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Configurations in abelian categories. II. Ringel–Hall algebras

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Abstract

This is the second in a series on *configurations* in an abelian category $\mathcal{A}$. Given a finite poset $(I,\preccurlyeq)$, an $(I,\preccurlyeq)$-*configuration* (σ,ι,π) is a finite collection of objects $\sigma(J)$ and morphisms $\iota(J,K)$ or $\pi(J,K):\sigma(J)\to\sigma(K)$ in $\mathcal{A}$ satisfying some axioms, where $J,K\subseteq I$. Configurations describe how an object X in $\mathcal{A}$ decomposes into subobjects.

The first paper defined configurations and studied moduli spaces of $(I,\preccurlyeq)$-configurations in $\mathcal{A}$, using the theory of Artin stacks. It showed well-behaved moduli stacks $\mathfrak{Obj}_{\mathcal{A}}, \mathfrak{M}(I,\preccurlyeq)_{\mathcal{A}}$ of objects and configurations in $\mathcal{A}$ exist when $\mathcal{A}$ is the abelian category $\mathrm{coh}(P)$ of coherent sheaves on a projective scheme P, or mod-$\mathbb{K}Q$ of representations of a quiver Q.

Write $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ for the vector space of $\mathbb{Q}$-valued constructible functions on the stack $\mathfrak{Obj}_{\mathcal{A}}$. Motivated by the idea of *Ringel–Hall algebras*, we define an associative multiplication $*$ on $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ using pushforwards and pullbacks along 1-morphisms between configuration moduli stacks, so that $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ is a $\mathbb{Q}$-*algebra*. We also study representations of $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$, the Lie subalgebra $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ of functions supported on indecomposables, and other algebraic structures on $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$.

Then we generalize all these ideas to *stack functions* $\underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}})$, a universal generalization of constructible functions, containing more information. When $\mathrm{Ext}^i(X,Y)=0$ for all $X,Y\in\mathcal{A}$ and $i>1$, or when $\mathcal{A}=\mathrm{coh}(P)$ for P a Calabi–Yau 3-fold, we construct (*Lie*) *algebra morphisms* from stack algebras to explicit algebras, which will be important in the sequels on invariants counting τ-semistable objects in $\mathcal{A}$.

Keywords: Configuration; Abelian category; Artin stack; Moduli space; Constructible function; Quiver; Coherent sheaf; Motivic

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1. Introduction

This is the second in a series of papers [12–14] developing the concept of *configuration* in an abelian category $\mathcal{A}$. Given a finite partially ordered set (poset) $(I, \preccurlyeq)$, we define an $(I, \preccurlyeq)$-*configuration* (σ, ι, π) in $\mathcal{A}$ to be a collection of objects $\sigma(J)$ and morphisms $\iota(J, K)$ or $\pi(J, K): \sigma(J) \to \sigma(K)$ in $\mathcal{A}$ satisfying certain axioms, where J, K are subsets of I. Configurations are a tool for describing *how an object X in $\mathcal{A}$ decomposes into subobjects*.

The first paper [12] defined configurations and developed their basic properties, and studied moduli spaces of $(I, \preccurlyeq)$-configurations in $\mathcal{A}$, using the theory of *Artin stacks*. It proved well-behaved moduli stacks $\mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}$ exist when $\mathcal{A}$ is the abelian category of coherent sheaves on a projective scheme P, or of representations of a quiver Q.

This paper develops versions of *Ringel–Hall algebras* [23,24] in the framework of configurations and Artin stacks. The idea of Ringel–Hall algebras is to make a $\mathbb{Q}$-algebra $\mathcal{H}$ from an abelian category $\mathcal{A}$. In the simplest version, the isomorphism classes $[X]$ of objects $X \in \mathcal{A}$ form a basis for $\mathcal{H}$ with multiplication $[X] * [Z] = \sum_{[Y]} g^Y_{XZ}[Y]$, where g^Y_{XZ} is the 'number' of exact sequences $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$. The important point is that $*$ is *associative*.

Ringel–Hall type algebras are defined in four main contexts:

- *Counting subobjects over finite fields*, as in Ringel [23,24].
- *Perverse sheaves on moduli spaces* are used by Lusztig [19].
- *Homology of moduli spaces*, as in Nakajima [20].
- *Constructible functions on moduli spaces* are used by Lusztig [19, §§10.18 and 10.19], Nakajima [20, §10], Frenkel, Malkin and Vybornov [5], Riedtmann [21] and others.

In the first half of the paper we follow the latter path. After some background on stacks and configurations in Sections 2 and 3, we begin in Section 4 with a detailed account of *Ringel–Hall algebras* $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ *of constructible functions on Artin stacks*, using the constructible functions theory developed in [10]. Using quiver representations $\mathcal{A} = \text{mod-}\mathbb{K}Q$ we construct examples of algebras of constructible functions isomorphic to universal enveloping algebras $U(\mathfrak{n}_+)$, for $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ a Kac–Moody algebra.

A distinctive feature of our treatment is the use of *configurations*. Working with configuration moduli stacks $\mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}$ and 1-morphisms between them makes the proofs more systematic, and also suggests new ideas. In particular we construct representations of Ringel–Hall algebras in a way that appears to be new, and define bialgebras and other algebraic structures. The only other paper known to the author using stacks in this way is the brief sketch in Kapranov and Vasserot [16, §3], but stacks appear to be the most natural setting.

The second half of our paper (Sections 5 and 6) studies various Ringel–Hall algebras of *stack functions* $\underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}), \underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda), \ldots$. Stack functions are a universal generalization of constructible functions on stacks introduced in [11], which contain much more information than constructible functions. Using quiver representations $\mathcal{A} = \text{mod-}\mathbb{K}Q$ we construct algebras of stack functions isomorphic to (or more generally quotients of) versions of *quantum groups* $U_\ell(\mathfrak{n}_+)$, for $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ a Kac–Moody algebra. This suggests stack functions can be regarded as a quantized version of constructible functions.

When $\mathrm{Ext}^i(X, Y) = 0$ for all $X, Y \in \mathcal{A}$ and $i > 1$, Section 6 constructs *algebra morphisms* Φ^Λ, Ψ^Λ, Ψ^{Λ°, Ψ^Ω from stack algebras $\underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}})$, $\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, *, *)$ to certain explicit algebras $A(\mathcal{A}, \Lambda, \chi)$, $B(\mathcal{A}, \Lambda, \chi)$, $B(\mathcal{A}, \Lambda^\circ, \chi)$, $C(\mathcal{A}, \Omega, \chi)$. When $\mathcal{A} = \mathrm{coh}(P)$ for P a Calabi–

Yau 3-fold, the same techniques give a *Lie algebra morphism* $\Psi^{\Omega} : \bar{\mathrm{SF}}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}}, \Theta, \Omega) \to C^{\mathrm{ind}}(\mathcal{A}, \Omega, \chi)$.

These ideas will be applied in the sequels [13,14]. Given a *stability condition* $(\tau, T, \leqslant)$ on $\mathcal{A}$, we will define stack functions $\bar{\delta}_{\mathrm{ss}}^{\alpha}(\tau)$ in $\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ parametrizing τ-semistable objects in class α. These satisfy many *identities* in the stack algebra $\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$. Applying $\Phi^{\Lambda}, \ldots, \Psi^{\Omega}$ to $\bar{\delta}_{\mathrm{ss}}^{\alpha}(\tau)$ yields *invariants* of $\mathcal{A}, (\tau, T, \leqslant)$ in $A(\mathcal{A}, \Lambda, \chi), \ldots, C(\mathcal{A}, \Omega, \chi)$, with interesting transformation laws.

2. Background material

We begin with some background material on Artin stacks, constructible functions, stack functions, and motivic invariants, drawn mostly from [10,11].

2.1. *Introduction to Artin $\mathbb{K}$-stacks*

Fix an algebraically closed field $\mathbb{K}$ throughout. There are four main classes of 'spaces' over $\mathbb{K}$ used in algebraic geometry, in increasing order of generality:

$$\mathbb{K}\text{-varieties} \subset \mathbb{K}\text{-schemes} \subset \text{algebraic } \mathbb{K}\text{-spaces} \subset \text{algebraic } \mathbb{K}\text{-stacks}.$$

Algebraic stacks (also known as Artin stacks) were introduced by Artin, generalizing *Deligne–Mumford stacks*. For a good introduction to algebraic stacks see Gómez [7], and for a thorough treatment see Laumon and Moret-Bailly [18]. We make the convention that all algebraic $\mathbb{K}$-stacks in this paper are *locally of finite type*, and $\mathbb{K}$-substacks are *locally closed*.

Algebraic $\mathbb{K}$-stacks form a 2-*category*. That is, we have *objects* which are $\mathbb{K}$-stacks $\mathfrak{F}, \mathfrak{G}$, and also two kinds of morphisms, 1-*morphisms* $\phi, \psi : \mathfrak{F} \to \mathfrak{G}$ between $\mathbb{K}$-stacks, and 2-*morphisms* $A : \phi \to \psi$ between 1-morphisms. An analogy to keep in mind is a 2-category of categories, where objects are categories, 1-morphisms are functors between the categories, and 2-morphisms are isomorphisms (natural transformations) between functors.

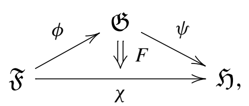
We define the set of $\mathbb{K}$-*points* of a stack.

Definition 2.1. Let $\mathfrak{F}$ be a $\mathbb{K}$-stack. Write $\mathfrak{F}(\mathbb{K})$ for the set of 2-isomorphism classes $[x]$ of 1-morphisms $x : \operatorname{Spec}\mathbb{K} \to \mathfrak{F}$. Elements of $\mathfrak{F}(\mathbb{K})$ are called $\mathbb{K}$-*points*, or *geometric points*, of $\mathfrak{F}$. If $\phi : \mathfrak{F} \to \mathfrak{G}$ is a 1-morphism then composition with ϕ induces a map of sets $\phi_* : \mathfrak{F}(\mathbb{K}) \to \mathfrak{G}(\mathbb{K})$.

For a 1-morphism $x : \operatorname{Spec}\mathbb{K} \to \mathfrak{F}$, the *stabilizer group* $\mathrm{Iso}_{\mathbb{K}}(x)$ is the group of 2-morphisms $x \to x$. When $\mathfrak{F}$ is an algebraic $\mathbb{K}$-stack, $\mathrm{Iso}_{\mathbb{K}}(x)$ is an *algebraic* $\mathbb{K}$-*group*. We say that $\mathfrak{F}$ *has affine geometric stabilizers* if $\mathrm{Iso}_{\mathbb{K}}(x)$ is an affine algebraic $\mathbb{K}$-group for all 1-morphisms $x : \operatorname{Spec}\mathbb{K} \to \mathfrak{F}$.

As an algebraic $\mathbb{K}$-group up to isomorphism, $\mathrm{Iso}_{\mathbb{K}}(x)$ depends only on the isomorphism class $[x] \in \mathfrak{F}(\mathbb{K})$ of x in $\mathrm{Hom}(\operatorname{Spec}\mathbb{K}, \mathfrak{F})$. If $\phi : \mathfrak{F} \to \mathfrak{G}$ is a 1-morphism, composition induces a morphism of algebraic $\mathbb{K}$-groups $\phi_* : \mathrm{Iso}_{\mathbb{K}}([x]) \to \mathrm{Iso}_{\mathbb{K}}(\phi_*([x]))$, for $[x] \in \mathfrak{F}(\mathbb{K})$.

One important difference in working with 2-categories rather than ordinary categories is that in diagram-chasing one only requires 1-morphisms to be 2-*isomorphic* rather than *equal*. The simplest kind of *commutative diagram* is:



by which we mean that $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$ are $\mathbb{K}$-stacks, ϕ, ψ, χ are 1-morphisms, and $F:\psi\circ\phi\to\chi$ is a 2-isomorphism. Usually we omit F, and mean that $\psi\circ\phi\cong\chi$.

Definition 2.2. Let $\phi:\mathfrak{F}\to\mathfrak{H}$, $\psi:\mathfrak{G}\to\mathfrak{H}$ be 1-morphisms of $\mathbb{K}$-stacks. Then one can define the *fibre product stack* $\mathfrak{F}\times_{\phi,\mathfrak{H},\psi}\mathfrak{G}$, or $\mathfrak{F}\times_{\mathfrak{H}}\mathfrak{G}$ for short, with 1-morphisms $\pi_{\mathfrak{F}}$, $\pi_{\mathfrak{G}}$ fitting into a commutative diagram:

$$\begin{array}{ccccc} & \overset{\pi_{\mathfrak{F}}}{\nearrow} & \mathfrak{F} & \overset{\phi}{\searrow} & \\ \mathfrak{F}\times_{\mathfrak{H}}\mathfrak{G} & & \Downarrow & & \mathfrak{H}. \\ & \underset{\pi_{\mathfrak{G}}}{\searrow} & \mathfrak{G} & \underset{\psi}{\nearrow} & \end{array} \tag{1}$$

A commutative diagram

$$\begin{array}{ccccc} & \overset{\theta}{\nearrow} & \mathfrak{F} & \overset{\phi}{\searrow} & \\ \mathfrak{E} & & \Downarrow & & \mathfrak{H} \\ & \underset{\eta}{\searrow} & \mathfrak{G} & \underset{\psi}{\nearrow} & \end{array}$$

is a *Cartesian square* if it is isomorphic to (1), so there is a 1-isomorphism $\mathfrak{E}\cong\mathfrak{F}\times_{\mathfrak{H}}\mathfrak{G}$. Cartesian squares may also be characterized by a universal property.

2.2. Constructible functions on stacks

Next we discuss *constructible functions* on $\mathbb{K}$-stacks, following [10]. For this section we need $\mathbb{K}$ to have *characteristic zero*.

Definition 2.3. Let $\mathfrak{F}$ be an algebraic $\mathbb{K}$-stack. We call $C\subseteq\mathfrak{F}(\mathbb{K})$ *constructible* if $C=\bigcup_{i\in I}\mathfrak{F}_i(\mathbb{K})$, where $\{\mathfrak{F}_i:\ i\in I\}$ is a finite collection of finite type algebraic $\mathbb{K}$-substacks $\mathfrak{F}_i$ of $\mathfrak{F}$. We call $S\subseteq\mathfrak{F}(\mathbb{K})$ *locally constructible* if $S\cap C$ is constructible for all constructible $C\subseteq\mathfrak{F}(\mathbb{K})$.

A function $f:\mathfrak{F}(\mathbb{K})\to\mathbb{Q}$ is called *constructible* if $f(\mathfrak{F}(\mathbb{K}))$ is finite and $f^{-1}(c)$ is a constructible set in $\mathfrak{F}(\mathbb{K})$ for each $c\in f(\mathfrak{F}(\mathbb{K}))\setminus\{0\}$. A function $f:\mathfrak{F}(\mathbb{K})\to\mathbb{Q}$ is called *locally constructible* if $f\cdot\delta_C$ is constructible for all constructible $C\subseteq\mathfrak{F}(\mathbb{K})$, where δ_C is the characteristic function of C. Write $\mathrm{CF}(\mathfrak{F})$ and $\mathrm{LCF}(\mathfrak{F})$ for the $\mathbb{Q}$-vector spaces of $\mathbb{Q}$-valued constructible and locally constructible functions on $\mathfrak{F}$.

Following [10, Definitions 4.8, 5.1 and 5.5] we define *pushforwards* and *pullbacks* of constructible functions along 1-morphisms.

Definition 2.4. Let $\mathfrak{F}$ be an algebraic $\mathbb{K}$-stack with affine geometric stabilizers and $C\subseteq\mathfrak{F}(\mathbb{K})$ be constructible. Then [10, Definition 4.8] defines the *naïve Euler characteristic* $\chi^{\mathrm{na}}(C)$ of C. It is called *naïve* as it takes no account of stabilizer groups. For $f\in\mathrm{CF}(\mathfrak{F})$, define $\chi^{\mathrm{na}}(\mathfrak{F},f)$ in $\mathbb{Q}$ by

$$\chi^{\mathrm{na}}(\mathfrak{F},f)=\sum_{c\in f(\mathfrak{F}(\mathbb{K}))\setminus\{0\}} c\,\chi^{\mathrm{na}}\bigl(f^{-1}(c)\bigr).$$

Let $\mathfrak{F}$, $\mathfrak{G}$ be algebraic $\mathbb{K}$-stacks with affine geometric stabilizers, and $\phi:\mathfrak{F}\to\mathfrak{G}$ a representable 1-morphism. Then for any $x\in\mathfrak{F}(\mathbb{K})$ we have an injective morphism $\phi_*:\mathrm{Iso}_{\mathbb{K}}(x)\to$

$\mathrm{Iso}_{\mathbb{K}}(\phi_*(x))$ of affine algebraic $\mathbb{K}$-groups. The image $\phi_*(\mathrm{Iso}_{\mathbb{K}}(x))$ is an affine algebraic $\mathbb{K}$-group closed in $\mathrm{Iso}_{\mathbb{K}}(\phi_*(x))$, so the quotient $\mathrm{Iso}_{\mathbb{K}}(\phi_*(x))/\phi_*(\mathrm{Iso}_{\mathbb{K}}(x))$ exists as a quasiprojective $\mathbb{K}$-variety. Define a function $m_\phi : \mathfrak{F}(\mathbb{K}) \to \mathbb{Z}$ by $m_\phi(x) = \chi(\mathrm{Iso}_{\mathbb{K}}(\phi_*(x))/\phi_*(\mathrm{Iso}_{\mathbb{K}}(x)))$ for $x \in \mathfrak{F}(\mathbb{K})$.

For $f \in \mathrm{CF}(\mathfrak{F})$, define $\mathrm{CF}^{\mathrm{stk}}(\phi) f : \mathfrak{G}(\mathbb{K}) \to \mathbb{Q}$ by

$$\mathrm{CF}^{\mathrm{stk}}(\phi) f(y) = \chi^{\mathrm{na}}(\mathfrak{F}, m_\phi \cdot f \cdot \delta_{\phi_*^{-1}(y)}) \quad \text{for } y \in \mathfrak{G}(\mathbb{K}),$$

where $\delta_{\phi_*^{-1}(y)}$ is the characteristic function of $\phi_*^{-1}(\{y\}) \subseteq \mathfrak{G}(\mathbb{K})$ on $\mathfrak{G}(\mathbb{K})$. Then $\mathrm{CF}^{\mathrm{stk}}(\phi) : \mathrm{CF}(\mathfrak{F}) \to \mathrm{CF}(\mathfrak{G})$ is a $\mathbb{Q}$-linear map called the *stack pushforward*.

Let $\theta : \mathfrak{F} \to \mathfrak{G}$ be a finite type 1-morphism. If $C \subseteq \mathfrak{G}(\mathbb{K})$ is constructible then so is $\theta_*^{-1}(C) \subseteq \mathfrak{F}(\mathbb{K})$. It follows that if $f \in \mathrm{CF}(\mathfrak{G})$ then $f \circ \theta_*$ lies in $\mathrm{CF}(\mathfrak{F})$. Define the *pullback* $\theta^* : \mathrm{CF}(\mathfrak{G}) \to \mathrm{CF}(\mathfrak{F})$ by $\theta^*(f) = f \circ \theta_*$. It is a linear map.

Here [10, Theorems 5.4, 5.6 and Definition 5.5] are some properties of these.

Theorem 2.5. *Let $\mathfrak{E}$, $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$ be algebraic $\mathbb{K}$-stacks with affine geometric stabilizers, and $\beta : \mathfrak{F} \to \mathfrak{G}$, $\gamma : \mathfrak{G} \to \mathfrak{H}$ be 1-morphisms. Then*

$$\mathrm{CF}^{\mathrm{stk}}(\gamma \circ \beta) = \mathrm{CF}^{\mathrm{stk}}(\gamma) \circ \mathrm{CF}^{\mathrm{stk}}(\beta) : \mathrm{CF}(\mathfrak{F}) \to \mathrm{CF}(\mathfrak{H}), \tag{2}$$

$$(\gamma \circ \beta)^* = \beta^* \circ \gamma^* : \mathrm{CF}(\mathfrak{H}) \to \mathrm{CF}(\mathfrak{F}), \tag{3}$$

supposing β, γ representable in (2), *and of finite type in* (3). *If*

$$\begin{array}{ccc} \mathfrak{E} & \xrightarrow{\ \eta\ } & \mathfrak{G} \\ \Big\downarrow{\scriptstyle\theta} & & {\scriptstyle\psi}\Big\downarrow \\ \mathfrak{F} & \xrightarrow{\ \phi\ } & \mathfrak{H} \end{array} \quad \begin{array}{c} \textit{is a Cartesian square with} \\ \eta,\ \phi \textit{ representable and} \\ \theta,\ \psi \textit{ of finite type, then} \\ \textit{the following commutes:} \end{array} \quad \begin{array}{ccc} \mathrm{CF}(\mathfrak{E}) & \xrightarrow{\ \mathrm{CF}^{\mathrm{stk}}(\eta)\ } & \mathrm{CF}(\mathfrak{G}) \\ \Big\uparrow{\scriptstyle\theta^*} & & {\scriptstyle\psi^*}\Big\uparrow \\ \mathrm{CF}(\mathfrak{F}) & \xrightarrow{\ \mathrm{CF}^{\mathrm{stk}}(\phi)\ } & \mathrm{CF}(\mathfrak{H}). \end{array} \tag{4}$$

As discussed in [10, §3.3] for the $\mathbb{K}$-scheme case, Eq. (2) is *false* for algebraically closed fields $\mathbb{K}$ of characteristic $p > 0$. This is our reason for restricting to $\mathbb{K}$ of characteristic zero in Section 4. In [10, §5.3] we extend Definition 2.4 and Theorem 2.5 to *locally constructible functions*.

2.3. Stack functions

Stack functions are a universal generalization of constructible functions introduced in [11, §3]. Here [11, Definition 3.1] is the basic definition. Throughout $\mathbb{K}$ is algebraically closed of arbitrary characteristic, except when we specify $\operatorname{char}\mathbb{K} = 0$.

Definition 2.6. Let $\mathfrak{F}$ be an algebraic $\mathbb{K}$-stack with affine geometric stabilizers. Consider pairs $(\mathfrak{R}, \rho)$, where $\mathfrak{R}$ is a finite type algebraic $\mathbb{K}$-stack with affine geometric stabilizers and $\rho : \mathfrak{R} \to \mathfrak{F}$ is a 1-morphism. We call two pairs $(\mathfrak{R}, \rho)$, $(\mathfrak{R}', \rho')$ *equivalent* if there exists a 1-isomorphism $\iota : \mathfrak{R} \to \mathfrak{R}'$ such that $\rho' \circ \iota$ and ρ are 2-isomorphic 1-morphisms $\mathfrak{R} \to \mathfrak{F}$. Write $[(\mathfrak{R}, \rho)]$ for the equivalence class of $(\mathfrak{R}, \rho)$. If $(\mathfrak{R}, \rho)$ is such a pair and $\mathfrak{S}$ is a closed $\mathbb{K}$-substack of $\mathfrak{R}$ then $(\mathfrak{S}, \rho|_{\mathfrak{S}})$, $(\mathfrak{R} \setminus \mathfrak{S}, \rho|_{\mathfrak{R}\setminus\mathfrak{S}})$ are pairs of the same kind.

Define $\underline{\mathrm{SF}}(\mathfrak{F})$ to be the $\mathbb{Q}$-vector space generated by equivalence classes $[(\mathfrak{R},\rho)]$ as above, with for each closed $\mathbb{K}$-substack $\mathfrak{S}$ of $\mathfrak{R}$ a relation

$$\big[(\mathfrak{R},\rho)\big]=\big[(\mathfrak{S},\rho|_{\mathfrak{S}})\big]+\big[(\mathfrak{R}\setminus\mathfrak{S},\rho|_{\mathfrak{R}\setminus\mathfrak{S}})\big]. \tag{5}$$

Define $\mathrm{SF}(\mathfrak{F})$ to be the $\mathbb{Q}$-vector space generated by $[(\mathfrak{R},\rho)]$ with ρ *representable*, with the same relations (5). Then $\mathrm{SF}(\mathfrak{F})\subseteq\underline{\mathrm{SF}}(\mathfrak{F})$.

Elements of $\underline{\mathrm{SF}}(\mathfrak{F})$ will be called *stack functions*. In [11, Definition 3.2] we relate $\mathrm{CF}(\mathfrak{F})$ and $\mathrm{SF}(\mathfrak{F})$.

Definition 2.7. Let $\mathfrak{F}$ be an algebraic $\mathbb{K}$-stack with affine geometric stabilizers, and $C\subseteq\mathfrak{F}(\mathbb{K})$ be constructible. Then $C=\coprod_{i=1}^n\mathfrak{R}_i(\mathbb{K})$, for $\mathfrak{R}_1,\dots,\mathfrak{R}_n$ finite type $\mathbb{K}$-substacks of $\mathfrak{F}$. Let $\rho_i:\mathfrak{R}_i\to\mathfrak{F}$ be the inclusion 1-morphism. Then $[(\mathfrak{R}_i,\rho_i)]\in\mathrm{SF}(\mathfrak{F})$. Define $\bar\delta_C=\sum_{i=1}^n[(\mathfrak{R}_i,\rho_i)]\in\mathrm{SF}(\mathfrak{F})$. We think of this stack function as the analogue of the characteristic function $\delta_C\in\mathrm{CF}(\mathfrak{F})$ of C. Define a $\mathbb{Q}$-linear map $\iota_{\mathfrak{F}}:\mathrm{CF}(\mathfrak{F})\to\mathrm{SF}(\mathfrak{F})$ by

$$\iota_{\mathfrak{F}}(f)=\sum_{0\neq c\in f(\mathfrak{F}(\mathbb{K}))}c\cdot\bar\delta_{f^{-1}(c)}.$$

For $\mathbb{K}$ of characteristic zero, define a $\mathbb{Q}$-linear map $\pi_{\mathfrak{F}}^{\mathrm{stk}}:\mathrm{SF}(\mathfrak{F})\to\mathrm{CF}(\mathfrak{F})$ by

$$\pi_{\mathfrak{F}}^{\mathrm{stk}}\left(\sum_{i=1}^n c_i\big[(\mathfrak{R}_i,\rho_i)\big]\right)=\sum_{i=1}^n c_i\,\mathrm{CF}^{\mathrm{stk}}(\rho_i)1_{\mathfrak{R}_i},$$

where $1_{\mathfrak{R}_i}$ is the function 1 in $\mathrm{CF}(\mathfrak{R}_i)$. Then [11, Proposition 3.3] shows $\pi_{\mathfrak{F}}^{\mathrm{stk}}\circ\iota_{\mathfrak{F}}$ is the identity on $\mathrm{CF}(\mathfrak{F})$. Thus, $\iota_{\mathfrak{F}}$ is injective and $\pi_{\mathfrak{F}}^{\mathrm{stk}}$ is surjective. In general $\iota_{\mathfrak{F}}$ is far from surjective, and $\underline{\mathrm{SF}}$, $\mathrm{SF}(\mathfrak{F})$ are much larger than $\mathrm{CF}(\mathfrak{F})$.

All the operations of constructible functions in Section 2.2 extend to stack functions.

Definition 2.8. Define *multiplication* '$\cdot$' on $\underline{\mathrm{SF}}(\mathfrak{F})$ by

$$\big[(\mathfrak{R},\rho)\big]\cdot\big[(\mathfrak{S},\sigma)\big]=\big[(\mathfrak{R}\times_{\rho,\mathfrak{F},\sigma}\mathfrak{S},\rho\circ\pi_{\mathfrak{R}})\big]. \tag{6}$$

This extends to a $\mathbb{Q}$-bilinear product $\underline{\mathrm{SF}}(\mathfrak{F})\times\underline{\mathrm{SF}}(\mathfrak{F})\to\underline{\mathrm{SF}}(\mathfrak{F})$ which is commutative and associative, and $\mathrm{SF}(\mathfrak{F})$ is closed under '$\cdot$'. Let $\phi:\mathfrak{F}\to\mathfrak{G}$ be a 1-morphism of algebraic $\mathbb{K}$-stacks with affine geometric stabilizers. Define the *pushforward* $\phi_*:\underline{\mathrm{SF}}(\mathfrak{F})\to\underline{\mathrm{SF}}(\mathfrak{G})$ by

$$\phi_*:\sum_{i=1}^m c_i\big[(\mathfrak{R}_i,\rho_i)\big]\mapsto\sum_{i=1}^m c_i\big[(\mathfrak{R}_i,\phi\circ\rho_i)\big]. \tag{7}$$

If ϕ is representable then ϕ_* maps $\mathrm{SF}(\mathfrak{F})\to\mathrm{SF}(\mathfrak{G})$. For ϕ of finite type, define *pullbacks* $\phi^*:\underline{\mathrm{SF}}(\mathfrak{G})\to\underline{\mathrm{SF}}(\mathfrak{F})$, $\phi^*:\mathrm{SF}(\mathfrak{G})\to\mathrm{SF}(\mathfrak{F})$ by

$$\phi^*:\sum_{i=1}^m c_i\big[(\mathfrak{R}_i,\rho_i)\big]\mapsto\sum_{i=1}^m c_i\big[(\mathfrak{R}_i\times_{\rho_i,\mathfrak{G},\phi}\mathfrak{F},\pi_{\mathfrak{F}})\big]. \tag{8}$$

The *tensor product* $\otimes : \underline{\mathrm{SF}}(\mathfrak{F}) \times \underline{\mathrm{SF}}(\mathfrak{G}) \to \underline{\mathrm{SF}}(\mathfrak{F} \times \mathfrak{G})$ or $\mathrm{SF}(\mathfrak{F}) \times \mathrm{SF}(\mathfrak{G}) \to \mathrm{SF}(\mathfrak{F} \times \mathfrak{G})$ is

$$\left(\sum_{i=1}^{m} c_i\big[(\mathfrak{R}_i, \rho_i)\big]\right) \otimes \left(\sum_{j=1}^{n} d_j\big[(\mathfrak{S}_j, \sigma_j)\big]\right) = \sum_{i,j} c_i d_j\big[(\mathfrak{R}_i \times \mathfrak{S}_j, \rho_i \times \sigma_j)\big]. \tag{9}$$

Here [11, Theorem 3.5] is the analogue of Theorem 2.5.

Theorem 2.9. *Let $\mathfrak{E}$, $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$ be algebraic $\mathbb{K}$-stacks with affine geometric stabilizers, and $\beta : \mathfrak{F} \to \mathfrak{G}$, $\gamma : \mathfrak{G} \to \mathfrak{H}$ be* 1*-morphisms. Then*

$$(\gamma \circ \beta)_* = \gamma_* \circ \beta_* : \underline{\mathrm{SF}}(\mathfrak{F}) \to \underline{\mathrm{SF}}(\mathfrak{H}), \qquad (\gamma \circ \beta)_* = \gamma_* \circ \beta_* : \mathrm{SF}(\mathfrak{F}) \to \mathrm{SF}(\mathfrak{H}),$$
$$(\gamma \circ \beta)^* = \beta^* \circ \gamma^* : \underline{\mathrm{SF}}(\mathfrak{H}) \to \underline{\mathrm{SF}}(\mathfrak{F}), \qquad (\gamma \circ \beta)^* = \beta^* \circ \gamma^* : \mathrm{SF}(\mathfrak{H}) \to \mathrm{SF}(\mathfrak{F}),$$

for β, γ representable in the second equation, and of finite type in the third and fourth. If $f, g \in \underline{\mathrm{SF}}(\mathfrak{G})$ and β is finite type then $\beta^*(f \cdot g) = \beta^*(f) \cdot \beta^*(g)$. If

$$\begin{array}{ccc} \mathfrak{E} & \xrightarrow{\;\eta\;} & \mathfrak{G} \\ \downarrow{\scriptstyle\theta} & & {\scriptstyle\psi}\downarrow \\ \mathfrak{F} & \xrightarrow{\;\phi\;} & \mathfrak{H} \end{array}$$

is a Cartesian square with θ, ψ of finite type, then the following commutes:

$$\begin{array}{ccc} \underline{\mathrm{SF}}(\mathfrak{E}) & \xrightarrow{\;\eta_*\;} & \underline{\mathrm{SF}}(\mathfrak{G}) \\ \uparrow{\scriptstyle\theta^*} & & {\scriptstyle\psi^*}\uparrow \\ \underline{\mathrm{SF}}(\mathfrak{F}) & \xrightarrow{\;\phi_*\;} & \underline{\mathrm{SF}}(\mathfrak{H}). \end{array}$$

The same applies for $\mathrm{SF}(\mathfrak{E}), \ldots, \mathrm{SF}(\mathfrak{H})$ if η, ϕ are representable.

In [11, Proposition 3.7 and Theorem 3.8] we relate pushforwards and pullbacks of stack and constructible functions using $\iota_{\mathfrak{F}}, \pi_{\mathfrak{F}}^{\mathrm{stk}}$.

Theorem 2.10. *Let $\mathbb{K}$ have characteristic zero, $\mathfrak{F}$, $\mathfrak{G}$ be algebraic $\mathbb{K}$-stacks with affine geometric stabilizers, and $\phi : \mathfrak{F} \to \mathfrak{G}$ be a* 1*-morphism. Then*

(a) $\phi^* \circ \iota_{\mathfrak{G}} = \iota_{\mathfrak{F}} \circ \phi^* : \mathrm{CF}(\mathfrak{G}) \to \mathrm{SF}(\mathfrak{F})$ *if ϕ is of finite type*;
(b) $\pi_{\mathfrak{G}}^{\mathrm{stk}} \circ \phi_* = \mathrm{CF}^{\mathrm{stk}}(\phi) \circ \pi_{\mathfrak{F}}^{\mathrm{stk}} : \mathrm{SF}(\mathfrak{F}) \to \mathrm{CF}(\mathfrak{G})$ *if ϕ is representable*; *and*
(c) $\pi_{\mathfrak{F}}^{\mathrm{stk}} \circ \phi^* = \phi^* \circ \pi_{\mathfrak{G}}^{\mathrm{stk}} : \mathrm{SF}(\mathfrak{G}) \to \mathrm{CF}(\mathfrak{F})$ *if ϕ is of finite type.*

In [11, §3] we extend all the material on $\underline{\mathrm{SF}}, \mathrm{SF}(\mathfrak{F})$ to *local stack functions* $\underline{\mathrm{LSF}}, \mathrm{LSF}(\mathfrak{F})$, the analogues of locally constructible functions. The main differences are in which 1-morphisms must be of finite type.

2.4. Motivic invariants of Artin stacks

In [11, §4] we extend *motivic* invariants of quasiprojective $\mathbb{K}$-varieties to Artin stacks. We need the following data, [11, Assumptions 4.1 and 6.1].

Assumption 2.11. Suppose Λ is a commutative $\mathbb{Q}$-algebra with identity 1, and

$$\Upsilon : \big\{\text{isomorphism classes } [X] \text{ of quasiprojective } \mathbb{K}\text{-varieties } X\big\} \to \Lambda$$

a map for $\mathbb{K}$ an algebraically closed field, satisfying the following conditions:

(i) If $Y \subseteq X$ is a closed subvariety then $\Upsilon([X]) = \Upsilon([X \setminus Y]) + \Upsilon([Y])$.
(ii) If X, Y are quasiprojective $\mathbb{K}$-varieties then $\Upsilon([X \times Y]) = \Upsilon([X])\Upsilon([Y])$.
(iii) Write $\ell = \Upsilon([\mathbb{K}])$ in Λ, regarding $\mathbb{K}$ as a $\mathbb{K}$-variety, the affine line (not the point $\operatorname{Spec}\mathbb{K}$). Then ℓ and $\ell^k - 1$ for $k = 1, 2, \ldots$ are invertible in Λ.

Suppose Λ° is a $\mathbb{Q}$-subalgebra of Λ containing the image of Υ and the elements ℓ^{-1} and $(\ell^k + \ell^{k-1} + \cdots + 1)^{-1}$ for $k = 1, 2, \ldots$, but *not* containing $(\ell - 1)^{-1}$. Let Ω be a commutative $\mathbb{Q}$-algebra, and $\pi : \Lambda^\circ \to \Omega$ a surjective $\mathbb{Q}$-algebra morphism, such that $\pi(\ell) = 1$. Define

$$\Theta : \{\text{isomorphism classes } [X] \text{ of quasiprojective } \mathbb{K}\text{-varieties } X\} \to \Omega$$

by $\Theta = \pi \circ \Upsilon$. Then $\Theta([\mathbb{K}]) = 1$.

We chose the notation 'ℓ' as in motivic integration $[\mathbb{K}]$ is called the *Tate motive* and written $\mathbb{L}$. We have $\Upsilon([\mathrm{GL}(m, \mathbb{K})]) = \ell^{m(m-1)/2} \prod_{k=1}^{m} (\ell^k - 1)$, so (iii) ensures $\Upsilon([\mathrm{GL}(m, \mathbb{K})])$ is invertible in Λ for all $m \geqslant 1$. Here [11, Examples 4.3 and 6.3] is an example of suitable $\Lambda, \Upsilon, \ldots$; more are given in [11, §§4.1 and 6.1].

Example 2.12. Let $\mathbb{K}$ be an algebraically closed field. Define $\Lambda = \mathbb{Q}(z)$, the algebra of rational functions in z with coefficients in $\mathbb{Q}$. For any quasiprojective $\mathbb{K}$-variety X, let $\Upsilon([X]) = P(X; z)$ be the *virtual Poincaré polynomial* of X. This has a complicated definition in [11, Example 4.3] which we do not repeat, involving Deligne's weight filtration when $\operatorname{char}\mathbb{K} = 0$ and the action of the Frobenius on l-adic cohomology when $\operatorname{char}\mathbb{K} > 0$. If X is smooth and projective then $P(X; z)$ is the ordinary Poincaré polynomial $\sum_{k=0}^{2\dim X} b^k(X) z^k$, where $b^k(X)$ is the kth Betti number in l-adic cohomology, for l coprime to $\operatorname{char}\mathbb{K}$. Also $\ell = P(\mathbb{K}; z) = z^2$.

Let Λ° be the subalgebra of $P(z)/Q(z)$ in Λ for which $z \pm 1$ do not divide $Q(z)$. Here are two possibilities for Ω, π. Assumption 2.11 holds in each case.

(a) Set $\Omega = \mathbb{Q}$ and $\pi : f(z) \mapsto f(-1)$. Then $\Theta([X]) = \pi \circ \Upsilon([X])$ is the *Euler characteristic* of X.
(b) Set $\Omega = \mathbb{Q}$ and $\pi : f(z) \mapsto f(1)$. Then $\Theta([X]) = \pi \circ \Upsilon([X])$ is the *sum of the virtual Betti numbers* of X.

We need a few facts about *algebraic* $\mathbb{K}$*-groups*. A good reference is Borel [2]. Following Borel, we define a $\mathbb{K}$*-variety* to be a $\mathbb{K}$-scheme which is reduced, separated, and of finite type, but *not* necessarily irreducible. An algebraic $\mathbb{K}$-group is then a $\mathbb{K}$-variety G with identity $1 \in G$, multiplication $\mu : G \times G \to G$ and inverse $i : G \to G$ (as morphisms of $\mathbb{K}$-varieties) satisfying the usual group axioms. We call G *affine* if it is an affine $\mathbb{K}$-variety. *Special* $\mathbb{K}$-groups are studied by Serre and Grothendieck in [3, §§1 and 5].

Definition 2.13. An algebraic $\mathbb{K}$-group G is called *special* if every principal G-bundle is Zariski locally trivial. Properties of special $\mathbb{K}$-groups can be found in [3, §§1.4, 1.5 and 5.5] and [11, §2.1]. In [11, Lemma 4.6] we show that if Assumption 2.11 holds and G is special then $\Upsilon([G])$ is invertible in Λ.

In [11, Theorem 4.9] we extend Υ to Artin stacks, using Definition 2.13.

Theorem 2.14. *Let Assumption* 2.11 *hold. Then there exists a unique morphism of* $\mathbb{Q}$*-algebras* $\Upsilon' : \underline{\mathrm{SF}}(\operatorname{Spec}\mathbb{K}) \to \Lambda$ *such that if* G *is a special algebraic* $\mathbb{K}$*-group acting on a quasiprojective* $\mathbb{K}$*-variety* X *then* $\Upsilon'([[X/G]]) = \Upsilon([X])/\Upsilon([G])$.

Thus, if $\mathfrak{R}$ is a finite type algebraic $\mathbb{K}$-stack with affine geometric stabilizers the theorem defines $\Upsilon'([\mathfrak{R}]) \in \Lambda$. Taking Λ, Υ as in Example 2.12 yields the *virtual Poincaré function* $P(\mathfrak{R}; z) = \Upsilon'([\mathfrak{R}])$ of $\mathfrak{R}$, a natural extension of virtual Poincaré polynomials to stacks. Clearly, Theorem 2.14 only makes sense if $\Upsilon([G])^{-1}$ exists for all special $\mathbb{K}$-groups G. This excludes the Euler characteristic $\Upsilon = \chi$, for instance, since $\chi([\mathbb{K}^\times]) = 0$ is not invertible. We overcome this in [11, §6] by defining a finer extension of Υ to stacks that keeps track of maximal tori of stabilizer groups, and allows $\Upsilon = \chi$. This can then be used with Θ, Ω in Assumption 2.11.

2.5. *Stack functions over motivic invariants*

In [11, §§4–6] we integrate the stack functions of Section 2.3 with the motivic invariant ideas of Section 2.4 to define more stack function spaces.

Definition 2.15. Let Assumption 2.11 hold, and $\mathfrak{F}$ be an algebraic $\mathbb{K}$-stack with affine geometric stabilizers. Consider pairs $(\mathfrak{R}, \rho)$, with equivalence, as in Definition 5. Define $\underline{\mathrm{SF}}(\mathfrak{F}, \Upsilon, \Lambda)$ to be the Λ-module generated by equivalence classes $[(\mathfrak{R}, \rho)]$, with the following relations:

- (i) Given $[(\mathfrak{R}, \rho)]$ as above and $\mathfrak{S}$ a closed $\mathbb{K}$-substack of $\mathfrak{R}$ we have $[(\mathfrak{R}, \rho)] = [(\mathfrak{S}, \rho|_{\mathfrak{S}})] + [(\mathfrak{R} \setminus \mathfrak{S}, \rho|_{\mathfrak{R}\setminus\mathfrak{S}})]$, as in (5).
- (ii) Let $\mathfrak{R}$ be a finite type algebraic $\mathbb{K}$-stack with affine geometric stabilizers, U a quasiprojective $\mathbb{K}$-variety, $\pi_{\mathfrak{R}} : \mathfrak{R} \times U \to \mathfrak{R}$ the natural projection, and $\rho : \mathfrak{R} \to \mathfrak{F}$ a 1-morphism. Then $[(\mathfrak{R} \times U, \rho \circ \pi_{\mathfrak{R}})] = \Upsilon([U])[(\mathfrak{R}, \rho)]$.
- (iii) Given $[(\mathfrak{R}, \rho)]$ as above and a 1-isomorphism $\mathfrak{R} \cong [X/G]$ for X a quasiprojective $\mathbb{K}$-variety and G a special algebraic $\mathbb{K}$-group acting on X, we have $[(\mathfrak{R}, \rho)] = \Upsilon([G])^{-1}[(X, \rho \circ \pi)]$, where $\pi : X \to \mathfrak{R} \cong [X/G]$ is the natural projection 1-morphism.

Define a $\mathbb{Q}$-linear projection $\Pi_{\mathfrak{F}}^{\Upsilon,\Lambda} : \underline{\mathrm{SF}}(\mathfrak{F}) \to \underline{\mathrm{SF}}(\mathfrak{F}, \Upsilon, \Lambda)$ by

$$\Pi_{\mathfrak{F}}^{\Upsilon,\Lambda} : \sum_{i \in I} c_i \big[(\mathfrak{R}_i, \rho_i)\big] \mapsto \sum_{i \in I} c_i \big[(\mathfrak{R}_i, \rho_i)\big],$$

using the embedding $\mathbb{Q} \subseteq \Lambda$ to regard $c_i \in \mathbb{Q}$ as an element of Λ.

We also define variants of these: spaces $\underline{\bar{\mathrm{SF}}}, \bar{\mathrm{SF}}(\mathfrak{F}, \Upsilon, \Lambda)$, $\underline{\bar{\mathrm{SF}}}, \bar{\mathrm{SF}}(\mathfrak{F}, \Upsilon, \Lambda^\circ)$ and $\underline{\bar{\mathrm{SF}}}, \bar{\mathrm{SF}}(\mathfrak{F}, \Theta, \Omega)$, which are the Λ-, Λ°- and Ω-modules respectively generated by $[(\mathfrak{R}, \rho)]$ as above, with ρ representable for $\bar{\mathrm{SF}}(\mathfrak{F}, *, *)$, and with relations (i), (ii) above but (iii) replaced by a finer, more complicated relation [11, Definition 5.17(iii)]. There are natural projections $\Pi_{\mathfrak{F}}^{\Upsilon,\Lambda}$, $\bar{\Pi}_{\mathfrak{F}}^{\Upsilon,\Lambda}$, $\bar{\Pi}_{\mathfrak{F}}^{\Upsilon,\Lambda^\circ}$, $\bar{\Pi}_{\mathfrak{F}}^{\Theta,\Omega}$ between various of the spaces. We can also define *local stack function* spaces $\underline{\mathrm{LSF}}$, $\underline{\mathrm{L}\bar{\mathrm{SF}}}$, $\mathrm{L}\bar{\mathrm{SF}}(\mathfrak{F}, *, *)$.

In [11] we give analogues of Definitions 2.7 and 2.8 and Theorems 2.9 and 2.10 for these spaces. For the analogue of $\pi_{\mathfrak{F}}^{\mathrm{stk}}$, suppose $\mathrm{X} : \Lambda^\circ \to \mathbb{Q}$ or $\mathrm{X} : \Omega \to \mathbb{Q}$ is an algebra morphism with $\mathrm{X}\circ\Upsilon([U])=\chi([U])$ or $\mathrm{X}\circ\Theta([U])=\chi([U])$ for varieties U, where χ is the Euler characteristic. Define $\bar\pi_{\mathfrak{F}}^{\mathrm{stk}}:\bar{\mathrm{SF}}(\mathfrak{F},\Upsilon,\Lambda^\circ)\to\mathrm{CF}(\mathfrak{F})$ or $\bar\pi_{\mathfrak{F}}^{\mathrm{stk}}:\bar{\mathrm{SF}}(\mathfrak{F},\Theta,\Omega)\to\mathrm{CF}(\mathfrak{F})$ by

$$\bar\pi_{\mathfrak{F}}^{\mathrm{stk}}\Biggl(\sum_{i=1}^n c_i\bigl[(\mathfrak{R}_i,\rho_i)\bigr]\Biggr)=\sum_{i=1}^n \mathrm{X}(c_i)\,\mathrm{CF}^{\mathrm{stk}}(\rho_i)1_{\mathfrak{R}_i}.$$

The operations '$\cdot$', ϕ_*, ϕ^*, $\otimes$ on $\underline{\mathrm{SF}}(*,\Upsilon,\Lambda),\dots,\bar{\mathrm{SF}}(*,\Theta,\Omega)$ are given by the same formulae. The important point is that (6)–(9) are compatible with the relations defining $\underline{\mathrm{SF}}(*,\Upsilon,\Lambda),\dots,\bar{\mathrm{SF}}(*,\Theta,\Omega)$, or they would not be well defined.

In [11, Proposition 4.14] we identify $\underline{\mathrm{SF}}(\mathrm{Spec}\,\mathbb{K},\Upsilon,\Lambda)$. The proof involves showing that Υ' in Theorem 2.14 is compatible with Definition 2.15(i)–(iii) and so descends to $\Upsilon':\underline{\mathrm{SF}}(\mathrm{Spec}\,\mathbb{K},\Upsilon,\Lambda)\to\Lambda$, which is the inverse of i_Λ.

Proposition 2.16. *The map $i_\Lambda:\Lambda\to\underline{\mathrm{SF}}(\mathrm{Spec}\,\mathbb{K},\Upsilon,\Lambda)$ taking $i_\Lambda:c\mapsto c[\mathrm{Spec}\,\mathbb{K}]$ is an isomorphism of algebras.*

Here [11, Propositions 5.21 and 5.22] is a useful way of representing these spaces.

