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A COUNTEREXAMPLE TO A CONJECTURE OF U. PINKALL

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A *vertex* of a smooth planar curve γ is a stationary point of the curvature of γ. Of course, for every closed curve the maxima and minima of the curvature ensure that there are at least two vertices. The classical four-vertex theorem states that every simple closed curve has at least four vertices (see [1] for a simple proof and more references). The figure eight is an obvious example of a curve that is not simple and only has two vertices. In [2], U. Pinkall proved a generalization of the four-vertex theorem by showing that a curve has at least 4 vertices if it bounds an immersed surface. (A curve γ is said to bound an immersed surface of genus g if there is an immersion of the g-holed torus minus a disk, into the plane, such that γ is the image of the boundary). For each natural number g, Pinkall gave a curve that bounds an immersed surface of genus g and has only $4g + 2$ vertices, and he conjectured that bounding curves always have at least $4g + 2$ vertices [2].

In this note we show that for each natural number g, there is a smooth planar curve that bounds an immersed surface of genus g and has only 6 vertices. Figure 1 gives an example of such a curve in the case $g = 1$. The reader may easily verify that the curve bounds an immersed surface. Visibly, it has only 6 vertices (we have marked the 3 local minima on the curve). Examples for higher genus may be obtained from this example by appropriately increasing the number of loops. Here is an explicit construction. Figure 2 is a rough sketch of the boundary. One runs down the big circle once in clockwise order, but at each small circle one loops $2g$ times. The curve can be formed by the union of 6 congruent spirals, so there is only a need of 6 vertices. The surface is seen in S^2 at the left hand side of the boundary curve. A bounded immersion is then obtained by stereographic projection. Figure 3 gives a list of needed patches. The last figure is a rule for gluing two patches at a certain edge. (Glue if three lines meet!)

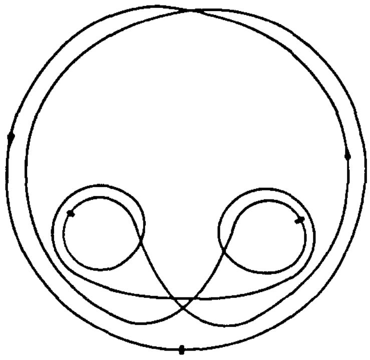

Fig. 1.

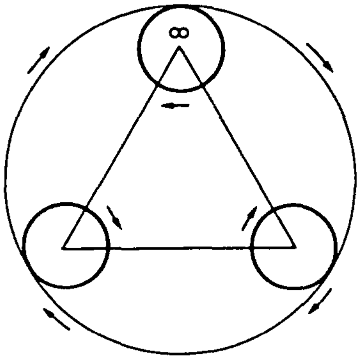

Fig. 2.

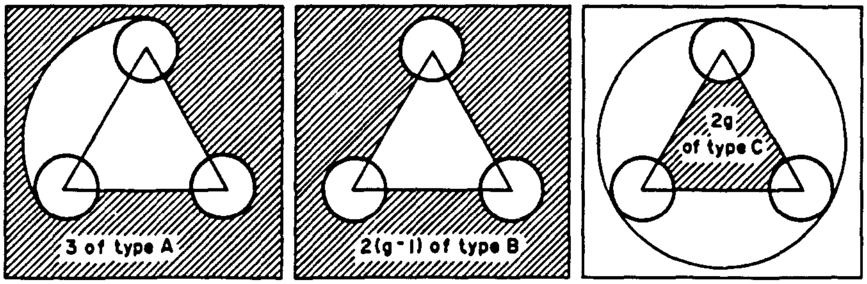

Fig. 3.

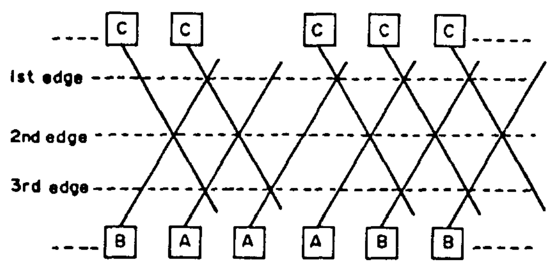

Fig. 4.

We are left with the following

CONJECTURE. *The number of vertices of a smooth closed planar curve is at least* 6, *if it bounds an immersed surface other than the disk.*

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