables,  $\phi(x)$  is a complex scalar field ( $\bar{\phi}(x)$  is its complex conjugate),  $\Psi(x)^T \equiv (\psi_0(x), \psi_1(x), \chi(x), (1/2)\eta(x))$  are fermionic fields and  $H(x)$  is the auxiliary field;  $a$  and  $\hat{\mu}$  respectively denote the lattice spacing and a unit vector along the direction  $\mu$  ( $\mu = 0$  or  $1$ ). One confirms that the square of above transformation (2.1) is a lattice gauge transformation with the parameter  $\phi(x)$ ,  $Q^2 = \delta_\phi$ ;  $Q$  is thus nilpotent on gauge invariant combinations. The exact invariance of the lattice action  $S_{2DSYM}^{\text{LAT}}$  under  $Q$  is then realized by defining it in a  $Q$ -exact form,

$$S_{2DSYM}^{\text{LAT}} = QX, \quad (2.2)$$

where  $X$  is a certain gauge invariant combination whose explicit form can be found in Ref. [2].

The super transformation in the target continuum theory, 2D  $\mathcal{N} = (2, 2)$  SYM, has four spinor components,  $(Q^{(0)}, Q^{(1)}, \tilde{Q}, Q)$ , and above transformation (2.1) is a lattice transcription of the continuum  $Q$  transformation. The lattice formulation however does not possess invariance under other three transformations,  $Q^{(0)}$ ,  $Q^{(1)}$  and  $\tilde{Q}$ , and a crucial issue is whether the invariance under these three transformations is restored in the continuum limit or not.

The present lattice formulation possesses two exact bosonic symmetries. Important in what follows is the  $U(1)_A$  symmetry, under which<sup>2,3</sup>

$$\begin{aligned} \Psi(x) &\rightarrow \exp(\alpha \Gamma_2 \Gamma_3) \Psi(x), \\ \phi(x) &\rightarrow \exp(2i\alpha) \phi(x), \quad \bar{\phi}(x) \rightarrow \exp(-2i\alpha) \bar{\phi}(x). \end{aligned} \quad (2.5)$$

From Eq. (2.1), we see that the  $Q$  transformation has the  $U(1)_A$  charge  $+1$ , i.e.,  $Q \rightarrow e^{i\alpha} Q$  under  $U(1)_A$ . Also, the combination  $X$  in Eq. (2.2) has  $U(1)_A$  charge  $-1$  and thus the lattice action  $S_{2DSYM}^{\text{LAT}}$  is neutral under  $U(1)_A$  as it should be ( $U(1)_A$  is a manifest lattice symmetry).<sup>4</sup> Although the target continuum theory possesses

<sup>1</sup> In this Letter, we adopt the convention that all lattice variables are dimensionless. The mass dimension of fields in the continuum theory is provided by multiplying appropriate powers of the lattice spacing.

<sup>2</sup> We adopt the convention

$$\begin{aligned} \Gamma_0 &= \begin{pmatrix} -i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & -i\sigma_3 \end{pmatrix}, \\ \Gamma_2 &= \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \Gamma_3 = C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \end{aligned} \quad (2.3)$$

and  $\Gamma_5 \equiv \Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}$ .

<sup>3</sup> Another manifest bosonic symmetry is the invariance under a “flip” of the 0- and 1-axes [2], under which

$$\begin{aligned} U_0(x) &\rightarrow U_1(\bar{x}), \quad U_1(x) \rightarrow U_0(\bar{x}), \quad H(x) \rightarrow -H(\bar{x}), \\ \phi(x) &\rightarrow \phi(\bar{x}), \quad \bar{\phi}(x) \rightarrow \bar{\phi}(\bar{x}), \\ \Psi(x) &\rightarrow \mathcal{F}\Psi(\bar{x}), \quad \mathcal{F} \equiv \frac{1}{2}(i + \Gamma_3)(\Gamma_0 - \Gamma_1), \end{aligned} \quad (2.4)$$

where  $\bar{x} \equiv (x_1, x_0)$  for  $x \equiv (x_0, x_1)$ . We, however, do not employ this 0–1 flip symmetry in the present analysis.

<sup>4</sup> Note that  $U(1)_A$  is not anomalous in 2D SYM.

other  $R$ -symmetries, the  $U(1)_V$  symmetry and a  $Z_2$  symmetry, the present lattice formulation is not invariant under these two.

Now, the most transparent way to examine the restoration of SUSY in the continuum limit would be to consider a WT identity associated with SUSY. To derive a corresponding identity in the present lattice formulation, we first define a lattice analogue of continuum fermionic transformations other than  $Q$ , i.e.,  $Q^{(0)}$ ,  $Q^{(1)}$  and  $\tilde{Q}$ .