Proposition 2.17. *$\underline{\bar{\mathrm{SF}}},\bar{\mathrm{SF}}(\mathfrak{F},\Upsilon,\Lambda)$, $\underline{\bar{\mathrm{SF}}},\bar{\mathrm{SF}}(\mathfrak{F},\Upsilon,\Lambda^\circ)$ and $\underline{\bar{\mathrm{SF}}},\bar{\mathrm{SF}}(\mathfrak{F},\Theta,\Omega)$ are generated over Λ, Λ° and Ω respectively by elements $[(U\times[\mathrm{Spec}\,\mathbb{K}/T],\rho)]$, for U a quasiprojective $\mathbb{K}$-variety and T an algebraic $\mathbb{K}$-group isomorphic to $(\mathbb{K}^\times)^k\times K$ for $k\geqslant 0$ and K finite abelian.*

Suppose $\sum_{i\in I}c_i[(U_i\times[\mathrm{Spec}\,\mathbb{K}/T_i],\rho_i)]=0$ in one of these spaces, where I is finite set, $c_i\in\Lambda,\Lambda^\circ$ or Ω, U_i is a quasiprojective $\mathbb{K}$-variety and T_i an algebraic $\mathbb{K}$-group isomorphic to $(\mathbb{K}^\times)^{k_i}\times K_i$ for $k_i\geqslant 0$ and K_i finite abelian, with $T_i\not\cong T_j$ for $i\neq j$. Then $c_j[(U_j\times[\mathrm{Spec}\,\mathbb{K}/T_j],\rho_j)]=0$ for all $j\in I$.

In [11, §5.2] we define operators Π^μ, Π_n^{vi}, $\hat\Pi_{\mathfrak{F}}^\nu$ on $\underline{\mathrm{SF}}(\mathfrak{F})$, $\underline{\bar{\mathrm{SF}}}(\mathfrak{F},*,*)$ (but not on $\underline{\mathrm{SF}}(\mathfrak{F},\Upsilon,\Lambda)$). Very roughly speaking, Π_n^{vi} projects $[(\mathfrak{R},\rho)]\in\underline{\mathrm{SF}}(\mathfrak{F})$ to $[(\mathfrak{R}_n,\rho)]$, where $\mathfrak{R}_n$ is the $\mathbb{K}$-substack of points $r\in\mathfrak{R}(\mathbb{K})$ whose stabilizer groups $\mathrm{Iso}_{\mathbb{K}}(r)$ have *rank* n, that is, maximal torus $(\mathbb{K}^\times)^n$.

Unfortunately, it is more complicated than this. The right notion is not the actual rank of stabilizer groups, but the *virtual rank*. This is a difficult idea which treats $r\in\mathfrak{R}(\mathbb{K})$ with nonabelian stabilizer group $G=\mathrm{Iso}_{\mathbb{K}}(r)$ as a linear combination of points with 'virtual ranks' in the range $\mathrm{rk}\,C(G)\leqslant n\leqslant\mathrm{rk}\,G$. Effectively this *abelianizes stabilizer groups*, that is, using virtual rank we can treat $\mathfrak{R}$ as though its stabilizer groups were all abelian, essentially tori $(\mathbb{K}^\times)^n$. These ideas will be key tools in Sections 5 and 6.

Here is a way to interpret the spaces of Definition 2.15, explained in [11]. In Section 2.2, pushforwards $\mathrm{CF}^{\mathrm{stk}}(\phi):\mathrm{CF}(\mathfrak{F})\to\mathrm{CF}(\mathfrak{G})$ are defined by 'integration' over the fibres of ϕ, using the Euler characteristic χ as measure. In the same way, given Λ, Υ as in Assumption 2.11 we could consider Λ-valued constructible functions $\mathrm{CF}(\mathfrak{F})_\Lambda$, and define a pushforward $\phi_*:\mathrm{CF}(\mathfrak{F})_\Lambda\to\mathrm{CF}(\mathfrak{G})_\Lambda$ by 'integration' using Υ as measure, instead of χ. But then $(\psi\circ\phi)_*=\psi_*\circ\phi_*$ may no longer hold, as this depends on properties of χ on non-Zariski-locally-trivial fibrations which are false for other Υ such as virtual Poincaré polynomials.

The space $\underline{\mathrm{SF}}(\mathfrak{F},\Upsilon,\Lambda)$ is very like $\mathrm{CF}(\mathfrak{F})_\Lambda$ with pushforwards ϕ_* defined using Υ, but satisfies $(\psi\circ\phi)_*=\psi_*\circ\phi_*$ and other useful functoriality properties. So we can use it as a substitute for $\mathrm{CF}(\mathfrak{F})$. The spaces $\underline{\bar{\mathrm{SF}}}, \bar{\mathrm{SF}}(\mathfrak{F}, *, *)$ are similar but also keep track of information on the maximal tori of stabilizer groups.

3. Background on configurations from [12]

We recall in Section 3.1 the main definitions and results on $(I, \preccurlyeq)$-configurations that we will need later, and in Section 3.2 some important facts on moduli stacks of configurations. For motivation and other results see [12], and for background on abelian categories, see Gelfand and Manin [6].

3.1. Basic definitions

Here is some notation for finite posets, taken from [12, Definitions 3.2 and 4.1].

Definition 3.1. A *finite partially ordered set* or *finite poset* $(I, \preccurlyeq)$ is a finite set I with a partial order I. Define $J \subseteq I$ to be an *f-set* if $i \in I$ and $h, j \in J$ and $h \preccurlyeq i \preccurlyeq j$ implies $i \in J$. Define $\mathcal{F}_{(I,\preccurlyeq)}$ to be the set of f-sets of I. Define $\mathcal{G}_{(I,\preccurlyeq)}$ to be the subset of $(J, K) \in \mathcal{F}_{(I,\preccurlyeq)} \times \mathcal{F}_{(I,\preccurlyeq)}$ such that $J \subseteq K$, and if $j \in J$ and $k \in K$ with $k \preccurlyeq j$, then $k \in J$. Define $\mathcal{H}_{(I,\preccurlyeq)}$ to be the subset of $(J, K) \in \mathcal{F}_{(I,\preccurlyeq)} \times \mathcal{F}_{(I,\preccurlyeq)}$ such that $K \subseteq J$, and if $j \in J$ and $k \in K$ with $k \preccurlyeq j$, then $j \in K$.

We define $(I, \preccurlyeq)$-*configurations*, [12, Definition 4.1].

Definition 3.2. Let $(I, \preccurlyeq)$ be a finite poset, and use the notation of Definition 3.1. Define an $(I, \preccurlyeq)$-*configuration* (σ, ι, π) in an abelian category $\mathcal{A}$ to be maps $\sigma : \mathcal{F}_{(I,\preccurlyeq)} \to \mathrm{Obj}(\mathcal{A})$, $\iota : \mathcal{G}_{(I,\preccurlyeq)} \to \mathrm{Mor}(\mathcal{A})$, and $\pi : \mathcal{H}_{(I,\preccurlyeq)} \to \mathrm{Mor}(\mathcal{A})$, where

(i) $\sigma(J)$ is an object in $\mathcal{A}$ for $J \in \mathcal{F}_{(I,\preccurlyeq)}$, with $\sigma(\emptyset) = 0$.
(ii) $\iota(J, K) : \sigma(J) \to \sigma(K)$ is injective for $(J, K) \in \mathcal{G}_{(I,\preccurlyeq)}$, and $\iota(J, J) = \mathrm{id}_{\sigma(J)}$.
(iii) $\pi(J, K) : \sigma(J) \to \sigma(K)$ is surjective for $(J, K) \in \mathcal{H}_{(I,\preccurlyeq)}$, and $\pi(J, J) = \mathrm{id}_{\sigma(J)}$.

These should satisfy the conditions:

(A) Let $(J, K) \in \mathcal{G}_{(I,\preccurlyeq)}$ and set $L = K \setminus J$. Then the following is exact in $\mathcal{A}$:

$$0 \longrightarrow \sigma(J) \xrightarrow{\iota(J,K)} \sigma(K) \xrightarrow{\pi(K,L)} \sigma(L) \longrightarrow 0.$$

(B) If $(J, K) \in \mathcal{G}_{(I,\preccurlyeq)}$ and $(K, L) \in \mathcal{G}_{(I,\preccurlyeq)}$ then $\iota(J, L) = \iota(K, L) \circ \iota(J, K)$.
(C) If $(J, K) \in \mathcal{H}_{(I,\preccurlyeq)}$ and $(K, L) \in \mathcal{H}_{(I,\preccurlyeq)}$ then $\pi(J, L) = \pi(K, L) \circ \pi(J, K)$.
(D) If $(J, K) \in \mathcal{G}_{(I,\preccurlyeq)}$ and $(K, L) \in \mathcal{H}_{(I,\preccurlyeq)}$ then

$$\pi(K, L) \circ \iota(J, K) = \iota(J \cap L, L) \circ \pi(J, J \cap L).$$

A *morphism* $\alpha : (\sigma, \iota, \pi) \to (\sigma', \iota', \pi')$ of $(I, \preccurlyeq)$-configurations in $\mathcal{A}$ is a collection of morphisms $\alpha(J) : \sigma(J) \to \sigma'(J)$ for each $J \in \mathcal{F}_{(I,\preccurlyeq)}$ satisfying

$$\alpha(K) \circ \iota(J, K) = \iota'(J, K) \circ \alpha(J) \quad \text{for all } (J, K) \in \mathcal{G}_{(I,\preccurlyeq)}, \quad \text{and}$$
$$\alpha(K) \circ \pi(J, K) = \pi'(J, K) \circ \alpha(J) \quad \text{for all } (J, K) \in \mathcal{H}_{(I,\preccurlyeq)}.$$

It is an *isomorphism* if $\alpha(J)$ is an isomorphism for all $J \in \mathcal{F}_{(I,\preccurlyeq)}$.

Here [12, Definitions 5.1 and 5.2] are two ways to construct new configurations.

Definition 3.3. Let $(I,\preccurlyeq)$ be a finite poset and $J\in\mathcal{F}_{(I,\preccurlyeq)}$. Then $(J,\preccurlyeq)$ is also a finite poset, and $\mathcal{F}_{(J,\preccurlyeq)},\mathcal{G}_{(J,\preccurlyeq)},\mathcal{H}_{(J,\preccurlyeq)}\subseteq\mathcal{F}_{(I,\preccurlyeq)},\mathcal{G}_{(I,\preccurlyeq)},\mathcal{H}_{(I,\preccurlyeq)}$. Let (σ,ι,π) be an $(I,\preccurlyeq)$-configuration in an abelian category $\mathcal{A}$. Define the *$(J,\preccurlyeq)$-subconfiguration* (σ',ι',π') of (σ,ι,π) by $\sigma'=\sigma|_{\mathcal{F}_{(J,\preccurlyeq)}}$, $\iota'=\iota|_{\mathcal{G}_{(J,\preccurlyeq)}}$ and $\pi'=\pi|_{\mathcal{H}_{(J,\preccurlyeq)}}$.

Let $(I,\preccurlyeq)$, $(K,\trianglelefteq)$ be finite posets, and $\phi:I\to K$ be surjective with $i\preccurlyeq j$ implies $\phi(i)\trianglelefteq\phi(j)$. Using ϕ^{-1} to pull subsets of K back to I maps $\mathcal{F}_{(K,\trianglelefteq)},\mathcal{G}_{(K,\trianglelefteq)},\mathcal{H}_{(K,\trianglelefteq)}\to\mathcal{F}_{(I,\preccurlyeq)},\mathcal{G}_{(I,\preccurlyeq)},\mathcal{H}_{(I,\preccurlyeq)}$. Let (σ,ι,π) be an $(I,\preccurlyeq)$-configuration in an abelian category $\mathcal{A}$. Define the *quotient $(K,\trianglelefteq)$-configuration* $(\tilde\sigma,\tilde\iota,\tilde\pi)$ by $\tilde\sigma(A)=\sigma(\phi^{-1}(A))$ for $A\in\mathcal{F}_{(K,\trianglelefteq)}$, $\tilde\iota(A,B)=\iota(\phi^{-1}(A),\phi^{-1}(B))$ for $(A,B)\in\mathcal{G}_{(K,\trianglelefteq)}$, and $\tilde\pi(A,B)=\pi(\phi^{-1}(A),\phi^{-1}(B))$ for $(A,B)\in\mathcal{H}_{(K,\trianglelefteq)}$.

3.2. Moduli stacks of configurations

Here are our initial assumptions.

Assumption 3.4. Fix an algebraically closed field $\mathbb{K}$. (Throughout Section 4 we will require $\mathbb{K}$ to have *characteristic zero*.) Let $\mathcal{A}$ be an abelian category with $\mathrm{Hom}(X,Y)=\mathrm{Ext}^0(X,Y)$ and $\mathrm{Ext}^1(X,Y)$ finite-dimensional $\mathbb{K}$-vector spaces for all $X,Y\in\mathcal{A}$, and all composition maps $\mathrm{Ext}^i(Y,Z)\times\mathrm{Ext}^j(X,Y)\to\mathrm{Ext}^{i+j}(X,Z)$ bilinear for $i,j,i+j=0$ or 1. Let $K(\mathcal{A})$ be the quotient of the Grothendieck group $K_0(\mathcal{A})$ by some fixed subgroup. Suppose that if $X\in\mathrm{Obj}(\mathcal{A})$ with $[X]=0$ in $K(\mathcal{A})$ then $X\cong 0$.

To define moduli stacks of objects or configurations in $\mathcal{A}$, we need some *extra data*, to tell us about algebraic families of objects and morphisms in $\mathcal{A}$, parametrized by a base scheme U. We encode this extra data as a *stack in exact categories* $\mathfrak{F}_{\mathcal{A}}$ on the *category of* $\mathbb{K}$-*schemes* $\mathrm{Sch}_{\mathbb{K}}$, made into a *site* with the *étale topology*. The $\mathbb{K}$, $\mathcal{A}$, $K(\mathcal{A})$, $\mathfrak{F}_{\mathcal{A}}$ must satisfy some complex additional conditions [12, Assumptions 7.1 and 8.1], which we do not give.

We define some notation, [12, Definition 7.3].

Definition 3.5. We work in the situation of Assumption 3.4. Define

$$\bar C(\mathcal{A})=\big\{[X]\in K(\mathcal{A}):\ X\in\mathcal{A}\big\}\subset K(\mathcal{A}).$$

That is, $\bar C(\mathcal{A})$ is the collection of classes in $K(\mathcal{A})$ of objects $X\in\mathcal{A}$. Note that $\bar C(\mathcal{A})$ is closed under addition, as $[X\oplus Y]=[X]+[Y]$. In [13,14] we shall make much use of $C(\mathcal{A})=\bar C(\mathcal{A})\setminus\{0\}$. We think of $C(\mathcal{A})$ as the 'positive cone' and $\bar C(\mathcal{A})$ as the 'closed positive cone' in $K(\mathcal{A})$, which explains the notation. For $(I,\preccurlyeq)$ a finite poset and $\kappa:I\to\bar C(\mathcal{A})$, define an *$(I,\preccurlyeq,\kappa)$-configuration* to be an $(I,\preccurlyeq)$-configuration (σ,ι,π) with $[\sigma(\{i\})]=\kappa(i)$ in $K(\mathcal{A})$ for all $i\in I$.

In the situation above, we define the following $\mathbb{K}$-stacks [12, Definitions 7.2 and 7.4]:

- The *moduli stacks* $\mathfrak{Obj}_{\mathcal{A}}$ of *objects in* $\mathcal{A}$, and $\mathfrak{Obj}^{\alpha}_{\mathcal{A}}$ of *objects in $\mathcal{A}$ with class α in $K(\mathcal{A})$*, for each $\alpha\in\bar C(\mathcal{A})$. They are algebraic $\mathbb{K}$-stacks, locally of finite type, with $\mathfrak{Obj}^{\alpha}_{\mathcal{A}}$ an open and closed $\mathbb{K}$-substack of $\mathfrak{Obj}_{\mathcal{A}}$. The underlying geometric spaces $\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$, $\mathfrak{Obj}^{\alpha}_{\mathcal{A}}(\mathbb{K})$ are the sets of isomorphism classes of objects X in $\mathcal{A}$, with $[X]=\alpha$ for $\mathfrak{Obj}^{\alpha}_{\mathcal{A}}(\mathbb{K})$.

- The *moduli stacks* $\mathfrak{M}(I,\preccurlyeq)_{\mathcal{A}}$ *of* $(I,\preccurlyeq)$*-configurations* and $\mathfrak{M}(I,\preccurlyeq,\kappa)_{\mathcal{A}}$ of $(I,\preccurlyeq,\kappa)$*-configurations in* $\mathcal{A}$, for all finite posets $(I,\preccurlyeq)$ and $\kappa: I\to\bar{C}(\mathcal{A})$. They are algebraic $\mathbb{K}$-stacks, locally of finite type, with $\mathfrak{M}(I,\preccurlyeq,\kappa)_{\mathcal{A}}$ an open and closed $\mathbb{K}$-substack of $\mathfrak{M}(I,\preccurlyeq)_{\mathcal{A}}$. Write $\mathcal{M}(I,\preccurlyeq)_{\mathcal{A}}$, $\mathcal{M}(I,\preccurlyeq,\kappa)_{\mathcal{A}}$ for the underlying geometric spaces $\mathfrak{M}(I,\preccurlyeq)_{\mathcal{A}}(\mathbb{K})$, $\mathfrak{M}(I,\preccurlyeq,\kappa)_{\mathcal{A}}(\mathbb{K})$. Then $\mathcal{M}(I,\preccurlyeq)_{\mathcal{A}}$ and $\mathcal{M}(I,\preccurlyeq,\kappa)_{\mathcal{A}}$ are the *sets of isomorphism classes of* $(I,\preccurlyeq)$*- and* $(I,\preccurlyeq,\kappa)$*-configurations in* $\mathcal{A}$, by [12, Proposition 7.6].

Each stabilizer group $\mathrm{Iso}_{\mathbb{K}}([X])$ or $\mathrm{Iso}_{\mathbb{K}}([(\sigma,\iota,\pi)])$ in $\mathfrak{Obj}_{\mathcal{A}}$ or $\mathfrak{M}(I,\preccurlyeq)_{\mathcal{A}}$ is the group of invertible elements in the finite-dimensional $\mathbb{K}$-algebra $\mathrm{End}(X)$ or $\mathrm{End}((\sigma,\iota,\pi))$. Thus $\mathfrak{Obj}_{\mathcal{A}}$, $\mathfrak{Obj}^{\alpha}_{\mathcal{A}}$, $\mathfrak{M}(I,\preccurlyeq)_{\mathcal{A}}$, $\mathfrak{M}(I,\preccurlyeq,\kappa)_{\mathcal{A}}$ have *affine geometric stabilizers*, which is required to use the results of Section 2.2.

In [12, Definition 7.7 and Proposition 7.8] we define 1-morphisms of $\mathbb{K}$-stacks, as follows:

- For $(I,\preccurlyeq)$ a finite poset, $\kappa: I\to\bar{C}(\mathcal{A})$ and $J\in\mathcal{F}_{(I,\preccurlyeq)}$, we define

$$\boldsymbol{\sigma}(J):\mathfrak{M}(I,\preccurlyeq)_{\mathcal{A}}\to\mathfrak{Obj}_{\mathcal{A}}\quad\text{or}\quad\boldsymbol{\sigma}(J):\mathfrak{M}(I,\preccurlyeq,\kappa)_{\mathcal{A}}\to\mathfrak{Obj}_{\mathcal{A}}^{\sum_{j\in J}\kappa(j)}.$$

  The induced maps $\boldsymbol{\sigma}(J)_*:\mathcal{M}(I,\preccurlyeq)_{\mathcal{A}}\to\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ or $\mathcal{M}(I,\preccurlyeq,\kappa)_{\mathcal{A}}\to\mathfrak{Obj}_{\mathcal{A}}^{\sum_{j\in J}\kappa(j)}(\mathbb{K})$ act by $\boldsymbol{\sigma}(J)_*:[(\sigma,\iota,\pi)]\mapsto[\sigma(J)]$.
- For $(I,\preccurlyeq)$ a finite poset, $\kappa: I\to\bar{C}(\mathcal{A})$ and $J\in\mathcal{F}_{(I,\preccurlyeq)}$, we define the $(J,\preccurlyeq)$*-subconfiguration* 1*-morphism*

$$S(I,\preccurlyeq,J):\mathfrak{M}(I,\preccurlyeq)_{\mathcal{A}}\to\mathfrak{M}(J,\preccurlyeq)_{\mathcal{A}}\quad\text{or}\quad S(I,\preccurlyeq,J):\mathfrak{M}(I,\preccurlyeq,\kappa)_{\mathcal{A}}\to\mathfrak{M}(J,\preccurlyeq,\kappa|_J)_{\mathcal{A}}.$$

  The induced maps $S(I,\preccurlyeq,J)_*$ act by $S(I,\preccurlyeq,J)_*:[(\sigma,\iota,\pi)]\mapsto[(\sigma',\iota',\pi')]$, where (σ,ι,π) is an $(I,\preccurlyeq)$-configuration in $\mathcal{A}$, and (σ',ι',π') its $(J,\preccurlyeq)$-subconfiguration.
- Let $(I,\preccurlyeq)$, $(K,\trianglelefteq)$ be finite posets, $\kappa: I\to\bar{C}(\mathcal{A})$, and $\phi: I\to K$ be surjective with $i\preccurlyeq j$ implies $\phi(i)\trianglelefteq\phi(j)$ for $i,j\in I$. Define $\mu: K\to\bar{C}(\mathcal{A})$ by $\mu(k)=\sum_{i\in\phi^{-1}(k)}\kappa(i)$. We define the *quotient* $(K,\trianglelefteq)$*-configuration* 1*-morphisms*

$$Q(I,\preccurlyeq,K,\trianglelefteq,\phi):\mathfrak{M}(I,\preccurlyeq)_{\mathcal{A}}\to\mathfrak{M}(K,\trianglelefteq)_{\mathcal{A}},\tag{10}$$

$$Q(I,\preccurlyeq,K,\trianglelefteq,\phi):\mathfrak{M}(I,\preccurlyeq,\kappa)_{\mathcal{A}}\to\mathfrak{M}(K,\trianglelefteq,\mu)_{\mathcal{A}}.\tag{11}$$

  The induced maps $Q(I,\preccurlyeq,K,\trianglelefteq,\phi)_*$ act by $Q(I,\preccurlyeq,K,\trianglelefteq,\phi)_*:[(\sigma,\iota,\pi)]\mapsto[(\tilde{\sigma},\tilde{\iota},\tilde{\pi})]$, where (σ,ι,π) is an $(I,\preccurlyeq)$-configuration in $\mathcal{A}$, and $(\tilde{\sigma},\tilde{\iota},\tilde{\pi})$ its quotient $(K,\trianglelefteq)$-configuration from ϕ.

Here [12, Theorem 8.4] are some properties of these 1-morphisms:

Theorem 3.6. *In the situation above*:

(a) $Q(I,\preccurlyeq,K,\trianglelefteq,\phi)$ *in* (10) *and* (11) *are representable, and* (11) *is finite type.*
(b) $\boldsymbol{\sigma}(I):\mathfrak{M}(I,\preccurlyeq,\kappa)_{\mathcal{A}}\to\mathfrak{Obj}^{\kappa(I)}_{\mathcal{A}}$ *is representable and of finite type, and* $\boldsymbol{\sigma}(I):\mathfrak{M}(I,\preccurlyeq)_{\mathcal{A}}\to\mathfrak{Obj}_{\mathcal{A}}$ *is representable.*
(c) $\prod_{i\in I}\boldsymbol{\sigma}(\{i\}):\mathfrak{M}(I,\preccurlyeq)_{\mathcal{A}}\to\prod_{i\in I}\mathfrak{Obj}_{\mathcal{A}}$ *is of finite type.*

In [12, §§9 and 10] we define the data $\mathcal{A}$, $K(\mathcal{A})$, $\mathfrak{F}_{\mathcal{A}}$ in some large classes of examples, and prove that Assumption 3.4 holds in each case.

4. Algebras of constructible functions on $\mathfrak{Obj}_{\mathcal{A}}$

We now generalize the idea of Ringel–Hall algebras to *constructible functions on stacks*. Let Assumption 3.4 hold. We show how to make the $\mathbb{Q}$-vector space $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ of constructible functions into a $\mathbb{Q}$-algebra.

We begin in Section 4.1 by defining the multiplication $*$ on $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$, and showing it is associative. Section 4.2 extends this to locally constructible functions, and Section 4.3 constructs left or right representations of the algebras $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$, using configuration moduli stacks. Section 4.4 shows the subspace $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ of functions *supported on indecomposables* is a Lie subalgebra of $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$, and Section 4.5 that under extra conditions on $\mathcal{A}$ the subspace $\mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$ of functions with *finite support* is a subalgebra of $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$.

Section 4.6 proves the $\mathbb{Q}$-algebra $\mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$ is isomorphic to the *universal enveloping algebra* $U(\mathrm{CF}^{\mathrm{ind}}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}))$. Section 4.7 defines a commutative comultiplication Δ on $\mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$ making it into a bialgebra, and Section 4.8 defines multilinear operations $P_{(I,\preccurlyeq)}$ on $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ for all finite posets $(I,\preccurlyeq)$, which satisfy an analogue of associativity. Finally, Section 4.9 gives some examples from quiver representations mod-$\mathbb{K}Q$.

Throughout this section we fix an algebraically closed field $\mathbb{K}$ of *characteristic zero*, so that we can apply the constructible functions theory of Section 2.2.

4.1. An associative algebra structure on $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$

We now extend the Ringel–Hall algebra idea to the stacks set up of Section 3.2. First we define the *identity* $\delta_{[0]}$ and *multiplication* $*$ on $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$.

Definition 4.1. Suppose Assumption 3.4 holds. Define $\delta_{[0]}\in\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ to be the characteristic function of the point $[0]\in\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$, so that $\delta_{[0]}([X])=1$ if $X\cong 0$, and $\delta_{[0]}([X])=0$ otherwise. For $f,g\in\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ we define $f\otimes g\in\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}\times\mathfrak{Obj}_{\mathcal{A}})$ by $(f\otimes g)([X],[Y])=f([X])g([Y])$ for all $([X],[Y])\in(\mathfrak{Obj}_{\mathcal{A}}\times\mathfrak{Obj}_{\mathcal{A}})(\mathbb{K})=\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})\times\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$.

Using the diagrams of 1-morphisms of stacks and pullbacks, pushforwards of constructible functions, explained in Remark 4.2:

$$\mathfrak{Obj}_{\mathcal{A}}\times\mathfrak{Obj}_{\mathcal{A}} \xleftarrow{\boldsymbol{\sigma}(\{1\})\times\boldsymbol{\sigma}(\{2\})} \mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}} \xrightarrow{\boldsymbol{\sigma}(\{1,2\})} \mathfrak{Obj}_{\mathcal{A}},$$

$$\begin{array}{ccccc} \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})\times\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) & \xrightarrow{(\boldsymbol{\sigma}(\{1\}))^*\cdot(\boldsymbol{\sigma}(\{2\}))^*} & & & \\ \downarrow{\scriptstyle\otimes} & & \searrow & & \\ \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}\times\mathfrak{Obj}_{\mathcal{A}}) & \xrightarrow{(\boldsymbol{\sigma}(\{1\})\times\boldsymbol{\sigma}(\{2\}))^*} & \mathrm{CF}(\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}}) & \xrightarrow{\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(\{1,2\}))} & \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}), \end{array} \tag{12}$$

define a bilinear operation $*:\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})\times\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})\to\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ by

$$\begin{aligned} f*g &= \mathrm{CF}^{\mathrm{stk}}\bigl(\boldsymbol{\sigma}\bigl(\{1,2\}\bigr)\bigr)\bigl[\boldsymbol{\sigma}\bigl(\{1\}\bigr)^*(f)\cdot\boldsymbol{\sigma}\bigl(\{2\}\bigr)^*(g)\bigr] \\ &= \mathrm{CF}^{\mathrm{stk}}\bigl(\boldsymbol{\sigma}\bigl(\{1,2\}\bigr)\bigr)\bigl[\bigl(\boldsymbol{\sigma}\bigl(\{1\}\bigr)\times\boldsymbol{\sigma}\bigl(\{2\}\bigr)\bigr)^*(f\otimes g)\bigr]. \end{aligned} \tag{13}$$

This is well defined as $\boldsymbol{\sigma}(\{1,2\})$ is representable and $\boldsymbol{\sigma}(\{1\}) \times \boldsymbol{\sigma}(\{2\})$ of finite type by Theorem 3.6(b),(c), so $\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(\{1,2\}))$ and $(\boldsymbol{\sigma}(\{1\}) \times \boldsymbol{\sigma}(\{2\}))^*$ are well-defined maps of constructible functions as in Section 2.2.

Remark 4.2. Here is what this means. In (12), $\mathfrak{Obj}_{\mathcal{A}}$ is the Artin $\mathbb{K}$-stack of objects $X \in \mathcal{A}$, and $\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}}$ is the Artin $\mathbb{K}$-stack of *short exact sequences* $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, except that using the configuration notation of [12] a short exact sequence is a $(\{1,2\},\leqslant)$-configuration (σ,ι,π), written $0 \to \sigma(\{1\}) \to \sigma(\{1,2\}) \to \sigma(\{2\}) \to 0$, where $\sigma(\cdots)$ are objects in $\mathcal{A}$.

The 1-morphisms $\boldsymbol{\sigma}(\{1\})$, $\boldsymbol{\sigma}(\{1,2\})$, $\boldsymbol{\sigma}(\{2\})$ in (12) take the configuration (σ,ι,π) to its objects $\sigma(\{1\})$, $\sigma(\{1,2\})$, $\sigma(\{2\})$. That is, they take a short exact sequence $0 \to X \to Y \to Z \to 0$ to the objects X, Y, Z respectively. Hence, the 1-morphisms $\boldsymbol{\sigma}(\{1\}) \times \boldsymbol{\sigma}(\{2\})$ and $\boldsymbol{\sigma}(\{1,2\})$ take isomorphism classes $[0 \to X \to Y \to Z \to 0]$ to isomorphism classes $([X],[Z])$ and $[Y]$ respectively.

Thus, interpreting pushforwards $\mathrm{CF}^{\mathrm{stk}}(\cdots)$ as in [10], for fixed $Y \in \mathcal{A}$ we can regard $(f * g)([Y])$ as the 'integral' over short exact sequences $0 \to X \to Y \to Z \to 0$ of $f([X])g([Z])$, or equivalently, the integral over subobjects $X \subset Y$ of $f([X])g([Y/X])$, with respect to a measure defined using the Euler characteristic of constructible sets. Our convention on the order of multiplication agrees with Frenkel et al. [5] and Riedtmann [21, §2]. However, Lusztig [19, §§3.1, 10.19 and 12.10] and Ringel use the opposite convention, with $(f * g)([Y])$ an integral over subobjects $X \subset Y$ of $f([Y/X])g([X])$.

Here is the basic result, saying $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ is a $\mathbb{Q}$-algebra. The proof is related to Ringel [23] and Lusztig [19, §10.19].

Theorem 4.3. *In the situation above,* $\delta_{[0]} * f = f * \delta_{[0]} = f$ *for all* f *in* $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$, *and* $*$ *is associative. Thus* $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ *is a* $\mathbb{Q}$*-algebra, with identity* $\delta_{[0]}$ *and multiplication* $*$.

Proof. By considering the $(\{1,2\},\leqslant)$-configuration (σ,ι,π) with $\sigma(\{1\}) = 0$ and $\sigma(\{2\}) = X$ for $X \in \mathcal{A}$, which is unique up to isomorphism, we find from Definition 4.1 that $(\delta_{[0]} * f)([X]) = f([X])$, so $\delta_{[0]} * f = f$ as this holds for all $X \in \mathcal{A}$, and similarly $f * \delta_{[0]} = f$.

Define $\alpha, \beta : \{1,2,3\} \to \{1,2\}$ by $\alpha(1) = \alpha(2) = 1$, $\alpha(3) = 2$, $\beta(1) = 1$, $\beta(2) = \beta(3) = 2$. Consider the commutative diagram of 1-morphisms, and the corresponding diagram of pullbacks and pushforwards:

$$
\begin{array}{ccccc}
\mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}} & \xleftarrow{\mathrm{id}_{\mathfrak{Obj}_{\mathcal{A}}} \times \boldsymbol{\sigma}(\{2\}) \times \boldsymbol{\sigma}(\{3\})} & \mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{M}(\{2,3\},\leqslant)_{\mathcal{A}} & \xrightarrow{\mathrm{id}_{\mathfrak{Obj}_{\mathcal{A}}} \times \boldsymbol{\sigma}(\{2,3\})} & \mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}} \\
\uparrow {\scriptstyle \boldsymbol{\sigma}(\{1\}) \times \boldsymbol{\sigma}(\{2\}) \times \mathrm{id}_{\mathfrak{Obj}_{\mathcal{A}}}} & & \uparrow {\scriptstyle \boldsymbol{\sigma}(\{1\}) \times S(\{1,2,3\},\leqslant,\{2,3\})} & & \uparrow {\scriptstyle \boldsymbol{\sigma}(\{1\}) \times \boldsymbol{\sigma}(\{2\})} \\
\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}} & \xleftarrow{S(\{1,2,3\},\leqslant,\{1,2\}) \times \boldsymbol{\sigma}(\{3\})} & \mathfrak{M}(\{1,2,3\},\leqslant)_{\mathcal{A}} & \xrightarrow{Q(\{1,2,3\},\leqslant,\{1,2\},\leqslant,\beta)} & \mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}} \\
\downarrow {\scriptstyle \boldsymbol{\sigma}(\{1,2\}) \times \mathrm{id}_{\mathfrak{Obj}_{\mathcal{A}}}} & & \downarrow {\scriptstyle Q(\{1,2,3\},\leqslant,\{1,2\},\leqslant,\alpha)} & & \downarrow {\scriptstyle \boldsymbol{\sigma}(\{1,2\})} \\
\mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}} & \xleftarrow{\boldsymbol{\sigma}(\{1\}) \times \boldsymbol{\sigma}(\{2\})} & \mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}} & \xrightarrow{\boldsymbol{\sigma}(\{1,2\})} & \mathfrak{Obj}_{\mathcal{A}},
\end{array}
\tag{14}
$$

$$
\begin{array}{ccccc}
\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}}) & \xrightarrow{(\mathrm{id}_{\mathfrak{Obj}_{\mathcal{A}}} \times \boldsymbol{\sigma}(\{2\}) \times \boldsymbol{\sigma}(\{3\}))^*} & \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{M}(\{2,3\},\leqslant)_{\mathcal{A}}) & \xrightarrow{\mathrm{CF}^{\mathrm{stk}}(\mathrm{id}_{\mathfrak{Obj}_{\mathcal{A}}} \times \boldsymbol{\sigma}(\{2,3\}))} & \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}}) \\
\Big\downarrow {\scriptstyle \boldsymbol{\sigma}(\{1\}) \times \boldsymbol{\sigma}(\{2\})) \times \mathrm{id}_{\mathfrak{Obj}_{\mathcal{A}}})^*} & & \Big\downarrow {\scriptstyle (\boldsymbol{\sigma}(\{1\}) \times S(\{1,2,3\},\leqslant,\{2,3\}))^*} & & \Big\downarrow {\scriptstyle (\boldsymbol{\sigma}(\{1\}) \times \boldsymbol{\sigma}(\{2\}))^*} \\
\mathrm{CF}(\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}}) & \xrightarrow{(S(\{1,2,3\},\leqslant,\{1,2\}) \times \boldsymbol{\sigma}(\{3\}))^*} & \mathrm{CF}(\mathfrak{M}(\{1,2,3\},\leqslant)_{\mathcal{A}}) & \xrightarrow{\mathrm{CF}^{\mathrm{stk}}(Q(\{1,2,3\},\leqslant,\{1,2\},\leqslant,\beta))} & \mathrm{CF}(\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}}) \\
\Big\downarrow {\scriptstyle \mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(\{1,2\}) \times \mathrm{id}_{\mathfrak{Obj}_{\mathcal{A}}})} & & \Big\downarrow {\scriptstyle \mathrm{CF}^{\mathrm{stk}}(Q(\{1,2,3\},\leqslant,\{1,2\},\leqslant,\alpha))} & & \Big\downarrow {\scriptstyle \mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(\{1,2\}))} \\
\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}}) & \xrightarrow{(\boldsymbol{\sigma}(\{1\}) \times \boldsymbol{\sigma}(\{2\}))^*} & \mathrm{CF}(\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}}) & \xrightarrow{\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(\{1,2\}))} & \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}).
\end{array}
\tag{15}
$$

Here $\mathfrak{M}(\{1,2,3\},\leqslant)_{\mathcal{A}}$ is the Artin $\mathbb{K}$-stack of $(\{1,2,3\},\leqslant)$-configurations in $\mathcal{A}$, which are essentially *chains of subobjects* $X \subset Y \subset Z$ in $\mathcal{A}$, except that they are written $\sigma(\{1\}) \subset \sigma(\{1,2\}) \subset \sigma(\{1,2,3\})$ in configuration notation. Also $\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}}$ is the Artin $\mathbb{K}$-stack of subobjects $X \subset Y$ in $\mathcal{A}$, as above. The four 1-morphisms from $\mathfrak{M}(\{1,2,3\},\leqslant)_{\mathcal{A}}$ in (14) act on an isomorphism class $[X \subset Y \subset Z]$ as follows: '$\uparrow$' takes it to $([X],[Y/X \subset Z/X])$, '$\leftarrow$' takes it to $([X \subset Y],[Z/Y])$, '$\rightarrow$' takes it to $([X \subset Z])$, and '$\downarrow$' takes it to $([Y \subset Z])$.

To show the maps in (15) are well defined we use Theorem 3.6(a)–(c) to show the corresponding 1-morphisms are representable or finite type, and note that $S(\{1,2,3\},\leqslant,\{1,2\}) \times \boldsymbol{\sigma}(\{3\})$, $\boldsymbol{\sigma}(\{1\}) \times S(\{1,2,3\},\leqslant,\{2,3\})$ are finite type by a similar proof to Theorem 3.6(c).

The top left square in (15) commutes by (3), and the bottom right by (2). Now [12, Theorem 7.10] implies that

$$
\begin{array}{ccc}
\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}} & \xleftarrow[S(\{1,2,3\},\leqslant,\{1,2\})]{} & \mathfrak{M}(\{1,2,3\},\leqslant)_{\mathcal{A}} \\
\Big\downarrow {\scriptstyle \boldsymbol{\sigma}(\{1,2\})} & & \Big\downarrow {\scriptstyle Q(\{1,2,3\},\leqslant,\{1,2\},\leqslant,\alpha)} \\
\mathfrak{Obj}_{\mathcal{A}} & \xleftarrow{\boldsymbol{\sigma}(\{1\})} & \mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}}
\end{array}
$$

is a Cartesian square. Taking fibre products with $\boldsymbol{\sigma}(\{2\}) : \mathfrak{M}(\{1,2\})_{\mathcal{A}} \to \mathfrak{Obj}_{\mathcal{A}}$ then shows the bottom left square in (14) is also Cartesian, so the bottom left square in (15) commutes by (4). Similarly the top right square in (15) commutes. Therefore (15) commutes. Now let $f, g, h \in \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$, so that $f \otimes g \otimes h \in \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}})$. As (15) commutes, applying the two routes round the outside of the square to $f \otimes g \otimes h$ shows that $(f * g) * h = f * (g * h)$. Thus $*$ is associative, and $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ is an algebra. □

Because of the use of Cartesian squares and Theorem 2.5 in the proof of Theorem 4.3, to make $*$ associative we must use the *stack pushforward* $\mathrm{CF}^{\mathrm{stk}}$ in (13), and other pushforwards such as the *naïve pushforward* $\mathrm{CF}^{\mathrm{na}}$ of [10, Section 4.3] will in general give nonassociative multiplications. In particular, as $\mathrm{CF}^{\mathrm{stk}}$ depends on the stabilizer groups $\mathrm{Iso}_{\mathbb{K}}(x)$, we cannot afford to forget this information by passing to coarse moduli schemes, if they existed. This is an important reason for working with Artin stacks, rather than some simpler class of spaces.

Define the *composition algebra* $\mathcal{C} \subseteq \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ to be the $\mathbb{Q}$-subalgebra generated by functions f supported on $[X] \in \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ with X a *simple* object in $\mathcal{A}$. In examples, the composition algebra $\mathcal{C}$ is usually more interesting than the Ringel–Hall algebra $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$. When there are only finitely many simple objects up to isomorphism $\mathcal{C}$ is *finitely generated.*

4.2. Extension to locally constructible functions

Next we observe that the associative multiplication $*$ in Section 4.1 extends to a large subspace $\dot{\mathrm{LCF}}(\mathfrak{Obj}_{\mathcal{A}})$ of the *locally constructible functions* $\mathrm{LCF}(\mathfrak{Obj}_{\mathcal{A}})$.

Definition 4.4. Suppose Assumption 3.4 holds. Define $\dot{\mathrm{LCF}}(\mathfrak{Obj}_{\mathcal{A}})$ to be the $\mathbb{Q}$-vector subspace of $\mathrm{LCF}(\mathfrak{Obj}_{\mathcal{A}})$ consisting of functions f supported on subsets $\coprod_{\alpha\in S}\mathfrak{Obj}^{\alpha}_{\mathcal{A}}(\mathbb{K})$ in $\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ for $S\subset\bar{C}(\mathcal{A})$ a *finite* subset. Following (13), define $*:\dot{\mathrm{LCF}}(\mathfrak{Obj}_{\mathcal{A}})\times\dot{\mathrm{LCF}}(\mathfrak{Obj}_{\mathcal{A}})\rightarrow\dot{\mathrm{LCF}}(\mathfrak{Obj}_{\mathcal{A}})$ by

$$f*g=\mathrm{LCF}^{\mathrm{stk}}\big(\boldsymbol{\sigma}\big(\{1,2\}\big)\big)\big[\boldsymbol{\sigma}\big(\{1\}\big)^{*}(f)\cdot\boldsymbol{\sigma}\big(\{2\}\big)^{*}(g)\big]. \tag{16}$$

To see this is well defined, recall the disjoint union of stacks [12, Theorem 7.5]

$$\mathfrak{M}\big(\{1,2\},\leqslant\big)_{\mathcal{A}}=\coprod_{\kappa:\{1,2\}\rightarrow\bar{C}(\mathcal{A})}\mathfrak{M}\big(\{1,2\},\leqslant,\kappa\big)_{\mathcal{A}}.$$

Let $f,g\in\dot{\mathrm{LCF}}(\mathfrak{Obj}_{\mathcal{A}})$ be supported on $\coprod_{\alpha\in S}\mathfrak{Obj}^{\alpha}_{\mathcal{A}}(\mathbb{K})$, $\coprod_{\alpha\in T}\mathfrak{Obj}^{\alpha}_{\mathcal{A}}(\mathbb{K})$ respectively for finite $S,T\subseteq\bar{C}(\mathcal{A})$. Then $\boldsymbol{\sigma}(\{1\})^{*}(f)\cdot\boldsymbol{\sigma}(\{2\})^{*}(g)$ in (16) is locally constructible and supported on the finite number of $\mathcal{M}(\{1,2\},\leqslant,\kappa)_{\mathcal{A}}$ for which $\kappa(1)\in S$ and $\kappa(2)\in T$. By Theorem 3.6(b), $\boldsymbol{\sigma}(\{1,2\}):\mathfrak{M}(\{1,2\},\leqslant,\kappa)_{\mathcal{A}}\rightarrow\mathfrak{Obj}^{\kappa(\{1,2\})}_{\mathcal{A}}$ is representable and of *finite type*. Thus $\mathrm{LCF}^{\mathrm{stk}}(\boldsymbol{\sigma}(\{1,2\}))[\cdots]$ is well defined in (16), and lies in $\mathrm{LCF}(\mathfrak{Obj}_{\mathcal{A}})$. But clearly $f*g$ is supported on $\bigcup_{\alpha\in S,\,\beta\in T}\mathfrak{Obj}^{\alpha+\beta}_{\mathcal{A}}(\mathbb{K})$, so $f*g\in\dot{\mathrm{LCF}}(\mathfrak{Obj}_{\mathcal{A}})$.

Actually, $*$ often makes sense on even larger subspaces of $\mathrm{LCF}(\mathfrak{Obj}_{\mathcal{A}})$. For $f*g$ to be well defined, all we need is that for each $\gamma\in\bar{C}(\mathcal{A})$, there should exist only finitely many pairs $\alpha,\beta\in\bar{C}(\mathcal{A})$ with $\gamma=\alpha+\beta$ and $f|_{\mathfrak{Obj}^{\alpha}_{\mathcal{A}}(\mathbb{K})}$, $g|_{\mathfrak{Obj}^{\beta}_{\mathcal{A}}(\mathbb{K})}$ both nonzero. If it happens that for all $\gamma\in\bar{C}(\mathcal{A})$ there are only finitely many pairs $\alpha,\beta\in\bar{C}(\mathcal{A})$ with $\alpha+\beta=\gamma$ then this holds automatically, and $f*g$ is well defined for all $f,g\in\mathrm{LCF}(\mathfrak{Obj}_{\mathcal{A}})$. In particular, this holds for all the quiver examples of [12, §10], so in these examples $\mathrm{LCF}(\mathfrak{Obj}_{\mathcal{A}})$ will be a $\mathbb{Q}$-algebra.

We shall deal only with $\dot{\mathrm{LCF}}(\mathfrak{Obj}_{\mathcal{A}})$, though, as it is sufficient for the applications in [13,14], where it is useful, for instance, that $\delta_{\mathfrak{Obj}^{\alpha}_{\mathcal{A}}(\mathbb{K})}$ lies in $\dot{\mathrm{LCF}}(\mathfrak{Obj}_{\mathcal{A}})$. Here is the analogue of Theorem 4.3. The proof follows that of Theorem 4.3, replacing $\mathrm{CF}(\cdots)$ by $\mathrm{LCF}(\cdots)$ and $\mathrm{CF}^{\mathrm{stk}}(\cdots)$ by $\mathrm{LCF}^{\mathrm{stk}}(\cdots)$, and arguing as in Definition 4.4 to show the operators $\mathrm{LCF}^{\mathrm{stk}}(\cdots)$ are well defined.

Theorem 4.5. *In the situation above* $\dot{\mathrm{LCF}}(\mathfrak{Obj}_{\mathcal{A}})$ *is a* $\mathbb{Q}$*-algebra, with identity* $\delta_{[0]}$ *and associative multiplication* $*$*, and* $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ *is a* $\mathbb{Q}$*-subalgebra.*

In the rest of the section we give many results for $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$. Mostly these have straightforward generalizations to $\dot{\mathrm{LCF}}(\mathfrak{Obj}_{\mathcal{A}})$, which we leave as exercises for the reader, just making the occasional comment. Here are two other remarks:

- The $\mathbb{Q}$-subalgebra of $\dot{\mathrm{LCF}}(\mathfrak{Obj}_{\mathcal{A}})$ generated by the characteristic functions $\delta_{\mathfrak{Obj}^{\alpha}_{\mathcal{A}}(\mathbb{K})}$ of $\mathfrak{Obj}^{\alpha}_{\mathcal{A}}(\mathbb{K})$ for $\alpha\in\bar{C}(\mathcal{A})$ may be an interesting algebra.

- One can also consider *infinite sums* in $\dot{\mathrm{LCF}}(\mathfrak{Obj}_{\mathcal{A}})$ or $\mathrm{LCF}(\mathfrak{Obj}_{\mathcal{A}})$. We call an infinite sum $\sum_{i\in I} f_i$ with $f_i \in \mathrm{LCF}(\mathfrak{Obj}_{\mathcal{A}})$ *convergent* if for all constructible $C \subseteq \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$, only finitely many $f_i|_C$ are nonzero. Then $\sum_{i\in I} f_i$ makes sense, and lies in $\mathrm{LCF}(\mathfrak{Obj}_{\mathcal{A}})$. In [14] we will prove identities which are convergent infinite sums of products in $\dot{\mathrm{LCF}}(\mathfrak{Obj}_{\mathcal{A}})$.

4.3. Representations of Ringel–Hall algebras

Here is a way to construct representations of the algebra $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ of Section 4.1. Although it is very simple, I did not find this method used explicitly or implicitly in the Ringel–Hall algebra literature.