For this, it is convenient to introduce two bosonic transformations  $R$  and  $S$ :  $R$  is defined by

$$\begin{aligned} R: \Psi(x) &\rightarrow i\Gamma_2 \Psi(x), \quad \phi(x) \rightarrow -\bar{\phi}(x), \quad \bar{\phi}(x) \rightarrow -\phi(x), \\ H(x) &\rightarrow -H(x) + i\hat{\phi}(x), \end{aligned} \quad (2.6)$$

where  $\hat{\phi}(x)$  is a particular combination [2] of the plaquette variables, whose continuum limit is the 2D field strength  $2a^2 F_{01}(x)$  ( $a^2 F_{01}(x) \equiv a\partial_0 A_1(x) - a\partial_1 A_0(x) + i[A_0(x), A_1(x)]$ ).  $S$  is defined by

$$S: \Psi(x) \rightarrow i\Gamma_5 \Psi(x). \quad (2.7)$$

In the continuum limit, these  $R$  and  $S$  are a part of  $R$ -symmetries in the continuum target theory (the former is a  $Z_2$  symmetry and the latter is the  $U(1)_V$  symmetry  $\Psi(x) \rightarrow \exp(i\alpha \Gamma_5) \Psi(x)$  with the angle  $\alpha = \pi/2$ ). We note that  $R$  flips the sign of the  $U(1)_A$  charge, while  $S$  does not change the  $U(1)_A$  charge. In the continuum target theory, fermionic transformations,  $Q^{(0)}$ ,  $Q^{(1)}$  and  $\tilde{Q}$ , are related to the  $Q$  transformation by (the continuum limit of)  $R$  and  $S$ , as

$$Q^{(0)} = RSQS^{-1}R^{-1}, \quad Q^{(1)} = RQR^{-1}, \quad \tilde{Q} = SQS^{-1}. \quad (2.8)$$

We can thus define  $Q^{(0)}$ ,  $Q^{(1)}$  and  $\tilde{Q}$  transformations on the lattice by applying relations (2.8) to lattice  $Q$  transformation (2.1). A virtue of this approach is that the covariance under  $U(1)_A$  becomes manifest. In fact, from Eq. (2.8), it immediately follows that  $(Q^{(0)}, Q^{(1)}, \tilde{Q}, Q) \rightarrow (e^{-i\alpha} Q^{(0)}, e^{-i\alpha} Q^{(1)}, e^{i\alpha} \tilde{Q}, e^{i\alpha} Q)$  under  $U(1)_A$  transformation (2.5). Also, from the nilpotency of  $Q$  and Eq. (2.8), the lattice  $Q^{(0)}$ ,  $Q^{(1)}$  and  $\tilde{Q}$  are individually nilpotent on gauge invariant combinations. However, since the lattice action is not invariant under  $R$  and  $S$ ,  $Q^{(0)}$ ,  $Q^{(1)}$  and  $\tilde{Q}$  are not lattice symmetries; we note

$$\begin{aligned} S_{2DSYM}^{\text{LAT}} &= QX \\ &= Q^{(0)}RSX + (1 - RS)S_{2DSYM}^{\text{LAT}} \\ &= Q^{(1)}RX + (1 - R)S_{2DSYM}^{\text{LAT}} \\ &= \tilde{Q}SX + (1 - S)S_{2DSYM}^{\text{LAT}}. \end{aligned} \quad (2.9)$$

In the second line above, for example, the first term  $Q^{(0)}RSX$  vanishes under the action of  $Q^{(0)}$  because  $Q^{(0)}$  is nilpotent. However, the second term  $(1 - RS)S_{2DSYM}^{\text{LAT}}$  is an  $O(a)$  quantity (because this combination vanishes in the naive continuum limit owing to  $R$ -symmetries in the continuum theory) that does not necessarily vanish under  $Q^{(0)}$ . We note that each term in Eq. (2.9), such as  $Q^{(0)}RSX$  or  $(1 - RS)S_{2DSYM}^{\text{LAT}}$ , is manifestly neutral under  $U(1)_A$ .