Definition 4.6. Let Assumption 3.4 hold. Define $\alpha, \beta : \{1,2,3\} \to \{1,2\}$ by

$$\alpha(1) = \alpha(2) = 1, \qquad \alpha(3) = 2, \qquad \beta(1) = 1, \qquad \beta(2) = \beta(3) = 2. \tag{17}$$

Using the diagram of pullbacks, pushforwards of constructible functions

$$\begin{array}{ccc} \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) \times \mathrm{CF}(\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}}) & \xrightarrow[\qquad]{(\boldsymbol{\sigma}(\{2\}))^*\cdot(Q(\{1,2,3\},\leqslant,\{1,2\},\leqslant,\alpha))^*} \searrow & \mathrm{CF}(\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}}) \\ \big\downarrow{\scriptstyle\otimes} & & \big\uparrow{\scriptstyle \mathrm{CF}^{\mathrm{stk}}(Q(\{1,2,3\},\leqslant,\{1,2\},\leqslant,\beta))} \\ \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}}) & \xrightarrow[(\boldsymbol{\sigma}(\{2\})\times Q(\{1,2,3\},\leqslant,\{1,2\},\leqslant,\alpha))^*]{} & \mathrm{CF}(\mathfrak{M}(\{1,2,3\},\leqslant)_{\mathcal{A}}), \end{array}$$

define $*_L : \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) \times \mathrm{CF}(\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}}) \to \mathrm{CF}(\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}})$ by

$$\begin{aligned} f *_L r = \mathrm{CF}^{\mathrm{stk}}\big(Q\big(\{1,2,3\},\leqslant,\{1,2\},\leqslant,\beta\big)\big)\big[\boldsymbol{\sigma}\big(\{2\}\big)^*(f) \\ \cdot \big(Q\big(\{1,2,3\},\leqslant,\{1,2\},\leqslant,\alpha\big)\big)^*(r)\big]. \end{aligned} \tag{18}$$

This is well defined as $Q(\{1,2,3\},\leqslant,\{1,2\},\leqslant,\beta)$ is representable by Theorem 3.6(a), and one can show $\boldsymbol{\sigma}(\{2\}) \times Q(\cdots,\alpha)$ is finite type. In the same way, define $*_R : \mathrm{CF}(\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}}) \times \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) \to \mathrm{CF}(\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}})$ by

$$\begin{aligned} r *_R f = \mathrm{CF}^{\mathrm{stk}}\big(Q\big(\{1,2,3\},\leqslant,\{1,2\},\leqslant,\alpha\big)\big)\big[\big(Q\big(\{1,2,3\},\leqslant,\{1,2\},\leqslant,\beta\big)\big)^*(r) \\ \cdot \boldsymbol{\sigma}\big(\{2\}\big)^*(f)\big]. \end{aligned}$$

For $Z \in \mathcal{A}$, write $V^{[Z]}$ for the vector subspace of $f \in \mathrm{CF}(\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}})$ supported on points $[(\sigma,\iota,\pi)]$ with $\sigma(\{1,2\}) \cong Z$.

Here is what this means, translated from configurations to subobjects. We are defining an action of $f \in \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$, which is a function on isomorphism classes $[X]$ of objects $X \in \mathcal{A}$, upon $r \in \mathrm{CF}(\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}})$, which is a function on isomorphism classes $[X \subset Y]$ of subobjects $X \subset Y$ in $\mathcal{A}$. The rule is that $(f *_L r)([X \subset Z])$ is the 'integral' over chains of subobjects $X \subset Y \subset Z$ of $f([Y/X])r([Y \subset Z])$, with respect to an Euler characteristic measure. In the finite field approach of Ringel [23,24], this 'integral' would just be done by counting.

Similarly, $(r *_R f)([Y \subset Z])$ is the integral over chains $X \subset Y \subset Z$ of $r([X \subset Z])f([Y/X])$. Note that both $(f *_L r)$ and $r *_R f$ at $[X \subset Z]$ depend only on $r([Y \subset Z])$ for other subobjects $Y \subset Z$ with the same Z. Thus, the subspace $V^{[Z]}$ of functions r supported on $[X \subset Z]$ for fixed Z is closed under $f *_L$ and $*_R f$ for all $f \in \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$.

The proof of the next theorem is modelled on that of Theorem 4.3. Note that we do *not* claim that $\mathrm{CF}(\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}})$, $V^{[Z]}$ are *two-sided* representations of $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$, only that $*_L$ and $*_R$ *separately* define left and right representations. That is, we do not claim that $(f *_L r) *_R g = f *_L (r *_R g)$ for $f, g \in \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ and $r \in \mathrm{CF}(\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}})$, and in general this is false.

To interpret the middle part of the proof in terms of subobjects, regard $\mathfrak{M}(\{1,2,3,4\},\leqslant)_{\mathcal{A}}$ in (19) as the Artin $\mathbb{K}$-stack of chains of subobjects $W \subset X \subset Y \subset Z$ in $\mathcal{A}$. Then the proof consists in showing that both $(f * g) *_L r$ and $f *_L (g *_L r)$ evaluated at $[W \subset Z]$ are the 'integral' over chains $W \subset X \subset Y \subset Z$ of $f(X/W)g(Y/X)r([Y \subset Z])$.

Theorem 4.7. *Above, if* $f, g \in \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ *and* $r \in \mathrm{CF}(\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}})$ *then* $\delta_{[0]} *_L r = r$, $(f * g) *_L r = f *_L (g *_L r)$ *and* $r *_R \delta_{[0]} = r$, $r *_R (f * g) = (r *_R g) *_R f$. *Thus,* $*_L$, $*_R$ *define left and right representations of the algebra* $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ *on the vector space* $\mathrm{CF}(\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}})$.

Furthermore, for $Z \in \mathcal{A}$ *the subspace* $V^{[Z]}$ *is closed under both actions* $*_L$, $*_R$ *of* $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$. *Hence* $V^{[Z]}$ *is a left and right representation of* $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$.

Proof. The first part of Theorem 4.3 generalizes easily to show that $\delta_{[0]} *_L r = r$. Define $\gamma, \delta : \{1,2,3,4\} \to \{1,2\}$ and $\epsilon, \zeta, \eta : \{1,2,3,4\} \to \{1,2,3\}$ by

$$\gamma : 1,2,3 \mapsto 1, \quad \gamma : 4 \mapsto 2, \quad \delta : 1 \mapsto 1, \quad \delta : 2,3,4 \mapsto 2, \quad \epsilon : 1,2 \mapsto 1, \quad \epsilon : 3 \mapsto 2,$$
$$\epsilon : 4 \mapsto 3, \quad \zeta : 1 \mapsto 1, \quad \zeta : 2,3 \mapsto 2, \quad \zeta : 4 \mapsto 3, \quad \eta : 1 \mapsto 1, \quad \eta : 2 \mapsto 2, \quad \eta : 3,4 \mapsto 3,$$

and α, β as (17). Consider the diagram of 1-morphisms

$$\begin{array}{ccccc}
\begin{array}{c}\mathfrak{Obj}_{\mathcal{A}}\times\\ \mathfrak{Obj}_{\mathcal{A}}\times\\ \mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}}\end{array} & \xleftarrow[Q(\{1,2,3\},\leqslant,\{1,2\},\leqslant,\alpha)]{\mathrm{id}_{\mathfrak{Obj}_{\mathcal{A}}}\times\sigma(\{2\})\times} & \begin{array}{c}\mathfrak{Obj}_{\mathcal{A}}\times\\ \mathfrak{M}(\{1,2,3\},\leqslant)_{\mathcal{A}}\end{array} & \xrightarrow[Q(\{1,2,3\},\leqslant,\{1,2\},\leqslant,\beta)]{\mathrm{id}_{\mathfrak{Obj}_{\mathcal{A}}}\times} & \begin{array}{c}\mathfrak{Obj}_{\mathcal{A}}\times\\ \mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}}\end{array} \\
\Big\uparrow{\scriptstyle (\sigma(\{2\})\times\sigma(\{3\}))\times\mathrm{id}_{\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}}}} & & \Big\uparrow{\scriptstyle \sigma(\{2\})\times Q(\{1,2,3,4\},\leqslant,\{1,2,3\},\leqslant,\epsilon)} & & \Big\uparrow{\scriptstyle \sigma(\{2\})\times Q(\{1,2,3\},\leqslant,\{1,2\},\leqslant,\alpha)} \\
\begin{array}{c}\mathfrak{M}(\{2,3\},\leqslant)_{\mathcal{A}}\times\\ \mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}}\end{array} & \xleftarrow[Q(\{1,2,3,4\},\leqslant,\{1,2\},\leqslant,\gamma)]{S(\{1,2,3,4\},\leqslant,\{2,3\})\times} & \mathfrak{M}(\{1,2,3,4\},\leqslant)_{\mathcal{A}} & \xrightarrow[Q(\{1,2,3,4\},\leqslant,\{1,2,3\},\leqslant,\eta)]{} & \mathfrak{M}(\{1,2,3\},\leqslant)_{\mathcal{A}} \\
\Big\downarrow{\scriptstyle \sigma(\{2,3\})\times\mathrm{id}_{\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}}}} & & \Big\downarrow{\scriptstyle Q(\{1,2,3,4\},\leqslant,\{1,2,3\},\leqslant,\zeta)} & & \Big\downarrow{\scriptstyle Q(\{1,2,3\},\leqslant,\{1,2\},\leqslant,\beta)} \\
\begin{array}{c}\mathfrak{Obj}_{\mathcal{A}}\times\\ \mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}}\end{array} & \xleftarrow{\begin{array}{c}\sigma(\{2\})\times\\ Q(\{1,2,3\},\leqslant,\{1,2\},\leqslant,\alpha)\end{array}} & \mathfrak{M}(\{1,2,3\},\leqslant)_{\mathcal{A}} & \xrightarrow{Q(\{1,2,3\},\leqslant,\{1,2\},\leqslant,\beta)} & \mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}},
\end{array} \tag{19}$$

analogous to (14). It is not difficult to show (19) commutes, and the top right and bottom left squares are Cartesian. Thus there is a commutative diagram of spaces $\mathrm{CF}(\cdots)$ and pullbacks/pushforwards analogous to (15). Applying this to $f \otimes g \otimes r$ in $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}})$ gives $(f * g) *_L r = f *_L (g *_L r)$, and $*_L$ is a left representation. The proof for $*_R$ is similar.

Finally, if $r \in V^{[Z]}$ then r is supported on points $[(\sigma,\iota,\pi)]$ with $\sigma(\{1,2\}) \cong Z$. If we have a diagram

$$[(\sigma,\iota,\pi)] \xleftarrow{Q(\{1,2,3\},\leqslant,\{1,2\},\leqslant,\alpha)_*} [(\sigma',\iota',\pi')] \xrightarrow{Q(\{1,2,3\},\leqslant,\{1,2\},\leqslant,\beta)_*} [(\tilde\sigma,\tilde\iota,\tilde\pi)],$$

then $Z \cong \sigma(\{1,2\}) \cong \sigma'(\{1,2,3\}) \cong \tilde\sigma(\{1,2\})$ as $\alpha(\{1,2,3\}) = \{1,2\} = \beta(\{1,2,3\})$. Hence in (18), $(Q(\{1,2,3\},\leqslant,\{1,2\},\leqslant,\alpha))^*(r)$ is supported on points $[(\sigma',\iota',\pi')]$ with $\sigma'(\{1,2,3\}) \cong Z$, and thus $f *_L r$ is supported on points $[(\tilde\sigma,\tilde\iota,\tilde\pi)]$ with $\tilde\sigma(\{1,2\}) \cong Z$. That is, $f *_L r$ lies in $V^{[Z]}$, so $V^{[Z]}$ is closed under $*_L$. The proof for $*_R$ is similar. □

This shows that the big representation $\mathrm{CF}(\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}})$ contains many smaller subrepresentations $V^{[Z]}$. In examples, this may be a useful tool for constructing *finite-dimensional representations* of interesting infinite-dimensional algebras, such as universal enveloping algebras.

The ideas above extend easily to representations $*_L$, $*_R$ of $\dot{\mathrm{LCF}}(\mathfrak{Obj}_{\mathcal{A}})$ in Section 4.2 on $\dot{\mathrm{LCF}}(\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}})$, the subspace of $f \in \mathrm{LCF}(\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}})$ supported on the union of $\mathcal{M}(\{1,2\},\leqslant,\kappa)_{\mathcal{A}}$ over finitely many $\kappa:\{1,2\} \to \bar{C}(\mathcal{A})$. They should also work for the other contexts for Ringel–Hall algebras referred to in Section 1, such as the finite field approach of Ringel [23,24], and may yield something new and interesting there.

4.4. Indecomposables and Lie algebras

We now give an analogue of ideas of Ringel [22] and Riedtmann [21], who both define a Lie algebra structure on spaces of functions on isomorphism classes of *indecomposable* objects in an abelian category.

Definition 4.8. Suppose Assumption 3.4 holds. An object $0 \ncong X \in \mathcal{A}$ is called *decomposable* if $X \cong Y \oplus Z$ for $0 \ncong Y, Z \in \mathcal{A}$. Otherwise X is *indecomposable*. Write $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ for the subspace of f in $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ *supported on indecomposables*, that is, $f([X]) \neq 0$ implies $0 \ncong X$ is indecomposable.

Decomposability can be characterized in terms of the finite-dimensional $\mathbb{K}$-algebra $\mathrm{End}(X) = \mathrm{Hom}(X,X)$: X is decomposable if and only if there exist $0 \neq e_1, e_2 \in \mathrm{End}(X)$ with $1 = e_1 + e_2$, $e_1^2 = e_1$, $e_2^2 = e_2$ and $e_1e_2 = e_2e_1 = 0$. Then e_1, e_2 are called *orthogonal idempotents*. Given such e_1, e_2 we can define nonzero objects $Y = \mathrm{Im}\, e_1$ and $Z = \mathrm{Im}\, e_2$ in $\mathcal{A}$, and there exists an isomorphism $X \cong Y \oplus Z$ identifying e_1, e_2 with id_Y, id_Z. By choosing a set of *primitive orthogonal idempotents* in $\mathrm{End}(X)$ one can show that each $0 \ncong X \in \mathcal{A}$ may be written $X \cong V_1 \oplus \cdots \oplus V_n$ for indecomposable $V_1, \ldots, V_n$, unique up to order and isomorphism.

Define a bilinear bracket $[\,,\,]:\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) \times \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) \to \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ by

$$[f,g] = f * g - g * f, \tag{20}$$

for $*$ defined in (13). Since $*$ is associative by Theorem 4.3, $[\,,\,]$ satisfies the *Jacobi identity*, and makes $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ into a *Lie algebra* over $\mathbb{Q}$.

The following result is related to Riedtmann [21, §2], and [5, Proposition 2.2.8].

Theorem 4.9. *In the situation above,* $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ *is closed under the Lie bracket* $[\,,\,]$, *and is a Lie algebra over* $\mathbb{Q}$.

Proof. Let $f, g \in \mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ and $Y \in \mathcal{A}$ with $(f * g)([Y]) \neq 0$. By a long but elementary calculation involving properties of the Euler characteristic, we can show that either

(i) Y is indecomposable;
(ii) $Y \cong X \oplus Z$ for $X, Z \in \mathcal{A}$ indecomposable with $X \ncong Z$ and

$$(f * g)\big([Y]\big) = \big(f\big([X]\big)g\big([Z]\big) + f\big([Z]\big)g\big([X]\big)\big) \cdot \chi\big(\mathrm{Aut}(X \oplus Z)/\mathrm{Aut}(X) \times \mathrm{Aut}(Z)\big); \quad \text{or} \tag{21}$$

(iii) $Y \cong X \oplus X$ for $X \in \mathcal{A}$ indecomposable and

$$(f * g)\big([Y]\big) = f\big([X]\big)g\big([X]\big) \cdot \chi\big(\mathrm{Aut}(X \oplus X)/\mathrm{Aut}(X) \times \mathrm{Aut}(X)\big).$$

Here is a sketch proof. Firstly, to show (i)–(iii) are the only possibilities with $(f*g)([Y]) \neq 0$, suppose Y has r indecomposable factors, so that $\mathrm{rk}\,\mathrm{Aut}(Y) = r$, that is, $\mathrm{Aut}(Y)$ has maximal torus $(\mathbb{K}^\times)^r$. Then $(f * g)([Y])$ is an 'integral' over subobjects $X \subset Y$ of $f([X])g([Y/X])$, which includes a factor $\chi(\mathrm{Aut}(Y)/\mathrm{Aut}(X \subset Y))$. For $f([X])g([Y/X]) \neq 0$ we must have X, Y/X indecomposable, and it easily follows that $\mathrm{rk}\,\mathrm{Aut}(X \subset Y) \leqslant 2$. Now if $r \geqslant 3$ then the action of the maximal torus of $\mathrm{Aut}(Y)$ on $\mathrm{Aut}(Y)/\mathrm{Aut}(X \subset Y)$ fibres it by tori $(\mathbb{K}^\times)^k$ for $k \geqslant 1$, which forces $\chi(\mathrm{Aut}(Y)/\mathrm{Aut}(X \subset Y)) = 0$. Hence $(f * g)([Y]) = 0$ if Y has $r \geqslant 3$ indecomposable factors.

By the same argument, when Y has 2 indecomposable factors, the only nonzero contributions to $(f * g)([Y])$ come from $X \subset Y$ with $\mathrm{rk}(\mathrm{Aut}(X \subset Y)) = 2$, which happens only when $Y \cong X \oplus Y/X$. Thus in (ii) the only nonzero contributions to $(f * g)([Y])$ are from $X \subset X \oplus Z$ and $Z \subset X \oplus Z$, which give the two terms in (21). For (iii) there is only one contribution, from $X \subset X \oplus X$. In (ii), (iii) we have $(f * g)([Y]) = (g * f)([Y])$, so that $[f, g]([Y]) = 0$ by (20). Hence the only possibility in which $[f, g]([Y]) \neq 0$ is (i), when Y is indecomposable. Thus $[f, g] \in \mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$, as we have to prove. □

In the same way we find $\mathrm{L\dot{C}F}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ is a Lie subalgebra of $\mathrm{L\dot{C}F}(\mathfrak{Obj}_{\mathcal{A}})$.

4.5. Constructible functions with finite support

Here is some more notation. As functions with finite support are always constructible, we do not generalize this to $\mathrm{L\dot{C}F}(\mathfrak{Obj}_{\mathcal{A}})$.

Definition 4.10. Write $\mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$ for the subspace of $f \in \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ with *finite support*, that is, f is nonzero on only finitely many points in $\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$. Define $\mathrm{CF}^{\mathrm{ind}}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}) = \mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}}) \cap \mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$. For each $[X] \in \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$, write $\delta_{[X]} : \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K}) \to \{0, 1\}$ for the characteristic function of $[X]$. Then the $\delta_{[X]}$ form a basis for $\mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$, and the $\delta_{[X]}$ for indecomposable X form a basis for $\mathrm{CF}^{\mathrm{ind}}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$.

We want $\mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$ to be a subalgebra of $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$. To prove this we need an extra assumption, which holds in Example 4.23 below.

Assumption 4.11. For all $X, Z \in \mathcal{A}$, there are only finitely isomorphism classes of $Y \in \mathcal{A}$ for which there exists an exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$.

If *all constructible sets in* $\mathfrak{Obj}_{\mathcal{A}}$ *are finite* then $\mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}) = \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ and $\mathrm{CF}^{\mathrm{ind}}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}) = \mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$, and Assumption 4.11 holds automatically.

Proposition 4.12. *If Assumptions* 3.4 *and* 4.11 *hold then* $\mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$ *is closed under* $*$, *and* $\mathrm{CF}^{\mathrm{ind}}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$ *is closed under* $[\,,\,]$.

Proof. For $X, Z \in \mathcal{A}$, $\delta_{[X]} * \delta_{[Z]}$ is supported on the set of $[Y] \in \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ for which there exists an exact sequence $0 \to X \to Y \to Z \to 0$. By Assumption 4.11 this set is finite, so $\delta_{[X]} * \delta_{[Z]}$ lies in $\mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$. As the $\delta_{[X]}$ form a basis of $\mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$, it is closed under $*$. $\square$

4.6. Universal enveloping algebras

We study the universal enveloping algebras of $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$, $\mathrm{CF}^{\mathrm{ind}}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$.

Definition 4.13. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{Q}$. The *universal enveloping algebra* $U(\mathfrak{g})$ is the $\mathbb{Q}$-algebra generated by $\mathfrak{g}$ with the relations $xy - yx = [x, y]$ for all $x, y \in \mathfrak{g}$. Multiplication in $U(\mathfrak{g})$ will be written as juxtaposition, $(x, y) \mapsto xy$. Each Lie algebra representation of $\mathfrak{g}$ extends uniquely to an algebra representation of $U(\mathfrak{g})$, so $U(\mathfrak{g})$ is a powerful tool for studying the representation theory of $\mathfrak{g}$. See Humphreys [8] for an introduction to these ideas.

In Theorem 4.9 the embedding of the Lie algebra $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ with bracket $[\,,\,]$ in the algebra $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ with multiplication $*$ satisfying (20) implies there is a unique $\mathbb{Q}$-algebra homomorphism

$$\Phi : U\big(\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})\big) \to \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) \tag{22}$$

with $\Phi(1) = \delta_{[0]}$ and $\Phi(f_1 \cdots f_n) = f_1 * \cdots * f_n$ for $f_1, \ldots, f_n \in \mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$. Similarly, in Proposition 4.12 there is a unique homomorphism

$$\Phi_{\mathrm{fin}} : U\big(\mathrm{CF}^{\mathrm{ind}}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})\big) \to \mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}). \tag{23}$$

The next two results are similar to Riedtmann [21, §3].

Proposition 4.14. Φ *and* Φ_{fin} *above are injective. Hence, the* $\mathbb{Q}$*-subalgebras of* $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$, $\mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$ *generated by* $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$, $\mathrm{CF}^{\mathrm{ind}}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$ *are isomorphic to* $U(\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}}))$, $U(\mathrm{CF}^{\mathrm{ind}}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}))$ *respectively.*

Proof. We do the case of Φ_{fin} first, so suppose Assumptions 3.4 and 4.11 hold. Let $V_1, V_2 \in \mathcal{A}$ be indecomposable. Applying (i)–(iii) in the proof of Theorem 4.9 to $f = \delta_{[V_1]}$, $g = \delta_{[V_2]}$ we find that $\delta_{[V_1]} * \delta_{[V_2]}$ is supported on points $[V_1 \oplus V_2]$ and $[Y]$ for $Y \in \mathcal{A}$ indecomposable, and $(\delta_{[V_1]} * \delta_{[V_2]})([V_1 \oplus V_2]) = \chi(\mathrm{Aut}(V_1 \oplus V_2)/\mathrm{Aut}(V_1) \times \mathrm{Aut}(V_2))$. That is, $\chi(\mathrm{Aut}(V_1 \oplus V_2)/$

$\mathrm{Aut}(V_1)\times\mathrm{Aut}(V_2))\cdot\delta_{[V_1\oplus V_2]}-\delta_{[V_1]}*\delta_{[V_2]}$ is supported on points $[Y]$ for indecomposable Y. It is not difficult to generalize this to show that if $V_1,\ldots,V_m\in\mathcal{A}$ are indecomposable then

$$\begin{aligned}&\chi\bigl(\mathrm{Aut}(V_1\oplus\cdots\oplus V_m)/\mathrm{Aut}(V_1)\times\cdots\times\mathrm{Aut}(V_m)\bigr)\cdot\delta_{[V_1\oplus\cdots\oplus V_m]}-\delta_{[V_1]}*\delta_{[V_2]}*\cdots*\delta_{[V_m]}\\&\quad\text{is supported on points }[W_1\oplus\cdots\oplus W_k]\\&\quad\text{for indecomposable }W_1,\ldots,W_k\in\mathcal{A}\text{ and }1\leqslant k<m.\end{aligned}\tag{24}$$

Let the $V_1,\ldots,V_m$ have a equivalence classes under isomorphism with sizes $m_1,\ldots,m_a$, so that $m=m_1+\cdots+m_a$. Using facts about the finite-dimensional algebras $\mathrm{End}(V_1\oplus\cdots\oplus V_m)$, $\mathrm{End}(V_1),\ldots,\mathrm{End}(V_m)$ and the *Jacobson radical* from Benson [1, §1] we find there is an isomorphism of $\mathbb{K}$-varieties

$$\mathrm{Aut}(V_1\oplus\cdots\oplus V_m)/\mathrm{Aut}(V_1)\times\cdots\times\mathrm{Aut}(V_m)\cong\mathbb{K}^l\times\prod_{i=1}^a\bigl(\mathrm{GL}(m_i,\mathbb{K})/\bigl(\mathbb{K}^\times\bigr)^{m_i}\bigr),$$

which allows us to compute the Euler characteristic

$$\chi\bigl(\mathrm{Aut}(V_1\oplus\cdots\oplus V_m)/\mathrm{Aut}(V_1)\times\cdots\times\mathrm{Aut}(V_m)\bigr)=\prod_{i=1}^a m_i!.\tag{25}$$

To show Φ_{fin} is injective, write $I=\{[X]\in\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})\colon X\text{ is indecomposable}\}$, and let $\leqslant$ be any arbitrary *total order* on I. Then $\{\delta_{[X]}\colon [X]\in I\}$ is a basis for $\mathrm{CF}^{\mathrm{ind}}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$, so the *Poincaré–Birkhoff–Witt Theorem* [8, Corollary C, p. 92] shows

$$\{1\}\cup\bigl\{\delta_{[X_1]}\delta_{[X_2]}\cdots\delta_{[X_n]}\colon n\geqslant 1,\ [X_1],\ldots,[X_n]\in I,\ [X_1]\leqslant\cdots\leqslant[X_n]\bigr\}\tag{26}$$

is a basis for $U(\mathrm{CF}^{\mathrm{ind}}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}))$. Suppose $u\in U(\mathrm{CF}^{\mathrm{ind}}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}))$ is nonzero. If $u=c\cdot 1$ for $c\in\mathbb{Q}\setminus\{0\}$ then $\Phi_{\mathrm{fin}}(u)=c\delta_{[0]}\neq 0$. Otherwise, there exists a basis element $\delta_{[X_1]}\delta_{[X_2]}\cdots\delta_{[X_n]}$ from (26) with n greatest such that the coefficient $u_{[X_1]\cdots[X_n]}$ of $\delta_{[X_1]}\delta_{[X_2]}\cdots\delta_{[X_n]}$ is nonzero.

We shall evaluate the constructible function $\Phi_{\mathrm{fin}}(u)$ at the point $[X_1\oplus\cdots\oplus X_n]$. Let $\delta_{[V_1]}\cdots\delta_{[V_m]}$ be any basis element from (26) with nonzero coefficient in u. Then $m\leqslant n$, by choice of n. We have

$$\begin{aligned}\Phi_{\mathrm{fin}}(\delta_{[V_1]}\cdots\delta_{[V_m]})\bigl([X_1\oplus\cdots\oplus X_n]\bigr)&=(\delta_{[V_1]}*\cdots*\delta_{[V_m]})\bigl([X_1\oplus\cdots\oplus X_n]\bigr)\\&=c\delta_{[V_1\oplus\cdots\oplus V_m]}\bigl([X_1\oplus\cdots\oplus X_n]\bigr)\\&=\begin{cases}c,&[V_1\oplus\cdots\oplus V_m]=[X_1\oplus\cdots\oplus X_n],\\0,&[V_1\oplus\cdots\oplus V_m]\neq[X_1\oplus\cdots\oplus X_n],\end{cases}\end{aligned}$$

where c is the nonzero integer (25) and in the second line we use (24) and $m\leqslant n$ to see that $c\delta_{[V_1\oplus\cdots\oplus V_m]}-\delta_{[V_1]}*\cdots*\delta_{[V_m]}$ is zero at $[X_1\oplus\cdots\oplus X_n]$.

As V_i, X_j are indecomposable, $[V_1\oplus\cdots\oplus V_m]=[X_1\oplus\cdots\oplus X_n]$ if and only if $m=n$ and $[V_1],\ldots,[V_m]$ and $[X_1],\ldots,[X_n]$ are the same up to a permutation of $1,\ldots,n$. But by assumption $[V_1]\leqslant\cdots\leqslant[V_m]$ and $[X_1]\leqslant\cdots\leqslant[X_n]$ in the total order $\leqslant$ on I, so $[V_1\oplus\cdots\oplus V_m]=[X_1\oplus\cdots\oplus X_n]$ only if $[V_i]=[X_i]$ for all i. Therefore $\Phi_{\mathrm{fin}}(u)([X_1\oplus\cdots\oplus X_n])=$

$c\,u_{[X_1]\cdots[X_n]} \neq 0$, as $\delta_{[X_1]}\cdots\delta_{[X_n]}$ is the only basis element making a nonzero contribution. Hence $\Phi_{\rm fin}(u) \neq 0$ for all $0 \neq u \in U(\mathrm{CF}^{\rm ind}_{\rm fin}(\mathfrak{Obj}_{\mathcal{A}}))$, and $\Phi_{\rm fin}$ is injective.

Showing Φ is injective uses essentially the same ideas, but is a little more tricky as we cannot choose a basis for $\mathrm{CF}^{\rm ind}(\mathfrak{Obj}_{\mathcal{A}})$ consisting of functions with *disjoint support*. We leave it as an exercise for the reader. For the last part, as Φ, $\Phi_{\rm fin}$ are injective they are isomorphisms with their images, which are the $\mathbb{Q}$-subalgebras generated by $\mathrm{CF}^{\rm ind}(\mathfrak{Obj}_{\mathcal{A}})$, $\mathrm{CF}^{\rm ind}_{\rm fin}(\mathfrak{Obj}_{\mathcal{A}})$. □

We shall show $\Phi_{\rm fin}$ is an isomorphism. This will enable us to identify the algebra $\mathrm{CF}_{\rm fin}(\mathfrak{Obj}_{\mathcal{A}})$ in examples.

Proposition 4.15. *Let Assumptions* 3.4 *and* 4.11 *hold. Then* $\Phi_{\rm fin}$ *in* (23) *is an isomorphism.*

Proof. As $\Phi_{\rm fin}$ is injective by Proposition 4.15, we need only show it is surjective. Suppose by induction that for some $m \geqslant 1$, $\operatorname{Im}\Phi_{\rm fin}$ contains $\delta_{[V_1\oplus\cdots\oplus V_n]}$ whenever $1 \leqslant n < m$ and $V_i \in \mathcal{A}$ are indecomposable. This is trivial for $m = 1$. Let $V_1, \ldots, V_m \in \mathcal{A}$ be indecomposable. Then by (24), $c\,\delta_{[V_1\oplus\cdots\oplus V_m]} - \delta_{[V_1]} * \cdots * \delta_{[V_m]}$ lies in the span of functions $\delta_{[W_1\oplus\cdots\oplus W_k]}$ for indecomposable $W_i \in \mathcal{A}$ and $1 \leqslant k < m$, where c is the nonzero integer (25).

By induction $\delta_{[W_1\oplus\cdots\oplus W_k]} \in \operatorname{Im}\Phi_{\rm fin}$, so $c\delta_{[V_1\oplus\cdots\oplus V_m]} - \delta_{[V_1]} * \cdots * \delta_{[V_m]}$ lies in $\operatorname{Im}\Phi_{\rm fin}$. But $\delta_{[V_1]} * \cdots * \delta_{[V_m]} = \Phi_{\rm fin}(\delta_{[V_1]}\cdots\delta_{[V_m]}) \in \operatorname{Im}\Phi_{\rm fin}$, so $\delta_{[V_1\oplus\cdots\oplus V_m]}$ lies in $\operatorname{Im}\Phi_{\rm fin}$. Thus the inductive hypothesis holds for $m+1$, and by induction $\delta_{[V_1\oplus\cdots\oplus V_n]} \in \operatorname{Im}\Phi_{\rm fin}$ whenever $n \geqslant 1$ and $V_i \in \mathcal{A}$ are indecomposable.

But each $0 \ncong X \in \mathcal{A}$ may be written $X \cong V_1 \oplus \cdots \oplus V_n$ for $V_i \in \mathcal{A}$ indecomposable, by Definition 4.8, so $\delta_{[X]} \in \operatorname{Im}\Phi_{\rm fin}$ from above. Also $\delta_{[0]} = \Phi_{\rm fin}(1) \in \operatorname{Im}\Phi_{\rm fin}$, so $\delta_{[X]} \in \operatorname{Im}\Phi_{\rm fin}$ for all $X \in \mathcal{A}$. As the $\delta_{[X]}$ are a basis for $\mathrm{CF}_{\rm fin}(\mathfrak{Obj}_{\mathcal{A}})$, this proves $\operatorname{Im}\Phi_{\rm fin} = \mathrm{CF}_{\rm fin}(\mathfrak{Obj}_{\mathcal{A}})$, and $\Phi_{\rm fin}$ is surjective. □

We shall see in Section 4.7 that with extra structures on $\mathrm{CF}_{\rm fin}(\mathfrak{Obj}_{\mathcal{A}})$, $\Phi_{\rm fin}$ is actually an isomorphism of $\mathbb{Q}$-bialgebras and of Hopf algebras.

4.7. Comultiplication and bialgebras

Next we define a *cocommutative comultiplication* on the $\mathbb{Q}$-algebra $\mathrm{CF}_{\rm fin}(\mathfrak{Obj}_{\mathcal{A}})$, making it into a *bialgebra*. Our treatment is based on Ringel [25], who defines a similar comultiplication on degenerate Ringel–Hall algebras at $q = 1$. For an introduction to bialgebras, see Joseph [9, §1].

Definition 4.16. Let Assumptions 3.4 and 4.11 hold, so that $\mathrm{CF}_{\rm fin}(\mathfrak{Obj}_{\mathcal{A}})$ is a $\mathbb{Q}$-algebra by Proposition 4.12. Let

$$\Psi : \mathrm{CF}_{\rm fin}(\mathfrak{Obj}_{\mathcal{A}}) \otimes \mathrm{CF}_{\rm fin}(\mathfrak{Obj}_{\mathcal{A}}) \to \mathrm{CF}_{\rm fin}(\mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}}) \tag{27}$$

be the unique linear map with $\Psi(f \otimes g) = f \otimes g$ for $f, g \in \mathrm{CF}_{\rm fin}(\mathfrak{Obj}_{\mathcal{A}})$, in the notation of Definition 4.1. Since $\delta_{([X],[Y])}$ for $X, Y \in \mathcal{A}$ form a basis for $\mathrm{CF}_{\rm fin}(\mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}})$ and $\Psi(\delta_{[X]} \otimes \delta_{[Y]}) = \delta_{[X]} \otimes \delta_{[Y]} = \delta_{([X],[Y])}$, we see that Ψ is an isomorphism of $\mathbb{Q}$-vector spaces.

For I any finite set, let $\bullet$ be the partial order on I with $i \bullet j$ if and only if $i = j$. Consider the diagram of 1-morphisms

$$\mathfrak{Obj}_{\mathcal{A}} \xleftarrow{\boldsymbol{\sigma}(\{1,2\})} \mathfrak{M}(\{1,2\},\bullet)_{\mathcal{A}} \xrightarrow{\boldsymbol{\sigma}(\{1\})\times\boldsymbol{\sigma}(\{2\})} \mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}}.$$

By [12, Proposition 7.9], $\boldsymbol{\sigma}(\{1\}) \times \boldsymbol{\sigma}(\{2\})$ is a 1-isomorphism, and so is representable. Now $\boldsymbol{\sigma}(\{1,2\})_* \circ (\boldsymbol{\sigma}(\{1\}) \times \boldsymbol{\sigma}(\{2\}))_*^{-1} : (\mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}})(\mathbb{K}) \to \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ maps $([Y],[Z]) \mapsto [Y \oplus Z]$. Since for any $X \in \mathcal{A}$ there are only finitely many pairs $Y, Z \in \mathcal{A}$ up to isomorphism with $Y \oplus Z \cong X$, and $(\boldsymbol{\sigma}(\{1\}) \times \boldsymbol{\sigma}(\{2\}))_*$ is a bijection, we see that $\boldsymbol{\sigma}(\{1,2\})_*$ takes only finitely many points to each point in $\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$. Thus the following maps are well defined:

$$\mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}) \xrightarrow{(\boldsymbol{\sigma}(\{1,2\}))^*} \mathrm{CF}_{\mathrm{fin}}\big(\mathfrak{M}(\{1,2\},\bullet)_{\mathcal{A}}\big) \xrightarrow{\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(\{1\}) \times \boldsymbol{\sigma}(\{2\}))} \mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}}).$$

As Ψ is an isomorphism Ψ^{-1} exists, so we may define the *comultiplication*

$$\begin{aligned} &\Delta : \mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}) \to \mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}) \otimes \mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}) \quad \text{by} \\ &\Delta = \Psi^{-1} \circ \mathrm{CF}^{\mathrm{stk}}\big(\boldsymbol{\sigma}\big(\{1\}\big) \times \boldsymbol{\sigma}\big(\{2\}\big)\big) \circ \big(\boldsymbol{\sigma}\big(\{1,2\}\big)\big)^*. \end{aligned} \tag{28}$$

Define the *counit* $\epsilon : \mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}) \to \mathbb{Q}$ by $\epsilon : f \mapsto f([0])$.

Theorem 4.17. *Let Assumptions* 3.4 *and* 4.11 *hold. Then* $\mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$ *with* $*, \Delta, \delta_{[0]}, \epsilon$ *is a cocommutative bialgebra.*

Proof. For the axioms of a bialgebra, see Joseph [9, §1.1]. First we show $\mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$, Δ, ϵ form a cocommutative coalgebra. As $(\{1,2\},\bullet)$ is preserved by exchanging 1, 2, Eq. (28) is unchanged by exchanging the factors in $\mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}) \otimes \mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$, so Δ is *cocommutative*. To show Δ is *coassociative* we must show the following commutes:

$$\begin{array}{ccc} \mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}) & \xrightarrow{\quad\Delta\quad} & \mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}) \otimes \mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}) \\ \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Delta\otimes\mathrm{id}} \\ \mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}) \otimes \mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}) & \xrightarrow{\mathrm{id}\otimes\Delta} & \mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}) \otimes \mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}) \otimes \mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}). \end{array}$$

Using [12, Proposition 7.9] and the fact Ψ is an isomorphism, this follows provided

$$\begin{array}{ccc} \mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}) & \xrightarrow{\quad\boldsymbol{\sigma}(\{1,2\})^*\quad} & \mathrm{CF}_{\mathrm{fin}}(\mathfrak{M}(\{1,2\},\bullet)_{\mathcal{A}}) \\ \downarrow{\scriptstyle\boldsymbol{\sigma}(\{1,2\})^*} & & \downarrow{\scriptstyle Q(\{1,2,3\},\bullet,\{1,2\},\bullet,\alpha)^*} \\ \mathrm{CF}_{\mathrm{fin}}(\mathfrak{M}(\{1,2\},\bullet)_{\mathcal{A}}) & \xrightarrow{Q(\{1,2,3\},\bullet,\{1,2\},\bullet,\beta)^*} & \mathrm{CF}_{\mathrm{fin}}(\mathfrak{M}(\{1,2,3\},\bullet)_{\mathcal{A}}) \end{array}$$

commutes, where α, β are as in (17). But this follows from (3) as $\boldsymbol{\sigma}(\{1,2\}) \circ Q(\{1,2,3\},\bullet,\{1,2\},\bullet,\alpha) = \boldsymbol{\sigma}(\{1,2\}) \circ Q(\{1,2,3\},\bullet,\{1,2\},\bullet,\beta)$.

Since $\mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$ is cocommutative, to show ϵ is a *counit* we need

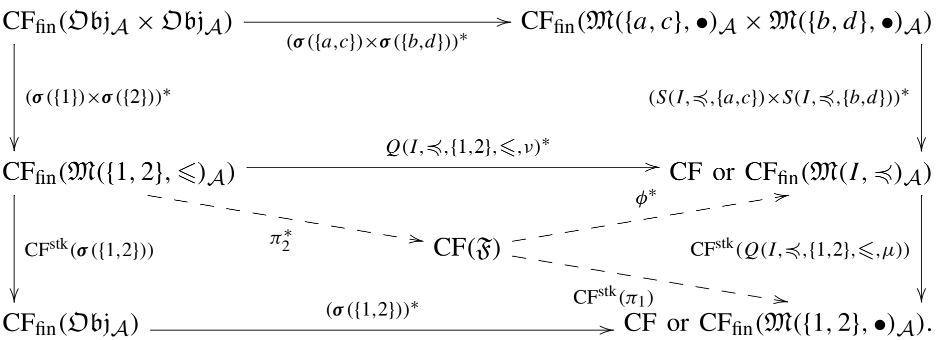
$$\begin{array}{ccc} \mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}) & \overset{f\mapsto 1\otimes f}{\searrow} & \\ \downarrow{\scriptstyle\Delta} & & \\ \mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}) \otimes \mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}) & \xrightarrow{\epsilon\otimes\mathrm{id}} & \mathbb{Q} \otimes \mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}) \end{array}$$

to commute. This holds if $(\Delta f)([0],[X]) = f([X])$ for all $f \in \mathrm{CF_{fin}}(\mathfrak{Obj}_{\mathcal{A}})$ and $[X] \in \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$. This is clear from (28), as there is just one point $[(\sigma,\iota,\pi)] \in \mathcal{M}(\{1,2\},\bullet)_{\mathcal{A}}$ with $\sigma(\{1\}) \cong 0$ and $\sigma(\{2\}) \cong X$, and it has $\sigma(\{1,2\}) \cong X$.

Next we prove Δ is *multiplicative*, that is, Δ is an algebra homomorphism from $\mathrm{CF_{fin}}(\mathfrak{Obj}_{\mathcal{A}})$ with multiplication $*$ to $\mathrm{CF_{fin}}(\mathfrak{Obj}_{\mathcal{A}}) \otimes \mathrm{CF_{fin}}(\mathfrak{Obj}_{\mathcal{A}})$ with multiplication $* \otimes *$. Define $I = \{a,b,c,d\}$ and a partial order $\preccurlyeq$ on I by $a \preccurlyeq b$, $c \preccurlyeq d$, and $i \preccurlyeq i$ for $i \in I$. Define maps $\mu,\nu : I \to \{1,2\}$ by

$$\mu(a)=\mu(b)=1, \quad \mu(c)=\mu(d)=2, \quad \nu(a)=\nu(c)=1, \quad \nu(b)=\nu(d)=2.$$

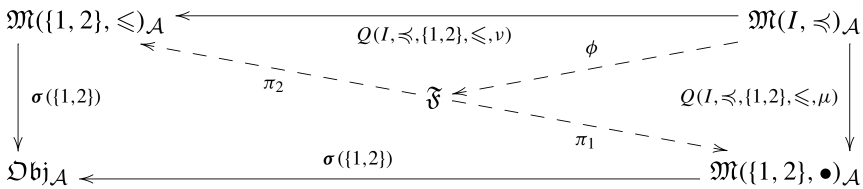
Then using [12, Proposition 7.9] and the fact that Ψ is an isomorphism, calculation shows Δ is multiplicative provided the unbroken arrows '$\rightarrow$' commute in



(29)

Here we write 'CF or $\mathrm{CF_{fin}}(\cdots)$' as the arrows '$\rightarrow$' map to $\mathrm{CF_{fin}}(\cdots)$, but the arrows '$\dashrightarrow$' defined below may map to $\mathrm{CF}(\cdots)$.

The top square commutes by (3) as the corresponding 1-morphisms do. The bottom square is more tricky, since although in



(30)

the square of 1-morphisms '$\rightarrow$' commutes, it is *not* a Cartesian square, and so Theorem 2.5 does not apply.

To get round this, define $\mathfrak{F}$ to be the *fibre product stack* of the bottom left corner of (30). Since the outer square of (30) commutes, there exist 1-morphisms π_1, π_2, ϕ in (30) unique up to 2-isomorphism, such that (30) commutes, and the bottom left quadrilateral is a Cartesian square. We can then add maps '$\dashrightarrow$' in (29). The bottom left quadrilateral of (29) commutes by (4), and the central triangle by (3). It remains only to show the right-hand triangle commutes.

We may justify this as follows. Points of $\mathfrak{F}(\mathbb{K})$ may be naturally identified with isomorphism classes of quadruples (X,S,T,U), where $X \in \mathcal{A}$ and $S,T,U \subset X$ are subobjects of X with $S,\ldots,X/U \in \mathcal{A}$, such that $X = S \oplus T$. An *isomorphism* $\phi : (X,S,T,U) \to (X',S',T',U')$ is an isomorphism $\phi : X \to X'$ in $\mathcal{A}$ such that $\phi(S) = S'$, $\phi(T) = T'$, $\phi(U) = U'$. Then π_1, π_2 act on $\mathfrak{F}(\mathbb{K})$ by

$$(\pi_1)_* : \big[(X,S,T,U)\big] \mapsto \big[(X,S,T)\big] \quad \text{and} \quad (\pi_2)_* : \big[(X,S,T,U)\big] \mapsto \big[(X,U)\big],$$

where we identify $[(\sigma,\iota,\pi)] \in \mathcal{M}(\{1,2\},\bullet)_{\mathcal{A}}$ and $[(\sigma',\iota',\pi')] \in \mathcal{M}(\{1,2\},\leqslant)_{\mathcal{A}}$ with

$$\begin{gathered}\Big[\big(\sigma\big(\{1,2\}\big), \iota\big(\{1\},\{1,2\}\big) : \sigma\big(\{1\}\big) \to \sigma\big(\{1,2\}\big), \iota\big(\{2\},\{1,2\}\big) : \sigma\big(\{2\}\big) \to \sigma\big(\{1,2\}\big)\big)\Big], \quad \text{and} \\ \Big[\big(\sigma'\big(\{1,2\}\big), \iota'\big(\{1\},\{1,2\}\big) : \sigma'\big(\{1\}\big) \to \sigma'\big(\{1,2\}\big)\big)\Big] \quad \text{respectively.}\end{gathered}$$

There is a closed substack $\mathfrak{G}$ of $\mathfrak{F}$ such that $\phi : \mathfrak{M}(I,\preccurlyeq)_{\mathcal{A}} \to \mathfrak{G}$ is a 1-isomorphism. A point $[(X,S,T,U)] \in \mathfrak{F}(\mathbb{K})$ lies in $\mathfrak{G}(\mathbb{K})$ if and only if $U = (S \cap U) \oplus (T \cap U)$. Here, since $X = S \oplus T$ we have $(S \cap U) \oplus (T \cap U) \subset U \subset X$, but it can happen that $(S \cap U) \oplus (T \cap U) \neq U$. To understand this, consider the case in which S, T, U are vector subspaces of a vector space X.

Write $\mathrm{CF}(\mathfrak{G})$, $\mathrm{CF}(\mathfrak{F} \setminus \mathfrak{G})$ for the subspaces of $\mathrm{CF}(\mathfrak{F})$ supported on $\mathfrak{G}(\mathbb{K})$, $\mathfrak{F}(\mathbb{K}) \setminus \mathfrak{G}(\mathbb{K})$ respectively. Then $\mathrm{CF}(\mathfrak{F}) = \mathrm{CF}(\mathfrak{G}) \oplus \mathrm{CF}(\mathfrak{F} \setminus \mathfrak{G})$, so it suffices to show the two triangles

$$\begin{array}{ccc} & & \mathrm{CF}(\mathfrak{M}(I,\preccurlyeq)_{\mathcal{A}}) \\ & \overset{\phi^*}{\nearrow} & \Big\downarrow {\scriptstyle \mathrm{CF}^{\mathrm{stk}}(Q(I,\preccurlyeq,\{1,2\},\leqslant,\mu))} \\ \mathrm{CF}(\mathfrak{G}) & & \\ & \underset{\mathrm{CF}^{\mathrm{stk}}(\pi_1)}{\searrow} & \\ & & \mathrm{CF}(\mathfrak{M}(\{1,2\},\bullet)_{\mathcal{A}}) \end{array} \qquad \begin{array}{ccc} & & \mathrm{CF}(\mathfrak{M}(I,\preccurlyeq)_{\mathcal{A}}) \\ & \overset{\phi^*}{\nearrow} & \Big\downarrow {\scriptstyle \mathrm{CF}^{\mathrm{stk}}(Q(I,\preccurlyeq,\{1,2\},\leqslant,\mu))} \\ \mathrm{CF}(\mathfrak{F} \setminus \mathfrak{G}) & & \\ & \underset{\mathrm{CF}^{\mathrm{stk}}(\pi_1)}{\searrow} & \\ & & \mathrm{CF}(\mathfrak{M}(\{1,2\},\bullet)_{\mathcal{A}}) \end{array} \tag{31}$$

commute. As $\phi : \mathfrak{M}(I,\preccurlyeq)_{\mathcal{A}} \to \mathfrak{G}$ is a 1-isomorphism we have $\phi^* = \mathrm{CF}^{\mathrm{stk}}(\phi^{-1})$ on $\mathrm{CF}(\mathfrak{G})$, so the left triangle commutes by (2).

In the right triangle, $\phi^* = 0$ as each $f \in \mathrm{CF}(\mathfrak{F} \setminus \mathfrak{G})$ is zero on $\mathfrak{G}(\mathbb{K}) = \phi_*(\mathcal{M}(I,\preccurlyeq)_{\mathcal{A}})$. We shall show $\mathrm{CF}^{\mathrm{stk}}(\pi_1) = 0$ too. Let $[(X,S,T,U)] \in \mathfrak{F}(\mathbb{K}) \setminus \mathfrak{G}(\mathbb{K})$, so that $(\pi_1)_* : [(X,S,T,U)] \mapsto [(X,S,T)]$. We have stabilizer groups

$$\mathrm{Iso}_{\mathbb{K}}\big(\big[(X,S,T,U)\big]\big) = \mathrm{Aut}(X,S,T,U) \quad \text{and} \quad \mathrm{Iso}_{\mathbb{K}}\big(\big[(X,S,T)\big]\big) = \mathrm{Aut}(X,S,T)$$

in $\mathrm{Aut}(X)$. Thus Definition 2.4 yields

$$m_{\pi_1}\big(\big[(X,S,T,U)\big]\big) = \chi\big(\mathrm{Aut}(X,S,T)/\mathrm{Aut}(X,S,T,U)\big). \tag{32}$$

As $X = S \oplus T$ there is a subgroup $\{\mathrm{id}_S + \alpha\, \mathrm{id}_T\colon \alpha \in \mathbb{K} \setminus \{0\}\} \cong \mathbb{K}^\times$ in the centre of $\mathrm{Aut}(X,S,T)$. Since $U \neq (S \cap U) \oplus (T \cap U)$, it is easy to see this group intersects $\mathrm{Aut}(X,S,T,U)$ in the identity. Thus $\mathbb{K}^\times$ acts *freely* on the left on $\mathrm{Aut}(X,S,T)/\mathrm{Aut}(X,S,T,U)$, fibring it by $\mathbb{K}^\times$ orbits. But $\chi(\mathbb{K}^\times) = 0$, so properties of χ show that $m_{\pi_1}([(X,S,T,U)]) = 0$ in (32). Definition 2.4 then shows that $\mathrm{CF}^{\mathrm{stk}}(\pi_1) f = 0$ for all $f \in \mathrm{CF}(\mathfrak{F} \setminus \mathfrak{G})$. Hence the right triangle in (31) commutes, and thus (29) commutes, and Δ is *multiplicative*.

By Proposition 4.12, to show $\mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$ is a *bialgebra* it remains only to verify some compatibilities [9, §1.1.3] between the unit $\delta_{[0]}$ and counit ϵ, which follow from the easy identities $\epsilon(\delta_{[0]}) = 1$, $\Delta\delta_{[0]} = \delta_{[0]} \otimes \delta_{[0]}$ and $(f * g)([0]) = f([0])g([0])$ for all $f, g \in \mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$. This completes the proof. $\square$

We can determine Δ, ϵ on the subspace $\mathrm{CF}^{\mathrm{ind}}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$.

Lemma 4.18. *If $f \in \mathrm{CF}^{\mathrm{ind}}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$ then $\Delta f = f \otimes \delta_{[0]} + \delta_{[0]} \otimes f$ and $\epsilon(f) = 0$.*

Proof. If $X \in \mathcal{A}$ is indecomposable then $\boldsymbol{\sigma}(\{1,2\})^*([X]) \subseteq \mathcal{M}(\{1,2\},\bullet)_{\mathcal{A}}$ is two points $[(\sigma,\iota,\pi)], [(\sigma',\iota',\pi')]$, where $\sigma(\{1\}) = \sigma(\{1,2\}) = X$, $\sigma(\{2\}) = 0$ and $\sigma'(\{1\}) = 0$, $\sigma'(\{2\}) = \sigma'(\{1,2\}) = X$. Thus $\Delta\,\delta_{[X]} = \delta_{[X]} \otimes \delta_{[0]} + \delta_{[0]} \otimes \delta_{[X]}$ by (28), proving the first equation as such $\delta_{[X]}$ form a basis for $\mathrm{CF}^{\mathrm{ind}}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$. Also $\epsilon(f) = f([0]) = 0$ as f is supported on $[X]$ for X indecomposable. □

Let $\mathfrak{g}$ be a Lie algebra. Then as in Joseph [9, §1.2.6], the universal enveloping algebra $U(\mathfrak{g})$ has the structure of a bialgebra with comultiplication Δ and counit ϵ satisfying $\Delta x = 1 \otimes x + x \otimes 1$, $\epsilon(x) = 0$ for all $x \in \mathfrak{g} \subset U(\mathfrak{g})$. Since $\mathfrak{g}$ generates $U(\mathfrak{g})$ as an algebra, Δ, ϵ are determined on the whole of $U(\mathfrak{g})$ by their values on $\mathfrak{g}$. So we deduce:

Corollary 4.19. *In Proposition* 4.15, *Φ_{fin} is an isomorphism of $\mathbb{Q}$-bialgebras.*

When we try to make $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ into a bialgebra in the same way, without assuming *finite support*, we run into the following problem: if we use spaces $\mathrm{CF}(\cdots)$ rather than $\mathrm{CF}_{\mathrm{fin}}(\cdots)$ in (27) then Ψ is injective, but generally *not* surjective. Thus Ψ^{-1} does not exist, and Δ in (28) is not well defined.

There are two natural solutions to this. The first is to omit Ψ^{-1} in (28), so Δ maps $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) \to \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}})$, where we regard $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}})$ as a *topological completion* of $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) \otimes \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$. Then the proof of Theorem 4.17 works with few changes, but what we get is not strictly a bialgebra.

The second is to restrict to a subalgebra $\mathcal{H}$ of $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ such that (28) yields a well-defined comultiplication $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$. That is, as Ψ is injective, Ψ^{-1} is well defined on $\mathrm{Im}\,\Psi$, so (28) makes sense if $\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(\{1\}) \times \boldsymbol{\sigma}(\{2\})) \circ (\boldsymbol{\sigma}(\{1,2\}))^*$ maps $\mathcal{H} \to \mathrm{Im}\,\Psi$. We take this approach in our next theorem.

Theorem 4.20. *Let Assumption* 3.4 *hold, $\mathcal{L} \subseteq \mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ be a Lie subalgebra, and $\mathcal{H}_{\mathcal{L}}$ the subalgebra of* $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ *generated by $\mathcal{L}$. In particular, $\mathcal{L}$ can be the Lie subalgebra generated by functions supported on points $[X]$ for $X \in \mathcal{A}$ simple, and then $\mathcal{H}_{\mathcal{L}} = \mathcal{C}$, the composition algebra of Section* 4.1.

Then (28) *yields a well-defined comultiplication $\Delta : \mathcal{H}_{\mathcal{L}} \to \mathcal{H}_{\mathcal{L}} \otimes \mathcal{H}_{\mathcal{L}}$, where*

$$\Psi : \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) \otimes \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) \to \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}}) \tag{33}$$

is injective and Ψ^{-1} defined on $\mathrm{Im}\,\Psi$, *and $\epsilon : f \mapsto f([0])$ defines a counit $\epsilon : \mathcal{H}_{\mathcal{L}} \to \mathbb{Q}$, which make $\mathcal{H}_{\mathcal{L}}$ into a cocommutative bialgebra isomorphic to $U(\mathcal{L})$.*

Proof. First we show Δ is well defined. Changing our point of view, omit Ψ^{-1} from (28) so that Δ maps $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) \to \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}})$ and is well defined, and regard Ψ in (33) as an *identification*, so that $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) \otimes \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ becomes a *vector subspace* of $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}})$. Then we must prove that

$$\Delta(\mathcal{H}_{\mathcal{L}}) \subseteq \mathcal{H}_{\mathcal{L}} \otimes \mathcal{H}_{\mathcal{L}} \subseteq \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) \otimes \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) \subseteq \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}}). \tag{34}$$

The proof of Theorem 4.17 still shows that Δ is *multiplicative* with respect to the natural product on $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}})$, which is essentially the right-hand column of (29). Furthermore, the subspaces $\mathcal{H}_{\mathcal{L}} \otimes \mathcal{H}_{\mathcal{L}}$ and $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) \otimes \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ are closed under this product, which equals $* \otimes *$ upon them. Let $f_1, \dots, f_n \in \mathcal{L}$. The proof of Lemma 4.18 shows that $\Delta f_i = f_i \otimes \delta_{[0]} + \delta_{[0]} \otimes f_i$ in $\mathcal{H}_{\mathcal{L}} \otimes \mathcal{H}_{\mathcal{L}}$, so by multiplicativity of Δ we have

$$\Delta(f_1 * \cdots * f_n) = (f_1 \otimes \delta_{[0]} + \delta_{[0]} \otimes f_1)(* \otimes *) \cdots (* \otimes *)(f_n \otimes \delta_{[0]} + \delta_{[0]} \otimes f_n).$$

Thus $\Delta(f_1 * \cdots * f_n)$ lies in $\mathcal{H}_{\mathcal{L}} \otimes \mathcal{H}_{\mathcal{L}}$. As $\mathcal{H}_{\mathcal{L}}$ is spanned by such $f_1 * \cdots * f_n$ and $\delta_{[0]}$, and $\Delta\delta_{[0]} = \delta_{[0]} \otimes \delta_{[0]} \in \mathcal{H}_{\mathcal{L}} \otimes \mathcal{H}_{\mathcal{L}}$ as in Theorem 4.17, we have proved (34), and $\Delta : \mathcal{H}_{\mathcal{L}} \to \mathcal{H}_{\mathcal{L}} \otimes \mathcal{H}_{\mathcal{L}}$ is well defined. The proof of Theorem 4.17 now shows $\mathcal{H}_{\mathcal{L}}$ is a cocommutative bialgebra. Finally, Proposition 4.14 shows that Φ in (22) restricts to an injective morphism $U(\mathcal{L}) \to \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$, which has image $\mathcal{H}_{\mathcal{L}}$. Hence $\mathcal{H}_{\mathcal{L}} \cong U(\mathcal{L})$ as an algebra, and the isomorphism of bialgebras follows as in Corollary 4.19. □

As in [9, §1.1.7], a *Hopf algebra* is a bialgebra A equipped with an *antipode* $S : A \to A$ satisfying certain conditions. If a bialgebra A admits an antipode S, then S is unique. Now for $\mathfrak{g}$ a Lie algebra, $U(\mathfrak{g})$ is actually a Hopf algebra [9, §1.2.6]. Therefore in the situations of Proposition 4.15 and Theorem 4.20, there must exist unique antipodes $S : \mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}) \to \mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$ and $S : \mathcal{H}_{\mathcal{L}} \to \mathcal{H}_{\mathcal{L}}$ making the bialgebras into Hopf algebras.

However, there does not appear to be a simple formula for S in terms of constructible functions. (The most obvious answer, that $(Sf)([X]) = (-1)^k f([X])$ if $X \in \mathcal{A}$ has k indecomposable factors, does not work.) So we shall not try to determine the antipodes S.

4.8. Other algebraic operations from finite posets

We define a family of *multilinear operations* $P_{(I,\preccurlyeq)}$ on $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$.

Definition 4.21. Let Assumption 3.4 hold and $(I, \preccurlyeq)$ be a finite poset. Using

$$\prod_{i\in I} \mathfrak{Obj}_{\mathcal{A}} \xleftarrow{\prod_{i\in I}\boldsymbol{\sigma}(\{i\})} \mathfrak{M}(I,\preccurlyeq)_{\mathcal{A}} \xrightarrow{\boldsymbol{\sigma}(I)} \mathfrak{Obj}_{\mathcal{A}},$$

define a multilinear operation $P_{(I,\preccurlyeq)} : \prod_{i\in I} \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) \to \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ by

$$P_{(I,\preccurlyeq)}(f_i : i \in I) = \mathrm{CF}^{\mathrm{stk}}\big(\boldsymbol{\sigma}(I)\big)\Big[\prod_{i\in I} \boldsymbol{\sigma}\big(\{i\}\big)^*(f_i)\Big],$$

which exists as $\boldsymbol{\sigma}(I)$ is representable and $\prod_{i\in I}\boldsymbol{\sigma}(\{i\})$ is of finite type by Theorem 3.6.

This generalizes Definition 4.1, as $P_{(\{1,2\},\leqslant)}(f_1, f_2) = f_1 * f_2$. In this notation, Theorem 4.3 shows $*$ is associative by proving that

$$\begin{aligned} P_{(\{1,2\},\leqslant)}\big(P_{(\{1,2\},\leqslant)}(f_1, f_2), f_3\big) &= P_{(\{1,2,3\},\leqslant)}(f_1, f_2, f_3) \\ &= P_{(\{1,2\},\leqslant)}\big(f_1, P_{(\{1,2\},\leqslant)}(f_2, f_3)\big). \end{aligned} \tag{35}$$

Here is a generalization, which shows that if we substitute one operation $P_{(J,\lesssim)}$ into another $P_{(K,\trianglelefteq)}$, we get a third $P_{(I,\preccurlyeq)}$. It is a constructible functions version of the notion [12, Definition 5.7] of *substitution* of configurations.

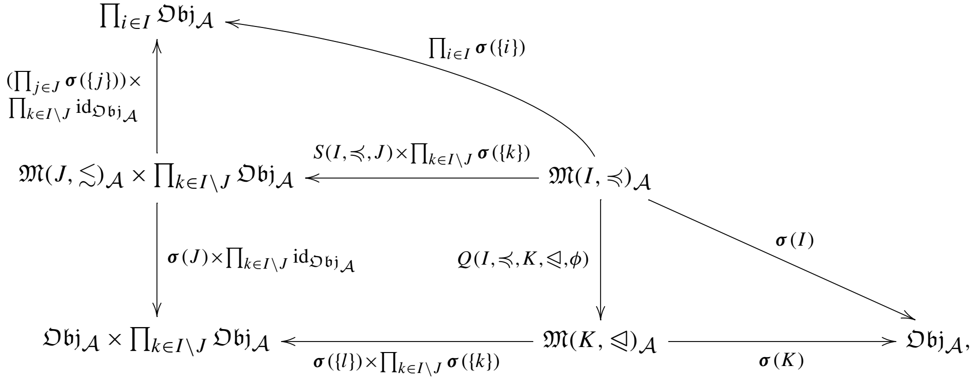
Theorem 4.22. *Let Assumption* 3.4 *hold,* $(J, \lesssim), (K, \trianglelefteq)$ *be nonempty finite posets with* $J \cap K = \emptyset$, *and* $l \in K$. *Set* $I = J \cup (K \setminus \{l\})$, *and define* $\precsim$ *on* I *by*

$$i \precsim j \quad \textit{for } i, j \in I \quad \textit{if} \quad \begin{cases} i \lesssim j, & i, j \in J, \\ i \trianglelefteq j, & i, j \in K \setminus \{l\}, \\ l \trianglelefteq j, & i \in J,\ j \in K \setminus \{l\}, \\ i \trianglelefteq l, & i \in K \setminus \{l\},\ j \in J. \end{cases}$$

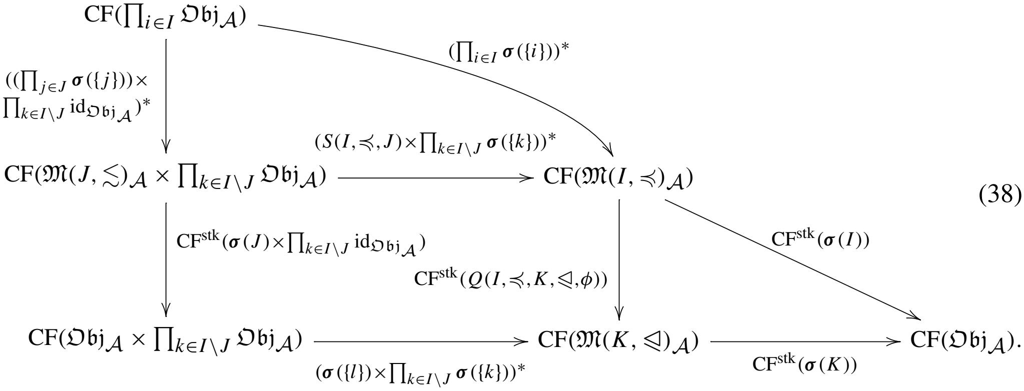
Let $f_j\colon j \in J$ *and* $g_k\colon k \in K \setminus \{l\}$ *lie in* $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$. *Then*

$$\begin{aligned} & P_{(I,\precsim)}\big(f_j\colon j \in J,\ g_k\colon k \in K \setminus \{l\}\big) \\ & \quad = P_{(K,\trianglelefteq)}\big(P_{(J,\lesssim)}(f_j\colon j \in J)_l,\ g_k\colon k \in K \setminus \{l\}\big). \end{aligned} \tag{36}$$

Proof. Define $\phi: I \to K$ by $\phi(i) = l$ if $i \in J$, and $\phi(i) = i$ if $i \in K \setminus \{l\}$. Consider the commutative diagram of 1-morphisms, and the corresponding diagram of pullbacks and pushforwards:



(37)



(38)

Calculation shows that (36) holds provided (38) commutes. The proof of this is similar to that for (15) in Theorem 4.3. By [12, Theorem 7.10]

$$\begin{array}{ccc} \mathfrak{M}(J,\lesssim)_{\mathcal{A}} & \xleftarrow[S(I,\preccurlyeq,J)]{} & \mathfrak{M}(I,\preccurlyeq)_{\mathcal{A}} \\ \Big\downarrow{\scriptstyle \sigma(J)} & & \Big\downarrow{\scriptstyle Q(I,\preccurlyeq,K,\trianglelefteq,\phi)} \\ \mathfrak{Obj}_{\mathcal{A}} & \xleftarrow{\sigma(\{l\})} & \mathfrak{M}(K,\trianglelefteq)_{\mathcal{A}} \end{array}$$

is a Cartesian square, so the square in (37) is Cartesian, and the square in (38) commutes by (4). We leave the details to the reader. □

Applying the theorem and induction shows that *any* multilinear operation on $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ obtained by combining operations $P_{(J,\lesssim)}$ is of the form $P_{(I,\preccurlyeq)}$ for some poset $(I,\preccurlyeq)$. For instance, the posets $(\{1,2\},\bullet)$, $(\{1,2,3\},\bullet)$ of Definition 4.16 and Theorem 4.22 give an analogue of (35):

$$\begin{aligned} P_{(\{1,2\},\bullet)}\big(P_{(\{1,2\},\bullet)}(f_1,f_2),f_3\big) &= P_{(\{1,2,3\},\bullet)}(f_1,f_2,f_3) \\ &= P_{(\{1,2\},\bullet)}\big(f_1,P_{(\{1,2\},\bullet)}(f_2,f_3)\big). \end{aligned}$$

This shows that $P_{(\{1,2\},\bullet)}$ gives an *associative, commutative multiplication* on $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$, different from $*$.

Often the operations $P_{(I,\preccurlyeq)}$ map $\prod_{i\in I}\mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})\to\mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}})$, which holds for $(I,\bullet)$, for example. Now Theorem 4.22 implies that Ringel–Hall algebras $\mathcal{H}$ admit many *extra algebraic operations* $P_{(I,\preccurlyeq)}$, which generalize multiplication $*$, and satisfy many compatibilities. It is an interesting question whether these operations may be useful tools in studying algebras which occur as Ringel–Hall algebras, such as certain $U(\mathfrak{g})$. See Remark 4.24 below on this.

Combining the ideas of Definitions 4.4 and 4.21 we may also define $P_{(I,\preccurlyeq)}:\prod_{i\in I}\mathrm{L\dot{C}F}(\mathfrak{Obj}_{\mathcal{A}})\to\mathrm{L\dot{C}F}(\mathfrak{Obj}_{\mathcal{A}})$, satisfying the analogue of Theorem 4.22.

4.9. Examples from quivers

Let Γ be a Dynkin diagram which is a disjoint union of diagrams of type A, D or E, and $\mathfrak{g}$ be the corresponding a finite-dimensional semisimple Lie algebra over $\mathbb{Q}$. Then we have a decomposition $\mathfrak{g}=\mathfrak{n}_+\oplus\mathfrak{h}\oplus\mathfrak{n}_-$, where $\mathfrak{h}$ is a Cartan subalgebra, and the nilpotent Lie subalgebra $\mathfrak{n}_+$ is a direct sum of one-dimensional subspaces indexed by the set Φ_+ of *positive roots* α of $\mathfrak{g}$.

Gabriel showed that if $Q=(Q_0,Q_1,b,e)$ is a quiver with underlying graph Γ, then isomorphism classes $[V]$ of indecomposable representations V of Q are in 1–1 correspondence with $\alpha\in\Phi_+$. Later, Ringel [23] used Ringel–Hall algebras over finite fields to recover the Lie bracket on $\mathfrak{n}_+$ on the vector space spanned by such $[V]$. Here is a constructible functions version of this, adapted from Riedtmann [21] and Frenkel et al. [5, §4].

Example 4.23. Let $\mathfrak{g}=\mathfrak{n}_+\oplus\mathfrak{h}\oplus\mathfrak{n}_-$, Γ and Q be as above. Set $\mathcal{A}=\text{mod-}\mathbb{K}Q$, the abelian category of representations of Q over $\mathbb{K}$, and define $K(\mathcal{A})=\mathbb{Z}^{Q_0}$, $\mathfrak{F}_{\mathcal{A}}$ as in [12, Example 10.5]. Then Assumption 3.4 holds by [12, §10]. Also Gabriel's result implies *all constructible sets in* $\mathfrak{Obj}_{\mathcal{A}}$ *are finite*, so $\mathrm{CF}(\mathfrak{Obj}_{\text{mod-}\mathbb{K}Q})=\mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\text{mod-}\mathbb{K}Q})$ and Assumption 4.11 holds automatically.

Now Riedtmann's Lie algebra $L(\mathbb{C}\vec{Q})\otimes_{\mathbb{Z}}\mathbb{Q}$ of [21, §2] coincides exactly with the Lie algebra $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\text{mod-}\mathbb{K}Q})=\mathrm{CF}^{\mathrm{ind}}_{\mathrm{fin}}(\mathfrak{Obj}_{\text{mod-}\mathbb{K}Q})$ defined in Section 4.4 above when $\mathbb{K}=\mathbb{C}$. So

[21, Proposition, p. 542] gives an isomorphism of Lie algebras

$$\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\text{mod-}\mathbb{K}Q}) = \mathrm{CF}^{\mathrm{ind}}_{\mathrm{fin}}(\mathfrak{Obj}_{\text{mod-}\mathbb{K}Q}) \cong \mathfrak{n}_+.$$

Proposition 4.15 and Corollary 4.19 thus yield an isomorphism of bialgebras

$$\mathrm{CF}(\mathfrak{Obj}_{\text{mod-}\mathbb{K}Q}) = \mathrm{CF}_{\mathrm{fin}}(\mathfrak{Obj}_{\text{mod-}\mathbb{K}Q}) \cong U(\mathfrak{n}_+). \tag{39}$$

Remark 4.24. We defined $P_{(I,\preccurlyeq)} : \prod_{i\in I} \mathrm{CF}(\mathfrak{Obj}_{\text{mod-}\mathbb{K}Q}) \to \mathrm{CF}(\mathfrak{Obj}_{\text{mod-}\mathbb{K}Q})$ for each finite poset $(I,\preccurlyeq)$ in Section 4.8, which by (39) yield operations $P_{(I,\preccurlyeq)}$ on $U(\mathfrak{n}_+)$ generalizing multiplication, and satisfying various compatibilities. The author does not know if these $P_{(I,\preccurlyeq)}$ on $U(\mathfrak{n}_+)$ are good for anything.

However, we can say one thing: in general they depend on the choice of *orientation* on Γ used to make Q. For example, if Γ is A_2: $\overset{i}{\bullet} - \overset{j}{\bullet}$, then calculation shows that the two possible choices $\overset{i}{\bullet} \to \overset{j}{\bullet}$ and $\overset{i}{\bullet} \leftarrow \overset{j}{\bullet}$ for Q give different answers for $P_{(\{1,2\},\bullet)}$ on $U(\mathfrak{n}_+)$. So the $P_{(I,\preccurlyeq)}$ on $U(\mathfrak{n}_+)$ do not seem to be canonical.

We generalize Example 4.23 to *affine Lie algebras* and *Kac–Moody Lie algebras*, based on Lusztig [19, §12] and Frenkel et al. [5, §5.6].

Example 4.25. Let Γ be a finite undirected graph with vertex set Q_0 and no edge joining a vertex with itself, and let $\mathfrak{g}$ be the corresponding *Kac–Moody algebra* over $\mathbb{Q}$, as in Kac [15]. Then $\mathfrak{g}$ is a $\mathbb{Q}$-Lie algebra with generators e_i, f_i, h_i for $i \in Q_0$, which satisfy certain relations. It has a decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$, where $\mathfrak{n}_+$, $\mathfrak{n}_-$, $\mathfrak{h}$ are the Lie subalgebras of $\mathfrak{g}$ generated by the e_i, the f_i and the h_i respectively, and $\mathfrak{h}$ is abelian, the Cartan subalgebra.

Let Q be a quiver with underlying graph Γ, and *without oriented cycles*. Take $\mathcal{A} = \text{mod-}\mathbb{K}Q$, and define $K(\mathcal{A}) = \mathbb{Z}^{Q_0}, \mathfrak{F}_{\mathcal{A}}$ as in [12, Example 10.5]. Then Assumption 3.4 holds by [12, §10], but in general Assumption 4.11 does not. There is up to isomorphism one simple object $V_i \in \text{mod-}\mathbb{K}Q$ for each $i \in Q_0$. Write $\mathcal{L}$ for the Lie subalgebra of $\mathrm{CF}(\mathfrak{Obj}_{\text{mod-}\mathbb{K}Q})$ generated by the isomorphism classes of simples $\delta_{[V_i]}$ for $i \in Q_0$. Then the subalgebra of $\mathrm{CF}(\mathfrak{Obj}_{\text{mod-}\mathbb{K}Q})$ generated by $\mathcal{L}$ is the composition algebra $\mathcal{C}$, and there is an isomorphism of bialgebras $U(\mathcal{L}) \cong \mathcal{C}$ by Theorem 4.20.

There are now natural identifications between the algebras $\mathrm{CF}(\mathfrak{Obj}_{\text{mod-}\mathbb{K}Q})$ above and $\mathcal{M}(\Omega)$ in Lusztig [19, §10.19], and between their complexification and $L(Q)$ in Frenkel et al. [5, §2.2]; and between the algebras $\mathcal{C}$ above and $\mathcal{M}_0(\Omega)$ in Lusztig [19, §10.19]; and between the Lie algebras $\mathcal{L} \otimes_{\mathbb{Q}} \mathbb{C}$ above and $\mathfrak{n}^*(Q)$ in Frenkel et al. [5, §2.2.5]; *except that* Lusztig uses the opposite order of multiplication to us, as in Remark 4.2.

From Lusztig [19, Proposition 10.20] (which he attributes to Schofield), there is a unique algebra isomorphism $\mathcal{C} \cong U(\mathfrak{n}_+)$ identifying the generators $\delta_{[V_i]}$ of $\mathcal{C}$ with the generators e_i of $\mathfrak{n}_+$ for $i \in Q_0$. This implies an isomorphism of the Lie subalgebras $\mathcal{L}$ and $\mathfrak{n}_+$ of $\mathcal{C}$ and $U(\mathfrak{n}_+)$.

When Γ is an *affine Dynkin diagram*, this isomorphism $\mathcal{L} \cong \mathfrak{n}_+$ also follows from Frenkel et al. [5, Corollary 5.6.27]. They also describe [5, §5.6] the *isomorphism classes of indecomposables* in $\text{mod-}\mathbb{K}Q$, and [5, Corollary 5.6.30] they define $\mathcal{L}$ explicitly as a subspace of $\mathrm{CF}^{\mathrm{ind}}_{\mathrm{fin}}(\mathfrak{Obj}_{\text{mod-}\mathbb{K}Q})$.

We return to these examples in Section 5.4.

5. Stack algebras

We now develop analogues of the Ringel–Hall algebras $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ of Section 4 in which we replace constructible functions by *stack functions*, as in Sections 2.3 and 2.5. We call the corresponding algebras *stack algebras*. Throughout this section $\mathbb{K}$ is an algebraically closed field of *arbitrary characteristic*, except when we specify characteristic zero for results comparing stack and constructible functions.

5.1. Different kinds of stack algebras

Following Definition 4.1, we define a *multiplication* $*$ on $\underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}})$.

Definition 5.1. Suppose Assumption 3.4 holds. Using the 1-morphism diagram

$$\mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}} \xleftarrow{\boldsymbol{\sigma}(\{1\})\times\boldsymbol{\sigma}(\{2\})} \mathfrak{M}\big(\{1,2\},\leqslant\big)_{\mathcal{A}} \xrightarrow{\boldsymbol{\sigma}(\{1,2\})} \mathfrak{Obj}_{\mathcal{A}},$$

define a bilinear operation $*:\underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}) \times \underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}) \to \underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}})$ by

$$f * g = \boldsymbol{\sigma}\big(\{1,2\}\big)_*\big[\big(\boldsymbol{\sigma}\big(\{1\}\big) \times \boldsymbol{\sigma}\big(\{2\}\big)\big)^*(f \otimes g)\big], \tag{40}$$

using operations $\phi_*, \phi^*, \otimes$ from Section 2.3. Here $(\boldsymbol{\sigma}(\{1\}) \times \boldsymbol{\sigma}(\{2\}))^*$ is well defined as $\boldsymbol{\sigma}(\{1\}) \times \boldsymbol{\sigma}(\{2\})$ is of finite type by Theorem 3.6(c). As $\boldsymbol{\sigma}(\{1,2\})$ is representable by Theorem 3.6(b), the restriction maps $*:\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}}) \times \mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}}) \to \mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$, that is, $\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$ is closed under $*$.

If Assumption 2.11 holds, (40) defines multiplications $*$ on $\underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$, $\bar{\underline{\mathrm{SF}}}$, $\bar{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$, $\bar{\underline{\mathrm{SF}}}$, $\bar{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda^\circ)$ and $\bar{\underline{\mathrm{SF}}}$, $\bar{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Theta, \Omega)$. Write $\bar\delta_{[0]} \in \mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}}), \ldots, \bar{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Theta, \Omega)$ for $\bar\delta_C$ in Definition 2.7 with $C = \{[0]\}$. Then $\bar\delta_{[0]} = [(\operatorname{Spec}\mathbb{K}, 0)]$, where $0:\operatorname{Spec}\mathbb{K} \to \mathfrak{Obj}_{\mathcal{A}}$ corresponds to $0 \in \mathcal{A}$.

Here is the analogue of Theorem 4.3.

Theorem 5.2. *If Assumptions* 2.11, 3.4 *hold* $\bar{\underline{\mathrm{SF}}}, \mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}}), \underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda), \ldots, \bar{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Theta, \Omega)$ *are algebras with identity* $\bar\delta_{[0]}$ *and multiplication* $*$. *Also*

$$\pi^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}}:\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}}) \to \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}), \qquad \bar\pi^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}}:\bar{\mathrm{SF}}\big(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda^\circ\big) \to \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}), \quad \textit{and}$$
$$\bar\pi^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}}:\bar{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Theta, \Omega) \to \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) \tag{41}$$

are morphisms of $\mathbb{Q}$*-algebras when* $\operatorname{char}\mathbb{K} = 0$ *and* $\pi^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}}, \bar\pi^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}}$ *are defined.*

Proof. To show $*$ is *associative* on $\underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}})$ we follow the proof of Theorem 4.3 replacing $\mathrm{CF}(\cdots)$ by $\underline{\mathrm{SF}}(\cdots)$, using Theorem 2.9 to show the analogue of (15) commutes. To show $\bar\delta_{[0]}$ is an identity on $\underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}})$ we can copy Theorem 4.3, or note that $\bar\delta_{[0]} = [(\operatorname{Spec}\mathbb{K}, 0)]$ and show directly that $[(\operatorname{Spec}\mathbb{K}, 0)] * [(\mathfrak{R}, \rho)] = [(\mathfrak{R}, \rho)]$ by constructing an explicit 1-isomorphism

$$i:\mathfrak{R} \to (\operatorname{Spec}\mathbb{K} \times \mathfrak{R})_{0\times\rho, \mathfrak{Obj}_{\mathcal{A}}\times\mathfrak{Obj}_{\mathcal{A}}, \boldsymbol{\sigma}(\{1\})\times\boldsymbol{\sigma}(\{2\})} \mathfrak{M}\big(\{1,2\},\leqslant\big)_{\mathcal{A}}$$

with $\boldsymbol{\sigma}(\{1,2\}) \circ \pi_{\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}}} \circ i$ 2-isomorphic to ρ. Thus $\underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}})$ is a $\mathbb{Q}$-algebra. Also $\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$ is a $\mathbb{Q}$-subalgebra as it is closed under $*$. The arguments for $\underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda), \ldots, \bar{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Theta, \Omega)$ are the same.

To prove (41) are $\mathbb{Q}$-algebra morphisms, note that $\bar{\delta}_{[0]} = \iota_{\mathfrak{Obj}_{\mathcal{A}}}(\delta_{[0]})$ and $\pi^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}} \circ \iota_{\mathfrak{Obj}_{\mathcal{A}}}$ is the identity by Definition 2.7, so

$$\pi^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}}(\bar{\delta}_{[0]}) = \delta_{[0]}.$$

For the $\pi^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}}$ case,

$$\pi^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}}(f * g) = \pi^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}}(f) * \pi^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}}(g)$$

follows from Eqs. (13), (40) and Theorem 2.10(b),(c). The $\bar{\pi}^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}}$ cases are the same. □

It is easy to show by example that $\iota_{\mathfrak{Obj}_{\mathcal{A}}} : \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) \to \mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$ is *not* in general an algebra morphism, as the $\iota_{\mathfrak{F}}$ do not commute with pushforwards. We will need the analogues of the $P_{(I,\preccurlyeq)}$ of Section 4.8 in Section 5.2, so we define them now.

Definition 5.3. Let Assumptions 2.11 and 3.4 hold and $(I, \preccurlyeq)$ be a finite poset. Define a multilinear operation $P_{(I,\preccurlyeq)} : \prod_{i\in I} \underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}) \to \underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}})$ on $\underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}})$ (or on $\underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$, $\bar{\underline{\mathrm{SF}}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$, $\bar{\underline{\mathrm{SF}}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda^{\circ})$, or $\bar{\underline{\mathrm{SF}}}(\mathfrak{Obj}_{\mathcal{A}}, \Theta, \Omega)$) by

$$P_{(I,\preccurlyeq)}(f_i\colon i \in I) = \boldsymbol{\sigma}(I)_* \left[\left(\prod_{i\in I} \boldsymbol{\sigma}(\{i\}) \right)^* \left(\bigotimes_{i\in I} f_i \right) \right], \tag{42}$$

using 1-morphisms from $\mathfrak{M}(I, \preccurlyeq)_{\mathcal{A}}$. Since $\boldsymbol{\sigma}(I)$ is representable, $\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$ and $\bar{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, *, *)$ are closed under $P_{(I,\preccurlyeq)}$. Also $P_{(\{1,2\},\leqslant)}$ is $*$ in Definition 5.1.

Here is the analogue of Theorem 4.22, proved as for Theorem 5.2.

Theorem 5.4. *Let Assumptions* 2.11 *and* 3.4 *hold and* $(J, \lesssim)$, $(K, \trianglelefteq)$, l *and* $(I, \preccurlyeq)$ *be as in Theorem* 4.22. *Let* $f_j\colon j \in J$ *and* $g_k\colon k \in K \setminus \{l\}$ *lie in* $\underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}})$ *or* $\underline{\mathrm{SF}}$, $\bar{\underline{\mathrm{SF}}}(\mathfrak{Obj}_{\mathcal{A}}, *, *)$. *Then* (36) *holds in the same space.*

Now apart from $\underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$ these algebras are all inconveniently large for our later work, and we will find it useful to define subalgebras $\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$, $\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, *, *)$ using the algebra structure on stabilizer groups in $\mathfrak{Obj}_{\mathcal{A}}$.

Definition 5.5. Suppose Assumptions 2.11 and 3.4 hold, and $[(\mathfrak{R}, \rho)]$ be a generator of $\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$. Let $r \in \mathfrak{R}(\mathbb{K})$ with $\rho_*(r) = [X] \in \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$, for $X \in \mathcal{A}$. Then ρ induces a morphism of stabilizer $\mathbb{K}$-groups $\rho_* : \mathrm{Iso}_{\mathbb{K}}(r) \to \mathrm{Iso}_{\mathbb{K}}([X]) \cong \mathrm{Aut}(X)$. As ρ is representable this is *injective*, and induces an isomorphism of $\mathrm{Iso}_{\mathbb{K}}(r)$ with a $\mathbb{K}$-subgroup of $\mathrm{Aut}(X)$.

Now $\mathrm{Aut}(X) = \mathrm{End}(X)^{\times}$ is the $\mathbb{K}$-group of invertible elements in a *finite-dimensional* $\mathbb{K}$-*algebra* $\mathrm{End}(X) = \mathrm{Hom}(X, X)$. We say the $[(\mathfrak{R}, \rho)]$ *has algebra stabilizers* if whenever $r \in \mathfrak{R}(\mathbb{K})$ with $\rho_*(r) = [X]$, the $\mathbb{K}$-subgroup $\rho_*(\mathrm{Iso}_{\mathbb{K}}(r))$ in $\mathrm{Aut}(X)$ is the $\mathbb{K}$-group $A^{\times}$ of invertible elements in a $\mathbb{K}$-subalgebra A in $\mathrm{End}(X)$. (Here when $X \cong 0$ we allow $\mathrm{End}(X) = \{0\}$ as a $\mathbb{K}$-algebra.)

Write $\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$, $\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$, $\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda^\circ)$, $\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, \Theta, \Omega)$ for the subspaces of $\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$, $\bar{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$, $\bar{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda^\circ)$, $\bar{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Theta, \Omega)$ respectively spanned over $\mathbb{Q}$, Λ, Λ° or Ω by $[(\mathfrak{R}, \rho)]$ with algebra stabilizers.

Remark 5.6. Definition 5.5 is an approximation to a more natural definition, which uses a more sophisticated kind of stack. For simplicity, and to be able to use [10,11], we have been working wherever possible with *stacks in groupoids*. But there exist natural definitions of (*Artin*) $\mathbb{K}$-*stacks in exact categories* and *in linear categories*, and $\mathfrak{Obj}_{\mathcal{A}}$ can be regarded as both. It would be more natural to define $\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ using generators $[(\mathfrak{R}, \rho)]$ where $\mathfrak{R}$ is an Artin $\mathbb{K}$-stack in exact (or linear) categories, and $\rho : \mathfrak{R} \to \mathfrak{Obj}_{\mathcal{A}}$ a morphism of such stacks. Then the 'algebra stabilizers' condition above would be automatic.

One of the relations [11, Definition 5.17(iii)] defining $\bar{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, *, *)$ mixes $[(\mathfrak{R}, \rho)]$ with and without algebra stabilizers, so $\bar{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, *, *)$ can contain expressions $\sum_i c_i [(\mathfrak{R}_i, \rho_i)]$ in which $[(\mathfrak{R}_i, \rho_i)]$ does not have algebra stabilizers. Propositions 5.8 and 5.9 below are tools for getting round the problems this causes. Also, $\iota_{\mathfrak{Obj}_{\mathcal{A}}}$ in Definition 2.7 maps $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) \to \mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$, as it involves $[(\mathfrak{R}, \rho)]$ with ρ an inclusion of substacks, so the condition of Definition 5.5 holds with $A = \mathrm{End}(X)$.

Proposition 5.7. $\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ *and* $\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, *, *)$ *are closed under* $*$ *and contain* $\bar{\delta}_{[0]}$, *and so are subalgebras. They are also closed under the* $P_{(I,\preccurlyeq)}$.

Proof. When $[(\sigma, \iota, \pi)] \in \mathcal{M}(\{1,2\}, \leqslant)_{\mathcal{A}}$ the stabilizer group maps

$$\boldsymbol{\sigma}\big(\{1,2\}\big)_* : \mathrm{Aut}\big((\sigma, \iota, \pi)\big) \to \mathrm{Aut}\big(\sigma\big(\{1,2\}\big)\big),$$
$$\big(\boldsymbol{\sigma}\big(\{1\}\big) \times \boldsymbol{\sigma}\big(\{2\}\big)\big)_* : \mathrm{Aut}\big((\sigma, \iota, \pi)\big) \to \mathrm{Aut}\big(\sigma\big(\{1\}\big)\big) \times \mathrm{Aut}\big(\sigma\big(\{2\}\big)\big)$$

are the restrictions to $\mathbb{K}$-groups of invertible elements of $\mathbb{K}$-algebra morphisms $\mathrm{End}((\sigma, \iota, \pi)) \to \mathrm{End}(\sigma(\{1,2\}))$, $\mathrm{End}((\sigma, \iota, \pi)) \to \mathrm{End}(\sigma(\{1\})) \times \mathrm{End}(\sigma(\{2\}))$. Using this and fibre products of $\mathbb{K}$-algebras, we find $\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ is closed under $*$. Replacing $(\{1,2\}, \leqslant)$ by $(I, \preccurlyeq)$ we also see $\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ is closed under $P_{(I,\preccurlyeq)}$. It contains $\bar{\delta}_{[0]}$ as $\bar{\delta}_{[0]} = \iota_{\mathfrak{Obj}_{\mathcal{A}}}(\delta_{[0]})$ and $\iota_{\mathfrak{Obj}_{\mathcal{A}}}$ maps to $\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$. The arguments for $\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, *, *)$ are the same. □

Now much of [11] involves taking $[(\mathfrak{R}, \rho)]$ with $\mathfrak{R} \cong [V/G]$ and then making linear combinations of $[([W/H], \rho \circ \iota^{W,H})]$ for certain $\mathbb{K}$-subgroups $H \subseteq G$ and H-invariant $\mathbb{K}$-subvarieties $W \subseteq V$. If $[(\mathfrak{R}, \rho)]$ has algebra stabilizers, for a general 1-isomorphism $\mathfrak{R} \cong [V/G]$ these $[([W/H], \rho \circ \iota^{W,H})]$ may *not* have algebra stabilizers, which is why [11, Definition 5.17(iii)] mixes $[(\mathfrak{R}, \rho)]$ with and without algebra stabilizers, as we said above.

The next two propositions construct special 1-isomorphisms $\mathfrak{R} \cong [V/A^\times]$ such that the $[([W/H], \rho \circ \iota^{W,H})]$ automatically have algebra stabilizers. Using these we can do the operations of [11] within $\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ and $\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, *, *)$.

Proposition 5.8. *Let* $S \subset \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ *be constructible. Then there exists a finite decomposition*

$$S = \coprod_{l \in L} \mathfrak{F}_l(\mathbb{K}),$$

where $\mathfrak{F}_l$ is a finite type $\mathbb{K}$-substack of $\mathfrak{Obj}_{\mathcal{A}}$, and 1-isomorphisms

$$\mathfrak{F}_l \cong [U_l / A_l^\times],$$

for A_l a finite-dimensional $\mathbb{K}$-algebra and U_l a quasiprojective $\mathbb{K}$-variety acted on by $A_l^\times$, such that if $u \in U_l(\mathbb{K})$ projects to $[X] \in \mathfrak{F}_l(\mathbb{K}) \subset \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ then there exists a subalgebra B_u of A_l with $\mathrm{Stab}_{A_l^\times}(u) = B_u^\times$ and an isomorphism $B_u \cong \mathrm{End}(X)$ compatible with the isomorphisms $\mathrm{Stab}_{A_l^\times}(u) \cong \mathrm{Iso}_{\mathbb{K}}([X]) \cong \mathrm{Aut}(X)$.

Proof. A result of Kresch [17, Proposition 3.5.9] used in [10,11] shows an algebraic $\mathbb{K}$-stack can be *stratified by global quotient stacks* if and only if it has *affine geometric stabilizers*. Applied to $S \subset \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$, this yields a finite decomposition $S = \coprod_{l \in L} \mathfrak{F}_l(\mathbb{K})$ for $\mathbb{K}$-substacks $\mathfrak{F}_l$ of $\mathfrak{Obj}_{\mathcal{A}}$, with $\mathfrak{F}_l \cong [U_l / \mathrm{GL}(m_l, \mathbb{K})]$ for U_l a quasiprojective $\mathbb{K}$-variety acted on by $\mathrm{GL}(m_l, \mathbb{K})$.

Here is how Kresch's proof works. He shows that S may be decomposed as above into disjoint, reduced $\mathbb{K}$-substacks $\mathfrak{F}_l$, such that $\mathfrak{F}_l$ has flat stabilizer and is a gerbe over a reduced $\mathbb{K}$-scheme S_l, with a finite flat morphism $T_l \to S_l$ from another reduced $\mathbb{K}$-scheme T_l, such that $\mathfrak{F}_l \times_{S_l} T_l \cong B(G_l \to T_l)$ for some flat $\mathbb{K}$-group scheme $\rho_l : G_l \to T_l$. Using affine geometric stabilizers we can take ρ_l to have a faithful linear representation upon a vector bundle $V_l \to B(G_l \to T_l) \cong \mathfrak{F}_l \times_{S_l} T_l$. Let the vector bundle $W_l \to \mathfrak{F}_l$ be the pushforward of V_l along the finite flat morphism $T_l \to S_l$, and let W_l have fibre $\mathbb{K}^{m_l}$. Let U_l be the principal bundle (frame bundle) of W_l. Then shrinking $\mathfrak{F}_l$ if necessary, U_l is a quasiprojective $\mathbb{K}$-variety acted on by $\mathrm{GL}(m_l, \mathbb{K})$, and $\mathfrak{F}_l \cong [U_l / \mathrm{GL}(m_l, \mathbb{K})]$.

In our situation we can say more than this. We generally consider $\mathfrak{Obj}_{\mathcal{A}}$ just as an Artin $\mathbb{K}$-stack in groupoids, but as in Remark 5.6 this is actually the substack of isomorphisms of a $\mathbb{K}$-*stack in exact categories*, and so of a $\mathbb{K}$-*stack in linear categories*, which are *Artin* stacks in exact and linear categories in a well-defined sense. So the stabilizers of $\mathfrak{Obj}_{\mathcal{A}}$, which are affine $\mathbb{K}$-groups, are also the groups of invertible elements of $\mathbb{K}$-algebras. In the argument above this means we can naturally take the group scheme $\rho_l : G_l \to T_l$ to be the invertible elements in an *algebra scheme* $\sigma_l : E_l \to T_l$, in a way compatible with the inclusions $\mathrm{Aut}(X) \subset \mathrm{End}(X)$ of $\mathbb{K}$-groups in $\mathbb{K}$-algebras for $[X] \in \mathfrak{F}_l(\mathbb{K})$.