We are now ready to derive the lattice SUSY WT identity. We define a would-be super transformation on the lattice  $\delta$  by

$$\begin{aligned} \delta &\equiv \frac{1}{a^{1/2}} (\varepsilon^{(0)} Q^{(0)} + \varepsilon^{(1)} Q^{(1)} + \tilde{\varepsilon} \tilde{Q} + \varepsilon Q), \\ \varepsilon &\equiv -(\varepsilon^{(0)}, \varepsilon^{(1)}, \tilde{\varepsilon}, \varepsilon), \end{aligned} \quad (2.10)$$

where  $(\varepsilon^{(0)}, \varepsilon^{(1)}, \tilde{\varepsilon}, \varepsilon)$  are Grassmann parameters. A WT identity can be derived as usual by employing a localized version of  $\delta$ , that is defined by  $\varepsilon \rightarrow \varepsilon(x)$  in Eq. (2.10). We note that the identity

$$\int [d(\text{fields})] \delta [e^{-S_{2\text{DSYM}}^{\text{LAT}} - S_{\text{mass}}^{\text{LAT}}} \mathcal{O}(y_1, \dots, y_n)] = 0, \quad (2.11)$$

holds for any multi-local operator  $\mathcal{O}(y_1, \dots, y_n)$ . As in Ref. [10], here we have introduced a scalar mass term

$$S_{\text{mass}}^{\text{LAT}} \equiv \frac{\mu^2}{g^2} \sum_x \text{tr}[\bar{\phi}(x)\phi(x)], \quad (2.12)$$

which explicitly breaks SUSY. Identity (2.11) holds because the functional integral measure  $[d(\text{fields})]$  (see Ref. [2]) is invariant under the shift of integration variables induced by the localized  $\delta$ ;  $[d(\text{fields})]$  is obviously invariant under  $R$  and  $S$  and it is invariant also under the shift of variables induced by  $Q$  [23].

We now set

$$\delta S_{2\text{DSYM}}^{\text{LAT}} \equiv -ia^2 \sum_x \epsilon(x)^T [-\partial_\mu^* s_\mu(x) + B(x)], \quad (2.13)$$

where  $\partial_\mu^*$  denotes the backward difference operator:  $\partial_\mu^* f(x) \equiv (1/a)(f(x) - f(x - a\hat{\mu}))$ .  $s_\mu(x)$  is a lattice counterpart of the supercurrent and the breaking term  $B(x)$  arises from the non-invariance of the lattice action  $S_{2\text{DSYM}}^{\text{LAT}}$  under  $\delta$ . The separation of  $\delta S_{2\text{DSYM}}^{\text{LAT}}$  into  $-\partial_\mu^* s_\mu(x)$  and  $B(x)$  in Eq. (2.13) is not unique and we fix this ambiguity as follows: In considering terms in  $\delta S_{2\text{DSYM}}^{\text{LAT}}$  that are proportional to  $\varepsilon^{(0)}(x)$ , for example, we use the decomposition in the second line of Eq. (2.9). A part of the Noether current  $-\partial_\mu^* s_\mu(x)$  is read off from the variation of the first term  $Q^{(0)}RSX$  (that is invariant under the global  $Q^{(0)}$  transformation), while the breaking effect  $B(x)$  is read off from the variation of the second term  $(1-RS)S_{2\text{DSYM}}^{\text{LAT}}$  that is  $O(a)$ . Similarly, for  $\varepsilon^{(1)}(x)$  (for  $\tilde{\varepsilon}(x)$ ), we use the decomposition in the third (fourth) line of Eq. (2.9). For  $\varepsilon(x)$ , since  $S_{2\text{DSYM}}^{\text{LAT}} = QX$  is manifestly invariant under  $Q$ , we can define a conserved Noether current without the breaking term. That is, the breaking term has the structure

$$B(x)^T = (*, *, *, 0). \quad (2.14)$$

Since, for example, both  $Q^{(0)}RSX$  and  $(1-RS)S_{2\text{DSYM}}^{\text{LAT}}$  are neutral under  $U(1)_A$ , and  $Q^{(0)}$  has a definite  $U(1)_A$  charge  $-1$ , the above prescription provides the supercurrent  $s_\mu(x)$  and the breaking term  $B(x)$  which are covariant under  $U(1)_A$ . That is, we have  $s_\mu(x) \rightarrow \exp(-\alpha\Gamma_2\Gamma_3)s_\mu(x)$  and  $B(x) \rightarrow \exp(-\alpha\Gamma_2\Gamma_3)B(x)$  under  $U(1)_A$ .<sup>5</sup> We do not need the (quite complicated) explicit expression of  $s_\mu(x)$  and  $B(x)$  in what follows. A naive continuum limit of the lattice supercurrent reads,

$$\begin{aligned} s_\mu(x) = & -\frac{1}{a^{7/2}} \frac{2}{g^2} C(-i\Gamma_0\Gamma_1\Gamma_\mu \text{tr}[H(x)\Psi(x)] \\ & - i\Gamma_\nu\Gamma_\uparrow\Gamma_\mu \text{tr}[aD_\nu\phi(x)\Psi(x)] \\ & - i\Gamma_\nu\Gamma_\downarrow\Gamma_\mu \text{tr}[aD_\nu\bar{\phi}(x)\Psi(x)] \\ & - \frac{i}{2}[\Gamma_\uparrow, \Gamma_\downarrow]\Gamma_\mu \text{tr}[[\phi(x), \bar{\phi}(x)]\Psi(x)] + O(a), \end{aligned} \quad (2.15)$$

where  $g$  is the 2D gauge coupling constant and  $\Gamma_{\uparrow,\downarrow} \equiv (i/2)(\Gamma_2 \mp i\Gamma_3)$ ;  $D_\mu$  denotes the covariant derivative with respect to the adjoint representation,  $aD_\mu \equiv a\partial_\mu + i[A_\mu, \cdot]$ .

For the scalar mass term, setting

$$\delta S_{\text{mass}}^{\text{LAT}} \equiv -ia^2 \sum_x \epsilon(x)^T \frac{\mu^2}{g^2} f(x), \quad (2.16)$$

we have

$$f(x) = \frac{1}{a^{5/2}} 2iC(\Gamma_\uparrow \text{tr}[\phi(x)\Psi(x)] + \Gamma_\downarrow \text{tr}[\bar{\phi}(x)\Psi(x)]). \quad (2.17)$$

By combining Eqs. (2.11), (2.13) and (2.16) and noting that the function  $\epsilon(x)$  is arbitrary, we have the lattice SUSY WT identity,

$$\begin{aligned} \partial_\mu^* \langle s_\mu(x) \mathcal{O}(y_1, \dots, y_n) \rangle \\ = \frac{\mu^2}{g^2} \langle f(x) \mathcal{O}(y_1, \dots, y_n) \rangle - i \frac{\delta}{\delta \epsilon(x)} \langle \mathcal{O}(y_1, \dots, y_n) \rangle \\ + \langle B(x) \mathcal{O}(y_1, \dots, y_n) \rangle, \end{aligned} \quad (2.18)$$

where  $\delta \mathcal{O}(y_1, \dots, y_n) \equiv a^2 \sum_x \epsilon(x)^T (\delta/\delta \epsilon(x)) \mathcal{O}(y_1, \dots, y_n)$ . We emphasize that this identity holds irrespective of the boundary conditions, because we could assume that the localized parameter  $\epsilon(x)$  has a compact support which does not overlap with the boundary.

Compared with the SUSY WT identity expected in the continuum target theory, lattice SUSY WT identity (2.18) has additional contribution owing to the breaking term  $B(x)$ .  $B(x)$  is an  $O(a)$  lattice artifact. However, it can generally become  $O(1)$  in correlation functions when combined with the ultraviolet divergence. In the next section, by employing formal perturbation theory, we discuss how  $B(x)$  behaves in the continuum limit.

### 3. Operator mixing and application of the lattice SUSY WT identity

In perturbation theory, one has to introduce the gauge fixing and the associated Faddeev–Popov ghost term (see, for example, Ref. [22]). Since these are not invariant under super transformations, they generally give rise to additional contribution to lattice SUSY WT identity (2.18). Also, if the multi-local operator  $\mathcal{O}(y_1, \dots, y_n)$  in Eq. (2.18) is not gauge invariant (just for a collection of elementary fields), one has to take into account the operator mixing with gauge non-invariant operators [6,7]. To avoid these complications, in the present Letter, we assume that the multi-local operator  $\mathcal{O}(y_1, \dots, y_n)$  in Eq. (2.18) is a collection of gauge invariant composite operators.<sup>6</sup>

We first consider the case in which the point  $x$  differs from  $y_1, \dots, y_n$  in Eq. (2.18). In this case, the contact term (the second term in the right-hand side of in Eq. (2.18)) is absent and, in the continuum limit, the operator  $B(x)$  may mix with gauge invariant fermionic local operators whose mass dimension is equal to or less than  $5/2$ .<sup>7</sup> Taking into account the covariance of  $B(x)$  under  $U(1)_A$  (2.5),  $B(x) \rightarrow \exp(-\alpha\Gamma_2\Gamma_3)B(x)$ , one sees that a possible operator with which  $B(x)$  can mix is a linear combination of the following eight operators (we have used the fact that  $\text{tr}[\Psi(x)] \equiv 0$  for the gauge group  $SU(k)$ )

$$\begin{aligned} \frac{1}{a^{5/2}} C\Gamma_\uparrow \text{tr}[\phi(x)\Psi(x)], & \quad \frac{1}{a^{5/2}} C\Gamma_\mu\Gamma_\uparrow \text{tr}[\phi(x)\Psi(x)], \\ \frac{1}{a^{5/2}} C\Gamma_5\Gamma_\uparrow \text{tr}[\phi(x)\Psi(x)], & \quad \frac{1}{a^{5/2}} C\Gamma_\downarrow \text{tr}[\bar{\phi}(x)\Psi(x)], \\ \frac{1}{a^{5/2}} C\Gamma_\mu\Gamma_\downarrow \text{tr}[\bar{\phi}(x)\Psi(x)], & \quad \frac{1}{a^{5/2}} C\Gamma_5\Gamma_\downarrow \text{tr}[\bar{\phi}(x)\Psi(x)]. \end{aligned} \quad (3.1)$$