Take the vector bundle $V_l \to B(G_l \to T_l)$ above to be the restriction to G_l of a faithful representation of the algebra scheme $E_l \to T_l$, which we are entitled to do. We now claim that the 1-isomorphisms $\mathfrak{F}_l \cong [U_l / \mathrm{GL}(m_l, \mathbb{K})]$ constructed above satisfy the conditions of the proposition, with $A_l = \mathrm{End}(\mathbb{K}^{m_l})$ and $A_l^\times = \mathrm{GL}(m_l, \mathbb{K})$. To see this, let $[X] \in \mathfrak{F}_l(\mathbb{K})$. Write $[Y] \in S_l(\mathbb{K})$ for the image of $[X]$, and $[Z_1], \ldots, [Z_k] \in T_l(\mathbb{K})$ for the preimages of $[Y]$ under the finite flat morphism $T_l \to S_l$. Then $Z_i : \mathrm{Spec}\,\mathbb{K} \to T_l$ are 1-morphisms, so $G_l|_{Z_i} = G_l \times_{\rho_l, T_l, Z_i} \mathrm{Spec}\,\mathbb{K}$ and $E_l|_{Z_i} = E_l \times_{\sigma_l, T_l, Z_i} \mathrm{Spec}\,\mathbb{K}$ are the fibres of G_l, E_l over $[Z_i]$, and $G_l|_{Z_i}$ is the $\mathbb{K}$-group of invertible elements in the finite-dimensional $\mathbb{K}$-algebra $E_l|_{Z_i}$.

The 1-isomorphism $\mathfrak{F}_l \times_{S_l} T_l \cong B(G_l \to T_l)$ and its algebra extension induce compatible isomorphisms $G_l|_{Z_i} \cong \mathrm{Aut}(X)$ and $E_l|_{Z_i} \cong \mathrm{End}(X)$. We can also form the fibres $V_l|_{Z_i}$, which are finite-dimensional $\mathbb{K}$-vector spaces, and by construction $G_l|_{Z_i}$ has a faithful $\mathbb{K}$-group representation on $V_l|_{Z_i}$ which is the restriction of a faithful $\mathbb{K}$-algebra representation of $E_l|_{Z_i}$. The fibre $W_l|_X$ of W_l at X is $\bigoplus_{i=1}^k V_l|_{Z_i}$, which must be isomorphic to $\mathbb{K}^{m_l}$.

The fibre $U_l|_X = U_l \cong_{\pi_l, \mathfrak{F}_l, X} \mathrm{Spec}\,\mathbb{K}$ of $\pi_l : U_l \to \mathfrak{F}_l$ over $[X] \in \mathfrak{F}_l(\mathbb{K})$ is the bundle of frames in $\bigoplus_{i=1}^k V_l|_{Z_i}$ divided by $\mathrm{Aut}(X)$ acting on each $V_l|_{Z_i}$ via the isomorphism $G_l|_{Z_i} \cong \mathrm{Aut}(X)$. Thus, at each point $u \in (U_l|_X)(\mathbb{K})$, the stabilizer group $\mathrm{Stab}_{\mathrm{GL}(m_l, \mathbb{K})}(u)$ is a $\mathbb{K}$-subgroup of

$\mathrm{GL}(m_l,\mathbb{K})$ isomorphic to $\mathrm{Aut}(X)$, determined by a choice of isomorphism $\mathbb{K}^{m_l}\cong\bigoplus_{i=1}^k V_l|_{Z_i}$. This is exactly $B_u^\times$, where $B_u\subseteq\mathrm{End}(\mathbb{K}^{m_l})$ is the $\mathbb{K}$-subalgebra isomorphic to $\mathrm{End}(X)$ determined by the action of $\mathrm{End}(X)$ on $\bigoplus_{i=1}^k V_l|_{Z_i}$ via the isomorphisms $E_l|_{Z_i}\cong\mathrm{End}(X)$, and the choice of isomorphism $\mathbb{K}^{m_l}\cong\bigoplus_{i=1}^k V_l|_{Z_i}$. The proposition follows. □

Proposition 5.9. *Let* $[(\mathfrak{R},\rho)]\in\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$ *and use the notation of Proposition* 5.8 *with* $S=\rho_*(\mathfrak{R}(\mathbb{K}))$. *Then* $[(\mathfrak{R},\rho)]=\sum_{l\in L}[([Y_l/A_l^\times],\rho_l)]$ *where* Y_l *is a quasiprojective* $\mathbb{K}$*-variety and* $\rho_l:[Y_l/A_l^\times]\to\mathfrak{F}_l\subseteq\mathfrak{Obj}_{\mathcal{A}}$ *is induced by an* $A_l^\times$*-equivariant morphism* $\phi_l:Y_l\to U_l$, *under the* 1*-isomorphism* $\mathfrak{F}_l\cong[U_l/A_l^\times]$. *Moreover* $[(\mathfrak{R},\rho)]$ *has algebra stabilizers if and only if* $\mathrm{Stab}_{A_l^\times}(y)=C_y^\times$ *for some subalgebra* $C_y\subseteq A_l$, *for all* $l\in L$ *and* $y\in Y_l(\mathbb{K})$.

Proof. Let $\pi_l:U_l\to\mathfrak{F}_l\subset\mathfrak{Obj}_{\mathcal{A}}$ be the natural projection from $\mathfrak{F}_l\cong[U_l/A_l^\times]$. As $\mathfrak{R}$, U_l are finite type and ρ is representable, $Z_l=\mathfrak{R}\times_{\rho,\mathfrak{Obj}_{\mathcal{A}},\pi_l}U_l$ is a *finite type algebraic* $\mathbb{K}$*-space*, with an action of $A_l^\times$. Thus there exists a finite decomposition $Z_l=\coprod_{i\in I_l}Z_l^i$ for $A_l^\times$-invariant quasiprojective $\mathbb{K}$-subvarieties Z_l^i. Define Y_l to be the scheme-theoretic disjoint union of the Z_l^i for $i\in I_l$.

By our convention in Section 2.1 that $\mathbb{K}$-varieties need not be irreducible, nor connected, Y_l is a quasiprojective $\mathbb{K}$-variety, acted on by $A_l^\times$. Let $\pi_{Z_l}:Y_l\to Z_l$ be the obvious morphism and $\pi_{\mathfrak{R}}:Z_l\to\mathfrak{R}$, $\pi_{U_l}:Z_l\to U_l$ the projections from the fibre product. Then $\pi_{\mathfrak{R}}$ and $\pi_{\mathfrak{R}}\circ\pi_{Z_l}$ are $A_l^\times$-invariant and so push down to 1-morphisms $\pi'_{\mathfrak{R}}:[Z_l/A_l^\times]\to\mathfrak{R}$ and $\pi''_{\mathfrak{R}}:[Y_l/A_l^\times]\to\mathfrak{R}$. Define $\rho_l=\rho\circ\pi''_{\mathfrak{R}}$ and $\phi_l=\pi_{U_l}\circ\pi_{Z_l}$. It is now easy to see that

$$\big[(\mathfrak{R},\rho)\big]=\sum_{l\in L}\big[\big(\big[Z_l/A_l^\times\big],\rho\circ\pi'_{\mathfrak{R}}\big)\big]=\sum_{l\in L}\big[\big(\big[Y_l/A_l^\times\big],\rho_l\big)\big],$$

since $\pi'_{\mathfrak{R}}$ embeds $[Z_l/A_l^\times]$ as the $\mathbb{K}$-substack of $\mathfrak{R}$ over $\mathfrak{F}_l$, and $[Z_l/A_l^\times]$ and $[Y_l/A_l^\times]$ split into the same pieces $[Z_l^i/A_l^\times]$ for $i\in I_l$. The first part follows.

For the second part, if $r\in\mathfrak{R}(\mathbb{K})$ with $\rho_*(r)=[X]$ then $[X]\in\mathfrak{F}_l(\mathbb{K})$ for unique $l\in L$, and $r=(\pi_{\mathfrak{R}})_*(z)$ for $z\in Z_l(\mathbb{K})$. As $(\pi_{Z_l})_*:Y_l(\mathbb{K})\to Z_l(\mathbb{K})$ is a bijection $z=(\pi_{Z_l})_*(y)$ for unique $y\in Y_l(\mathbb{K})$. Let $u=(\phi_l)_*(y)\in U_l(\mathbb{K})$. Then $B_u^\times=\mathrm{Stab}_{A_l^\times}(u)\cong\mathrm{Aut}(X)$ as u projects to $[X]$, for some subalgebra $B_u\subseteq A_l$ with compatible isomorphism $B_u\cong\mathrm{End}(X)$. Now $\mathrm{Iso}_{\mathbb{K}}(r)\cong\mathrm{Stab}_{A_l^\times}(y)$, and $\rho_*:\mathrm{Iso}_{\mathbb{K}}(r)\to\mathrm{Iso}_{\mathbb{K}}([X])$ is just inclusion $\mathrm{Stab}_{A_l^\times}(y)\subseteq B_u^\times$ as $\mathbb{K}$-subgroups of $A_l^\times$. Thus, $[(\mathfrak{R},\rho)]$ has algebra stabilizers if and only if $\mathrm{Stab}_{A_l^\times}(y)=C_y^\times$ for some subalgebra $C_y\subseteq B_u$, for all l, u, y. But a subalgebra of B_u is the same as a subalgebra of A_l lying in B_u, and the proof is complete. □

Corollary 5.10. $\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ *and* $\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}},*,*)$ *are closed under the operators* Π^μ, Π_n^{vi}, $\hat{\Pi}_{\mathfrak{F}}^\nu$ *of* [11, §5.2].

Proof. Let $[(\mathfrak{R},\rho)]$ have algebra stabilizers, and use the notation of Proposition 5.9. Then

$$\Pi^\mu\big(\big[(\mathfrak{R},\rho)\big]\big)=\sum_{l\in L}\Pi^\mu\big(\big[\big(\big[Y_l/A_l^\times\big],\rho_l\big)\big]\big).$$

The definition [11, Definition 5.10] of $\Pi^\mu([([Y_l/A_l^\times],\rho_l)])$ gives a linear combination of $[([W_l/H_l],\rho_l\circ\iota^{W_l,H_l})]$ for certain $\mathbb{K}$-subgroups $H_l\subseteq A_l^\times$ and H_l-invariant $\mathbb{K}$-subvarieties

$W_l \subseteq Y_l$. We may take $A_l = \mathrm{End}(\mathbb{K}^{m_l})$, and then [11, Example 5.7] implies the H_l appearing in the sum are of the form $B_l^\times$ for subalgebras $B_l \subseteq A_l$. It easily follows that $[([W_l/H_l], \rho_l \circ \iota^{W_l,H_l})]$ has algebra stabilizers, which proves what we want. The Π_n^{vi}, $\hat{\Pi}_{\mathfrak{F}}^{\nu}$ cases are the same. □

Combining Proposition 5.9 with the proof of the first part of Proposition 2.17 we find $\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, *, *)$ are generated by $[(U \times [\mathrm{Spec}\,\mathbb{K}/T], \rho)]$ as in Proposition 2.17 with algebra stabilizers. But $T \cong (\mathbb{K}^\times)^k \times K$ for K finite abelian can only be of the form $B^\times$ for a $\mathbb{K}$-algebra B if K is trivial, giving:

Corollary 5.11. $\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$*,* $\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda^\circ)$ *and* $\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, \Theta, \Omega)$ *are generated over* Λ, Λ° *and* Ω *respectively by elements* $[(U \times [\mathrm{Spec}\,\mathbb{K}/T], \rho)]$ *with algebra stabilizers, for* U *a quasiprojective* $\mathbb{K}$*-variety and* $T \cong (\mathbb{K}^\times)^k$.

We define projections $\Pi_{[I,\kappa]}$ on $\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$, $\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, *, *)$.

Definition 5.12. Let Assumptions 2.11 and 3.4 hold. Write $C(\mathcal{A}) = \bar{C}(\mathcal{A}) \setminus \{0\}$, as in Definition 3.5. Consider pairs (I, κ) with I a finite set and $\kappa : I \to C(\mathcal{A})$ a map. Define an equivalence relation '$\approx$' on such (I, κ) by $(I, \kappa) \approx (I', \kappa')$ if there exists a bijection $i : I \to I'$ with $\kappa' \circ i = \kappa$. Write $[I, \kappa]$ for the $\approx$-equivalence class of (I, κ). For such an $[I, \kappa]$ we will define projections

$$\begin{aligned} \Pi_{[I,\kappa]} &: \mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}) \to \mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}), \\ \Pi_{[I,\kappa]} &: \bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, *, *) \to \bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, *, *), \end{aligned} \tag{43}$$

for $*, * = \Upsilon, \Lambda$ or Υ, Λ° or Θ, Ω, using the operators $\hat{\Pi}_{\mathfrak{F}}^{\nu}$ of [11, Definition 5.15]. Define

$$\begin{aligned} \nu : \big\{\big(T, [X], \phi\big)&:\ T \text{ a } \mathbb{K}\text{-group isomorphic to } \big(\mathbb{K}^\times\big)^k \times K,\ K \text{ finite abelian}, \\ &[X] \in \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K}),\ \phi : T \to \mathrm{Iso}_{\mathbb{K}}\big([X]\big) = \mathrm{Aut}(X) \text{ a } \mathbb{K}\text{-group morphism}\big\} \to \mathbb{Q} \end{aligned}$$

by $\nu(T, [X], \phi) = 1$ if $T \cong (\mathbb{K}^\times)^{|I|}$, ϕ is injective, and there exists a splitting $X \cong \bigoplus_{i \in I} X_i$ in $\mathcal{A}$ with $[X_i] = \kappa(i)$ for all $i \in I$, such that $\phi(T)$ is the $\mathbb{K}$-subgroup $\{\sum_{i \in I} \lambda_i \,\mathrm{id}_{X_i}\colon\ \lambda_i \in \mathbb{K}^\times\}$ in $\mathrm{Aut}(X)$, and $\nu(T, [X], \phi) = 0$ otherwise. This depends only on the equivalence class $[I, \kappa]$ of (I, κ). Note too that the $\mathbb{K}$-subgroup of $\mathrm{Aut}(X)$ above is $A^\times$, where A is the subalgebra $\{\sum_{i \in I} \lambda_i \,\mathrm{id}_{X_i}\colon\ \lambda_i \in \mathbb{K}\}$ in $\mathrm{End}(X)$. Then ν is an $\mathfrak{Obj}_{\mathcal{A}}$*-weight function* in the sense of [11, Definition 5.15]. Define projections $\Pi_{[I,\kappa]}$ on $\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ and $\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, *, *)$ to be the operators $\hat{\Pi}_{\mathfrak{Obj}_{\mathcal{A}}}^{\nu}$ of [11, Definition 5.15]. Corollary 5.10 implies these map as in (43).

Here is what this means. The $\Pi_{[I,\kappa]}$ are refinements of the Π_n^{vi} of [11, §5.2], which project to components with *virtual rank n*. Now if $[X] \in \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ then $\mathrm{Iso}_{\mathbb{K}}([X]) \cong \mathrm{Aut}(X)$. Maximal tori in $\mathrm{Aut}(X)$ are of the form $\{\sum_{i \in I} \lambda_i \,\mathrm{id}_{X_i}\colon\ \lambda_i \in \mathbb{K}^\times\}$, where $X = \bigoplus_{i \in I} X_i$ with $0 \ncong X_i$ *indecomposable*. Thus the rank of $\mathrm{Iso}_{\mathbb{K}}([X])$ is the *number of indecomposable factors of* X.

Now $[(\mathfrak{R}, \rho)] \in \mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ also has stabilizers of the form $A^\times$ for subalgebras $A \subseteq \mathrm{End}(X)$, so we can treat A as like $\mathrm{End}(Y)$ for some 'object' Y, and $\mathrm{rk}\, A^\times$ as the 'number of indecomposables' in Y. As the Π_n^{vi} project to components with 'virtual rank' n, and in $\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ we equate rank with number of indecomposables, so on $\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ we should think of Π_n^{vi} as projecting to stack functions *supported on objects with n virtual indecomposable factors*. That is, the idea of 'virtual rank' in $\mathrm{SF}(\mathfrak{F})$ translates to 'number of virtual indecomposables' in $\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$.

Each $X \in \mathcal{A}$ splits as $X = \bigoplus_{i \in I} X_i$ with $0 \ncong X_i$ indecomposable, uniquely up to bijective changes of indexing set I and isomorphisms of X_i. Defining $\kappa : I \to C(\mathcal{A})$ by $\kappa(i) = [X_i]$, we see that the equivalence class $[I, \kappa]$ depends only on $[X]$, and X has $|I|$ indecomposables. In the same way, the $\Pi_{[I,\kappa]}$ project to components whose virtual indecomposables are of equivalence class $[I, \kappa]$.

Proposition 5.13. *In the situation above* $\Pi^2_{[I,\kappa]} = \Pi_{[I,\kappa]}$, *and* $\Pi_{[I,\kappa]}\Pi_{[J,\lambda]} = 0$ *if* $[I, \kappa] \neq [J, \lambda]$. *Also, if* $f \in \mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ *or* $\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, *, *)$ *then*

$$f = \sum_{\text{eq. classes } [I,\kappa]} \Pi_{[I,\kappa]}(f) \quad \text{and} \quad \Pi^{\mathrm{vi}}_n(f) = \sum_{\text{eq. classes } [I,\kappa]:\ |I|=n} \Pi_{[I,\kappa]}(f), \tag{44}$$

where the sums make sense as only finitely many $\Pi_{[I,\kappa]}(f)$ *are nonzero.*

Proof. The analogue of [11, Theorem 5.12(c)] for the $\hat{\Pi}^{\nu}_{\mathfrak{F}}$ says that $\hat{\Pi}^{\nu_1}_{\mathfrak{F}} \circ \hat{\Pi}^{\nu_2}_{\mathfrak{F}} = \hat{\Pi}^{\nu_1\nu_2}_{\mathfrak{F}}$. If ν_1, ν_2 are the $\mathfrak{Obj}_{\mathcal{A}}$-weight functions defined in Definition 5.12 using $[I, \kappa]$ and $[J, \lambda]$ then $\nu_1\nu_2$ is ν_1 if $[I, \kappa] = [J, \lambda]$ and 0 otherwise, so the first part follows. For the second, let $[(\mathfrak{R}, \rho)]$ have algebra stabilizers, use the notation of Proposition 5.9, and define $\Pi_{[I,\kappa]}([([Y_l/A^{\times}_l], \rho_l)])$ using [11, Definition 5.15]. This gives a finite sum over subtori P, Q, R in $A^{\times}_l$ with $(\mathbb{K}^{\times})^{|I|} \cong R \subseteq P \cap Q$ of a term involving the subset $(Y_l)^{P,R}_{\nu,1}$ of $y \in Y^P_l(\mathbb{K})$ such that R induces a decomposition of type $[I, \kappa]$ of the image point in $\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$.

Each $y \in Y^P_l(\mathbb{K})$ induces such a decomposition for a unique $[I, \kappa]$, and the map $y \mapsto [I, \kappa]$ is constructible, and so realizes only finitely many $[I, \kappa]$ on $Y^P_l(\mathbb{K})$. Hence

$$\Pi_{[I,\kappa]}\big(\big[\big(\big[Y_l/A^{\times}_l\big], \rho_l\big)\big]\big) \neq 0$$

for only finitely many $[I, \kappa]$, proving the last line. Summing over all $[I, \kappa]$ yields a sum over P, Q, R of a term involving the whole of $Y^P_l(\mathbb{K})$. Comparing with [11, Definition 5.10] and using [11, Theorem 5.12(a)] gives

$$\sum_{[I,\kappa]} \Pi_{[I,\kappa]}\big(\big[\big(\big[Y_l/A^{\times}_l\big], \rho_l\big)\big]\big) = \Pi^1\big(\big[\big(\big[Y_l/A^{\times}_l\big], \rho_l\big)\big]\big) = \big[\big(\big[Y_l/A^{\times}_l\big], \rho_l\big)\big].$$

Restricting to $|I| = n$ fixes $\dim R = n$, and the sum reduces to $\Pi^{\mathrm{vi}}_n([([Y_l/A^{\times}_l], \rho_l)])$. Equation (44) follows. □

The $\Pi_{[I,\kappa]}$ are also defined on $\underline{\mathrm{SF}}, \mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$ and $\underline{\bar{\mathrm{SF}}}, \bar{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, *, *)$ with $\Pi^2_{[I,\kappa]} = \Pi_{[I,\kappa]}$ and $\Pi_{[I,\kappa]}\Pi_{[J,\lambda]} = 0$ if $[I, \kappa] \neq [J, \lambda]$, but on these larger spaces (44) does not hold. Using Proposition 2.17 and Corollary 5.11 we can show

$$\sum_{\text{eq. classes } [I,\kappa]} \Pi_{[I,\kappa]} : \bar{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, *, *) \to \bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, *, *)$$

is a surjective projection. But the same is not true for $\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}}), \mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$.

5.2. The Lie algebras $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ *and* $\bar{\mathrm{SF}}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}},*,*)$

Next we study stack function analogues of the Lie algebra $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ of Section 4.4.

Definition 5.14. Let Assumptions 2.11 and 3.4 hold. Define $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$, $\bar{\mathrm{SF}}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}},\Upsilon,\Lambda)$, $\bar{\mathrm{SF}}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}},\Upsilon,\Lambda^\circ)$ and $\bar{\mathrm{SF}}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}},\Theta,\Omega)$ to be the subspaces of $f\in\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ or $\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}},*,*)$ satisfying $\Pi_1^{\mathrm{vi}}(f)=f$.

We interpreted Π_n^{vi} above as projecting to f 'supported on objects with n virtual indecomposable factors.' So $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$, $\bar{\mathrm{SF}}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}},*,*)$ should be thought of as stack functions *supported on virtual indecomposables*, and are good analogues of $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$. Our goal is to prove that $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$, $\bar{\mathrm{SF}}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}},*,*)$ are *Lie subalgebras* of $\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$, $\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}},*,*)$. To do this we must study the relationship between multiplication $*$ and projections Π_n^{vi}, that is, express $\Pi_n^{\mathrm{vi}}(f*g)$ in terms of $\Pi_l^{\mathrm{vi}}(f)$ and $\Pi_m^{\mathrm{vi}}(g)$.

Proposition 5.15. *Let* $T\subseteq\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})\times\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ *be constructible. Then there exist a finite decomposition* $T=\coprod_{m\in M}\mathfrak{G}_m(\mathbb{K})$*, where* $\mathfrak{G}_m$ *is a finite type* $\mathbb{K}$*-substack of* $\mathfrak{Obj}_{\mathcal{A}}\times\mathfrak{Obj}_{\mathcal{A}}$*, 1-isomorphisms* $\mathfrak{G}_m\cong[V_m/G_m]$ *for* G_m *a special* $\mathbb{K}$*-group and* V_m *a quasiprojective* $\mathbb{K}$*-variety, finite-dimensional representations* E_m^0,E_m^1 *of* G_m*, and morphisms* $\jmath_m:(\mathbb{K}^\times)^2\to G_m$ *for all* $m\in M$*, satisfying*:

(a) *Let* $v\in V_m(\mathbb{K})$ *project to* $([X],[Y])\in\mathfrak{G}_m(\mathbb{K})\subset\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})\times\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$*, so*

$$\mathrm{Stab}_{G_m}(v)\cong\mathrm{Iso}_{\mathbb{K}}\big([X],[Y]\big)\cong\mathrm{Aut}(X)\times\mathrm{Aut}(Y).\tag{45}$$

Then there are isomorphisms $E_m^0\cong\mathrm{Hom}(Y,X)$ *and* $E_m^1\cong\mathrm{Ext}^1(Y,X)$ *such that* (45) *identifies the action of* $\mathrm{Stab}_{G_m}(v)$ *on* E_m^i *with the action of* $\mathrm{Aut}(X)\times\mathrm{Aut}(Y)$ *on* $\mathrm{Hom}(Y,X),\mathrm{Ext}^1(Y,X)$ *given by* $(\alpha,\beta)\cdot e=\alpha\circ e\circ\beta^{-1}$.

(b) $\jmath_m$ *maps into the centre of* G_m*, and* $\jmath_m((\mathbb{K}^\times)^2)$ *acts freely on* V_m*. Thus, in* (a) $\jmath_m$ *maps* $(\mathbb{K}^\times)^2\to\mathrm{Stab}_{G_m}(v)$*. Composing this with* (45) *gives the map* $(\delta,\epsilon)\mapsto(\delta\,\mathrm{id}_X,\epsilon\,\mathrm{id}_Y)$*, for* $\delta,\epsilon\in\mathbb{K}^\times$.

(c) *Write* $i_m:\mathfrak{G}_m\to\mathfrak{Obj}_{\mathcal{A}}\times\mathfrak{Obj}_{\mathcal{A}}$ *for the inclusion 1-morphism. Then there is a 1-isomorphism*

$$\mathfrak{G}_m\times_{i_m,\mathfrak{Obj}_{\mathcal{A}}\times\mathfrak{Obj}_{\mathcal{A}},\sigma(\{1\})\times\sigma(\{2\})}\mathfrak{M}\big(\{1,2\},\leqslant\big)_{\mathcal{A}}\cong\big[V_m\times E_m^1/G_m\ltimes E_m^0\big].\tag{46}$$

Here multiplication on $G_m\ltimes E_m^0$ *is* $(\gamma,e)\cdot(\gamma',e')=(\gamma\gamma',e+\gamma\cdot e')$*, and* E_m^0 *acts trivially on* $V_m\times E_m^1$*, and* G_m *acts in the given way on* V_m,E_m^1.

(d) *Equation* (46) *is a substack of* $\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}}$*, so its* $\mathbb{K}$*-points are* $[(\sigma,\iota,\pi)]$ *for* $(\{1,2\},\leqslant)$*-configurations* (σ,ι,π)*. Let* X*,* Y*,* v *be as in* (a)*, and* $(v,e)\in(V_m\times E_m^1)(\mathbb{K})$ *project to* $[(\sigma,\iota,\pi)]$*. Then we can choose* $\sigma(\{1\})=X$*,* $\sigma(\{2\})=Y$*, and* $e\in\mathrm{Ext}^1(Y,X)$ *corresponds to the exact sequence*

$$0\longrightarrow\sigma\big(\{1\}\big)\xrightarrow{\iota(\{1\},\{1,2\})}\sigma\big(\{1,2\}\big)\xrightarrow{\pi(\{1,2\},\{2\})}\sigma\big(\{2\}\big)\longrightarrow 0.\tag{47}$$

Proof. The proof of Proposition 5.8 easily generalizes to give a finite decomposition $T = \coprod_{m\in M} \mathfrak{G}_m(\mathbb{K})$ and 1-isomorphisms $\mathfrak{G}_m \cong [U_m/A_m^\times \times B_m^\times]$, for A_m, B_m finite-dimensional $\mathbb{K}$-algebras, such that if $u \in U_m(\mathbb{K})$ projects to $([X],[Y])$ then

$$\mathrm{Stab}_{A_m^\times\times B_m^\times}(u) = C_u^\times \times D_u^\times$$

for subalgebras $C_u \subseteq A_m$, $D_u \subseteq B_m$ with isomorphisms $C_u \cong \mathrm{End}(X)$, $D_u \cong \mathrm{End}(Y)$ inducing the isomorphism $\mathrm{Stab}_{A_m^\times\times B_m^\times}(u) \cong \mathrm{Aut}(X)\times \mathrm{Aut}(Y)$.

The functions $([X],[Y]) \mapsto \dim\mathrm{Hom}(Y,X)$ or $\dim\mathrm{Ext}^1(Y,X)$ are locally constructible on $(\mathfrak{Obj}_{\mathcal{A}}\times\mathfrak{Obj}_{\mathcal{A}})(\mathbb{K})$, and so take finitely many values on T. Refining $T = \coprod_{m\in M}\mathfrak{G}_m(\mathbb{K})$, we can make $\dim\mathrm{Hom}(Y,X)$, $\dim\mathrm{Ext}^1(Y,X)$ constant on each $\mathfrak{G}_m(\mathbb{K})$. Refining further, $\mathrm{Hom}(Y,X)$, $\mathrm{Ext}^1(Y,X)$ are the fibres over $([X],[Y]) \in \mathfrak{G}_m(\mathbb{K})$ of *vector bundles* over $\mathfrak{G}_m$, in the stack sense. These pull back under the projection $U_m \to \mathfrak{G}_m$ to vector bundles $\mathcal{E}_m^0$, $\mathcal{E}_m^1$ over U_m, with fibres $\mathbb{K}$-vector spaces E_m^0, E_m^1, and the $A_m^\times \times B_m^\times$-action on U_m lifts to actions on $\mathcal{E}_m^0, \mathcal{E}_m^1$ preserving the vector bundle structure.

Define V_m to be the quasiprojective $\mathbb{K}$-variety of triples (u,α^0,α^1), for $u \in U_m(\mathbb{K})$ and $\alpha^i : (\mathcal{E}_m^i)_v \to E_m^i$ vector space isomorphisms between the fibre of $\mathcal{E}_m^i$ over v for $i = 0,1$ and E_m^i. Then V_m is a principal bundle over $\tilde V_m$ with structure group $\mathrm{Aut}(E_m^0)\times\mathrm{Aut}(E_m^1)$, and the $A_m^\times\times B_m^\times$-action lifts to V_m and commutes with the $\mathrm{Aut}(E_m^0)\times\mathrm{Aut}(E_m^1)$-action. Define

$$G_m = A_m^\times \times B_m^\times \times \mathrm{Aut}\bigl(E_m^0\bigr)\times\mathrm{Aut}\bigl(E_m^1\bigr),$$

which is special as it is a product of groups of the form $A^\times$ for finite-dimensional algebras A. It acts on V_m with

$$[V_m/G_m] \cong \bigl[U_m/A_m^\times \times B_m^\times\bigr] \cong \mathfrak{G}_m.$$

Define actions of G_m on E_m^i for $i = 0,1$ by $(a,b,\beta^0,\beta^1)\cdot e^i = \beta^i e^i$. It is now easy to see that (a) holds for V_m, G_m.

For (b), define

$$\jmath_m(\delta,\epsilon) = \bigl(\delta\,\mathrm{id}_{A_m}, \epsilon\,\mathrm{id}_{B_m}, \delta\epsilon^{-1}\,\mathrm{id}_{E_m^0}, \delta\epsilon^{-1}\,\mathrm{id}_{E_m^1}\bigr).$$

This is clearly a $\mathbb{K}$-group morphism to the centre of G_m. If $v = (u,\alpha^0,\alpha^1)$ in $V_m(\mathbb{K})$ then

$$\mathrm{Stab}_{A_m^\times\times B_m^\times}(u) = C_u^\times \times D_u^\times$$

as above, and $\delta\,\mathrm{id}_{A_m} \in C_u^\times$, $\epsilon\,\mathrm{id}_{B_m} \in D_u^\times$ as C_u, D_u are subalgebras, so $(\delta\,\mathrm{id}_{A_m}, \epsilon\,\mathrm{id}_{B_m})$ fixes u. The identification of actions in (a) then shows $(\delta\,\mathrm{id}_{A_m},\dots,\delta\epsilon^{-1}\,\mathrm{id}_{E_m^1})$ fixes v. Thus, $\jmath_m((\mathbb{K}^\times)^2)$ fixes each $v\in V_m(\mathbb{K})$, and acts freely on V_m. Composing with (45) gives $(\delta,\epsilon)\mapsto(\delta\,\mathrm{id}_X, \epsilon\,\mathrm{id}_Y)$, as $\delta\,\mathrm{id}_{A_m} = \delta\,\mathrm{id}_{C_u}$, $\epsilon\,\mathrm{id}_{B_m} = \epsilon\,\mathrm{id}_{D_u}$ are identified with $\delta\,\mathrm{id}_X, \epsilon\,\mathrm{id}_Y$ under $C_u \cong \mathrm{End}(X)$, $D_u \cong \mathrm{End}(Y)$.

For (c) and (d), note that the fibre of $\boldsymbol{\sigma}(\{1\})\times\boldsymbol{\sigma}(\{2\}) : \mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}} \to \mathfrak{Obj}_{\mathcal{A}}\times\mathfrak{Obj}_{\mathcal{A}}$ over $([X],[Y]) \in (\mathfrak{Obj}_{\mathcal{A}}\times\mathfrak{Obj}_{\mathcal{A}})(\mathbb{K})$ is the family of $[(\sigma,\iota,\pi)]$ for (σ,ι,π) a $(\{1,2\},\leqslant)$-configuration with $\sigma(\{1\})\cong X$ and $\sigma(\{2\})\cong Y$. This is equivalent to a short exact sequence (47), which are classified by $\mathrm{Ext}^1(Y,X)$. But as the isomorphisms $X \cong \sigma(\{1\})$, $Y \cong \sigma(\{2\})$ are not prescribed we divide by $\mathrm{Aut}(X) \times \mathrm{Aut}(Y)$, so as sets we have

$$\mathcal{M}\big(\{1,2\},\leqslant\big)_{\mathcal{A}} \supset \big(\boldsymbol{\sigma}\big(\{1\}\big) \times \boldsymbol{\sigma}\big(\{2\}\big)\big)_*^{-1}\big(([X],[Y])\big) \cong \frac{\mathrm{Ext}^1(Y,X)}{\mathrm{Aut}(X)\times\mathrm{Aut}(Y)}.$$

To describe this fibre as a stack we must take stabilizer groups into account. One can show that if (47) corresponds to $e \in \mathrm{Ext}^1(Y,X)$ then $\mathrm{Aut}((\sigma,\iota,\pi)) \cong H_e \ltimes \mathrm{Hom}(Y,X)$, where H_e is the $\mathbb{K}$-subgroup of $\mathrm{Aut}(X)\times\mathrm{Aut}(Y)$ fixing e. If $([X],[Y])$ is the image of $v \in V_m(\mathbb{K})$, we have 1-isomorphisms

$$\begin{aligned}&\operatorname{Spec}\mathbb{K} \times_{X\times Y,\mathfrak{Obj}_{\mathcal{A}}\times\mathfrak{Obj}_{\mathcal{A}},\boldsymbol{\sigma}(\{1\})\times\boldsymbol{\sigma}(\{2\})} \mathfrak{M}\big(\{1,2\},\leqslant\big)_{\mathcal{A}}\\ &\quad\cong \big[\mathrm{Ext}^1(Y,X)/\big(\mathrm{Aut}(X)\times\mathrm{Aut}(Y)\big)\ltimes\mathrm{Hom}(Y,X)\big] \cong \big[E_m^1/\operatorname{Stab}_{G_m\ltimes E_m^0}(v)\big].\end{aligned}$$

Here $\mathrm{Hom}(Y,X)$ acts trivially on $\mathrm{Ext}^1(Y,X)$, but contributes to the stabilizers, and similarly E_m^0 acts trivially on E_m^1. Parts (c) and (d) follow by a families version of this argument. □

Corollary 5.16. *Let f,g lie in $\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ or $\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}},*,*)$, choose $T \subseteq \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K}) \times \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ constructible with $f\otimes g$ supported on T, and use the notation of Proposition* 5.15. *Then arguing as in Proposition* 5.9*, we may write*

$$f\otimes g = \sum_{m\in M,\,n\in N_m} c_{mn}\big[\big([W_{mn}/G_m],\tau_{mn}\big)\big], \tag{48}$$

where N_m is finite, $c_{mn} \in \mathbb{Q},\Lambda,\Lambda^\circ$ or Ω, W_{mn} is a quasiprojective $\mathbb{K}$-variety acted on by G_m, and $\tau_{mn} : [W_{mn}/G_m] \to \mathfrak{G}_m \subseteq \mathfrak{Obj}_{\mathcal{A}}\times\mathfrak{Obj}_{\mathcal{A}}$ is induced by a G_m-equivariant morphism $\phi_{mn} : W_{mn} \to V_m$. Moreover

$$\big(\boldsymbol{\sigma}\big(\{1\}\big)\times\boldsymbol{\sigma}\big(\{2\}\big)\big)^*\big(\big[\big([W_{mn}/G_m],\tau_{mn}\big)\big]\big) = \big[\big(\big[W_{mn}\times E_m^1/G_m\ltimes E_m^0\big],\xi_{mn}\big)\big] \tag{49}$$

in $\mathrm{SF}(\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}})$ or $\bar{\mathrm{SF}}(\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}},*,*)$, where ϕ_{mn} induces

$$\xi_{mn} : \big[W_{mn}\times E_m^1/G_m\ltimes E_m^0\big] \to \big[V_m\times E_m^1/G_m\ltimes E_m^0\big], \tag{50}$$

using (46) *to regard the right-hand side of* (50) *as a substack of $\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}}$. Combining* (40)*,* (48) *and* (49) *gives*

$$f*g = \sum_{m\in M,\,n\in N_m} c_{mn}\big[\big(\big[W_{mn}\times E_m^1/G_m\ltimes E_m^0\big],\boldsymbol{\sigma}\big(\{1,2\}\big)\circ\xi_{mn}\big)\big]. \tag{51}$$

Here we have used a formula for the fibre product of quotient stacks from the proof of [11, Theorem 4.12] to deduce (49). Our next theorem, which will be important in [13, §8], proves a relationship between the operators Π_k^{vi} and $P_{(I,\preccurlyeq)}$.

Theorem 5.17. *Let Assumptions* 2.11 *and* 3.4 *hold, $(I,\trianglelefteq)$ be a finite poset, $k \geqslant 0$, and f_i for $i\in I$ lie in $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ or $\bar{\mathrm{SF}}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}},*,*)$. Then*

$$\Pi_k^{\mathrm{vi}}\big[P_{(I,\trianglelefteq)}(f_i\colon i\in I)\big]=\sum_{\substack{\text{iso. classes of}\\ \text{finitesets } K,\\ k\leqslant|K|\leqslant|I|}}\;\sum_{\substack{\text{surjective }\phi:I\to K;\\ \text{define }\precsim\text{ on }I\text{ by }i\precsim j\\ \text{if }i\trianglelefteq j\text{ and }\phi(i)=\phi(j)}} N_{I,K,\phi,k}\cdot P_{(I,\precsim)}(f_i\colon i\in I),\tag{52}$$

where $N_{I,K,\phi,k}\in\mathbb{Q}$ depends only on I, K, ϕ up to isomorphism and k.

Proof. It is easy to partially generalize Proposition 5.15(a),(b) from $(\{1,2\},\leqslant)$ to $(I,\precsim)$ as follows. Choose $T\subseteq\prod_{i\in I}\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ constructible with $\bigotimes_{i\in I}f_i$ supported on T. Then there exist a finite decomposition $T=\coprod_{m\in M}\mathfrak{G}_m(\mathbb{K})$, where $\mathfrak{G}_m$ is a finite type $\mathbb{K}$-substack of $\prod_{i\in I}\mathfrak{Obj}_{\mathcal{A}}$, 1-isomorphisms $\mathfrak{G}_m\cong[V_m/G_m]$ for G_m a special $\mathbb{K}$-group and V_m a quasiprojective $\mathbb{K}$-variety, and morphisms $\jmath_m:(\mathbb{K}^\times)^I\to G_m$ for all $m\in M$, satisfying:

(a) Let $v\in V_m(\mathbb{K})$ project to $\prod_{i\in I}[X_i]\in\mathfrak{G}_m(\mathbb{K})\subset\prod_{i\in I}\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$, so that

$$\mathrm{Stab}_{G_m}(v)\cong\mathrm{Iso}_{\mathbb{K}}\Big(\prod_{i\in I}[X_i]\Big)\cong\prod_{i\in I}\mathrm{Aut}(X_i).\tag{53}$$

(b) $\jmath_m$ maps into the centre of G_m, and $\jmath_m((\mathbb{K}^\times)^I)$ acts freely on V_m. Thus, in (a) $\jmath_m$ maps $(\mathbb{K}^\times)^I\to\mathrm{Stab}_{G_m}(v)$. Composing this with (53) gives the map $\delta\mapsto\prod_{i\in I}\delta(i)\,\mathrm{id}_{X_i}$, for $\delta\in(\mathbb{K}^\times)^I$.

Generalizing part (c) is more tricky. Write $i_m:\mathfrak{G}_m\to\prod_{i\in I}\mathfrak{Obj}_{\mathcal{A}}$ for the inclusion 1-morphism. Then we can form

$$\mathfrak{G}_m\times_{i_m,\prod_{i\in I}\mathfrak{Obj}_{\mathcal{A}},\prod_{i\in I}\boldsymbol{\sigma}(\{i\})}\mathfrak{M}(I,\trianglelefteq)_{\mathcal{A}},$$

regarded as a $\mathbb{K}$-substack of $\mathfrak{M}(I,\trianglelefteq)_{\mathcal{A}}$. It is of *finite type*, as $\mathfrak{G}_m$ is and $\prod_{i\in I}\boldsymbol{\sigma}(\{i\})$ is by Theorem 3.6(c). Also $\mathfrak{M}(I,\trianglelefteq)_{\mathcal{A}}$ has affine geometric stabilizers. So using [17, Proposition 3.5.9] as in Proposition 5.8, we may write

$$\big(\mathfrak{G}_m\times_{i_m,\prod_{i\in I}\mathfrak{Obj}_{\mathcal{A}},\prod_{i\in I}\boldsymbol{\sigma}(\{i\})}\mathfrak{M}(I,\trianglelefteq)_{\mathcal{A}}\big)(\mathbb{K})=\coprod_{p\in P_m}\mathfrak{H}_{mp}(\mathbb{K}),$$

where P_m is finite and $\mathfrak{H}_{mp}$ a $\mathbb{K}$-substack of $\mathfrak{M}(I,\trianglelefteq)_{\mathcal{A}}$ with a 1-isomorphism

$$\mathfrak{H}_{mp}\cong[Y_{mp}/G_m\ltimes K_{mp}],\tag{54}$$

where Y_{mp} is a quasiprojective $\mathbb{K}$-variety and K_{mp} a *nilpotent* $\mathbb{K}$-group acted on by G_m, such that $\prod_{i\in I}\boldsymbol{\sigma}(\{i\}):\mathfrak{H}_{mp}\to\mathfrak{G}_m$ is induced by a morphism $\psi_{mp}:Y_{mp}\to V_m$ equivariant w.r.t. the natural projection $G_m\ltimes K_{mp}\to G_m$.

The only nontrivial claim here is that we can take the quotient group in (54) to be $G_m\ltimes K_{mp}$ with K_{mp} *nilpotent*, rather than an arbitrary $\mathbb{K}$-group with a morphism to G_m. When $(I,\trianglelefteq)=(\{1,2\},\leqslant)$ this follows from Proposition 5.15(c) with $K_{mp}=E_m^0$. The general case can be proved by the inductive argument on $|I|$ in the proof of Theorem 6.2 below, which builds up $\mathfrak{M}(I,\trianglelefteq)_{\mathcal{A}}$ by repeated fibre products with $\boldsymbol{\sigma}(\{1\})\times\boldsymbol{\sigma}(\{2\}):\mathfrak{M}(\{1,2\})_{\mathcal{A}}\to\mathfrak{Obj}_{\mathcal{A}}\times\mathfrak{Obj}_{\mathcal{A}}$. The point of this is that as K_{mp} is nilpotent we can use the same maximal torus for G_m and $G_m\ltimes K_{mp}$, which will be important when we come to apply Π_k^{vi}.

Now we can generalize Corollary 5.16 to write

$$\bigotimes_{i\in I} f_i = \sum_{m\in M,\, n\in N_m} c_{mn}\big[\big([W_{mn}/G_m], \tau_{mn}\big)\big], \tag{55}$$

where N_m is finite, $c_{mn} \in \mathbb{Q}, \Lambda, \Lambda^\circ$ or Ω, W_{mn} is a quasiprojective $\mathbb{K}$-variety acted on by G_m, and $\tau_{mn}:[W_{mn}/G_m] \to \mathfrak{G}_m \subseteq \mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}}$ is induced by a G_m-equivariant morphism $\phi_{mn}: W_{mn} \to V_m$. Moreover

$$\begin{aligned}&\Big(\prod_{i\in I}\boldsymbol{\sigma}\big(\{i\}\big)\Big)^*\big(\big[\big([W_{mn}/G_m], \tau_{mn}\big)\big]\big)\\ &\quad= \sum_{p\in P_m}\big[\big([W_{mn}\times_{\phi_{mn},V_m,\psi_{mp}} Y_{mp}/G_m \ltimes K_{mp}], \xi_{mnp}\big)\big]\end{aligned} \tag{56}$$

in $\mathrm{SF}(\mathfrak{M}(I,\trianglelefteq)_{\mathcal{A}})$ or $\bar{\mathrm{SF}}(\mathfrak{M}(I,\trianglelefteq)_{\mathcal{A}},*,*)$, where $\pi_{Y_{mp}}$ induces

$$\xi_{mnp}:[W_{mn}\times_{\phi_{mn},V_m,\psi_{mp}} Y_{mp}/G_m \ltimes K_{mp}] \to [Y_{mp}/G_m \ltimes K_{mp}], \tag{57}$$

using (54) to regard the right-hand side of (57) as a substack of $\mathfrak{M}(I,\trianglelefteq)_{\mathcal{A}}$. Combining (42), (55) and (56) gives

$$P_{(I,\trianglelefteq)}(f_i:\ i\in I) = \sum_{\substack{m\in M,\\ n\in N_m,\, p\in P_m}} c_{mn}\big[\big([W_{mn}\times_{\phi_{mn},V_m,\psi_{mp}} Y_{mp}/G_m \ltimes K_{mp}], \boldsymbol{\sigma}(I)\circ\xi_{mnp}\big)\big]. \tag{58}$$

Since $\Pi^{\mathrm{vi}}_1(f_i) = f_i$, we deduce from [11, Proposition 5.14(iv)] that

$$\Pi^{\mathrm{vi}}_k\Big[\bigotimes_{i\in I} f_i\Big] = \begin{cases}\bigotimes_{i\in I} f_i, & k=|I|,\\ 0, & \text{otherwise.}\end{cases} \tag{59}$$

Let T_m be a maximal torus in G_m, so that $T_m \times \{0\} = T_m$ is a maximal torus in $G_m \ltimes K_{mp}$. Applying [11, Definitions 5.10 and 5.13] to (55) we find that

$$\begin{aligned}&\Pi^{\mathrm{vi}}_k\Big[\bigotimes_{i\in I} f_i\Big]\\ &\quad= \sum_{\substack{m\in M,\, n\in N_m,\, P\in\mathcal{P}(W_{mn},T_m),\\ Q\in\mathcal{Q}(G_m,T_m),\, R\in\mathcal{R}(W_{mn},G_m,T_m):\\ R\subseteq P\cap Q,\, M^{W_{mn}}_{G_m}(P,Q,R)\neq 0,\, \dim R=k}} c_{mn}\, M^{W_{mn}}_{G_m}(P,Q,R)\cdot\big[\big(\big[W^P_{mn}/C_{G_m}(Q)\big], \tau_{mn}\circ\iota^{P\cap Q}\big)\big].\end{aligned} \tag{60}$$

Let $m' \in M$ and $R' \subseteq T_{m'}$ lie in $\mathcal{R}(W_{m'n}, G_{m'}, T_{m'})$ for at least one $n \in N_{m'}$. By [11, Definition 5.15] we can define $\hat{\Pi}^{\nu}_{\Pi_{i\in I}\mathfrak{Obj}_{\mathcal{A}}}$ on $\mathrm{SF}(\Pi_{i\in I}\mathfrak{Obj}_{\mathcal{A}})$ or $\bar{\mathrm{SF}}(\Pi_{i\in I}\mathfrak{Obj}_{\mathcal{A}},*,*)$ with the $\Pi_{i\in I}\mathfrak{Obj}_{\mathcal{A}}$-weight function ν given by $\nu(T,g,\phi)=1$ if ϕ is injective and $\phi_*(T)\subset \mathrm{Iso}_{\mathbb{K}}(g)$ is identified with $R' \subseteq \mathrm{Stab}_{G_{m'}}(w)$ for some $n \in N_{m'}$ and $w \in W_{m'n}(\mathbb{K})$ projecting to $g \in \mathfrak{G}_{m'}(\mathbb{K}) \subseteq (\Pi_{i\in I}\mathfrak{Obj}_{\mathcal{A}})(\mathbb{K})$ under $\mathfrak{G}_{m'} \cong [W_{m'n}/G_{m'}]$, and $\nu(T,g,\phi)=0$ otherwise.

Applying $\hat{\Pi}^{\nu}_{\Pi_{i\in I}\mathfrak{Obj}_{\mathcal{A}}}$ to (60) projects to those components in the sum with $m=m'$ and R conjugate to R' under the Weyl group W of G_m. By (59), Eq. (60) is zero for $k\neq|I|$. So, if $\dim R'\neq|I|$ then the sum of components in (60) with $m=m'$ and R conjugate to R' is zero. But by symmetry in W each of the conjugates of R' give the same answer. Hence, for any fixed $m\in M$ and R with $\dim R\neq|I|$, the sum of components in (60) with these m, R is zero.