<sup>6</sup> We can regard the Faddeev–Popov ghosts and the Nakanishi–Lautrup auxiliary field as SUSY singlet. Then, since the operations  $Q$ ,  $R$  and  $S$  possess gauge-invariant meaning, lattice super transformation (2.10) and the lattice BRST transformation [22] commute. This implies that SUSY variation of the gauge fixing and the Faddeev–Popov terms is BRST exact and does not contribute to lattice SUSY WT identity (2.18) if the operator  $\mathcal{O}$  is gauge (and thus BRST) invariant.

<sup>7</sup>  $B(x)$  has the structure that  $1/g^2$  times a dimension  $9/2$  operator. Since the loop expansion parameter in the present system is  $g^2$  and it has the mass dimension 2, in the continuum limit,  $B(x)$  mixes with operators whose mass dimension is equal to or less than  $5/2$ , as a result of radiative corrections in 1PI diagrams containing  $B(x)$ .

<sup>5</sup> It turns out that the supercurrent and the breaking term are covariant also under flip transformation (2.4) as,  $s_\mu(x) \rightarrow \mathcal{F}s_\mu(\bar{x})$  and  $B(x) \rightarrow \mathcal{F}B(\bar{x})$ .

We further *assume* that supersymmetry itself has no intrinsic anomaly. That is, we assume that in the continuum limit the breaking effect can be removed by local counterterms. Then only possible mixing turns to be  $B(x) \xrightarrow{a \rightarrow 0} cf(x)$ , where  $c$  is a constant and  $f(x)$  is given by Eq. (2.17). In fact, this combination may be removed by the super transformation of a scalar mass term. However, because of structure (2.14) (that follows from the  $Q$ -invariance of the formulation), the constant  $c$  must vanish. In this way, we see that  $B(x) \xrightarrow{a \rightarrow 0} 0$  and the continuum limit of the lattice SUSY WT identity becomes

$$\partial_\mu \langle s_\mu(x) \mathcal{O}(y_1, \dots, y_n) \rangle = \frac{\mu^2}{g^2} \langle f(x) \mathcal{O}(y_1, \dots, y_n) \rangle, \quad (3.2)$$

when the point  $x$  differs from  $y_1, \dots, y_n$ . This relation shows that the lattice supercurrent  $s_\mu(x)$ , without any renormalization, reproduces in the continuum limit a relation expected in the target continuum theory.<sup>8</sup> Such a supercurrent on the lattice is however not unique. In fact, let  $s'_\mu(x)$  be an appropriately-chosen another lattice supercurrent such that  $\Delta s_\mu(x) \equiv s'_\mu(x) - s_\mu(x) = O(a)$  is gauge invariant. Then  $\Delta s_\mu(x)$  can mix with gauge invariant dimension 3/2 fermionic local operators. Only possible operator mixing is thus  $\Delta s_\mu(x) \xrightarrow{a \rightarrow 0} M \text{tr}[\Psi(x)] \equiv 0$  ( $M$  being a certain  $4 \times 4$  matrix) for the gauge group  $SU(k)$ . This shows that a precise choice of a lattice supercurrent is not relevant for identity (3.2) to hold in the continuum limit.

This corresponds precisely to the situation studied in Ref. [10]. There, the authors employed an appropriately-chosen lattice supercurrent  $s'_\mu(x)$  that is different from  $s_\mu(x)$  by an  $O(a)$  amount. The composite operator was

$$\mathcal{O}(y) = f_\nu(y) \equiv -\frac{1}{2g^2} \Gamma_\nu C^{-1} f(y), \quad (3.3)$$

and the restoration of relation (3.2) with  $x \neq y$  in the continuum limit was observed by means of the Monte Carlo simulation. This demonstrated the SUSY restoration in a nonperturbative level.

Usually, from a WT identity such as (3.2) that does not contain the contact term, i.e., the second term of the right-hand side of Eq. (2.18), one cannot conclude that the current operator  $s_\mu(x)$  is finite or correctly-normalized. In our present 2D case, fortunately, we can directly see that the supercurrent  $s_\mu(x)$  and the operator  $f(x)$  are finite operators which do not require nontrivial renormalization. One can readily see that 1PI diagrams that contain  $s_\mu(x)$  or  $f(x)$  are ultraviolet finite except one-loop diagrams being proportional to  $\text{tr}[\Psi(x)] \equiv 0$ . Thus, the above supercurrent,  $s_\mu(x)$  or  $s'_\mu(x)$ , is a correctly-normalized, finite operator.

As an interesting application of lattice SUSY WT identity (2.18) is obtained by taking an appropriately-chosen lattice supercurrent  $s'_\mu(y)$  itself as the composite operator:

$$\mathcal{O}(y) = (s'_0)_{i=1}(y), \quad (3.4)$$

where  $i$  refers to the spinor index. The  $i=1$  component of the supercurrent corresponds to a Noether current associated with the fermionic transformation  $Q^{(0)}$ . Then, assuming that a naive  $\mu^2 \rightarrow 0$  limit can be taken in lattice SUSY WT identity (2.18), we have

$$\partial_\mu^* \langle (s_\mu)_{i=4}(x) (s'_0)_{i=1}(y) \rangle = i \frac{1}{a^2} \delta_{x,y} \langle Q (s'_0)_{i=1}(x) \rangle. \quad (3.5)$$

Note that we have focused especially on the  $i=4$  spinor component of the lattice supercurrent  $s_\mu(x)$ . Since the  $i=4$  component corresponds to the  $Q$  transformation, we do not have the breaking term  $B(x)$  in Eq. (3.5) even with finite lattice spacings (recall Eq. (2.14)). Now, in the target continuum theory *in classical level*, the  $Q$  transformation of the time component of the Noether current associated with the  $Q^{(0)}$  transformation is the Hamiltonian density,  $Q (s'_0)_{i=1}(x) = 2\mathcal{H}(x)$ , as is consistent with the SUSY algebra,  $\{Q, Q^{(0)}\} = -2i\partial_0 + 2\delta_{A_0}$ . Therefore, it is quite natural to regard the right-hand side of Eq. (3.5) as the expectation value of the Hamiltonian density *in quantum theory*:

$$\langle Q (s'_0)_{i=1}(x) \rangle \equiv 2\langle \mathcal{H}(x) \rangle. \quad (3.6)$$

This is precisely the prescription advocated in Refs. [11,12] for the Hamiltonian density in the present lattice formulation. The reasoning for this prescription in Refs. [11,12] was based on a topological property of the Witten index. Here, we arrived at the identical prescription from an argument of the operator algebra among correctly-normalized supercurrents. This provides another justification for the prescription in Refs. [11,12].

One might wonder to what extent the definition of the Hamiltonian density  $\mathcal{H}(x)$  in Eq. (3.6) is affected by a choice of the supercurrent  $s'_0(y)$  in Eqs. (3.5) and (3.6). Let  $\Delta s'_0(y) \equiv s''_0(y) - s'_0(y) = O(a)$ , where  $s''_0(y)$  denotes a yet another (gauge invariant) lattice supercurrent. An argument similar to above then shows that this does not contribute to the left-hand side of Eq. (3.5),  $\Delta s'_0(y) \xrightarrow{a \rightarrow 0} 0$  when  $x \neq y$ .  $\Delta s'_0(y)$  can contribute only when the positions of two composite operators coincide, i.e., when  $x = y$ . From a dimensional analysis, a possible effect of the difference in the left-hand side of (3.5) is thus  $\partial_\mu^* \langle (s_\mu)_{i=4}(x) (\Delta s'_0)_{i=1}(y) \rangle \xrightarrow{a \rightarrow 0} (d_{00}(\partial_0)^2 + d_{01}\partial_0\partial_1 + d_{11}(\partial_1)^2)\delta^2(x-y)$ , where  $d_{\alpha\beta}$  are constants. However, since the continuum limit of the difference in the right-hand side of Eq. (3.5) is proportional to  $\delta^2(x-y)$  without derivative, we conclude that  $d_{00} = d_{01} = d_{11} = 0$ ; the continuum limit of the Hamiltonian density is not affected by a choice of  $s'_0(y)$ .

On the basis of this prescription for the Hamiltonian density, in Refs. [11,12] and more extensively in Ref. [14], the vacuum energy density of 2D  $\mathcal{N} = (2, 2)$  SYM has been numerically computed. This would provide a possible clue for a conjectured spontaneous SUSY breaking in this system [24]. Note that Eqs. (3.5) and (3.6) show that  $\langle \mathcal{H}(x) \rangle$  is precisely the order parameter of the SUSY breaking, in the sense that its non-zero (positive) value ensures the massless Nambu–Goldstone fermion in the channel of the left-hand side of Eq. (3.5).