By (b) above, $\jmath_m((\mathbb{K}^\times)^I)$ lies in the centre of G_m and acts trivially on V_m, and we can use algebra stabilizers to show that it also acts trivially on each W_{mn}. Thus $\jmath_m((\mathbb{K}^\times)^I)\subseteq R$ for each $R\in\mathcal{R}(W_{mn},G_m,T_m)$. Also, as G_m is special and each f_i has algebra stabilizers one can show that each such R is a torus (rather than the product of a torus with a finite group). And as each f_i is supported over *nonzero* elements $[X_i]$, as composing $\jmath_m$ with (53) takes $\delta\mapsto(\delta(i)\,\mathrm{id}_{X_i})_{i\in I}$, we see $\jmath_m$ is *injective*. Therefore $\jmath_m((\mathbb{K}^\times)^I)\cong(\mathbb{K}^\times)^{|I|}$ is the minimal element of $\mathcal{R}(W_{mn},G_m,T_m)$, and the unique element R with $\dim R=|I|$. By (55) and (59), the sum of terms in (60) with these m, R is $\sum_{n\in N_m}c_{mn}[([W_{mn}/G_m],\tau_{mn})]$.

As for (60), Eq. (58) yields an expression for $\Pi^{\mathrm{vi}}_k[P_{(I,\trianglelefteq)}(f_i\colon i\in I)]$, but it is not in the form we want. Applying Π^{vi}_k to $[([W_{mn}\times_{V_m}Y_{mp}/G_m\ltimes K_{mp}],\boldsymbol{\sigma}(I)\circ\xi_{mnp})]$ involves summing over the three finite sets:

$$\begin{aligned}
&\mathcal{P}(W_{mn}\times_{V_m}Y_{mp},T_m)\subseteq\bigl\{P\cap\dot P\colon\ P\in\mathcal{P}(W_{mn},T_m),\ \dot P\in\mathcal{P}(Y_{mp},T_m)\bigr\},\\
&\mathcal{Q}(G_m\ltimes K_{mp},T_m)=\bigl\{Q\cap\dot Q\colon\ Q\in\mathcal{Q}(G_m,T_m),\ \dot Q\in\mathcal{Q}(G_m\ltimes K_{mp},T_m)\bigr\},\\
&\mathcal{R}(W_{mn}\times_{V_m}Y_{mp},G_m\ltimes K_{mp},T_m)\\
&\quad\subseteq\bigl\{R\cap\dot R\colon\ R\in\mathcal{R}(W_{mn},G_m,T_m),\ \dot R\in\mathcal{R}(Y_{mp},G_m\ltimes K_{mp},T_m)\bigr\}.
\end{aligned}$$

It follows from the proof in [11, §5] that the definition of Π^{vi}_k is independent of choices, that if we replace $\mathcal{P}(W_{mn}\times_{V_m}Y_{mp},T_m)$ by $\mathcal{P}(W_{mn},T_m)\times\mathcal{P}(Y_{mp},T_m)$, and $\mathcal{Q}(G_m\ltimes K_{mp},T_m)$ by $\mathcal{Q}(G_m,T_m)\times\mathcal{Q}(G_m\ltimes K_{mp},T_m)$, and $\mathcal{R}(W_{mn}\times_{V_m}Y_{mp},G_m\ltimes K_{mp},T_m)$ by $\mathcal{R}(W_{mn},G_m,T_m)\times\mathcal{R}(Y_{mp},G_m\ltimes K_{mp},T_m)$ throughout the definition, taking $(P,\dot P)$ to act as $P\cap\dot P$ on $W_{mn}\times_{V_m}Y_{mp}$ and so on, we get the same answer. But defined using these sets we easily find that

$$M^{W_{mn}\times_{V_m}Y_{mp}}_{G_m\ltimes K_{mp}}\bigl((P,\dot P),(Q,\dot Q),(R,\dot R)\bigr)=M^{W_{mn}}_{G_m}(P,Q,R)\cdot M^{Y_{mp}}_{G_m\ltimes K_{mp}}(\dot P,\dot Q,\dot R).$$

This yields:

$$\begin{aligned}
&\Pi^{\mathrm{vi}}_k\bigl[P_{(I,\trianglelefteq)}(f_i\colon i\in I)\bigr]\\
&\quad=\sum_{\substack{m\in M,\ p\in P_m,\ \dot P\in\mathcal{P}(Y_{mp},T_m),\\ \dot Q\in\mathcal{Q}(G_m\ltimes K_{mp},T_m),\ \dot R\in\mathcal{R}(Y_{mp},G_m\ltimes K_{mp},T_m):\\ \dot R\subseteq\dot P\cap\dot Q,\ M^{Y_{mp}}_{G_m\ltimes K_{mp}}(\dot P,\dot Q,\dot R)\neq0}} M^{Y_{mp}}_{G_m\ltimes K_{mp}}(\dot P,\dot Q,\dot R)\\
&\qquad\cdot\Biggl[\sum_{\substack{n\in N_{nm},\ P\in\mathcal{P}(W_{mn},T_m),\\ Q\in\mathcal{Q}(G_m,T_m),\ R\in\mathcal{R}(W_{mn},G_m,T_m):\\ R\subseteq P\cap Q,\ M^{W_{mn}}_{G_m}(P,Q,R)\neq0,\ \dim R\cap\dot R=k}} c_{mn}M^{W_{mn}}_{G_m}(P,Q,R)\\
&\qquad\cdot\bigl[\bigl(\bigl[W^P_{mn}\times_{V_m}Y^{\dot P}_{mp}/C_{G_m}(Q)\ltimes(K_{mp})^{\dot Q}\bigr],\boldsymbol{\sigma}(I)\circ\xi_{mnp}\circ\iota^{P\cap Q,\dot P\cap\dot Q}\bigr)\bigr]\Biggr].
\end{aligned}\tag{61}$$

Actually, according to the argument we gave the last term should be

$$\big[\big(\big[W_{mn}^{P\cap\dot P}\times_{V_m} Y_{mp}^{P\cap\dot P}/C_{G_m}(Q\cap\dot Q)\ltimes(K_{mp})^{Q\cap\dot Q}\big],\boldsymbol{\sigma}(I)\circ\xi_{mnp}\circ\iota^{P\cap\dot P\cap Q\cap\dot Q}\big)\big].$$

However, using [11, Lemma 5.9] we can show that for fixed $P,\dot P,Q,\dot Q$ in (61), unless $P,\dot P$ are the smallest elements of $\mathcal{P}(W_{mn},T_m)$, $\mathcal{P}(Y_{mp},T_m)$ containing $P\cap\dot P$, and $Q,\dot Q$ are the smallest elements of $\mathcal{Q}(G_m,T_m)$, $\mathcal{Q}(G_m\ltimes K_{mp},T_m)$ containing $Q\cap\dot Q$, then the sum of $M_{G_m\ltimes K_{mp}}^{Y_{mp}}(\dot P,\dot Q,\dot R)\cdot M_{G_m}^{W_{mn}}(P,Q,R)$ over all $R,\dot R$ with fixed $R\cap\dot R$ is zero. But if $P,\dot P,Q,\dot Q$ are the smallest elements containing $P\cap\dot P$, $Q\cap\dot Q$ then

$$W_{mn}^{P}=W_{mn}^{P\cap\dot P},\quad Y_{mp}^{\dot P}=Y_{mp}^{P\cap\dot P},\quad C_{G_m}(Q)=C_{G_m}(Q\cap\dot Q)\quad\text{and}\quad(K_{mp})^{\dot Q}=(K_{mp})^{Q\cap\dot Q},$$

by [11, Lemmas 5.4(iii) and 5.6(iii)]. So (61) is correct.

The important point about the way we have written (61) is that the last lines $[\cdots]$ is a sum over certain R of a linear operation applied to the terms in (60) with fixed m,R. But we have already shown that the sum of terms in (60) with fixed m,R is $\sum_{n\in N_m}c_{mn}[([W_{mn}/G_m],\tau_{mn})]$ if $R=\jmath_m((\mathbb{K}^\times)^I)$, and 0 otherwise. This proves that

$$\Pi_k^{\mathrm{vi}}\big[P_{(I,\trianglelefteq)}(f_i\colon i\in I)\big]=\sum_{\substack{m\in M,\,n\in N_{nm},\,p\in P_m,\\ \dot P\in\mathcal{P}(Y_{mp},T_m),\,\dot Q\in\mathcal{Q}(G_m\ltimes K_{mp},T_m),\\ \dot R\in\mathcal{R}(Y_{mp},G_m\ltimes K_{mp},T_m)\colon\ \dot R\subseteq\dot P\cap\dot Q,\\ M_{G_m\ltimes K_{mp}}^{Y_{mp}}(\dot P,\dot Q,\dot R)\neq0,\ \dim \jmath_m((\mathbb{K}^\times)^I)\cap\dot R=k}} M_{G_m\ltimes K_{mp}}^{Y_{mp}}(\dot P,\dot Q,\dot R)\cdot c_{mn}\cdot\big[\big(\big[W_{mn}\times_{V_m}Y_{mp}^{\dot P}/G_m\ltimes(K_{mp})^{\dot Q}\big],\boldsymbol{\sigma}(I)\circ\xi_{mnp}\big)\big].\tag{62}$$

Now using the argument of [11, Lemma 5.9] again, we find that unless $\dot P,\dot Q\subseteq \jmath_m((\mathbb{K}^\times)^I)$, the sum of over $\dot R$ of terms with these $\dot P,\dot Q$ in (62) is zero. So we may restrict to $\dot P,\dot Q,\dot R\subseteq \jmath_m((\mathbb{K}^\times)^I)$, and then as $\jmath_m((\mathbb{K}^\times)^I)$ lies in the centre of G_m we may replace G_m,T_m by $\jmath_m((\mathbb{K}^\times)^I)$ in the definitions of $\dot P,\dot Q,\dot R$ and $M_{G_m\ltimes K_{mp}}^{Y_{mp}}(\dot P,\dot Q,\dot R)$, yielding:

$$\Pi_k^{\mathrm{vi}}\big[P_{(I,\trianglelefteq)}(f_i\colon i\in I)\big]=\sum_{\substack{m\in M,\,n\in N_{nm},\,p\in P_m,\\ \dot P\in\mathcal{P}(Y_{mp},\,\jmath_m((\mathbb{K}^\times)^I)),\\ \dot Q\in\mathcal{Q}(\jmath_m((\mathbb{K}^\times)^I)\ltimes K_{mp},\,\jmath_m((\mathbb{K}^\times)^I)),\\ \dot R\in\mathcal{R}(Y_{mp},\jmath_m((\mathbb{K}^\times)^I)\ltimes K_{mp},\jmath_m((\mathbb{K}^\times)^I))\colon\\ \dot R\subseteq\dot P\cap\dot Q,\ M_{\jmath_m((\mathbb{K}^\times)^I)\ltimes K_{mp}}^{Y_{mp}}(\dot P,\dot Q,\dot R)\neq0,\ \dim\dot R=k}} M_{\jmath_m((\mathbb{K}^\times)^I)\ltimes K_{mp}}^{Y_{mp}}(\dot P,\dot Q,\dot R)\cdot c_{mn}\cdot\big[\big(\big[W_{mn}\times_{V_m}Y_{mp}^{\dot P}/G_m\ltimes(K_{mp})^{\dot Q}\big],\boldsymbol{\sigma}(I)\circ\xi_{mnp}\big)\big].\tag{63}$$

Suppose that $m\in M$, $p\in P_m$, $\dot P\subseteq \jmath_m((\mathbb{K}^\times)^I)$ is a $\mathbb{K}$-subgroup, and $y\in Y_{mp}(\mathbb{K})$ is fixed by $\dot P$. Then y projects to $[(\sigma,\iota,\pi)]\in\mathcal{M}(I,\trianglelefteq)_{\mathcal{A}}$ with a commutative diagram of $\mathbb{K}$-groups

$$\begin{array}{ccccc}\dot P & & \longrightarrow & & \mathrm{Aut}(\sigma,\iota,\pi)\\ \downarrow{\scriptstyle\subseteq} & & & & \downarrow{\scriptstyle\prod_{i\in I}\boldsymbol{\sigma}(\{i\})_*}\\ \jmath_m((\mathbb{K}^\times)^I) & \xrightarrow{\ \jmath_m^{-1}\ } & (\mathbb{K}^\times)^I & \xrightarrow{\ \delta\mapsto\prod_{i\in I}\delta(i)\,\mathrm{id}_{X_i}\ } & \prod_{i\in I}\mathrm{Aut}(\sigma(\{i\})).\end{array}\tag{64}$$

Now the bottom right corner extends naturally to algebra morphisms between the algebras $\mathbb{K}^I$, $\prod_{i\in I}\mathrm{End}(\sigma(\{i\}))$ and $\mathrm{End}(\sigma,\iota,\pi)$. Let $A\subseteq\mathbb{K}^I$ be the subalgebra of $\mathbb{K}^I$ generated by $\mathbb{K}$-subgroup $\jmath_m^{-1}(\dot P)$, and $A^\times\subseteq(\mathbb{K}^\times)^I$ the $\mathbb{K}$-subgroup of invertible elements in A. Then $\jmath_m(A^\times)$ is a $\mathbb{K}$-subgroup of $\jmath_m((\mathbb{K}^\times)^I)$ with $\dot P\subseteq\jmath_m(A^\times)\subseteq\jmath_m((\mathbb{K}^\times)^I)$, and it is easy to see that we can extend (64) to replace $\dot P$ by $\jmath_m(A^\times)$. Hence $y\in Y_{mp}(\mathbb{K})$ is fixed by $\jmath_m(A^\times)$.

From [11, Definition 5.3] we now see that each $\dot P\in\mathcal{P}(Y_{mp},\jmath_m((\mathbb{K}^\times)^I))$ is of the form $\jmath_m(A^\times)$ for some subalgebra $A\subseteq\mathbb{K}^I$. A related proof shows the same holds for each $\dot Q\in\mathcal{Q}(\jmath_m((\mathbb{K}^\times)^I)\ltimes K_{mp},\jmath_m((\mathbb{K}^\times)^I))$. Let K be a finite set and $\phi:I\to K$ a surjective map, and define

$$A_{I,K,\phi}=\bigl\{\delta\in\mathbb{K}^I:\ \phi(i)=\phi(j)\text{ implies }\delta(i)=\delta(j),\ i,j\in I\bigr\}.$$

Then $A_{I,K,\phi}$ is a $\mathbb{K}$-subalgebra of $\mathbb{K}^I$, and every subalgebra is of this form. Thus

$$\begin{aligned}&\mathcal{P}\bigl(Y_{mp},\jmath_m\bigl((\mathbb{K}^\times)^I\bigr)\bigr),\mathcal{Q}\bigl(\jmath_m\bigl((\mathbb{K}^\times)^I\bigr)\ltimes K_{mp},\jmath_m\bigl((\mathbb{K}^\times)^I\bigr)\bigr)\\&\quad\subseteq\bigl\{A^\times_{I,K,\phi}:\ K\text{ finite},\ \phi:I\to K\text{ surjective}\bigr\}.\end{aligned}$$

It is a consequence of the proof in [11, §5] that the definition of Π_k^{vi} is independent of choices, that if in (63) we replace $\mathcal{P}(Y_{mp},\jmath_m((\mathbb{K}^\times)^I))$, $\mathcal{Q}(\jmath_m((\mathbb{K}^\times)^I)\ltimes K_{mp},\jmath_m((\mathbb{K}^\times)^I))$, $\mathcal{R}(Y_{mp},\jmath_m((\mathbb{K}^\times)^I)\ltimes K_{mp},\jmath_m((\mathbb{K}^\times)^I))$ by larger finite sets closed under intersection, and compute $M^{Y_{mp}}_{\jmath_m((\mathbb{K}^\times)^I)\ltimes K_{mp}}(\dot P,\dot Q,\dot R)$ using these larger sets, then we get the same answer for $\Pi_k^{\mathrm{vi}}[\cdots]$. Replacing all three sets by $\{A^\times_{I,K,\phi}:\ K\text{ finite},\ \phi:I\to K\text{ surjective}\}$, Eq. (63) becomes:

$$\begin{aligned}\Pi_k^{\mathrm{vi}}\bigl[P_{(I,\trianglelefteq)}(f_i:\ i\in I)\bigr]=&\sum_{\substack{m\in M,\,n\in N_{nm},\,p\in P_m,\\ \dot P,\dot R\in\{A^\times_{I,K,\phi}:\ K\text{ finite},\\ \phi:I\to K\text{ surjective}\},\\ \dot R\subseteq\dot P,\,M_I(\dot P,\dot P,\dot R)\neq0,\,\dim\dot R=k}}M_I(\dot P,\dot P,\dot R)\cdot c_{mn}\\&\cdot\bigl[\bigl(\bigl[W_{mn}\times_{V_m}Y_{mp}^{\dot P}/G_m\ltimes(K_{mp})^{\dot P}\bigr],\boldsymbol{\sigma}(I)\circ\xi_{mnp}\bigr)\bigr].\end{aligned}\tag{65}$$

Here $M_I(\dot P,\dot P,\dot R)$ is $M^{Y_{mp}}_{\jmath_m((\mathbb{K}^\times)^I)\ltimes K_{mp}}(\dot P,\dot P,\dot R)$ computed using $\{A^\times_{I,K,\phi}:\ K\text{ finite},\phi:I\to K$ surjective$\}$ in place of $\mathcal{P},\mathcal{Q},\mathcal{R}(\cdots)$, and may be written explicitly as in (76) below. We have also used the fact [11, Lemma 5.9] that $M_I(\dot P,\dot Q,\dot R)=0$ unless $\dot P,\dot Q$ are the smallest elements containing $\dot P\cap\dot Q$, which forces $\dot P=\dot Q$ as $\dot P,\dot Q$ take values in the same set. We may rewrite (65) as

$$\begin{aligned}&\Pi_k^{\mathrm{vi}}\bigl[P_{(I,\trianglelefteq)}(f_i:\ i\in I)\bigr]\\&\quad=\sum_{\substack{\text{iso. classes of finite}\\ \text{sets }K,\,k\leqslant|K|\leqslant|I|}}\ \sum_{\substack{\phi:I\to K\\ \text{surjective}}}N_{I,K,\phi,k}\\&\qquad\cdot\Biggl[\sum_{m\in M,\,n\in N_{nm},\,p\in P_m,}c_{mn}\bigl[\bigl(\bigl[W_{mn}\times_{V_m}Y_{mp}^{A^\times_{I,K,\phi}}/G_m\ltimes(K_{mp})^{A^\times_{I,K,\phi}}\bigr],\boldsymbol{\sigma}(I)\circ\xi_{mnp}\bigr)\bigr]\Biggr],\end{aligned}\tag{66}$$

where

$$N_{I,K,\phi,k} = \frac{1}{|K|!} \cdot \sum_{\substack{\dot R \in \{A^\times_{I,L,\psi}:\ L \text{ finite}, \psi: I \to L \text{ surjective}\}: \\ \dot R \subseteq A^\times_{I,K,\phi},\ M_I(A^\times_{I,K,\phi}, A^\times_{I,K,\phi}, \dot R) \neq 0,\ \dim \dot R = k}} M_I\big(A^\times_{I,K,\phi}, A^\times_{I,K,\phi}, \dot R\big), \qquad (67)$$

setting $\dot P = A^\times_{I,K,\phi}$ and replacing the sum over $\dot P$ in (65) with sums over K, ϕ. The factor $1/|K|!$ in (67) compensates for the fact that if $\iota : K \to K$ is a bijection then

$$A^\times_{I,K,\iota\circ\phi} = A^\times_{I,K,\phi},$$

and there are $|K|!$ such bijections ι, so each $A^\times_{I,K,\phi}$ is represented by $|K|!$ choices of K, ϕ in (66). Note that $N_{I,K,\phi,k}$ in (67) lies in $\mathbb{Q}$ and depends only on I, K, ϕ up to isomorphism and k, as we want.

Now let I, K, ϕ be as in (66) and define a partial order $\preccurlyeq$ on I as in (52). Then $\trianglelefteq$ dominates $\preccurlyeq$ and we have a 1-morphism $Q(I,\preccurlyeq,\trianglelefteq) : \mathfrak{M}(I,\preccurlyeq)_{\mathcal{A}} \to \mathfrak{M}(I,\trianglelefteq)_{\mathcal{A}}$. In a similar way to (54), we claim there is a natural 1-isomorphism

$$\mathfrak{H}_{mp} \times_{\iota_{mp},\mathfrak{M}(I,\trianglelefteq)_{\mathcal{A}},Q(I,\preccurlyeq,\trianglelefteq)} \mathfrak{M}(I,\preccurlyeq)_{\mathcal{A}} \cong \big[Y_{mp}^{A^\times_{I,K,\phi}} / G_m \ltimes (K_{mp})^{A^\times_{I,K,\phi}}\big], \qquad (68)$$

where $\iota_{mp} : \mathfrak{H}_{mp} \to \mathfrak{M}(I,\trianglelefteq)_{\mathcal{A}}$ is the inclusion. To see this, observe that a point of the r.h.s. of (68) is equivalent to a point $[(\sigma,\iota,\pi)]$ in $\mathfrak{H}_{mp}(\mathbb{K}) \subseteq \mathcal{M}(I,\trianglelefteq)_{\mathcal{A}}$ together with a choice of commutative diagram (64) in which $\dot P = \jmath_m^{-1}(A^\times_{I,K,\phi})$.

That is, the $(I,\trianglelefteq)$-configuration (σ,ι,π) is equipped with a choice of $\mathbb{K}$-group morphism $\rho : A^\times_{I,K,\phi} \to \mathrm{Aut}(\sigma,\iota,\pi)$ such that $\prod_{i\in I} \boldsymbol{\sigma}(\{i\})_* \circ \rho : A^\times_{I,K,\phi} \to \prod_{i\in I} \mathrm{Aut}(\sigma(\{i\}))$ maps $\delta \mapsto (\delta(i)\,\mathrm{id}_{\sigma(\{i\})})_{i\in I}$. It is not difficult to show that choosing such a ρ is equivalent, up to canonical isomorphism, to choosing an $(I,\preccurlyeq)$-*improvement* of (σ,ι,π), in the sense of [12, §6], and (68) follows. Using (55), (56) and (68) we see that the last line $[\cdots]$ of (66) is

$$\boldsymbol{\sigma}(I)_* \circ Q(I,\preccurlyeq,\trianglelefteq)^* \circ \Big(\prod_{i\in I} \boldsymbol{\sigma}\big(\{i\}\big)\Big)^* \Big(\bigotimes_{i\in I} f_i\Big) = P_{(I,\preccurlyeq)}(f_i : i \in I),$$

and so (66) implies (52), completing the proof of Theorem 5.17. □

Write $[f,g] = f * g - g * f$ for $f, g \in \mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ or $\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, *, *)$. Then $[\,,]$ satisfies the *Jacobi identity* and is a *Lie bracket* by Theorem 5.2. We shall prove an analogue of Theorem 4.9:

Theorem 5.18. *Let Assumptions* 2.11 *and* 3.4 *hold. Then* $\mathrm{SF}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ *and* $\bar{\mathrm{SF}}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, *, *)$ *are closed under* $[\,,]$, *and are Lie algebras, and* (41) *restricts to Lie algebra morphisms* $\mathrm{SF}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}), \bar{\mathrm{SF}}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, *, *) \to \mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ *when* $\mathbb{K}$ *has characteristic zero.*

Proof. If $f, g \in \mathrm{SF}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ or $\bar{\mathrm{SF}}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, *, *)$ then as $* = P_{(\{1,2\},\leqslant)}$ we have

$$\begin{aligned}\Pi^{\mathrm{vi}}_1\big([f,g]\big) &= \Pi^{\mathrm{vi}}_1\big(P_{(\{1,2\},\leqslant)}(f,g)\big) - \Pi^{\mathrm{vi}}_1\big(P_{(\{1,2\},\leqslant)}(g,f)\big) \\ &= \big(P_{(\{1,2\},\leqslant)}(f,g) - P_{(\{1,2\},\bullet)}(f,g)\big) - \big(P_{(\{1,2\},\leqslant)}(g,f) - P_{(\{1,2\},\bullet)}(g,f)\big) \\ &= P_{(\{1,2\},\leqslant)}(f,g) - P_{(\{1,2\},\leqslant)}(g,f) = [f,g]\end{aligned}$$

by Theorem 5.17, where $\bullet$ is the partial order on $\{1,2\}$ with $i \bullet j$ if $i=j$, so that by symmetry $P_{(\{1,2\},\bullet)}(f,g)=P_{(\{1,2\},\bullet)}(g,f)$. Therefore $[f,g]$ also lies in $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ or $\bar{\mathrm{SF}}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}},*,*)$, proving the first part.

For the second part, combining Corollary 5.11 with the fact that Π_k^{vi} is the identity on $[(U\times[\operatorname{Spec}\mathbb{K}/(\mathbb{K}^\times)^l],\rho)]$ if $k=l$ and 0 otherwise, we see that $\bar{\mathrm{SF}}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}},*,*)$ is generated over Λ, Λ° or Ω by elements $[(U\times[\operatorname{Spec}\mathbb{K}/\mathbb{K}^\times],\rho)]$ for U a quasiprojective $\mathbb{K}$-variety. If $u\in U(\mathbb{K})$ with $\rho_*(u)=[X]\in\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ then $\rho_*:\mathbb{K}^\times\to\operatorname{Aut}(X)$ is injective and u contributes $\chi(\operatorname{Aut}(X)/\rho_*(\mathbb{K}^\times))$ to $\bar\pi^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}}([(U\times[\operatorname{Spec}\mathbb{K}/\mathbb{K}^\times],\rho)])$ at $[X]$. Since $\mathbb{K}^\times\cong\rho(\mathbb{K}^\times)\subseteq\operatorname{Aut}(X)$ we see that $\operatorname{rk}\operatorname{Aut}(X)\geqslant 1$, and if $\operatorname{rk}\operatorname{Aut}(X)>1$ then the action of a maximal torus of $\operatorname{Aut}(X)$ fibres $\operatorname{Aut}(X)/\rho_*(\mathbb{K}^\times)$ by tori, forcing $\chi(\operatorname{Aut}(X)/\rho_*(\mathbb{K}^\times))=0$.

Thus u makes a nonzero contribution at $[X]$ only if $\operatorname{rk}\operatorname{Aut}(X)=1$, that is, if X is indecomposable. Hence $\bar\pi^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}}([(U\times[\operatorname{Spec}\mathbb{K}/\mathbb{K}^\times],\rho)])$ is supported on indecomposables, and as such $[(U\times[\operatorname{Spec}\mathbb{K}/\mathbb{K}^\times],\rho)]$ generate $\bar{\mathrm{SF}}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}},*,*)$ we see that $\bar\pi^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}}$ maps to $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$, and is a Lie algebra morphism by Theorem 5.2. The result for $\pi^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}}$ follows as $\pi^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}}=\bar\pi^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}}\circ\bar\Pi^{*,*}_{\mathfrak{Obj}_{\mathcal{A}}}$. □

5.3. *Relations between $*$ and $\Pi_{[I,\kappa]}$ in $\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}},*,*)$*

Theorem 5.18 gives a compatibility between multiplication $*$ and the projections Π_n^{vi}, $\Pi_{[I,\kappa]}$, in that subspaces $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$, $\bar{\mathrm{SF}}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}},*,*)$ defined using the Π_n^{vi} are closed under $(f,g)\mapsto f*g-g*f$. This is one consequence of a deeper relationship, in which we can write $\Pi_{[K,\mu]}(f*g)$ explicitly in terms of the components $\Pi_{[I,\kappa]}(f)$, $\Pi_{[J,\lambda]}(g)$. This deeper relationship is very complicated to write down, so for simplicity we do so only for $f,g\in\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}},*,*)$, when we can modify Proposition 2.17 to represent f, g and $f\otimes g$ in a special way.

Theorem 5.19. *Suppose Assumptions* 2.11 *and* 3.4 *hold,* $[I,\kappa]$, $[J,\lambda]$, $[K,\mu]$ *are as in Definition* 5.12*, and* f, g *lie in* $\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}},\Upsilon,\Lambda)$, $\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}},\Upsilon,\Lambda^\circ)$ *or* $\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}},\Theta,\Omega)$ *with* $\Pi_{[I,\kappa]}(f)=f$ *and* $\Pi_{[J,\lambda]}(g)=g$. *Choose constructible* $T\subseteq\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})\times\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ *with* $f\otimes g$ *supported on* T*, and use the notation of Proposition* 5.15. *Write* $(\mathbb{K}^\times)^I,\ldots$ *for the* $\mathbb{K}$*-groups of functions* $I,\ldots\to\mathbb{K}^\times$.

Then we may represent $f\otimes g\in\bar{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}\times\mathfrak{Obj}_{\mathcal{A}},*,*)$ *as*

$$f\otimes g=\sum_{m\in M,\,n\in N_m} c_{mn}\bigl[\bigl(W_{mn}\times\bigl[\operatorname{Spec}\mathbb{K}/\bigl(\mathbb{K}^\times\bigr)^I\times\bigl(\mathbb{K}^\times\bigr)^J\bigr],\tau_{mn}\bigr)\bigr],\tag{69}$$

where N_m *is finite,* $c_{mn}\in\Lambda$, Λ° *or* Ω, W_{mn} *is a quasiprojective* $\mathbb{K}$*-variety, and*

$$\tau_{mn}:W_{mn}\times\bigl[\operatorname{Spec}\mathbb{K}/\bigl(\mathbb{K}^\times\bigr)^I\times\bigl(\mathbb{K}^\times\bigr)^J\bigr]\to\mathfrak{G}_m\subseteq\mathfrak{Obj}_{\mathcal{A}}\times\mathfrak{Obj}_{\mathcal{A}}$$

is induced, using the 1*-isomorphism* $\mathfrak{G}_m\cong[V_m/G_m]$*, by an injective* $\mathbb{K}$*-group morphism* $\rho_{mn}:(\mathbb{K}^\times)^I\times(\mathbb{K}^\times)^J\to G_m$ *and a morphism* $\sigma_{mn}:W_{mn}\to V_m^{\rho_{mn}((\mathbb{K}^\times)^I\times(\mathbb{K}^\times)^J)}\subseteq V_m$.

These have the property that if $w\in W_{mn}(\mathbb{K})$ *projects to* $v\in V_m(\mathbb{K})$ *and* $([X],[Y])\in\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})\times\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ *then* ρ_{mn} *maps to* $\operatorname{Stab}_{G_m}(v)\subseteq G_m$*, and there exist splittings*

$$X\cong\bigoplus_{i\in I}X_i\quad\text{and}\quad Y\cong\bigoplus_{j\in J}Y_j$$

in $\mathcal{A}$ with $[X_i]=\kappa(i)$ and $[Y_j]=\lambda(j)$ in $C(\mathcal{A})$ for all i, j, such that composing ρ_{mn} with (45) *yields the $\mathbb{K}$-group morphism $(\mathbb{K}^\times)^I\times(\mathbb{K}^\times)^J\to\mathrm{Aut}(X)\times\mathrm{Aut}(Y)$ given by*

$$(\gamma,\delta)\mapsto\left(\sum_{i\in I}\gamma(i)\,\mathrm{id}_{X_i},\sum_{j\in J}\delta(j)\,\mathrm{id}_{Y_j}\right).$$

In this representation we have

$$f*g=\sum_{m\in M,\,n\in N_m}c_{mn}\bigl[\bigl(W_{mn}\times\bigl[E_m^1/\bigl(\bigl(\mathbb{K}^\times\bigr)^I\times\bigl(\mathbb{K}^\times\bigr)^J\bigr)\ltimes E_m^0\bigr],\boldsymbol{\sigma}\bigl(\{1,2\}\bigr)\circ\xi_{mn}\bigr)\bigr],\quad(70)$$

where $(\mathbb{K}^\times)^I\times(\mathbb{K}^\times)^J$ acts on E_m^0, E_m^1 via ρ_{mn} and the G_m-actions, and E_m^0 acts trivially on E_m^1, and σ_{mn}, ρ_{mn} induce

$$\xi_{mn}:W_{mn}\times\bigl[E_m^1/\bigl(\bigl(\mathbb{K}^\times\bigr)^I\times\bigl(\mathbb{K}^\times\bigr)^J\bigr)\ltimes E_m^0\bigr]\to\bigl[V_m\times E_m^1/G_m\ltimes E_m^0\bigr],\quad(71)$$

using (46) *to regard the r.h.s. of* (71) *as a substack of $\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}}$.*

If L is a finite set and $\phi:I\to L$, $\psi:J\to L$ maps with $\phi\amalg\psi:I\amalg J\to L$ surjective, define $T_{L,\phi,\psi}\subseteq(\mathbb{K}^\times)^I\times(\mathbb{K}^\times)^J$ to be the subtorus of $(\gamma,\delta)\in(\mathbb{K}^\times)^I\times(\mathbb{K}^\times)^J$ for which there exists $\epsilon:L\to\mathbb{K}^\times$ with $\gamma(i)=\epsilon\circ\phi(i)$ and $\delta(j)=(\epsilon\circ\psi(j))^{-1}$ for all $i\in I$ and $j\in J$. Then ϵ determines γ,δ uniquely, so that $T_{L,\phi,\psi}\cong(\mathbb{K}^\times)^L$. For each $m\in M$, $n\in N_m$ and $i=0,1$ write $(E_m^i)^{T_{L,\phi,\psi}}$ for the vector subspace of E_m^i fixed by $\rho_{mn}(T_{L,\phi,\psi})$. Write $\mathrm{Aut}(K,\mu)$ for the finite group of bijections $\iota:K\to K$ with $\mu=\mu\circ\iota$. Then

$$\begin{aligned}&\Pi_{[K,\mu]}(f*g)\\&\quad=\frac{1}{|\mathrm{Aut}(K,\mu)|}\sum_{\substack{\text{iso. classes}\\\text{of finite sets }L}}\frac{(-1)^{|L|-|K|}}{|L|!}\sum_{\substack{\phi:I\to L,\,\psi:J\to L\text{ and}\\\theta:L\to K:\,\phi\amalg\psi\text{ surjective,}\\\mu(k)=\kappa((\theta\circ\phi)^{-1}(k))+\\\lambda((\theta\circ\psi)^{-1}(k)),\,k\in K}}\prod_{k\in K}\bigl(\bigl|\theta^{-1}(k)\bigr|-1\bigr)!\\&\qquad\sum_{m\in M,\,n\in N_m}c_{mn}\bigl[\bigl(W_{mn}\times\bigl[\bigl(E_m^1\bigr)^{T_{L,\phi,\psi}}/\bigl(\bigl(\mathbb{K}^\times\bigr)^I\times\bigl(\mathbb{K}^\times\bigr)^J\bigr)\ltimes\bigl(E_m^0\bigr)^{T_{L,\phi,\psi}}\bigr],\boldsymbol{\sigma}\bigl(\{1,2\}\bigr)\circ\xi_{mn}\bigr)\bigr].\end{aligned}\quad(72)$$

Proof. By Corollary 5.16 we can write $f\otimes g$ in the form (48). The proofs of the first part of Proposition 2.17 in [11, §5.3] and Corollary 5.11 then show we can write $f\otimes g$ as a sum of terms $c_{mn}[(W_{mn}\times[\mathrm{Spec}\,\mathbb{K}/T],\tau_{mn})]$, where $T\cong(\mathbb{K}^\times)^k$ and τ_{mn} maps to $\mathfrak{G}_m\subseteq\mathfrak{Obj}_{\mathcal{A}}\times\mathfrak{Obj}_{\mathcal{A}}$ and is induced by a $\mathbb{K}$-group morphism $\rho_{mn}:T\to G_m$ and a morphism $\sigma_{mn}:W_{mn}\to V_m^{\rho_{mn}(T)}\subseteq V_m$.

As $\Pi_{[I,\kappa]}(f)=f$ and $\Pi_{[J,\lambda]}(g)=g$ we can show using Definition 5.12 that the sum can be chosen such that when $w\in W_{mn}(\mathbb{K})$ projects to $v\in V_m(\mathbb{K})$ and $([X],[Y])\in\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})\times\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$, there is an isomorphism $T\cong(\mathbb{K}^\times)^I\times(\mathbb{K}^\times)^J$ for which the second and third paragraphs hold. This isomorphism may depend on w, but it does so constructibly, so refining the sum we can take the isomorphism to be constant on $W_{mn}(\mathbb{K})$, and identify T with $(\mathbb{K}^\times)^I\times(\mathbb{K}^\times)^J$. This gives the first three paragraphs of the theorem.

Equation (70) now follows from (69) as for (51). To prove (72) we apply $\Pi_{[K,\mu]}$ to (70) and use Definition 5.12 and [11, §5.2]. Deleting terms with $W_{mn}(\mathbb{K})=\emptyset$, let $m\in M$ and $n\in N_m$ and pick $w \in W_{mn}(\mathbb{K})$, projecting to $v \in V_m(\mathbb{K})$ and $([X],[Y]) \in \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K}) \times \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$. Then the first part of the theorem gives splittings $X \cong \bigoplus_{i\in I} X_i$ and $Y \cong \bigoplus_{j\in J} Y_j$. By Proposition 5.15(a) we have isomorphisms

$$E_m^0 \cong \mathrm{Hom}(Y,X) \cong \bigoplus_{i\in I,\, j\in J} \mathrm{Hom}(Y_j,X_i),$$

$$E_m^1 \cong \mathrm{Ext}^1(Y,X) \cong \bigoplus_{i\in I,\, j\in J} \mathrm{Ext}^1(Y_j,X_i).$$

Under these isomorphisms, $(\gamma,\delta) \in (\mathbb{K}^\times)^I \times (\mathbb{K}^\times)^J$ acts on E_m^0, E_m^1 via ρ_{mn} by multiplying by $\gamma(i)\delta(j)$ in $\mathrm{Hom}(Y_j,X_i)$, $\mathrm{Ext}^1(Y_j,X_i)$. It follows that for L,ϕ,ψ and $T_{L,\phi,\psi}$ as in the theorem we have

$$\big(E_m^0\big)^{T_{L,\phi,\psi}} \cong \bigoplus_{\substack{i\in I,\, j\in J:\\ \phi(i)=\psi(j)}} \mathrm{Hom}(Y_j,X_i), \qquad \big(E_m^1\big)^{T_{L,\phi,\psi}} \cong \bigoplus_{\substack{i\in I,\, j\in J:\\ \phi(i)=\psi(j)}} \mathrm{Ext}^1(Y_j,X_i). \qquad (73)$$

In applying $\Pi_{[K,\mu]}$ to (70) we can take the W_{mn} factor outside, as $((\mathbb{K}^\times)^I \times (\mathbb{K}^\times)^J) \ltimes E_m^0$ acts trivially upon it. If $\mathrm{Hom}(Y_j,X_i) \ncong 0 \ncong \mathrm{Ext}^1(Y_j,X_i)$ for all i,j then using (73) and the notation of [11, §5.2] we find that

$$\begin{aligned}
&\mathcal{P}\big(E_m^1, (\mathbb{K}^\times)^I \times (\mathbb{K}^\times)^J\big)\\
&\quad = \mathcal{Q}\big(\big((\mathbb{K}^\times)^I \times (\mathbb{K}^\times)^J\big) \ltimes E_m^0, (\mathbb{K}^\times)^I \times (\mathbb{K}^\times)^J\big)\\
&\quad = \mathcal{R}\big(E_m^1, \big((\mathbb{K}^\times)^I \times (\mathbb{K}^\times)^J\big) \ltimes E_m^0, (\mathbb{K}^\times)^I \times (\mathbb{K}^\times)^J\big)\\
&\quad = \{T_{L,\phi,\psi}\colon L \text{ a finite set},\ \phi: I \to L,\ \psi: J \to L,\ \phi \amalg \psi \text{ surjective}\}. \qquad (74)
\end{aligned}$$

If some $\mathrm{Hom}(Y_j,X_i)$, $\mathrm{Ext}^1(Y_j,X_i)$ are zero then $\mathcal{P},\mathcal{Q},\mathcal{R}(\cdots)$ may be subsets of the values in (74). However, since the formulae in [11, §5.2] give the same answers if we replace $\mathcal{P}(X,T^G)$, $\mathcal{Q}(G,T^G)$, $\mathcal{R}(X,G,T^G)$ by larger finite sets of $\mathbb{K}$-subgroups of T^G closed under intersections, the values (74) give the correct answer for computing $\Pi_{[K,\mu]}(\cdots)$, so we shall still use them.

Now let $P \in \mathcal{P}(\cdots)$ and $R \in \mathcal{R}(\cdots)$ with $R \subseteq P$ and $0 \neq c \in \mathbb{Q}$, and ν be the $\mathfrak{Obj}_{\mathcal{A}}$-weight function of Definition 5.12 defining $\Pi_{[K,\mu]}$. Then [11, Definition 5.15] defines a constructible set $(W_{mn} \times E_m^1)_{\nu,c}^{P,R}$ in $(W_{mn} \times E_m^1)(\mathbb{K})$, which we will evaluate. Let $(w,e) \in (W_{mn} \times E_m^1)(\mathbb{K})$ project to $([X],[Y]) \in \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K}) \times \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ under $(\boldsymbol{\sigma}(\{1\}) \times \boldsymbol{\sigma}(\{2\}))_*$ and to $[Z] \in \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ under $\boldsymbol{\sigma}(\{1,2\})_*$. Let $R = T_{L,\phi,\psi}$ for some L,ϕ,ψ. Then there is an exact sequence $0 \to X \to Z \to Y \to 0$ invariant under R. The splittings $X \cong \bigoplus_{i\in I} X_i$, $Y \cong \bigoplus_{j\in J} Y_j$ correspond to a splitting $Z \cong \bigoplus_{l\in L} Z_l$ with Z_l in an exact sequence

$$0 \to \bigoplus_{i\in I:\ \phi(i)=l} X_i \to Z_l \to \bigoplus_{j\in J:\ \psi(j)=l} Y_j \to 0.$$

It follows that $[Z_l] = \kappa(\phi^{-1}(l)) + \lambda(\psi^{-1}(l))$ in $C(\mathcal{A})$.

Under the natural isomorphism $R \cong (\mathbb{K}^\times)^L$, $\epsilon \in (\mathbb{K}^\times)^L$ acts on Z as $\sum_{l\in L} \epsilon(l)\,\mathrm{id}_{Z_l}$. From Definition 5.12 we find $\nu(R,[Z],(\boldsymbol{\sigma}(\{1,2\}) \circ \xi_{mn})_*)$ is 1 if there is a bijection $\theta: L \to K$ with

$\mu(k)=\kappa(\bar\phi^{-1}(k))+\lambda(\bar\psi^{-1}(k))$ for $k\in K$, where $\bar\phi=\theta\circ\phi$ and $\bar\psi=\theta\circ\psi$, and 0 otherwise. Then $R=T_{K,\bar\phi,\bar\psi}$. So [11, Definition 5.15] gives

$$\bigl(W_{mn}\times E_m^1\bigr)^{P,R}_{\nu,c}=W_{mn}(\mathbb{K})\times\bigl(E_m^1\bigr)^P(\mathbb{K})$$

if $c=1$ and $R=T_{K,\bar\phi,\bar\psi}$ as above, and

$$\bigl(W_{mn}\times E_m^1\bigr)^{P,R}_{\nu,c}=\emptyset$$

otherwise for $c\neq 0$. The definitions now yield:

$$\begin{aligned}
&\Pi_{[K,\mu]}\bigl(\bigl[\bigl(W_{mn}\times\bigl[E_m^1/\bigl((\mathbb{K}^\times)^I\times(\mathbb{K}^\times)^J\bigr)\ltimes E_m^0\bigr],\boldsymbol{\sigma}\bigl(\{1,2\}\bigr)\circ\xi_{mn}\bigr)\bigr]\bigr)\\
&\quad=\sum_{\substack{P,R\in\mathcal{P}(E_m^1,(\mathbb{K}^\times)^I\times(\mathbb{K}^\times)^J):\ R\subseteq P,\\ R=T_{K,\bar\phi,\bar\psi},\ \bar\phi:I\to K,\ \bar\psi:J\to K,\\ \mu(k)=\kappa(\bar\phi^{-1}(k))+\lambda(\bar\psi^{-1}(k)),\ k\in K,\\ M^{E_m^1}_{((\mathbb{K}^\times)^I\times(\mathbb{K}^\times)^J)\ltimes E_m^0}(P,P,R)\neq 0,}} M^{E_m^1}_{((\mathbb{K}^\times)^I\times(\mathbb{K}^\times)^J)\ltimes E_m^0}(P,P,R)\\
&\qquad\cdot\bigl[\bigl(W_{mn}\times\bigl[\bigl(E_m^1\bigr)^P/\bigl((\mathbb{K}^\times)^I\times(\mathbb{K}^\times)^J\bigr)\ltimes\bigl(E_m^0\bigr)^P\bigr],\boldsymbol{\sigma}\bigl(\{1,2\}\bigr)\circ\xi_{mn}\bigr)\bigr].
\end{aligned}\tag{75}$$

Here [11, §5.2] actually defines $\Pi^\nu_{\mathfrak{F}}(\cdots)$ as a sum over triples $P\in\mathcal{P}(X,T^G)$, $Q\in\mathcal{Q}(G,T^G)$ and $R\in\mathcal{R}(X,G,T^G)$ with $R\subseteq P\cap Q$ and $M_G^X(P,Q,R)\neq 0$. But P, Q are the smallest elements of $\mathcal{P}(X,T^G)$, $\mathcal{Q}(G,T^G)$ containing $P\cap Q$ by [11, Lemma 5.9], so (74) implies that $P=Q$, which we have included in (75).

To get from (75) to (72) we set $P=T_{L,\phi,\psi}$ and $R=T_{K,\bar\phi,\bar\psi}$ in (72). Then $R\subseteq P$ if and only if there exists a surjective $\theta:L\to K$ with $\bar\phi=\theta\circ\phi$ and $\bar\psi=\theta\circ\psi$, which is then unique. So we replace the sums over P, R in (75) with sums over isomorphism classes of L and maps ϕ,ψ,θ, as in (72). The combinatorial factors in the second line of (72) are then the product of

$$M^{E_m^1}_{((\mathbb{K}^\times)^I\times(\mathbb{K}^\times)^J)\ltimes E_m^0}(P,P,R)=(-1)^{|L|-|K|}\prod_{k\in K}\bigl(\bigl|\theta^{-1}(k)\bigr|-1\bigr)!\tag{76}$$

in (75), which can be computed explicitly using (74) and [11, Definition 5.8], and factors $1/|\mathrm{Aut}(K,\mu)|$, $1/|L|!$ to compensate for the fact that each pair (P,R) in (75) is $(T_{L,\phi,\psi},T_{K,\theta\circ\phi,\theta\circ\psi})$ for exactly $|\mathrm{Aut}(K,\mu)|\cdot|L|!$ quadruples (L,ϕ,ψ,θ) in (72). This completes the proof. $\square$

This theorem will be very useful in Section 6 when we impose extra assumptions on $\mathcal{A}$ implying formulae for $\dim E_m^0-\dim E_m^1$ and $\dim(E_m^0)^{T_{L,\phi,\psi}}-\dim(E_m^1)^{T_{L,\phi,\psi}}$, as then (72) will enable us to construct *algebra morphisms* from $\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}},*,*)$ to certain explicit algebras $B(\mathcal{A},\Lambda,\chi)$, $B(\mathcal{A},\Lambda^\circ,\chi)$ and $C(\mathcal{A},\Omega,\chi)$.

Theorems 5.17–5.19 are an important reason for introducing *virtual rank* in [11, §5] and the operators Π_n^{vi}, $\Pi_{[I,\kappa]}$. They show these operators have a useful compatibility with $P_{(I,\preccurlyeq)}$ and $*$, although this is difficult to state. For comparison, the simpler idea of *real rank* and operators Π_n^{re} in [11, §5.1] have no such compatibility with $*$, as far as the author knows.

5.4. Generalization of other parts of Section 4

So far we have generalized the material of Sections 4.1 and 4.8 to stack algebras. We can also generalize much of the rest of Section 4, using the techniques of Sections 5.1 and 5.2. This is straightforward, so we just sketch the main ideas.