#### 4. Confirmation of a SUSY WT identity in small volume lattices

Our discussion on the operator mixing in the previous section is somewhat formal because perturbation theory in 2D gauge theory suffers from the infrared divergence. For generic quantities, one cannot trust perturbation theory in infinite volume, even if the dimensionless loop expansion parameter  $(ag)^2$  becomes very small in the continuum limit.<sup>9</sup> The infrared divergence can be avoided

<sup>8</sup> To show this, we thus used the  $Q$  and  $U(1)_A$  symmetries of the lattice formulation and the absence of an intrinsic SUSY anomaly in the target theory. It might appear that we needed a further assumption on the absence of SUSY anomaly compared with the argument in Ref. [1]. However, one should note that this assumption is implicitly made also in Ref. [1]. Actually, in Ref. [1], the possibility of SUSY breaking arising from *non-local* terms is not taken into account from the beginning. If one does not like to accept a priori the absence of SUSY anomaly in this system, it would be possible to confirm this by explicit (one-loop) perturbative consideration.

<sup>9</sup> For example, the expectation value of the action density  $\mathcal{L}$  in 2D  $SU(2)$  Yang–Mills theory, defined by the plaquette action, is given by  $\langle \mathcal{L} \rangle = (3/2)(1/a^2) - (3/32)g^2$  in the continuum limit; this is an *exact* expression obtained by the character expansion. On the other hand, perturbation theory in infinite volume (see, for example, Ref. [25]) yields  $\langle \mathcal{L} \rangle = (3/2)(1/a^2) + (1/32)g^2$  to the first nontrivial order and this is wrong. There is no real paradox here, because higher-order perturbative corrections are infrared diverging and perturbation theory in infinite volume itself is meaningless for this quantity.

by putting the system into a finite box of size  $L$  (we set the one-dimensional number of lattice points  $N \equiv L/a$ ) that introduces a physical energy scale to the problem. Then perturbation theory turns out to be an asymptotic expansion with respect to  $(Lg)^2$ , rather than  $(ag)^2$  (the infrared divergence is reproduced as a divergence in  $L \rightarrow \infty$ ). Therefore, we may always employ perturbation theory, if volume of the system is small enough measured in the gauge coupling. Certainly, perturbation theory cannot completely substitute Monte Carlo simulations, if one is interested in low-energy physics in large physical volume.

In perturbation theory in a finite box, however, another complication arises; depending on the boundary condition, constant modes of various (perturbatively) massless fields may survive. One cannot apply the conventional perturbation theory to those constant modes because they do not have a quadratic kinetic term; they are rather subject of nonperturbative integrations. In the context of a lattice formulation of 2D  $\mathcal{N} = (2, 2)$  SYM of Ref. [18], the two-point correlation function of scalar fields at zero momentum has been studied by combining one-loop perturbation theory and nonperturbative integrations over constant modes [16]. (For the nonperturbative integration, the technique in Ref. [26] was employed.) In what follows, we confirm a SUSY WT identity examined in Ref. [10] by using this “semi-perturbative” treatment to the first nontrivial order. This analytical study supplements the formal argument in the previous section. Compared with the Monte Carlo study [10], this analytical study is advantageous in that it is free from statistical/systematic errors. We consider the case in which fermionic fields obey the periodic boundary condition along the temporal direction; for this case no definite conclusion was obtained in Ref. [10] owing to large statistical errors.

We thus first parametrize the link variables by gauge potentials as  $U_\mu(x) = \exp(iA_\mu(x))$ . We introduce the measure term [27] and the gauge fixing and the Faddeev–Popov ghost terms [22]. We then decompose lattice fields as<sup>10</sup>

$$A_\mu(x) = \sum_k e^{ikx/a} \tilde{A}_\mu(k), \quad k_\mu \equiv \frac{2\pi n_\mu}{N},$$

$$n_\mu = 0, 1, 2, \dots, N-1, \quad (4.1)$$

and similar expressions for other fields. For modes with  $k_\mu \neq 0$ , we can apply the perturbative expansion. For constant modes with which  $k_\mu = 0$ , a perturbative expansion is impossible and one has to generally carry out the integration in a nonperturbative way. It can be seen from the lattice action, the expectation value of  $\tilde{A}_\mu(0)$  and  $\tilde{\phi}(0)$  is  $O((ag)^{1/2})$  while the expectation value of  $\tilde{\Psi}(0)$  (that is present for the periodic boundary condition) is  $O((ag)^{3/4})$ .

Now, we are interested in whether a SUSY WT identity of the form of Eq. (3.2) [10]

$$\partial_\mu \langle s_\mu(x) f_\nu(y) \rangle = \frac{\mu^2}{g^2} \langle f(x) f_\nu(y) \rangle, \quad \text{for } x \neq y, \quad (4.2)$$

where the operators  $s_\mu(x)$ ,  $f_\nu(y)$  and  $f(x)$  are given by Eqs. (2.15), (3.3) and (2.17), respectively, holds in the continuum limit or not. We thus decompose composite operators in the left-hand side  $\langle s_\mu(x) f_\nu(y) \rangle$  into constant modes and non-constant modes. We neglect ultraviolet finite diagrams because these should not modify the identity in the continuum limit.<sup>11</sup> Then taking into account the order-counting elucidated above, it turns out that the lowest non-trivial order contribution to this function is  $O((ag)^{3/2})$ . It is given

by: Fermion fields  $\Psi(x)$  and  $\Psi(y)$  in composite operators are replaced by the constant mode  $\tilde{\Psi}(0)$  and scalar fields in composite operators are connected by the scalar two-point function with one-loop self-energy corrections. By applying  $\partial_\mu$  to this lowest-order term, one finds

$$\partial_\mu \langle s_\mu(x) f_\nu(y) \rangle = \left( \frac{\mu^2}{g^2} + C \right) \langle f(x) f_\nu(y) \rangle, \quad \text{for } x \neq y, \quad (4.3)$$

to  $O((ag)^{3/2})$ , where the constant  $C$  is given by the one-loop self-energy of scalar fields arising from integrations over non-constant modes. Although the self-energy itself depends on the external momentum, the dependence is higher order in  $(ag)^2$  for a dimensional reason; we can thus set the external momentum zero and regard the self-energy as a constant. In the function  $\langle f(x) f_\nu(y) \rangle$  in the right-hand side of Eq. (4.3), fermion fields  $\Psi(x)$  and  $\Psi(y)$  in composite operators are also replaced by the constant mode  $\tilde{\Psi}(0)$  and scalar fields in composite operators are connected by the scalar two-point function to the one-loop order.

Eq. (4.3) shows that if  $C \neq 0$  in the continuum limit then the expected SUSY WT identity is not restored. A straightforward one-loop calculation yields

$$C = k \frac{2}{N^2} \sum_{(n_0, n_1) \neq (0,0)} \left[ \frac{1}{2} \left( 1 + \frac{1}{\lambda} \right) \frac{1}{\hat{k}^2} + \frac{1}{2} \left( 1 - \frac{1}{\lambda} \right) \frac{1}{\hat{k}^2 + a^2 \mu^2} - \frac{1}{\hat{k}^2} \right], \quad (4.4)$$

where  $\lambda$  denotes the gauge parameter,  $\mu^2$  is the scalar mass-squared,  $\hat{k}^2 \equiv \sum_{\mu=0}^1 (k_\mu)^2$  and  $\hat{k}_\mu \equiv 2 \sin(k_\mu/2)$ . In the square brackets of Eq. (4.4), the first term is the contribution of the gauge loop, the second is the scalar-gauge loop and the third is the fermions' contribution. In the second term, we can neglect  $a^2 \mu^2 = (\mu^2/g^2)(ag)^2$  because this is higher order in  $(ag)^2$ . In this way, we have  $C = 0$ .<sup>12</sup> Combined with Eq. (4.3), this demonstrates expected identity (4.2) with the periodic boundary condition to  $O((ag)^{3/2})$ .

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<sup>12</sup> Although it is not relevant to Eq. (4.2) in the lowest order, the one-loop self-energy of the gauge field is also of interest because power counting tells that it is also ultraviolet diverging. Writing the one-loop effective action of  $\tilde{A}(0)$  as  $S_{\text{eff}} = \sum_{\mu, \nu=0}^1 C_{\mu\nu} N^2 \text{tr}[\tilde{A}_\mu(0) A_\nu(0)]$ , a somewhat lengthy calculation shows

$$C_{\mu\nu} = k \delta_{\mu\nu} \frac{1}{N^2} \sum_{(n_0, n_1) \neq (0,0)} \sum_{\rho=0}^1 \frac{\partial}{\partial k_\rho} \left[ 0 + \frac{\hat{k}_\rho}{\hat{k}^2 + a^2 \mu^2} - \frac{\hat{k}_\rho}{\hat{k}^2} \right], \quad (4.5)$$

where  $\hat{k}_\rho \equiv \sin k_\rho$ . Thus  $C_{\mu\nu} = 0$ , if  $a^2 \mu^2$  in the second term is neglected as a higher order correction. It is interesting to note that, despite the underlying gauge invariance, this expression itself could not vanish for finite  $N$ , if the field content was not supersymmetric.

<sup>10</sup> For simplicity of calculation, we assumed that  $N$  is an odd integer.

<sup>11</sup> Note that the integrations over constant modes do not produce the ultraviolet divergence.

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