For Section 4.2, we can use the idea of *local stack functions* $\underline{\mathrm{LSF}}, \mathrm{LSF}(\mathfrak{F})$ in [11, §4] to define spaces $\underline{\mathrm{L\dot{S}F}}, \mathrm{L\dot{S}F}(\mathfrak{Obj}_{\mathcal{A}})$ and $\underline{\mathrm{L\dot{S}F}}, \underline{\mathrm{L\dot{\bar{S}}F}}, \mathrm{L\dot{\bar{S}}F}(\mathfrak{Obj}_{\mathcal{A}}, *, *)$ of stack functions f in $\underline{\mathrm{LSF}}$, $\mathrm{LSF}(\mathfrak{Obj}_{\mathcal{A}})$ and $\underline{\mathrm{LSF}}, \underline{\mathrm{L\bar{S}F}}, \mathrm{L\bar{S}F}(\mathfrak{Obj}_{\mathcal{A}}, *, *)$ supported on $\coprod_{\alpha\in S} \mathfrak{Obj}_{\mathcal{A}}^{\alpha}$ for finite $S \subseteq \bar{C}(\mathcal{A})$, and subspaces $\mathrm{L\dot{S}F_{al}}(\mathfrak{Obj}_{\mathcal{A}})$, $\mathrm{L\dot{\bar{S}}F_{al}}(\mathfrak{Obj}_{\mathcal{A}}, *, *)$ analogous to $\mathrm{SF_{al}}(\mathfrak{Obj}_{\mathcal{A}})$, $\mathrm{\bar{S}F_{al}}(\mathfrak{Obj}_{\mathcal{A}}, *, *)$. Multiplication $*$ is well defined on all of these, making them into associative algebras.

For Section 4.3, define $*_L : \underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}) \times \underline{\mathrm{SF}}(\mathfrak{M}(\{1,2\}, \leqslant)_{\mathcal{A}}) \to \underline{\mathrm{SF}}(\mathfrak{M}(\{1,2\}, \leqslant)_{\mathcal{A}})$ by

$$f *_L r = Q\big(\{1,2,3\}, \leqslant, \{1,2\}, \leqslant, \beta\big)_*\big[\big(\sigma\big(\{2\}\big) \times Q\big(\{1,2,3\}, \leqslant, \{1,2\}, \leqslant, \alpha\big)\big)^*(f \otimes r)\big],$$

by analogy with (18), and define $*_R$ in a similar way. Then the analogue of Theorem 4.7 holds and shows $*_L, *_R$ are left and right representations of the algebra $\underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}})$ on $\underline{\mathrm{SF}}(\mathfrak{M}(\{1,2\}, \leqslant)_{\mathcal{A}})$. The same result holds if we work with any of the spaces SF, $\underline{\mathrm{SF}}$, $\underline{\mathrm{\bar{S}F}}$, $\mathrm{\bar{S}F}(*)$ or their local stack function versions. If $\mathbb{K}$ has characteristic zero, the linear maps $\pi^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}}, \pi^{\mathrm{stk}}_{\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}}}$ intertwine the representations $*_L, *_R$ of $\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$ on $\mathrm{SF}(\mathfrak{M}(\{1,2\}, \leqslant)_{\mathcal{A}})$ with the representations $*_L, *_R$ of $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ on $\mathrm{CF}(\mathfrak{M}(\{1,2\}, \leqslant)_{\mathcal{A}})$ of Section 4.3.

The generalization of Section 4.5 is immediate, with Assumption 4.11 implying that the subspaces $\underline{\mathrm{SF}}_{\mathrm{fin}}, \mathrm{SF_{fin}}(\mathfrak{Obj}_{\mathcal{A}})$ and $\underline{\mathrm{SF}}_{\mathrm{fin}}, \underline{\mathrm{\bar{S}F}}_{\mathrm{fin}}, \mathrm{\bar{S}F_{fin}}(\mathfrak{Obj}_{\mathcal{A}}, *, *)$ of stack functions with *finite support* are closed under $*$. Each stack function in one of these subspaces is a sum of stack functions supported over single points $[X] \in \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$. So using the relations in $\underline{\mathrm{SF}}_{\mathrm{fin}}, \underline{\mathrm{\bar{S}F}}_{\mathrm{fin}}$, $\mathrm{\bar{S}F_{fin}}(\mathfrak{Obj}_{\mathcal{A}}, *, *)$ we can write down simple representations of these subspaces. For instance:

Lemma 5.20. *The subspace $\underline{\mathrm{SF}}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$ of f in $\underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$ with finite support has Λ-basis $[(\operatorname{Spec}\mathbb{K}, X)]$ for $[X] \in \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$. Thus, $\iota_{\mathfrak{Obj}_{\mathcal{A}}} : \mathrm{CF_{fin}}(\mathfrak{Obj}_{\mathcal{A}}) \otimes_{\mathbb{Q}} \Lambda \to \underline{\mathrm{SF}}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$ is an isomorphism of Λ-modules.*

If Assumption 4.11 holds then $\mathrm{CF_{fin}}(\mathfrak{Obj}_{\mathcal{A}}) \otimes_{\mathbb{Q}} \Lambda$ and $\underline{\mathrm{SF}}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$ are both Λ-algebras, but in general $\iota_{\mathfrak{Obj}_{\mathcal{A}}}$ is *not* an isomorphism of Λ-algebras. Rather, $\underline{\mathrm{SF}}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$ may be thought of as a 'quantized' version of the 'classical' algebra $\mathrm{CF_{fin}}(\mathfrak{Obj}_{\mathcal{A}}) \otimes_{\mathbb{Q}} \Lambda$.

To generalize Section 4.6, let Assumptions 2.11, 3.4 and 4.11 hold, and consider the *Λ-universal enveloping algebra* $U^{\Lambda}(\mathrm{\bar{S}F^{ind}_{al,fin}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda))$ of the Λ-Lie subalgebra $\mathrm{\bar{S}F^{ind}_{al,fin}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$ of stack functions with finite support in $\mathrm{\bar{S}F^{ind}_{al}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$. Then

$$\Phi_{\mathrm{fin}} : U^{\Lambda}\big(\mathrm{\bar{S}F^{ind}_{al,fin}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)\big) \to \mathrm{\bar{S}F_{al,fin}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$$

is an *isomorphism*. The same holds for $U^{\Lambda^\circ}(\mathrm{\bar{S}F^{ind}_{al,fin}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda^\circ))$ and $U^{\Omega}(\mathrm{\bar{S}F^{ind}_{al,fin}}(\mathfrak{Obj}_{\mathcal{A}}, \Theta, \Omega))$. The author is not sure whether Φ is injective in the non-finite-support case.

We do not generalize Section 4.7, as there are problems with the analogue of the proof that Δ is multiplicative in Theorem 4.17. We have already discussed the analogue of Section 4.8 in

Section 5.1. In the quiver examples of Section 4.9, we can use the algebra $\underline{\mathrm{SF}}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$ to construct examples of *quantum groups*.

Example 5.21. Suppose Assumption 2.11 holds for $\mathbb{K}$, Υ, Λ, and let $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$, Γ and $Q = (Q_0, Q_1, b, e)$ be as in Example 4.23 or 4.25. Then Assumption 3.4 holds for $\mathcal{A} = \text{mod-}\mathbb{K}Q$, and there are simple elements $V_i \in \mathcal{A}$ for $i \in Q_0$. Write $\bar{\delta}_{[V_i]}$ for the stack function in $\underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$ associated to the constructible set $\{[V_i]\} \subseteq \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$, and let $\bar{\mathcal{C}}$ be the Λ-subalgebra of $\underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$ generated by the $\bar{\delta}_{[V_i]}$ for $i \in Q_0$. This is a stack function version of the *composition algebra* $\mathcal{C}$ of Section 4.1. Define e_{ij} for $i, j \in Q_0$ by $e_{ii} = 1$ and $-e_{ij}$ is the number of edges $\overset{i}{\bullet} \to \overset{j}{\bullet}$ in Q. Then $\dim \mathrm{Ext}^1(V_i, V_j) = -e_{ij}$ for $i \neq j$ in Q_0. Define $a_{ij} = e_{ij} + e_{ji}$.

Now the isomorphism $\mathcal{C} \cong U(\mathfrak{n}_+)$ in Example 4.25 identifies $\delta_{[V_i]}$ with $e_i \in U(\mathfrak{n}_+)$, and holds because the $\delta_{[V_i]}$ satisfy the identity

$$\big(\mathrm{ad}(\delta_{[V_i]})\big)^{1-a_{ij}} \delta_{[V_j]} \quad \text{in } \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) \quad \text{if } i \neq j \in Q_0, \tag{77}$$

known as the *Serre relations*. We will explain the corresponding relations for the $\bar{\delta}_{[V_i]}$ in $\underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$. For $0 \leqslant k \leqslant n$ define the *Gauss polynomial*

$$\binom{n}{k}_{\ell} = \frac{(\ell^n - 1)(\ell^{n-1} - 1) \cdots (\ell^{n-k+1} - 1)}{(\ell^k - 1)(\ell^{k-1} - 1) \cdots (\ell - 1)}.$$

With this notation, for $i \neq j \in Q_0$ we claim that

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{\ell} \ell^{-k(1-a_{ij}-k)/2} \bar{\delta}_{[V_i]}^{*^k} * \bar{\delta}_{[V_j]} * \bar{\delta}_{[V_i]}^{*^{1-a_{ij}-k}} = 0. \tag{78}$$

Here f^{*^k} means $f * f * \cdots * f$, with f occurring k times. Equation (78) is known as the *quantum Serre relations*, and with q in place of ℓ and X_i^+ in place of $\bar{\delta}_{[V_i]}$ it was introduced by Drinfeld [4, Example 6.2] as the defining relations of the *quantum group* $U_q(\mathfrak{n}_+)$. We recover (77) from (78) by replacing $\bar{\delta}_{[V_i]}$ by $\delta_{[V_i]}$ and taking the limit $\ell \to 1$.

To prove (78) we can adapt the proof in Ringel [24, §2]. Ringel works over finite fields with q elements, so that to define his Ringel–Hall multiplication he simply counts numbers of filtrations satisfying some conditions, giving answers which are polynomials in q. In our case, using the ideas of Sections 5.1 and 5.2 we translate Ringel's manipulation of finite counts q^k into addition and subtraction of constructible sets of the form $\mathbb{K}^k$, which become factors ℓ^k by the relations in $\underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$. We leave the details to the reader.

Let the 'quantum group' $U_\ell(\mathfrak{n}_+)$ be defined by the usual quantum Serre relations over the algebra Λ. Then from (78) we obtain a unique, surjective algebra morphism $U_\ell(\mathfrak{n}_+) \to \bar{\mathcal{C}}$. In the case of Example 4.23, when $\underline{\mathrm{SF}}_{\mathrm{fin}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda) = \underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$, we can use Lemma 5.20 and $U(\mathfrak{n}_+) \cong \mathcal{C}$ to show this is an *isomorphism*. This is a Ringel–Hall-type realization of $U_\ell(\mathfrak{n}_+)$ using stack functions, parallel to those of Ringel [24] using finite fields, and Lusztig [19, Theorem 10.17] using perverse sheaves. We state this in the following theorem:

Theorem 5.22. *Let $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be a Kac–Moody algebra constructed from an undirected graph Γ, and Q be a quiver with underlying graph Γ and no oriented cycles. Set $\mathcal{A} =$* mod-$\mathbb{K}Q$

and construct the stack composition algebra $\bar{\mathcal{C}}$ *in* $\underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$ *as above. Then there is a natural surjective algebra morphism* $U_\ell(\mathfrak{n}_+) \to \bar{\mathcal{C}}$ *from the quantum group* $U_\ell(\mathfrak{n}_+)$ *over* Λ *of the positive part of* $\mathfrak{g}$*. If* Γ *is of type A, D or E, so that* $\mathfrak{g}$ *is finite-dimensional, this is an isomorphism.*

Remark 5.23. This suggests that stack algebras such as $\underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$ can be regarded as quantized versions of $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$, with quantum parameter $\ell = \Upsilon([\mathbb{K}])$. However, the theorem is less satisfactory than it first appears. In it we have defined the 'quantum group' $U_\ell(\mathfrak{n}_+)$ by the quantum Serre relations *over the algebra* Λ of Assumption 2.11. Now by definition $\ell - 1$ is *invertible* in Λ, so we cannot take $\Lambda = \mathbb{C}[[\ell - 1]]$, the algebra of formal power series in $\ell - 1$, conventionally used to define quantum groups as in Drinfeld [4]. Thus $U_\ell(\mathfrak{n}_+)/(\ell - 1)U_\ell(\mathfrak{n}_+)$ is 0, rather than $U(\mathfrak{n}_+)$ as in the usual definition. Also, we would have liked to recover the classical case of Section 4.9 by an algebra morphism $\underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda) \to \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$, but there is no such morphism.

The obvious resolution to both these objections is to work instead in the space $\bar{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda^\circ)$, over the algebra Λ° in which $\ell - 1$ is not invertible, which does have an algebra morphism $\bar{\pi}^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}} : \bar{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda^\circ) \to \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$. Then we could define $U_\ell(\mathfrak{n}_+)$ over Λ° rather than Λ, which would in fact allow $\Lambda^\circ = \mathbb{C}[[\ell - 1]]$. Unfortunately, though, the quantum Serre relations (78) do not hold in $\bar{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda^\circ)$, because its proof mixes terms with stabilizer groups of different ranks, which are separated in $\bar{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda^\circ)$.

One way out may be to define a new space $\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda^\circ)$ to be the Λ°-submodule of $\underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$ generated by equivalence classes $[(\mathfrak{R}, \rho)]$ with ρ representable. It is a Λ°-subalgebra of $\underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$, so (78) holds in it. Define $\bar{\mathcal{C}}^\circ$ to be the Λ°-subalgebra of $\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda^\circ)$ generated by the $\bar{\delta}_{[V_i]}$, and $U_\ell(\mathfrak{n}_+)^\circ$ to be the quantum group defined over Λ°. Then there is a surjective morphism $U_\ell(\mathfrak{n}_+)^\circ \to \bar{\mathcal{C}}^\circ$. There should also be an algebra morphism $\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda^\circ) \to \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ defined in the same way as $\bar{\pi}^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}}$, which projects $\bar{\mathcal{C}}^\circ \to \mathcal{C}$ commuting with the natural projection $U_\ell(\mathfrak{n}_+)^\circ \to U(\mathfrak{n}_+)$. If we knew that $\bar{\mathcal{C}}^\circ$ was a *free* Λ°-submodule we could use this and the isomorphism $\mathcal{C} \cong U(\mathfrak{n}_+)$ of Example 4.25 to show that $\bar{\mathcal{C}}^\circ \cong U_\ell(\mathfrak{n}_+)^\circ$.

6. Morphisms from stack (Lie) algebras

If $\mathcal{A} = \text{mod-}\mathbb{K}Q$ for a quiver Q, or $\mathcal{A} = \mathrm{coh}(P)$ for a smooth projective curve P, there is a biadditive $\chi : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$ called the *Euler form* with

$$\dim_{\mathbb{K}} \mathrm{Hom}(X, Y) - \dim_{\mathbb{K}} \mathrm{Ext}^1(X, Y) = \chi\big([X], [Y]\big) \quad \text{for all } X, Y \in \mathcal{A}.$$

Assuming this, Sections 6.1–6.5 construct *algebra morphisms* Φ^Λ, Ψ^Λ, Ψ^{Λ°, Ψ^Ω from $\underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}})$, $\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, *, *)$ to *explicit algebras* $A(\mathcal{A}, \Lambda, \chi)$, $B(\mathcal{A}, \Lambda \text{ or } \Lambda^\circ, \chi)$ and $C(\mathcal{A}, \Omega, \chi)$ depending only on $C(\mathcal{A})$, χ, Λ, Λ°, Ω, which restrict to *Lie algebra morphisms* from $\bar{\mathrm{SF}}{}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, *, *)$ to $B^{\mathrm{ind}}(\mathcal{A}, \Lambda \text{ or } \Lambda^\circ, \chi)$, $C^{\mathrm{ind}}(\mathcal{A}, \Omega, \chi)$.

In a similar way, if $\mathcal{A} = \mathrm{coh}(P)$ for P a *Calabi–Yau* 3*-fold* then

$$\begin{aligned}&\big(\dim_{\mathbb{K}} \mathrm{Hom}(X, Y) - \dim_{\mathbb{K}} \mathrm{Ext}^1(X, Y)\big) - \big(\dim_{\mathbb{K}} \mathrm{Hom}(Y, X) - \dim_{\mathbb{K}} \mathrm{Ext}^1(Y, X)\big)\\ &\quad = \bar{\chi}\big([X], [Y]\big) \quad \text{for all } X, Y \in \mathcal{A},\end{aligned}$$

for an antisymmetric bilinear $\bar{\chi}: K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$. Assuming this, Section 6.6 constructs a *Lie algebra morphism* $\Psi^{\Omega}: \bar{\mathrm{SF}}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}}, \Theta, \Omega) \to C^{\mathrm{ind}}(\mathcal{A}, \Omega, \frac{1}{2}\bar{\chi})$.

These (Lie) algebra morphisms will be essential tools in the sequels [13,14] on *stability conditions*. Given a permissible weak stability condition $(\tau, T, \leqslant)$ on $\mathcal{A}$, in [13] we will construct interesting subalgebras $\bar{\mathcal{H}}^{\mathrm{pa}}(\tau), \bar{\mathcal{H}}^{\mathrm{to}}(\tau)$ of $\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ and Lie subalgebras $\bar{\mathcal{L}}^{\mathrm{pa}}(\tau), \bar{\mathcal{L}}^{\mathrm{to}}(\tau)$ of $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$. Under mild conditions, [14] shows these (Lie) subalgebras are *independent of the choice of* $(\tau, T, \leqslant)$, and gives combinatorial *basis change formulae* relating bases of $\bar{\mathcal{H}}^{\mathrm{pa}}(\tau), \ldots, \bar{\mathcal{L}}^{\mathrm{to}}(\tau)$ associated to $(\tau, T, \leqslant)$ and $(\tilde{\tau}, \tilde{T}, \leqslant)$.

Thus Φ^{Λ}, Ψ^{Λ}, $\Psi^{\Lambda^{\circ}}$, Ψ^{Ω} induce morphisms $\bar{\mathcal{H}}^{\mathrm{pa}}(\tau), \bar{\mathcal{H}}^{\mathrm{to}}(\tau) \to A, B, C(\mathcal{A}, *, \chi)$ and $\bar{\mathcal{L}}^{\mathrm{pa}}(\tau), \bar{\mathcal{L}}^{\mathrm{to}}(\tau) \to B^{\mathrm{ind}}, C^{\mathrm{ind}}(\mathcal{A}, *, \chi)$. We shall regard these as encoding interesting *families of invariants* of $\mathcal{A}$, $(\tau, T, \leqslant)$, which 'count' τ-(semi)stable objects and configurations of objects in $\mathcal{A}$ with fixed classes in $C(\mathcal{A})$. The fact that $\Phi^{\Lambda}, \ldots, \Psi^{\Omega}$ are (Lie) algebra morphisms imply *multiplicative identities* on the invariants, and the basis change formulae imply *transformation laws* for the invariants from $(\tau, T, \leqslant)$ to $(\tilde{\tau}, \tilde{T}, \leqslant)$.

6.1. Identities relating i_{Λ} and $*$, $P_{(I,\preccurlyeq)}$

Recall that a $\mathbb{K}$-linear abelian category $\mathcal{A}$ is called of *finite type* if $\mathrm{Ext}^i(X, Y)$ is a finite-dimensional $\mathbb{K}$-vector space for all $X, Y \in \mathcal{A}$ and $i \geqslant 0$, and $\mathrm{Ext}^i(X, Y) = 0$ for $i \gg 0$. Then there is a unique biadditive map $\chi: K_0(\mathcal{A}) \times K_0(\mathcal{A}) \to \mathbb{Z}$ on the Grothendieck group $K_0(\mathcal{A})$ known as the *Euler form*, satisfying

$$\chi\big([X], [Y]\big) = \sum_{i \geqslant 0} (-1)^i \dim_{\mathbb{K}} \mathrm{Ext}^i(X, Y) \quad \text{for all } X, Y \in \mathcal{A}. \tag{79}$$

We shall suppose $K(\mathcal{A})$ in Assumption 3.4 is chosen such that χ factors through the projection $K_0(\mathcal{A}) \to K(\mathcal{A})$, and so descends to $\chi: K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$. All this holds for the examples $\mathcal{A} = \mathrm{coh}(P)$ in [12, Example 9.1] with P a smooth projective $\mathbb{K}$-scheme, for the quiver examples $\mathcal{A} = \text{mod-}\mathbb{K}Q$ in [12, Example 10.5], and for many of the other examples of [12, §10].

Now assume $\mathrm{Ext}^i(X, Y) = 0$ for all $X, Y \in \mathcal{A}$ and $i > 1$. Then (79) becomes

$$\dim_{\mathbb{K}} \mathrm{Hom}(X, Y) - \dim_{\mathbb{K}} \mathrm{Ext}^1(X, Y) = \chi\big([X], [Y]\big) \quad \text{for all } X, Y \in \mathcal{A}. \tag{80}$$

This happens for $\mathcal{A} = \mathrm{coh}(P)$ in [12, Example 9.1] with P a smooth projective curve, and for $\mathcal{A} = \text{mod-}\mathbb{K}Q$ in [12, Example 10.5]. We shall prove that multiplication $*$ on $\underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$ and i_{Λ} in Proposition 2.16 satisfy an important identity.

Theorem 6.1. *Let Assumptions* 2.11 *and* 3.4 *hold, and i_{Λ} be as in Proposition* 2.16. *Suppose $\chi: K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$ is biadditive and satisfies* (80). *Let $f, g \in \underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$ be supported on $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}$, $\mathfrak{Obj}_{\mathcal{A}}^{\beta}$ respectively, for $\alpha, \beta \in \bar{C}(\mathcal{A})$. Write $\Pi: \mathfrak{Obj}_{\mathcal{A}} \to \mathrm{Spec}\,\mathbb{K}$ for the projection. Then*

$$i_{\Lambda}^{-1} \circ \Pi_*(f * g) = \ell^{-\chi(\beta,\alpha)} \big(i_{\Lambda}^{-1} \circ \Pi_*(f)\big)\big(i_{\Lambda}^{-1} \circ \Pi_*(g)\big) \quad \text{in } \Lambda. \tag{81}$$

Proof. Choose a constructible set $T \subseteq \mathfrak{Obj}_{\mathcal{A}}^{\alpha}(\mathbb{K}) \times \mathfrak{Obj}_{\mathcal{A}}^{\beta}(\mathbb{K})$ with $f \otimes g$ supported on T, and use the notation of Proposition 5.15. Since $\underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$ is generated by $[(U, \rho)]$ with U

a variety and ρ representable, arguing as in Proposition 5.9 and Corollary 5.16 we may write $f \otimes g$ and $f * g$ in the forms (48) and (51), for N_m finite and $c_{mn} \in \Lambda$.

As i_Λ is an algebra isomorphism we have

$$\big(i_\Lambda^{-1} \circ \Pi_*(f)\big)\big(i_\Lambda^{-1} \circ \Pi_*(g)\big) = i_\Lambda^{-1}\big(\Pi_*(f) \cdot \Pi_*(g)\big) = i_\Lambda^{-1} \circ (\Pi \times \Pi)_*(f \otimes g). \tag{82}$$

Equations (48) and (51) and relations in $\underline{\mathrm{SF}}(\operatorname{Spec}\mathbb{K}, \Upsilon, \Lambda)$ imply that

$$i_\Lambda^{-1} \circ (\Pi \times \Pi)_*(f \otimes g) = \sum_{m\in M,\, n\in N_m} c_{mn}\Upsilon\big([W_{mn}]\big)\Upsilon\big([G_m]\big)^{-1}, \tag{83}$$

$$i_\Lambda^{-1} \circ \Pi_*(f * g) = \sum_{m\in M,\, n\in N_m} c_{mn}\Upsilon\big([W_{mn}]\big)\Upsilon\big(\big[E_m^1\big]\big)\Upsilon\big([G_m]\big)^{-1}\Upsilon\big(\big[E_m^0\big]\big)^{-1}, \tag{84}$$

using Assumption 2.11(ii) and that $G_m \ltimes E_m^0 \cong G_m \times E_m^0$ as $\mathbb{K}$-varieties. But

$$\Upsilon\big(\big[E_m^1\big]\big)\Upsilon\big(\big[E_m^0\big]\big)^{-1} = \ell^{\dim E_m^1}\ell^{-\dim E_m^0} = \ell^{-\chi(\beta,\alpha)}, \tag{85}$$

by Assumption 2.11, Proposition 5.15(a), (80) and the fact that $T \subseteq \mathfrak{Obj}_{\mathcal{A}}^{\alpha}(\mathbb{K}) \times \mathfrak{Obj}_{\mathcal{A}}^{\beta}(\mathbb{K})$. Equation (81) now follows from (82)–(85). □

We generalize this to the operations $P_{(I,\preccurlyeq)}$ of Definition 5.3. Theorem 6.1 is the case $(I,\preccurlyeq) = (\{1,2\},\leqslant)$ of Theorem 6.2.

Theorem 6.2. *Suppose Assumptions* 2.11 *and* 3.4 *hold, and* $\chi : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$ *is biadditive and satisfies* (80)*. Let* $(I,\preccurlyeq)$ *be a finite poset,* $\kappa : I \to \bar{C}(\mathcal{A})$ *and* $f_i \in \underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$ *be supported on* $\mathfrak{Obj}_{\mathcal{A}}^{\kappa(i)}$ *for all* $i \in I$*. Then*

$$i_\Lambda^{-1} \circ \Pi_*\big(P_{(I,\preccurlyeq)}(f_i\colon i \in I)\big) = \Bigg[\prod_{i\neq j\in I:\ i\preccurlyeq j} \ell^{-\chi(\kappa(j),\kappa(i))}\Bigg] \cdot \Bigg[\prod_{i\in I} i_\Lambda^{-1} \circ \Pi_*(f_i)\Bigg]. \tag{86}$$

Proof. When $|I| = 0$ or 1, Eq. (86) is obvious. Suppose by induction that (86) holds for $|I| \leqslant n$, and let I, $\preccurlyeq$, κ be as above with $|I| = n+1$. Choose $k \in I$ to be $\preccurlyeq$-maximal, and define $J = I \setminus \{k\}$, $K = \{i \in I\colon i \preccurlyeq k\}$, $L = J \cap K$, and $\phi : K \to \{1,2\}$ by $\phi(i) = 1$ for $i \in L$ and $\phi(k) = 2$. Then a similar proof to [12, Theorem 7.10] shows the following is a *Cartesian square*:

$$\begin{array}{ccc}
\mathfrak{M}(I,\preccurlyeq)_{\mathcal{A}} & \xrightarrow[Q(L,\preccurlyeq,\{1,2\},\leqslant,\phi)\circ S(I,\preccurlyeq,K)]{} & \mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}} \\
\Big\downarrow {\scriptstyle S(I,\preccurlyeq,J)\times\sigma(\{k\})} & & {\scriptstyle \boldsymbol{\sigma}(\{1\})\times\boldsymbol{\sigma}(\{2\})} \Big\downarrow \\
\mathfrak{M}(J,\preccurlyeq,\kappa)_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}} & \xrightarrow{\boldsymbol{\sigma}(L)\times\mathrm{id}_{\mathfrak{Obj}_{\mathcal{A}}}} & \mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}}.
\end{array} \tag{87}$$

If $f_i \in \underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$ for $i \in I$ are as in the theorem we have

$$\begin{aligned}
&i_\Lambda^{-1}\circ\Pi_*\big(P_{(I,\preccurlyeq)}(f_i\colon i\in I)\big)\\
&\quad=i_\Lambda^{-1}\circ\Pi_*\circ\boldsymbol\sigma(I)_*\Big[\Big(\prod_{i\in I}\boldsymbol\sigma\big(\{i\}\big)\Big)^*\Big(\bigotimes_{i\in I}f_i\Big)\Big]\\
&\quad=i_\Lambda^{-1}\circ\Pi_*\circ\boldsymbol\sigma\big(\{1,2\}\big)_*\circ\big(Q\big(L,\preccurlyeq,\{1,2\},\leqslant,\phi\big)\circ S(I,\preccurlyeq,K)\big)_*\\
&\qquad\circ\big(S(I,\preccurlyeq,J)\times\sigma\big(\{k\}\big)\big)^*\Big[\Big(\prod_{j\in J}\boldsymbol\sigma\big(\{j\}\big)\Big)^*\Big(\bigotimes_{j\in J}f_j\Big)\otimes f_k\Big]\\
&\quad=i_\Lambda^{-1}\circ\Pi_*\circ\boldsymbol\sigma\big(\{1,2\}\big)_*\circ\big(\boldsymbol\sigma\big(\{1\}\big)\times\boldsymbol\sigma\big(\{2\}\big)\big)^*\\
&\qquad\circ\big(\boldsymbol\sigma(L)\times\mathrm{id}_{\mathfrak{Obj}_{\mathcal A}}\big)_*\Big[\Big(\prod_{j\in J}\boldsymbol\sigma\big(\{j\}\big)\Big)^*\Big(\bigotimes_{j\in J}f_j\Big)\otimes f_k\Big]\\
&\quad=i_\Lambda^{-1}\circ\Pi_*\Big\{\Big(\boldsymbol\sigma(L)_*\Big[\Big(\prod_{j\in J}\boldsymbol\sigma\big(\{j\}\big)\Big)^*\Big(\bigotimes_{j\in J}f_j\Big)\Big]\Big)*f_k\Big\}\\
&\quad=\ell^{-\chi(\kappa(k),\kappa(L))}\Big(i_\Lambda^{-1}\circ\Pi_*\circ\boldsymbol\sigma(L)_*\Big[\Big(\prod_{j\in J}\boldsymbol\sigma\big(\{j\}\big)\Big)^*\Big(\bigotimes_{j\in J}f_j\Big)\Big]\Big)\big(i_\Lambda^{-1}\circ\Pi_*(f_k)\big)\\
&\quad=\Big[\prod_{i\in L}\ell^{-\chi(\kappa(k),\kappa(i))}\Big]\big(i_\Lambda^{-1}\circ\Pi_*\circ P_{(J,\preccurlyeq)}(f_j\colon j\in J)\big)\big(i_\Lambda^{-1}\circ\Pi_*(f_k)\big).
\end{aligned}\tag{88}$$

Here we have used (42) in the first step, 2-isomorphisms

$$\Pi\circ\boldsymbol\sigma(I)\cong\Pi\circ\boldsymbol\sigma\big(\{1,2\}\big)\circ Q\big(L,\preccurlyeq,\{1,2\},\leqslant,\phi\big)\circ S(I,\preccurlyeq,K)\quad\text{and}$$
$$\prod_{i\in I}\boldsymbol\sigma\big(\{i\}\big)\cong\Big(\Big(\prod_{j\in J}\boldsymbol\sigma\big(\{j\}\big)\Big)\times\mathrm{id}_{\mathfrak{Obj}_{\mathcal A}}\Big)\circ\big(S(I,\preccurlyeq,J)\times\sigma\big(\{k\}\big)\big)$$

in the second, Theorem 2.9 and (87) Cartesian in the third, (40) in the fourth, Theorem 6.1 and $\boldsymbol\sigma(L)_*[(\prod_{j\in J}\boldsymbol\sigma(\{j\}))^*(\bigotimes_{j\in J}f_j)]$ supported on $\mathfrak{Obj}_{\mathcal A}^{\kappa(L)}$ in the fifth, and $\Pi\circ\boldsymbol\sigma(L)\cong\Pi\circ\boldsymbol\sigma(J)$ and (42) in the sixth. Since $|J|=n$ we can expand $i_\Lambda^{-1}\circ\Pi_*\circ P_{(J,\preccurlyeq)}(f_j\colon j\in J)$ in the last line of (88) using (86) with J in place of I, and this proves (86) for I. The theorem follows by induction. □

6.2. *Algebras $A(\mathcal A,\Lambda,\chi)$ and morphisms to them*

If the factor $\ell^{-\chi(\beta,\alpha)}$ were not there, Eq. (81) would say $i_\Lambda^{-1}\circ\Pi_*$ is a morphism of $\mathbb Q$-algebras. We can make an algebra morphism $\Phi^\Lambda:\underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal A},\Upsilon,\Lambda)\to A(\mathcal A,\Lambda,\chi)$ by introducing generators a^α in $A(\mathcal A,\Lambda,\chi)$ for $\alpha\in\bar C(\mathcal A)$, and twisting multiplication $a^\alpha\star a^\beta$ in $A(\mathcal A,\Lambda,\chi)$ by $\ell^{-\chi(\beta,\alpha)}$.

Definition 6.3. Let Assumptions 2.11 and 3.4 hold. Then $K(\mathcal A)$ is an abelian group, $\bar C(\mathcal A)\subseteq K(\mathcal A)$ closed under addition, Λ a commutative $\mathbb Q$-algebra, and $\ell\in\Lambda$ is invertible. Suppose $\chi:K(\mathcal A)\times K(\mathcal A)\to\mathbb Z$ is a biadditive map. Using only this data $K(\mathcal A),\bar C(\mathcal A),\Lambda,\ell,\chi$ we will define a $\mathbb Q$-*algebra* $A(\mathcal A,\Lambda,\chi)$.

Let a^α for $\alpha \in \bar{C}(\mathcal{A})$ be formal symbols, and $A(\mathcal{A}, \Lambda, \chi)$ the Λ-module with basis $\{a^\alpha: \alpha \in \bar{C}(\mathcal{A})\}$. That is, $A(\mathcal{A}, \Lambda, \chi)$ is the set of sums $\sum_{\alpha\in\bar{C}(\mathcal{A})} \lambda^\alpha a^\alpha$ with $\lambda^\alpha \in \Lambda$ nonzero for only finitely many α. Addition, and multiplication by $\mathbb{Q}$, are defined in the obvious way. Define a *multiplication* $\star$ on $A(\mathcal{A}, \Lambda, \chi)$ by

$$\Bigl(\sum_{i\in I}\lambda_i a^{\alpha_i}\Bigr)\star\Bigl(\sum_{j\in J}\mu_j a^{\beta_j}\Bigr)=\sum_{i\in I}\sum_{j\in J}\lambda_i\mu_j\ell^{-\chi(\beta_j,\alpha_i)}a^{\alpha_i+\beta_j}, \tag{89}$$

where I, J are finite indexing sets, $\lambda_i, \mu_j \in \Lambda$ and $\alpha_i, \beta_j \in \bar{C}(\mathcal{A})$. Using the biadditivity of χ it is easy to verify $\star$ is associative, and makes $A(\mathcal{A}, \Lambda, \chi)$ into a $\mathbb{Q}$-algebra (in fact, a Λ-algebra), with identity a^0.

For $(I, \preccurlyeq)$ a finite poset, define $P_{(I,\preccurlyeq)} : \prod_{i\in I} A(\mathcal{A}, \Lambda, \chi) \to A(\mathcal{A}, \Lambda, \chi)$ by

$$P_{(I,\preccurlyeq)}\Bigl[\sum_{c_i\in C_i}\mu_i^{c_i}a^{\alpha_i^{c_i}} : i\in I\Bigr]=\sum_{\substack{\text{choices of}\\ c_i\in C_i \text{ for}\\ \text{all } i\in I}}\Bigl[\prod_{i\in I}\mu_i^{c_i}\Bigr]\cdot\Bigl[\prod_{i\neq j\in I:\, i\preccurlyeq j}\ell^{-\chi(\alpha_j^{c_j},\alpha_i^{c_i})}\Bigr]a^{\sum_{i\in I}\alpha_i^{c_i}}, \tag{90}$$

for C_i finite indexing sets, $\mu_i^{c_i} \in \Lambda$ and $\alpha_i^{c_i} \in \bar{C}(\mathcal{A})$. These $P_{(I,\preccurlyeq)}$ satisfy (36).

Now for each $\alpha \in \bar{C}(\mathcal{A})$ write $i_\alpha : \mathfrak{Obj}^\alpha_{\mathcal{A}} \to \mathfrak{Obj}_{\mathcal{A}}$ for the inclusion and $\Pi^\alpha : \mathfrak{Obj}^\alpha_{\mathcal{A}} \to \operatorname{Spec}\mathbb{K}$ for the projection 1-morphisms, and let i_Λ be as in Proposition 2.16. Define $\Phi^\Lambda : \underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda) \to A(\mathcal{A}, \Lambda, \chi)$ by

$$\Phi^\Lambda(f)=\sum_{\alpha\in\bar{C}(\mathcal{A})}\bigl[i_\Lambda^{-1}\circ\Pi^\alpha_*\circ i_\alpha^*(f)\bigr]a^\alpha\quad\text{for } f\in\underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}},\Upsilon,\Lambda). \tag{91}$$

This is well defined as f is supported on the disjoint union of $\mathfrak{Obj}^\alpha_{\mathcal{A}}$ over finitely many $\alpha \in \bar{C}(\mathcal{A})$, so $[i_\Lambda^{-1} \circ \Pi^\alpha_* \circ i_\alpha^*(f)] \neq 0$ in Λ for only finitely many α.

We can think of $\Phi^\Lambda(f)$ as encoding the 'integral' of f over $\mathfrak{Obj}^\alpha_{\mathcal{A}}$ for all α.

Theorem 6.4. *Let Assumptions* 2.11 *and* 3.4 *hold and* $\chi : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$ *be biadditive and satisfy* (80). *Then* $\Phi^\Lambda : \underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda) \to A(\mathcal{A}, \Lambda, \chi)$ *is a* Λ*-algebra morphism. If* $(I, \preccurlyeq)$ *is a finite poset and* $f_i \in \underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$ *for* $i \in I$ *then* $\Phi^\Lambda(P_{(I,\preccurlyeq)}[f_i: i \in I]) = P_{(I,\preccurlyeq)}[\Phi^\Lambda(f_i): i \in I]$.

Proof. Suppose $f^\alpha, g^\beta \in \underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$ are supported on $\mathfrak{Obj}^\alpha_{\mathcal{A}}$, $\mathfrak{Obj}^\beta_{\mathcal{A}}$ respectively, for $\alpha, \beta \in \bar{C}(\mathcal{A})$. Then for $\gamma \in \bar{C}(\mathcal{A})$ we have

$$\Pi^\gamma_*\circ i_\gamma^*\bigl(f^\alpha\bigr)=\begin{cases}\Pi_*(f^\alpha), & \alpha=\gamma,\\ 0, & \alpha\neq\gamma,\end{cases}\qquad\text{so}\quad \Phi^\Lambda\bigl(f^\alpha\bigr)=\bigl[i_\Lambda^{-1}\circ\Pi_*\bigl(f^\alpha\bigr)\bigr]a^\alpha\quad\text{by (91).}$$

Similarly

$$\Phi^\Lambda\bigl(g^\beta\bigr)=\bigl[i_\Lambda^{-1}\circ\Pi_*(g^\beta)\bigr]a^\beta\quad\text{and}\quad\Phi^\Lambda\bigl(f^\alpha*g^\beta\bigr)=\bigl[i_\Lambda^{-1}\circ\Pi_*\bigl(f^\alpha*g^\beta\bigr)\bigr]a^{\alpha+\beta},$$

as $f^\alpha * g^\beta$ is supported on $\mathfrak{Obj}_{\mathcal{A}}^{\alpha+\beta}$. Thus

$$\Phi^\Lambda\big(f^\alpha * g^\beta\big) = \Phi^\Lambda\big(f^\alpha\big) \star \Phi^\Lambda\big(g^\beta\big)$$

follows from Eqs. (81) and (89). For the general case, any $f, g \in \underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$ may be written as

$$f = \sum_{\alpha \in S} f^\alpha, \qquad g = \sum_{\beta \in T} g^\beta$$

with $S, T \subset \bar{C}(\mathcal{A})$ finite and f^α, g^β supported on $\mathfrak{Obj}_{\mathcal{A}}^\alpha, \mathfrak{Obj}_{\mathcal{A}}^\beta$, so

$$\Phi^\Lambda(f * g) = \Phi^\Lambda(f) \star \Phi^\Lambda(g)$$

follows by linearity. Clearly Φ^Λ is Λ-linear and $\Phi^\Lambda(\bar{\delta}_{[0]}) = a^0$, so Φ^Λ is an algebra morphism. The $P_{(I,\preccurlyeq)}$ equation is proved in the same way, using (86) rather than (81). □

Remark 6.5. (a) Since

$$\Pi_{\mathfrak{Obj}_{\mathcal{A}}}^{\Upsilon,\Lambda} : \underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}) \to \underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$$

is an algebra morphism,

$$\Phi^\Lambda \circ \Pi_{\mathfrak{Obj}_{\mathcal{A}}}^{\Upsilon,\Lambda} : \underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}) \to A(\mathcal{A}, \Lambda, \chi)$$

is also an algebra morphism, which commutes with the $P_{(I,\preccurlyeq)}$. The same applies to morphisms $\Phi^\Lambda \circ \Pi_{\mathfrak{Obj}_{\mathcal{A}}}^{\Upsilon,\Lambda}$ from $\bar{\underline{\mathrm{SF}}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$ and $\bar{\underline{\mathrm{SF}}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda^\circ)$, and all the subalgebras $\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$, $\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}), \ldots$. We defined Φ^Λ on $\underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$ as it is the coarsest choice.

(b) Suppose ℓ admits a *square root* $\wp$ in Λ, that is, $\wp^2 = \ell$. In Example 2.12 we can take $\wp = z$. Define elements $\tilde{a}^\alpha = \wp^{-\chi(\alpha,\alpha)} a^\alpha$ in $A(\mathcal{A}, \Lambda, \chi)$ for $\alpha \in \bar{C}(\mathcal{A})$. Then the $\tilde{a}^\alpha$ are an alternative basis for $A(\mathcal{A}, \Lambda, \chi)$ over Λ, and

$$\tilde{a}^\alpha \star \tilde{a}^\beta = \wp^{\chi(\alpha,\beta) - \chi(\beta,\alpha)} \tilde{a}^{\alpha+\beta},$$

by (89). This depends only on the *antisymmetrization* of χ. (This is not true for the $P_{(I,\preccurlyeq)}$ in the $\tilde{a}^\alpha$ basis, though.) If χ is symmetric, with $\chi(\alpha, \beta) \equiv \chi(\beta, \alpha)$, then $A(\mathcal{A}, \Lambda, \chi)$ is commutative with $\tilde{a}^\alpha \star \tilde{a}^\beta = \tilde{a}^{\alpha+\beta}$.

6.3. Algebras $B(\mathcal{A}, \Lambda, \chi)$ and morphisms to them

Theorem 5.19 gave a compatibility between multiplication $*$ in $\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$ and the projections $\Pi_{[I,\kappa]}$. We now exploit this to construct a larger algebra $B(\mathcal{A}, \Lambda, \chi)$, with an algebra morphism $\Psi^\Lambda : \bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda) \to B(\mathcal{A}, \Lambda, \chi)$.

Definition 6.6. Let Assumptions 2.11 and 3.4 hold, and $\chi : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$ be a biadditive map. Using only the data $K(\mathcal{A}), C(\mathcal{A}), \Lambda, \ell, \chi$ we will define a $\mathbb{Q}$-*algebra* $B(\mathcal{A}, \Lambda, \chi)$. Consider pairs (I, κ) for $\kappa : I \to C(\mathcal{A})$ and $\approx$-equivalence classes $[I, \kappa]$ as in Definition 5.12.

Introduce formal symbols $b_{[I,\kappa]}$ for all such equivalence classes $[I,\kappa]$. Let $B(\mathcal{A},\Lambda,\chi)$ be the Λ-module with basis the $b_{[I,\kappa]}$. That is, $B(\mathcal{A},\Lambda,\chi)$ is the set of sums $\sum_{\text{classes }[I,\kappa]}\beta_{[I,\kappa]}b_{[I,\kappa]}$ with $\beta_{[I,\kappa]}\in\Lambda$ nonzero for only finitely many $[I,\kappa]$. Addition, and multiplication by $\mathbb{Q}$, are defined in the obvious way. Define a *multiplication* $\star$ on $B(\mathcal{A},\Lambda,\chi)$ by

$$\begin{aligned} &b_{[I,\kappa]}\star b_{[J,\lambda]}\\ &\quad=\sum_{\text{eq. classes }[K,\mu]} b_{[K,\mu]}\cdot\frac{(\ell-1)^{|K|-|I|-|J|}}{|\mathrm{Aut}(K,\mu)|}\\ &\qquad\cdot\Biggl[\sum_{\substack{\text{iso. classes}\\ \text{of finite}\\ \text{sets }L}}\frac{(-1)^{|L|-|K|}}{|L|!}\sum_{\substack{\phi:I\to L,\,\psi:J\to L\text{ and}\\ \theta:L\to K:\ \phi\amalg\psi\text{ surjective},\\ \mu(k)=\kappa((\theta\circ\phi)^{-1}(k))+\\ \lambda((\theta\circ\psi)^{-1}(k)),\,k\in K}}\prod_{k\in K}\bigl(\bigl|\theta^{-1}(k)\bigr|-1\bigr)!\prod_{\substack{i\in I,\,j\in J:\\ \phi(i)=\psi(j)}}\ell^{-\chi(\lambda(j),\kappa(i))}\Biggr],\end{aligned} \tag{92}$$

extended Λ-bilinearly. An elementary but lengthy combinatorial calculation shows $\star$ is *associative*, and makes $B(\mathcal{A},\Lambda,\chi)$ into a $\mathbb{Q}$-*algebra* (in fact, a Λ-*algebra*), with identity $b_{[\emptyset,\emptyset]}$, writing $\emptyset$ for the trivial map $\emptyset\to C(\mathcal{A})$. Define $B^{\text{ind}}(\mathcal{A},\Lambda,\chi)$ to be the subspace of $\sum_{[I,\kappa]}\beta_{[I,\kappa]}b_{[I,\kappa]}$ in $B(\mathcal{A},\Lambda,\chi)$ with $\beta_{[I,\kappa]}=0$ unless $|I|=1$. Equation (92) implies $B^{\text{ind}}(\mathcal{A},\Lambda,\chi)$ is closed under the Lie bracket $[b,c]=b\star c-c\star b$, and so is a $\mathbb{Q}$- or Λ-*Lie algebra*.

Let $f\in\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}},\Upsilon,\Lambda)$, so that $\Pi_{[I,\kappa]}(f)\in\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}},\Upsilon,\Lambda)$ and $\Pi_*\circ\Pi_{[I,\kappa]}(f)\in\underline{\mathrm{SF}}(\operatorname{Spec}\mathbb{K},\Upsilon,\Lambda)$. Using the explicit form [11, Proposition 5.23] for $\underline{\mathrm{SF}}(\operatorname{Spec}\mathbb{K},\Upsilon,\Lambda)$ and properties of the $\Pi_{[I,\kappa]}$ we find that

$$\Pi_*\circ\Pi_{[I,\kappa]}(f)=\beta_{[I,\kappa]}\bigl[\operatorname{Spec}\mathbb{K}/\bigl(\mathbb{K}^\times\bigr)^{|I|}\bigr],\tag{93}$$

for some unique $\beta_{[I,\kappa]}\in\Lambda$. Now define $\Psi^\Lambda:\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}},\Upsilon,\Lambda)\to B(\mathcal{A},\Lambda,\chi)$ by

$$\Psi^\Lambda(f)=\sum_{\text{eq. classes }[I,\kappa]}\beta_{[I,\kappa]}b_{[I,\kappa]}.\tag{94}$$

Definition 5.14 implies that Ψ^Λ maps $\bar{\mathrm{SF}}{}^{\text{ind}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}},\Upsilon,\Lambda)$ to $B^{\text{ind}}(\mathcal{A},\Lambda,\chi)$.

Theorem 6.7. *Let Assumptions* 2.11 *and* 3.4 *hold and* $\chi:K(\mathcal{A})\times K(\mathcal{A})\to\mathbb{Z}$ *be biadditive and satisfy* (80). *Then* $\Psi^\Lambda:\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}},\Upsilon,\Lambda)\to B(\mathcal{A},\Lambda,\chi)$ *and* $\Psi^\Lambda:\bar{\mathrm{SF}}{}^{\text{ind}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}},\Upsilon,\Lambda)\to B^{\text{ind}}(\mathcal{A},\Lambda,\chi)$ *are* (*Lie*) *algebra morphisms.*

Proof. Clearly Ψ^Λ is bilinear and $\Psi^\Lambda(\bar{\delta}_{[0]})=b_{[\emptyset,\emptyset]}$, so that Ψ^Λ takes the identity to the identity. Thus the theorem follows from $\Psi^\Lambda(f*g)=\Psi^\Lambda(f)\star\Psi^\Lambda(g)$ for all $f,g\in\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}},\Upsilon,\Lambda)$. By Proposition 5.13 and bilinearity it is enough to show that $\Psi^\Lambda(f*g)=\Psi^\Lambda(f)\star\Psi^\Lambda(g)$ when $\Pi_{[I,\kappa]}(f)=f$ and $\Pi_{[J,\lambda]}(g)=g$ for some $[I,\kappa]$ and $[J,\lambda]$. Thus we can apply Theorem 5.19 to get representations (69) and (72) for $f\otimes g$ and $\Pi_{[K,\mu]}(f*g)$.

We have

$$\Psi^\Lambda(f)=\beta_{[I,\kappa]}b_{[I,\kappa]}\quad\text{and}\quad\Psi^\Lambda(g)=\gamma_{[J,\lambda]}b_{[J,\lambda]}$$

for some $\beta_{[I,\kappa]}, \gamma_{[J,\lambda]} \in \Lambda$. Then $\Pi_*(f \otimes g) = \beta_{[I,\kappa]}\gamma_{[J,\lambda]}[\operatorname{Spec}\mathbb{K}/(\mathbb{K}^\times)^I \times (\mathbb{K}^\times)^J]$ in $\underline{\bar{\mathrm{SF}}}(\operatorname{Spec}\mathbb{K}, \Upsilon, \Lambda)$. Projecting (69) to $\underline{\bar{\mathrm{SF}}}(\operatorname{Spec}\mathbb{K}, \Upsilon, \Lambda)$, we deduce that

$$\beta_{[I,\kappa]}\gamma_{[J,\lambda]} = \sum_{m\in M,\, n\in N_m} c_{mn}\Upsilon\big([W_{mn}]\big). \tag{95}$$

Now write $\Psi^\Lambda(f * g) = \sum_{[K,\mu]} \delta_{[K,\mu]} b_{[K,\mu]}$ for $\delta_{[K,\mu]} \in \Lambda$. Then $\Pi_* \circ \Pi_{[K,\mu]}(f * g) = \delta_{[K,\mu]}[\operatorname{Spec}\mathbb{K}/(\mathbb{K}^\times)^K]$ in $\underline{\bar{\mathrm{SF}}}(\operatorname{Spec}\mathbb{K}, \Upsilon, \Lambda)$. Applying $\Pi^{\Upsilon,\Lambda}_{\operatorname{Spec}\mathbb{K}}$ to map to $\underline{\mathrm{SF}}(\operatorname{Spec}\mathbb{K}, \Upsilon, \Lambda)$ and i_Λ^{-1} to map to Λ gives

$$i_\Lambda^{-1} \circ \Pi^{\Upsilon,\Lambda}_{\operatorname{Spec}\mathbb{K}} \circ \Pi_* \circ \Pi_{[K,\mu]}(f * g) = \delta_{[K,\mu]}(\ell - 1)^{-|K|}.$$

So applying $i_\Lambda^{-1} \circ \Pi^{\Upsilon,\Lambda}_{\operatorname{Spec}\mathbb{K}} \circ \Pi_*$ to (72) and using relations in $\underline{\mathrm{SF}}(\operatorname{Spec}\mathbb{K}, \Upsilon, \Lambda)$ gives

$$\delta_{[K,\mu]} = \frac{(\ell-1)^{|K|-|I|-|J|}}{|\mathrm{Aut}(K,\mu)|} \sum_{\substack{\text{iso. classes}\\ \text{of finite sets } L}} \frac{(-1)^{|L|-|K|}}{|L|!} \sum_{\substack{\phi: I\to L,\ \psi: J\to L \text{ and}\\ \theta: L\to K:\ \phi\amalg\psi \text{ surjective},\\ \mu(k)=\kappa((\theta\circ\phi)^{-1}(k))+\\ \lambda((\theta\circ\psi)^{-1}(k)),\, k\in K}} \prod_{k\in K}\big(\big|\theta^{-1}(k)\big|-1\big)! \sum_{m\in M,\, n\in N_m} c_{mn}\Upsilon\big([W_{mn}]\big)\Upsilon\big(\big[\big(E_m^1\big)^{T_{L,\phi,\psi}}\big]\big)\Upsilon\big(\big[\big(E_m^0\big)^{T_{L,\phi,\psi}}\big]\big)^{-1}. \tag{96}$$

Let m, n, L, ϕ, ψ be as in (96), and pick $w \in W_{mn}(\mathbb{K})$ projecting to $v \in V_m(\mathbb{K})$ and $([X],[Y]) \in \mathfrak{Obj}_\mathcal{A}(\mathbb{K}) \times \mathfrak{Obj}_\mathcal{A}(\mathbb{K})$. Then Theorem 5.19 gives splittings $X \cong \bigoplus_{i\in I} X_i$ and $Y \cong \bigoplus_{j\in J} Y_j$ in $\mathcal{A}$ with $[X_i] = \kappa(i)$ and $[Y_j] = \lambda(j)$ in $C(\mathcal{A})$ for all i, j, and isomorphisms (73). Combining these with (80) yields

$$\Upsilon\big(\big[\big(E_m^1\big)^{T_{L,\phi,\psi}}\big]\big)\Upsilon\big(\big[\big(E_m^0\big)^{T_{L,\phi,\psi}}\big]\big)^{-1} = \prod_{i\in I,\ j\in J:\ \phi(i)=\psi(j)} \ell^{-\chi(\lambda(j),\kappa(i))}. \tag{97}$$

We now see that $\Psi^\Lambda(f * g) = \Psi^\Lambda(f) \star \Psi^\Lambda(g)$ by comparing (92) and (95)–(97). □

Remark 6.8. (a) We can also generalize (90) to $P_{(I,\preccurlyeq)}: \prod_{i\in I} B(\mathcal{A},\Lambda,\chi) \to B(\mathcal{A},\Lambda,\chi)$ satisfying (36) and $\Psi^\Lambda(P_{(I,\preccurlyeq)}[f_i\colon i\in I]) = P_{(I,\preccurlyeq)}[\Psi^\Lambda(f_i)\colon i \in I]$ for $f_i \in \bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_\mathcal{A}, \Upsilon, \Lambda)$, as in Theorem 6.4. But since the definition and proof are rather complicated, we omit them.

(b) As $\bar\Pi^{\Upsilon,\Lambda}_{\mathfrak{Obj}_\mathcal{A}}: \mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_\mathcal{A}) \to \bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_\mathcal{A}, \Upsilon, \Lambda)$ is an algebra morphism taking

$$\mathrm{SF}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{Obj}_\mathcal{A}) \to \bar{\mathrm{SF}}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{Obj}_\mathcal{A}, \Upsilon, \Lambda),$$

$\Psi^\Lambda \circ \bar\Pi^{\Upsilon,\Lambda}_{\mathfrak{Obj}_\mathcal{A}}: \mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_\mathcal{A}) \to B(\mathcal{A},\Lambda,\chi)$ is too, and restricts to a Lie algebra morphism

$$\mathrm{SF}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{Obj}_\mathcal{A}) \to B^{\mathrm{ind}}(\mathcal{A},\Lambda,\chi).$$

The same holds for $\bar{\Pi}^{\Upsilon,\Lambda}_{\mathfrak{Obj}_{\mathcal{A}}}:\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}},\Upsilon,\Lambda^\circ)\to\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}},\Upsilon,\Lambda)$. We defined Ψ^Λ on $\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}},\Upsilon,\Lambda)$ as it is the coarsest choice.

(c) Here is an alternative description of $B(\mathcal{A},\Lambda,\chi)$, $B^{\mathrm{ind}}(\mathcal{A},\Lambda,\chi)$. Define $b^\alpha\in B^{\mathrm{ind}}(\mathcal{A},\Lambda,\chi)$ for $\alpha\in C(\mathcal{A})$ by $b^\alpha=b_{[\{1\},\alpha']}$, where $\alpha'(1)=\alpha$. Then $B^{\mathrm{ind}}(\mathcal{A},\Lambda,\chi)$ is the Λ-module with basis b^α for $\alpha\in C(\mathcal{A})$, and (92) yields

$$\big[b^\alpha,b^\beta\big]=\frac{\ell^{-\chi(\beta,\alpha)}-\ell^{-\chi(\alpha,\beta)}}{\ell-1}\,b^{\alpha+\beta}. \tag{98}$$

Given $\kappa:\{1,\dots,n\}\to C(\mathcal{A})$, we can use (92) to show that

$$b^{\kappa(1)}\star b^{\kappa(2)}\star\dots\star b^{\kappa(n)}=\frac{1}{|\mathrm{Aut}(\{1,\dots,n\},\kappa)|}\cdot b_{[\{1,\dots,n\},\kappa]}+\big(\text{terms in } b_{[J,\lambda]} \text{ for } |J|<n\big).$$

By induction on n we find the b^α generate $B(\mathcal{A},\Lambda,\chi)$ over Λ, and (98) are the only relations on the b^α over Λ. Also (98) satisfies the Jacobi identity.

Thus, $B(\mathcal{A},\Lambda,\chi)$ is the Λ-algebra generated by the b^α for $\alpha\in C(\mathcal{A})$, with relations (98). Equivalently, $B^{\mathrm{ind}}(\mathcal{A},\Lambda,\chi)$ is the Λ-Lie algebra with basis b^α for $\alpha\in C(\mathcal{A})$ and relations (98), and $B(\mathcal{A},\Lambda,\chi)$ is the *universal enveloping Λ-algebra* of $B^{\mathrm{ind}}(\mathcal{A},\Lambda,\chi)$. Note this does *not* mean $B(\mathcal{A},\Lambda,\chi)$ is the universal enveloping $\mathbb{Q}$-algebra of $B^{\mathrm{ind}}(\mathcal{A},\Lambda,\chi)$ as a $\mathbb{Q}$-Lie algebra.

(d) Suppose as in Remark 6.5(b) that ℓ admits a square root $\wp$ in Λ. Define elements $\tilde b^\alpha=\wp^{1-\chi(\alpha,\alpha)}b^\alpha$ in $B^{\mathrm{ind}}(\mathcal{A},\Lambda,\chi)$ for $\alpha\in C(\mathcal{A})$. These are another Λ-basis for $B^{\mathrm{ind}}(\mathcal{A},\Lambda,\chi)$, and

$$\big[\tilde b^\alpha,\tilde b^\beta\big]=\frac{\wp^{\chi(\alpha,\beta)-\chi(\beta,\alpha)}-\wp^{\chi(\beta,\alpha)-\chi(\alpha,\beta)}}{\wp-\wp^{-1}}\tilde b^{\alpha+\beta}$$

by (98), which depends only on the antisymmetrization of χ, and is also unchanged by replacing $\wp$ by $\wp^{-1}$.

(e) Define a Λ-algebra morphism $\Delta:B(\mathcal{A},\Lambda,\chi)\to A(\mathcal{A},\Lambda,\chi)$ by

$$\Delta(b_{[I,\kappa]})=(\ell-1)^{-|I|}a^{\kappa(I)}.$$

Then

$$\Phi^\Lambda\circ\Pi^{\Upsilon,\Lambda}_{\mathfrak{Obj}_{\mathcal{A}}}=\Delta\circ\Psi^\Lambda:\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}},\Upsilon,\Lambda)\to A(\mathcal{A},\Lambda,\chi).$$

6.4. Multiplication in $B(\mathcal{A},\Lambda,\chi)$ as a sum over graphs

We now rewrite the multiplication law (92) in $B(\mathcal{A},\Lambda,\chi)$ as a sum over *directed graphs Γ*, with vertex set $I\amalg J$. In (99), 'no multiples' means there are *no multiple edges*, that is, at most one edge joins any two vertices in Γ. By the *connected components* of Γ we mean the sets of vertices of connected components, which are subsets of $I\amalg J$. And $b_1(\Gamma)$ is the *first Betti number* of Γ.

We find graphs helpful as we can use topological ideas like connected, simply-connected and $b_1(\Gamma)$. The transformation laws for Calabi–Yau 3-fold invariants in [14] will also be written in terms of sums over graphs, and the author believes these may have something to do with *Feynman diagrams* in physics. Since $b_1(\Gamma)\geqslant 0$, the rational functions of ℓ appearing in (99) are *continuous* at $\ell=1$, and lie in Λ°. This will be important in Section 6.5.

Theorem 6.9. *Equation* (92) *is equivalent to*

$$b_{[I,\kappa]} \star b_{[J,\lambda]} = \sum_{\textit{eq. classes } [K,\mu]} b_{[K,\mu]} \cdot \frac{1}{|\mathrm{Aut}(K,\mu)|} \sum_{\substack{\eta: I\rightarrow K,\, \zeta: J\rightarrow K:\\ \mu(k)=\kappa(\eta^{-1}(k))+\lambda(\zeta^{-1}(k))}}$$
$$\Biggl[\sum_{\substack{\textit{directed graphs } \Gamma\textit{: vertices } I\amalg J,\\ \textit{edges } \overset{i}{\bullet}\rightarrow\overset{j}{\bullet},\, i\in I,\, j\in J,\textit{ no multiples},\\ \textit{conn. components } \eta^{-1}(k)\amalg\zeta^{-1}(k),\, k\in K}} (\ell-1)^{b_1(\Gamma)} \prod_{\substack{\textit{edges } \overset{i}{\bullet}\rightarrow\overset{j}{\bullet}\\ \textit{in } \Gamma}} \frac{\ell^{-\chi(\lambda(j),\kappa(i))}-1}{\ell-1}\Biggr]. \tag{99}$$

Proof. First we rewrite (92) as far as possible as a product over $k \in K$. To do this, for L, ϕ, ψ, θ as in (92) write $\eta = \theta \circ \phi$ and $\zeta = \theta \circ \psi$, and for all $k \in K$ set $L_k = \theta^{-1}(\{k\})$ and $\phi_k : \eta^{-1}(\{k\}) \to L_k$, $\psi_k : \zeta^{-1}(\{k\}) \to L_k$ to be the restrictions of ϕ, ψ to $\eta^{-1}(\{k\})$, $\zeta^{-1}(\{k\})$. Then $L = \coprod_{k\in K} L_k$. The number of surjective maps $\theta : L \to K$ such that $|\theta^{-1}(\{k\})| = |L_k|$ for $k \in K$ is $|L|!/\prod_{k\in K} |L_k|!$. Thus, replacing the choice of L, θ in (92) by the choice of sets L_k for $k \in K$, we replace the factor $1/|L|!$ in (92) by $1/\prod_{k\in K} |L_k|!$. Writing other terms in (92) as products over $k \in K$, we deduce:

$$b_{[I,\kappa]} \star b_{[J,\lambda]} = \sum_{\text{eq. classes } [K,\mu]} b_{[K,\mu]} \cdot \frac{1}{|\mathrm{Aut}(K,\mu)|} \sum_{\substack{\eta: I\rightarrow K,\, \zeta: J\rightarrow K:\\ \mu(k)=\kappa(\eta^{-1}(k))+\lambda(\zeta^{-1}(k))}} \tag{100}$$
$$\prod_{k\in K} \Biggl[(\ell-1)^{1-|\eta^{-1}(k)|-|\zeta^{-1}(k)|} \sum_{\substack{\text{iso. classes}\\ \text{of finite}\\ \text{sets } L_k}} \frac{(-1)^{|L_k|-1}}{|L_k|} \sum_{\substack{\phi_k: \eta^{-1}(k)\rightarrow L_k,\\ \psi_k: \zeta^{-1}(k)\rightarrow L_k:\\ \phi_k\amalg\psi_k \text{ surjective}}} \prod_{\substack{i\in\eta^{-1}(k),\\ j\in\zeta^{-1}(k):\\ \phi_k(i)=\psi_k(j)}} \ell^{-\chi(\lambda(j),\kappa(i))} \Biggr].$$

Next we prove the bottom lines of (100) and (99) are equal. We can rewrite the bottom line of (99) as a product over $k \in K$ of sums of weighted, connected digraphs Γ_k with vertices $\eta^{-1}(\{k\}) \amalg \zeta^{-1}(\{k\})$, so it is enough to show the terms in each product over $k \in K$ are equal. This is equivalent to proving the case $|K| = 1$, so that $K = \{k\}$. Dropping subscripts k, we have to prove that

$$(\ell-1)^{1-|I|-|J|} \cdot \sum_{\substack{\text{iso. classes of}\\ \text{finite sets } L}} \frac{(-1)^{|L|-1}}{|L|} \sum_{\substack{\phi: I\rightarrow L,\, \psi: J\rightarrow L:\\ \phi\amalg\psi \text{ surjective}}} \prod_{\substack{i\in I,\, j\in J:\\ \phi(i)=\psi(j)}} \ell^{-\chi(\lambda(j),\kappa(i))}$$
$$= \sum_{\substack{\text{connected directed graphs } \Gamma:\\ \text{vertices } I\amalg J,\text{ edges } \overset{i}{\bullet}\rightarrow\overset{j}{\bullet},\\ i\in I,\, j\in J,\text{ no multiples}}} (\ell-1)^{b_1(\Gamma)} \prod_{\substack{\text{edges } \overset{i}{\bullet}\rightarrow\overset{j}{\bullet}\\ \text{in } \Gamma}} \frac{\ell^{-\chi(\lambda(j),\kappa(i))}-1}{\ell-1}. \tag{101}$$

Rewrite the top line of (101) as a sum over Γ as follows. Replace $\ell^{-\chi(\lambda(j),\kappa(i))}$ by $(\ell^{-\chi(\lambda(j),\kappa(i))} - 1) + 1$ and multiply out the product in i, j to get a sum of products of $\ell^{-\chi(\lambda(j),\kappa(i))} - 1$ or 1. Associate a digraph Γ to each of these by putting in an edge $\overset{i}{\bullet} \rightarrow \overset{j}{\bullet}$ for a factor $\ell^{-\chi(\lambda(j),\kappa(i))} - 1$, and no edge for a factor 1. Then edges only join i, j with $\phi(i) = \psi(j)$, so each connected component of Γ lies in $\phi^{-1}(\{l\}) \amalg \psi^{-1}(\{l\})$ for some $l \in L$. Thus, reversing the sums over Γ and L, ϕ, ψ, the top line of (101) becomes

$$(\ell-1)^{1-|I|-|J|} \sum_{\substack{\text{directed graphs } \Gamma: \\ \text{vertices } I\amalg J,\text{ edges } \overset{i}{\bullet}\to\overset{j}{\bullet}, \\ i\in I,\ j\in J,\text{ no multiples}}} \prod_{\substack{\text{edges } \overset{i}{\bullet}\to\overset{j}{\bullet} \\ \text{in } \Gamma}} \left(\ell^{-\chi(\lambda(j),\kappa(i))}-1\right)$$
$$\cdot\left[\sum_{\substack{\text{iso. classes of} \\ \text{finite sets } L}} \frac{(-1)^{|L|-1}}{|L|} \sum_{\substack{\phi: I\to L,\ \psi: J\to L:\ \phi\amalg\psi \text{ surjective, conn.} \\ \text{components of } \Gamma \text{ lie in } \phi^{-1}(l)\amalg\psi^{-1}(l),\ l\in L}} 1\right]. \tag{102}$$

We shall show the bottom line $[\cdots]$ of (102) is 1 if Γ is connected, and 0 otherwise. If Γ is connected then $I \amalg J$ must lie in some $\phi^{-1}(\{l\}) \amalg \psi^{-1}(\{l\})$, so as $\phi \amalg \psi$ is surjective the only possibility is $L = \{l\}$ and $|L| = 1$, and the sum reduces to 1. If Γ is not connected, fix one connected component $I_0 \amalg J_0$ of Γ, $I_0 \subseteq I$, $J_0 \subseteq J$. If a triple (L, ϕ, ψ) appears in the bottom line of (102) then for some $l \in L$ we have $\phi|_{I_0} \equiv l$, $\psi|_{J_0} \equiv l$. Divide such (L, ϕ, ψ) into two cases: (a) $\phi^{-1}(\{l\}) = I_0$ and $\psi^{-1}(\{l\}) = J_0$, and (b) otherwise.

If (L, ϕ, ψ) satisfy (b) then define (L', ϕ', ψ') satisfying (a) by $L' = L \amalg \{l'\}$ for some $l' \notin L$, and $\phi'(i) = l'$ for $i \in I_0$ and $\phi'(i) = \phi(i)$ for $i \notin I_0$, and $\psi'(j) = l'$ for $j \in J_0$ and $\psi'(j) = \psi(j)$ for $j \notin J_0$. Conversely, if (L', ϕ', ψ') satisfy (a) with $\phi'|_{I_0} \equiv \psi'|_{J_0} \equiv l'$, then define (L, ϕ, ψ) satisfying (b) by $L = L' \setminus \{l'\}$, and choosing some $l \in L$ define $\phi(i) = l$ for $i \in I_0$ and $\phi(i) = \phi'(i)$ for $i \notin I_0$, and $\psi(j) = l$ for $j \in J_0$ and $\psi(j) = \psi'(j)$ for $j \notin J_0$.

This establishes a 1–1 correspondence between (L, ϕ, ψ) satisfying (b), and quadruples L', ϕ', ψ', l with special element $l' \in L'$, such that (L', ϕ', ψ') satisfies (a) with $\phi'|_{I_0} \equiv \psi'|_{J_0} \equiv l' \in L'$, and $l \in L' \setminus \{l'\}$. We also have $|L'| = |L| + 1$. Thus, the terms in the bottom line of (102) corresponding to (L, ϕ, ψ) and (L', ϕ', ψ') differ by a factor $-|L|/|L'|$.

Now the equation $\phi'|_{I_0} \equiv \psi'|_{J_0} \equiv l' \in L$ constrains the choice of ϕ', ψ', fixing l' out of $|L'|$ choices of points in L', and this accounts for the factor $1/|L'|$. And given L', ϕ', ψ' there are $|L|$ possible choices for $l \in L' \setminus \{l'\}$, which accounts for the factor $|L|$. Because of these, for disconnected Γ the contributions of L, ϕ, ψ of types (a) and (b) in the bottom line of (102) cancel, giving zero.

Thus (102) reduces to the bottom line of (101), except for the powers of $\ell - 1$, which are $b_1(\Gamma) - n$ in the bottom line of (101), where n is the number of edges in Γ, and $1 - |I| - |J|$ in (102). But Γ is connected with $|I| + |J|$ vertices and n edges, so $b_1(\Gamma) - n = 1 - |I| - |J|$. This proves (101), and hence (99). □

6.5. Algebras $B(\mathcal{A}, \Lambda^\circ, \chi)$, $C(\mathcal{A}, \Omega, \chi)$ and morphisms to them

We define algebras $B(\mathcal{A}, \Lambda^\circ, \chi)$, $C(\mathcal{A}, \Omega, \chi)$ and a morphism Π between them.

Definition 6.10. Let Assumptions 2.11 and 3.4 hold, $\chi : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$ or $\mathbb{Q}$ be bi-additive, and use the notation of Definition 6.6. Define $B(\mathcal{A}, \Lambda^\circ, \chi)$ to be the subspace of $\sum_{[I,\kappa]} \beta_{[I,\kappa]} b_{[I,\kappa]}$ in $B(\mathcal{A}, \Lambda, \chi)$ with all $\beta_{[I,\kappa]} \in \Lambda^\circ$. Since the bottom line of (99) lies in Λ° we see that $B(\mathcal{A}, \Lambda^\circ, \chi)$ is closed under $\star$, and so is a Λ°-*subalgebra* of $B(\mathcal{A}, \Lambda, \chi)$, with Λ°-*basis* the $b_{[I,\kappa]}$. Define $B^{\rm ind}(\mathcal{A}, \Lambda^\circ, \chi) = B^{\rm ind}(\mathcal{A}, \Lambda, \chi) \cap B(\mathcal{A}, \Lambda^\circ, \chi)$. Then $B^{\rm ind}(\mathcal{A}, \Lambda^\circ, \chi)$ is a Λ°-*Lie subalgebra* of $B(\mathcal{A}, \Lambda^\circ, \chi)$, since $B^{\rm ind}(\mathcal{A}, \Lambda, \chi)$ is a Λ-Lie subalgebra in $B(\mathcal{A}, \Lambda, \chi)$.

As in Definition 6.6, introduce symbols $c_{[I,\kappa]}$ for all equivalence classes $[I, \kappa]$, and let $C(\mathcal{A}, \Omega, \chi)$ be the Ω-module with basis the $c_{[I,\kappa]}$. That is, $C(\mathcal{A}, \Omega, \chi)$ is the set of sums

$\sum_{\text{classes }[I,\kappa]} \gamma_{[I,\kappa]} c_{[I,\kappa]}$ with $\gamma_{[I,\kappa]} \in \Omega$ nonzero for only finitely many $[I,\kappa]$. Define a *multiplication* $\star$ on $C(\mathcal{A},\Omega,\chi)$ by

$$c_{[I,\kappa]} \star c_{[J,\lambda]} = \sum_{\text{eq. classes }[K,\mu]} c_{[K,\mu]} \cdot \frac{1}{|\mathrm{Aut}(K,\mu)|} \sum_{\substack{\eta: I\to K,\ \zeta: J\to K:\\ \mu(k)=\kappa(\eta^{-1}(k))+\lambda(\zeta^{-1}(k))}} \Biggl[\sum_{\substack{\text{simply-connected directed graphs }\Gamma:\\ \text{vertices } I\amalg J,\ \text{edges } \overset{i}{\bullet}\to\overset{j}{\bullet},\ i\in I,\ j\in J,\\ \text{conn. components } \eta^{-1}(k)\amalg\zeta^{-1}(k),\ k\in K}} \prod_{\substack{\text{edges } \overset{i}{\bullet}\to\overset{j}{\bullet}\\ \text{in } \Gamma}} \bigl(-\chi\bigl(\lambda(j),\kappa(i)\bigr)\bigr)\Biggr], \tag{103}$$

extended Ω-bilinearly in the usual way. This is still well defined if χ takes values in $\mathbb{Q}$ rather than $\mathbb{Z}$, and in Section 6.6 we allow $\chi = \frac{1}{2}\bar{\chi}$ to take values in $\frac{1}{2}\mathbb{Z}$.

Define a $\mathbb{Q}$-linear map $\Pi : B(\mathcal{A},\Lambda^\circ,\chi) \to C(\mathcal{A},\Omega,\chi)$ by

$$\Pi: \sum_{\text{eq. classes }[I,\kappa]} \beta_{[I,\kappa]} b_{[I,\kappa]} \mapsto \sum_{\text{eq. classes }[I,\kappa]} \pi(\beta_{[I,\kappa]}) c_{[I,\kappa]}, \tag{104}$$

for $\pi : \Lambda^\circ \to \Omega$ as in Assumption 2.11. Then $\Pi(b_{[I,\kappa]}) = c_{[I,\kappa]}$, and Π is *surjective* as π is. Comparing (99) and (103) shows that

$$\Pi(b_{[I,\kappa]}) \star \Pi(b_{[J,\lambda]}) = c_{[I,\kappa]} \star c_{[J,\lambda]} = \Pi(b_{[I,\kappa]} \star b_{[J,\lambda]}). \tag{105}$$

Here we effectively take the limit $\ell \to 1$ in the bottom line $[\cdots]$ of (99) to get the bottom line $[\cdots]$ of (103), since $\pi(\ell) = 1$ in Ω. The factor $(\ell-1)^{b_1(\Gamma)}$ in (99) shows that only Γ with $b_1(\Gamma) = 0$ contribute in the limit, that is, only *simply-connected* Γ. We drop the condition 'no multiples' from (99), as this is implied by Γ simply-connected.

Using (105), the associativity of $\star$ in $B(\mathcal{A},\Lambda^\circ,\chi)$, and π an algebra morphism, we see that $\star$ in $C(\mathcal{A},\Omega,\chi)$ is *associative*, so $C(\mathcal{A},\Omega,\chi)$ is an *Ω-algebra*, with identity $c_{[\emptyset,\emptyset]}$, and Π in (104) is a *$\mathbb{Q}$-algebra morphism*. Define $C^{\mathrm{ind}}(\mathcal{A},\Omega,\chi)$ to be the subspace of $\sum_{[I,\kappa]} \gamma_{[I,\kappa]} c_{[I,\kappa]}$ in $C(\mathcal{A},\Omega,\chi)$ with $\gamma_{[I,\kappa]} = 0$ unless $|I| = 1$. From (103) we see $C^{\mathrm{ind}}(\mathcal{A},\Omega,\chi)$ is an *Ω-Lie subalgebra* of $C(\mathcal{A},\Omega,\chi)$. Also, Π restricts to a *$\mathbb{Q}$-Lie algebra morphism* $\Pi : B^{\mathrm{ind}}(\mathcal{A},\Lambda^\circ,\chi) \to C^{\mathrm{ind}}(\mathcal{A},\Omega,\chi)$.

Now let $f \in \bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}},\Upsilon,\Lambda^\circ)$. As in Definition 6.6, Eq. (93) holds, but this time for $\beta_{[I,\kappa]} \in \Lambda^\circ$. Define $\Psi^{\Lambda^\circ} : \bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}},\Upsilon,\Lambda^\circ) \to B(\mathcal{A},\Lambda^\circ,\chi)$ by

$$\Psi^{\Lambda^\circ}(f) = \sum_{[I,\kappa]} \beta_{[I,\kappa]} b_{[I,\kappa]},$$

as in (94). In the same way, if $f \in \bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}},\Theta,\Omega)$ then (93) holds for $\beta_{[I,\kappa]} \in \Omega$. Define $\Psi^{\Omega} : \bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}},\Theta,\Omega) \to C(\mathcal{A},\Omega,\chi)$ by

$$\Psi^{\Omega}(f) = \sum_{[I,\kappa]} \beta_{[I,\kappa]} c_{[I,\kappa]}.$$

These restrict to $\Psi^{\Lambda^\circ} : \bar{\mathrm{SF}}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}},\Upsilon,\Lambda^\circ) \to B^{\mathrm{ind}}(\mathcal{A},\Lambda^\circ,\chi)$ and $\Psi^{\Omega} : \bar{\mathrm{SF}}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}},\Theta,\Omega) \to C^{\mathrm{ind}}(\mathcal{A},\Omega,\chi)$.

Here is the analogue of Theorem 6.7.

Theorem 6.11. *Let Assumptions* 2.11 *and* 3.4 *hold and* $\chi : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$ *be biadditive and satisfy* (80). *Then*

$$\Psi^{\Lambda^\circ} : \bar{\mathrm{SF}}_{\mathrm{al}}\big(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda^\circ\big) \to B(\mathcal{A}, \Lambda^\circ, \chi) \quad \textit{and} \quad \Psi^{\Omega} : \bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, \Theta, \Omega) \to C(\mathcal{A}, \Omega, \chi)$$

are Λ°, Ω*-algebra morphisms, and*

$$\Psi^{\Lambda^\circ} : \bar{\mathrm{SF}}{}^{\mathrm{ind}}_{\mathrm{al}}\big(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda^\circ\big) \to B^{\mathrm{ind}}(\mathcal{A}, \Lambda^\circ, \chi) \quad \textit{and}$$
$$\Psi^{\Omega} : \bar{\mathrm{SF}}{}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, \Theta, \Omega) \to C^{\mathrm{ind}}(\mathcal{A}, \Omega, \chi)$$

are Λ°, Ω*-Lie algebra morphisms.*

Proof. Ψ^{Λ° is the restriction of Ψ^{Λ} to $\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda^\circ) \subset \bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda)$, so Ψ^{Λ° is a (Lie) algebra morphism by Theorem 6.7. For Ψ^{Ω}, note that

$$\Pi \circ \Psi^{\Lambda^\circ} = \Psi^{\Omega} \circ \bar{\Pi}^{\Theta,\Omega}_{\mathfrak{Obj}_{\mathcal{A}}} : \bar{\mathrm{SF}}_{\mathrm{al}}\big(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda^\circ\big) \to C(\mathcal{A}, \Omega, \chi). \tag{106}$$

Let $f, g \in \bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, \Theta, \Omega)$. Since $\bar{\Pi}^{\Theta,\Omega}_{\mathfrak{Obj}_{\mathcal{A}}}$ is surjective we can lift them to $f', g' \in \bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda^\circ)$ with $f, g = \bar{\Pi}^{\Theta,\Omega}_{\mathfrak{Obj}_{\mathcal{A}}}(f', g')$. Then

$$\begin{aligned}
\Psi^{\Omega}(f * g) &= \Psi^{\Omega}\big(\bar{\Pi}^{\Theta,\Omega}_{\mathfrak{Obj}_{\mathcal{A}}}(f') * \bar{\Pi}^{\Theta,\Omega}_{\mathfrak{Obj}_{\mathcal{A}}}(g')\big) = \Psi^{\Omega} \circ \bar{\Pi}^{\Theta,\Omega}_{\mathfrak{Obj}_{\mathcal{A}}}(f' * g') \\
&= \Pi \circ \Psi^{\Lambda^\circ}(f' * g') = \big(\Pi \circ \Psi^{\Lambda^\circ}(f')\big) \star \big(\Pi \circ \Psi^{\Lambda^\circ}(g')\big) \\
&= \big(\Psi^{\Omega} \circ \bar{\Pi}^{\Theta,\Omega}_{\mathfrak{Obj}_{\mathcal{A}}}(f')\big) \star \big(\Psi^{\Omega} \circ \bar{\Pi}^{\Theta,\Omega}_{\mathfrak{Obj}_{\mathcal{A}}}(g')\big) = \Psi^{\Omega}(f) \star \Psi^{\Omega}(g),
\end{aligned}$$

using (106) and that Π, Ψ^{Λ° and $\bar{\Pi}^{\Theta,\Omega}_{\mathfrak{Obj}_{\mathcal{A}}}$ are algebra morphisms. Also $\Psi^{\Omega}(\bar{\delta}_{[0]}) = c_{[\emptyset,\emptyset]}$, so Ψ^{Ω} is a (Lie) algebra morphism. □

The analogue of Remark 6.8 holds. In particular, $\Psi^{\Lambda^\circ} \circ \bar{\Pi}^{\Upsilon,\Lambda^\circ}_{\mathfrak{Obj}_{\mathcal{A}}}$, $\Psi^{\Omega} \circ \bar{\Pi}^{\Theta,\Omega}_{\mathfrak{Obj}_{\mathcal{A}}}$ are algebra morphisms from $\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}) \to B(\mathcal{A}, \Lambda^\circ, \chi), C(\mathcal{A}, \Omega, \chi)$. Define $c^{\alpha} \in C^{\mathrm{ind}}(\mathcal{A}, \Omega, \chi)$ for $\alpha \in C(\mathcal{A})$ by $c^{\alpha} = c_{[\{1\},\alpha']}$, where $\alpha'(1) = \alpha$. Then $C^{\mathrm{ind}}(\mathcal{A}, \Omega, \chi)$ is the Ω-module with basis c^{α} for $\alpha \in C(\mathcal{A})$, and (103) yields

$$\big[c^{\alpha}, c^{\beta}\big] = \big(\chi(\alpha, \beta) - \chi(\beta, \alpha)\big) c^{\alpha+\beta}. \tag{107}$$

It will be important in Section 6.6 that this depends only on the *antisymmetrization* of χ. The argument of Remark 6.8(c) shows that $B(\mathcal{A}, \Lambda^\circ, \chi)$ and $C(\mathcal{A}, \Omega, \chi)$ are the Λ°- and Ω-enveloping algebras of $B^{\mathrm{ind}}(\mathcal{A}, \Lambda^\circ, \chi)$ and $C^{\mathrm{ind}}(\mathcal{A}, \Omega, \chi)$.

6.6. Calabi–Yau 3-folds and Lie algebra morphisms

Let P be a smooth projective $\mathbb{K}$-scheme of dimension m, and $\mathcal{A} = \mathrm{coh}(P)$ and $\mathfrak{F}_{\mathcal{A}}, K(\mathcal{A})$ be as in [12, Example 9.1]. Then as in Section 6.1 there is a bilinear map $\bar\chi : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$ called the *Euler form* satisfying

$$\bar\chi\big([X],[Y]\big) = \sum_{i=0}^{m} (-1)^i \dim_{\mathbb{K}} \mathrm{Ext}^i(X,Y) \quad \text{for all } X, Y \in \mathcal{A}. \tag{108}$$

We denote it $\bar\chi$ to distinguish it from χ in Sections 6.1–6.5. By *Serre duality* we have natural isomorphisms

$$\mathrm{Ext}^i(X,Y)^* \cong \mathrm{Ext}^{m-i}(Y, X \otimes K_P) \quad \text{for all } X, Y \in \mathcal{A} \text{ and } i = 0, \ldots, m, \tag{109}$$

where K_P is the *canonical line bundle* of P. We call P a *Calabi–Yau m-fold* if K_P is trivial, so that (109) reduces to $\mathrm{Ext}^i(X,Y)^* \cong \mathrm{Ext}^{m-i}(Y,X)$.

In particular, if P is a *Calabi–Yau* 3*-fold* then (108) and (109) imply that

$$\begin{aligned} &\big(\dim_{\mathbb{K}} \mathrm{Hom}(X,Y) - \dim_{\mathbb{K}} \mathrm{Ext}^1(X,Y)\big) - \big(\dim_{\mathbb{K}} \mathrm{Hom}(Y,X) - \dim_{\mathbb{K}} \mathrm{Ext}^1(Y,X)\big) \\ &\quad = \bar\chi\big([X],[Y]\big) \quad \text{for all } X, Y \in \mathcal{A}, \end{aligned} \tag{110}$$

where $\bar\chi$ is antisymmetric. This is similar to Eq. (80), which we used to construct the algebra morphisms $\Phi^\Lambda, \Psi^\Lambda, \Psi^{\Lambda^\circ}, \Psi^\Omega$ in Sections 6.2–6.5. In fact, (80) implies (110) with $\bar\chi(\alpha,\beta) = \chi(\alpha,\beta) - \chi(\beta,\alpha)$, so (110) is a *weakening* of (80), which holds when $\mathcal{A} = \mathrm{coh}(P)$ for P a smooth projective curve and when $\mathcal{A} = \text{mod-}\mathbb{K}Q$, as in Section 6.1, and also when $\mathcal{A} = \mathrm{coh}(P)$ for P a Calabi–Yau 3-fold.

We shall show that (110) implies $\Psi^\Omega : \bar{\mathrm{SF}}{}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, \Theta, \Omega) \to C^{\mathrm{ind}}(\mathcal{A}, \Omega, \frac{1}{2}\bar\chi)$ is a Lie algebra morphism, generalizing Theorem 6.11. First we explain why this *only* works for the restriction of Ψ^Ω to $\bar{\mathrm{SF}}{}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, \Theta, \Omega)$. That is, (110) is too weak an assumption to make Ψ^Ω an algebra morphism on $\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, \Theta, \Omega)$, nor to make Φ^Λ, Ψ^Λ or Ψ^{Λ° into (Lie) algebra morphisms.

Consider whether (110) could imply $\Phi^\Lambda : \underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda) \to A(\mathcal{A}, \Lambda, \frac{1}{2}\bar\chi)$ is an algebra morphism. Let $\chi : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$ be bilinear with $\bar\chi(\alpha,\beta) = \chi(\alpha,\beta) - \chi(\beta,\alpha)$; note that for fixed $\bar\chi$ there will be *many* such χ, differing by symmetric bilinear forms. As a Λ-module $A(\mathcal{A}, \Lambda, \chi)$ depends only on $\mathcal{A}, \Lambda$, so that $A(\mathcal{A}, \Lambda, \chi) = A(\mathcal{A}, \Lambda, \frac{1}{2}\bar\chi)$, and Φ^Λ also depends only on $\mathcal{A}, \Lambda$. It is only the multiplication $\star$ in $A(\mathcal{A}, \Lambda, \chi)$ which depends on the choice of χ.

Now (80) for χ implies (110) for $\bar\chi$, as above. If (80) holds then

$$\Phi^\Lambda : \underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda) \to A(\mathcal{A}, \Lambda, \chi)$$

is an algebra morphism by Theorem 6.4. Therefore

$$\Phi^\Lambda : \underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda) \to A\big(\mathcal{A}, \Lambda, \tfrac{1}{2}\bar\chi\big)$$

cannot be an algebra morphism in general, since the multiplications (89) on $A(\mathcal{A},\Lambda,\chi) = A(\mathcal{A},\Lambda,\frac{1}{2}\bar\chi)$ associated to χ and $\frac{1}{2}\bar\chi$ are different. The many choices of χ giving the same $\bar\chi$ mean there is no one choice of $\star$ for which Φ^Λ is an algebra morphism whenever (110) holds.

Similarly $\Psi^\Lambda, \Psi^{\Lambda^\circ}, \Psi^\Omega$ cannot be algebra morphisms on $\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}},*,*)$, nor $\Psi^\Lambda, \Psi^{\Lambda^\circ}$ Lie algebra morphisms on $\bar{\mathrm{SF}}{}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}},*,*)$, as in each case $\star$ or $[\,,\,]$ in the image varies nontrivially with χ giving the same $\bar\chi$. But $\Psi^\Omega : \bar{\mathrm{SF}}{}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}},\Theta,\Omega) \to C^{\mathrm{ind}}(\mathcal{A},\Omega,\frac{1}{2}\bar\chi)$ is different, since by (107) the Lie bracket $[\,,\,]$ on $C^{\mathrm{ind}}(\mathcal{A},\Omega,\chi)$ depends only on $\bar\chi(\alpha,\beta)=\chi(\alpha,\beta)-\chi(\beta,\alpha)$, as we want.

Theorem 6.12. *Let Assumptions* 2.11 *and* 3.4 *hold and* $\bar\chi : K(\mathcal{A})\times K(\mathcal{A})\to\mathbb{Z}$ *be biadditive and satisfy* (110). *Then* $\Psi^\Omega : \bar{\mathrm{SF}}{}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}},\Theta,\Omega) \to C^{\mathrm{ind}}(\mathcal{A},\Omega,\frac{1}{2}\bar\chi)$ *in Definition* 6.10 *is an* Ω*-Lie algebra morphism.*

Proof. We must show

$$\Psi^\Omega\big([f,g]\big) = \big[\Psi^\Omega(f),\Psi^\Omega(g)\big] \quad \text{for } f,g\in \bar{\mathrm{SF}}{}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}},\Theta,\Omega).$$

It is enough to prove this for f,g supported on $\mathfrak{Obj}_{\mathcal{A}}^\alpha(\mathbb{K}),\mathfrak{Obj}_{\mathcal{A}}^\beta(\mathbb{K})$ respectively for $\alpha,\beta\in C(\mathcal{A})$. Lift f,g to $f',g'\in \bar{\mathrm{SF}}{}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}},\Upsilon,\Lambda^\circ)$ with $\bar\Pi^{\Theta,\Omega}_{\mathfrak{Obj}_{\mathcal{A}}}(f',g')=f,g$, which is possible as $\bar\Pi^{\Theta,\Omega}_{\mathfrak{Obj}_{\mathcal{A}}}$ is surjective.

Choose constructible $T\subseteq \mathfrak{Obj}_{\mathcal{A}}^\alpha(\mathbb{K})\times\mathfrak{Obj}_{\mathcal{A}}^\beta(\mathbb{K})$ with $f\otimes g$ and $f'\otimes g'$ supported on T. Generalizing Proposition 5.15, we can find a finite decomposition $T=\coprod_{m\in M}\mathfrak{G}_m(\mathbb{K})$, 1-isomorphisms $\mathfrak{G}_m\cong[V_m/G_m]$, and finite-dimensional G_m-representations $E_m^0, E_m^1, \tilde E_m^0, \tilde E_m^1$ for $m\in M$, satisfying analogues of Proposition 5.15(a)–(d), where in (a) we have isomorphisms:

$$\begin{aligned}\mathrm{Hom}(Y,X)&\cong E_m^0, &\quad \mathrm{Ext}^1(Y,X)&\cong E_m^1,\\ \mathrm{Hom}(X,Y)&\cong \tilde E_m^0, &\quad \mathrm{Ext}^1(X,Y)&\cong \tilde E_m^1.\end{aligned} \tag{111}$$

The proof of Eqs. (69) and (70) in Theorem 5.19 now shows we may write

$$f'\otimes g' = \sum_{m\in M,\,n\in N_m} c_{mn}\big[\big(W_{mn}\times\big[\operatorname{Spec}\mathbb{K}/(\mathbb{K}^\times)^2\big],\tau_{mn}\big)\big], \tag{112}$$

$$\begin{aligned}[f',g'] = \sum_{m\in M,\,n\in N_m} c_{mn}\Big\{&\big[\big(W_{mn}\times\big[E_m^1/(\mathbb{K}^\times)^2\ltimes E_m^0\big],\boldsymbol{\sigma}(\{1,2\})\circ\xi_{mn}\big)\big]\\ &-\big[\big(W_{mn}\times\big[\tilde E_m^1/(\mathbb{K}^\times)^2\ltimes \tilde E_m^0\big],\boldsymbol{\sigma}(\{1,2\})\circ\tilde\xi_{mn}\big)\big]\Big\},\end{aligned} \tag{113}$$

for N_m finite, $c_{mn}\in\Lambda^\circ$, and W_{mn} quasiprojective $\mathbb{K}$-varieties.

Since $f',g',[f',g']$ lie in $\bar{\mathrm{SF}}{}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}},\Upsilon,\Lambda^\circ)$ and are supported on $\mathfrak{Obj}_{\mathcal{A}}^\alpha(\mathbb{K}),\mathfrak{Obj}_{\mathcal{A}}^\beta(\mathbb{K})$ and $\mathfrak{Obj}_{\mathcal{A}}^{\alpha+\beta}(\mathbb{K})$ respectively, we have

$$\Psi^{\Lambda^\circ}(f')=\delta' b^\alpha,\qquad \Psi^{\Lambda^\circ}(g')=\epsilon' b^\beta,\qquad \Psi^{\Lambda^\circ}\big([f',g']\big)=\zeta' b^{\alpha+\beta}, \tag{114}$$

for some $\delta', \epsilon', \zeta' \in \Lambda^\circ$, and b^α, b^β, $b^{\alpha+\beta}$ as in Remark 6.8(c). Projecting (112) and (113) to $\underline{\bar{\mathrm{SF}}}(\operatorname{Spec}\mathbb{K}, \Upsilon, \Lambda)$ as in the proofs of (95) and (96) then shows that

$$\delta'\epsilon' = \sum_{m\in M,\, n\in N_m} c_{mn}\Upsilon\bigl([W_{mn}]\bigr), \tag{115}$$

$$\zeta' = \sum_{m\in M,\, n\in N_m} c_{mn}\Upsilon\bigl([W_{mn}]\bigr)\frac{\Upsilon([E_m^1])\Upsilon([E_m^0])^{-1} - \Upsilon([\tilde E_m^1])\Upsilon([\tilde E_m^0])^{-1}}{\ell - 1}. \tag{116}$$

For $m \in M$ we can find $[X] \in \mathfrak{Obj}_{\mathcal{A}}^\alpha(\mathbb{K})$ and $[Y] \in \mathfrak{Obj}_{\mathcal{A}}^\beta(\mathbb{K})$ such that (111) holds, and then in (116) we have

$$\frac{\Upsilon([E_m^1])\Upsilon([E_m^0])^{-1} - \Upsilon([\tilde E_m^1])\Upsilon([\tilde E_m^0])^{-1}}{\ell - 1} = \ell^{\dim\operatorname{Ext}^1(X,Y) - \dim\operatorname{Hom}(X,Y)}$$
$$\cdot \frac{\ell^{\dim\operatorname{Ext}^1(Y,X) - \dim\operatorname{Hom}(Y,X) - \dim\operatorname{Ext}^1(X,Y) + \dim\operatorname{Hom}(X,Y)} - 1}{\ell - 1}.$$

Composing with $\pi : \Lambda^\circ \to \Omega$ and using (110) and $\pi(\ell) = 1$ then gives

$$\pi\left(\frac{\Upsilon([E_m^1])\Upsilon([E_m^0])^{-1} - \Upsilon([\tilde E_m^1])\Upsilon([\tilde E_m^0])^{-1}}{\ell - 1}\right) = \bar\chi(\alpha, \beta) \in \mathbb{Z} \subset \Omega. \tag{117}$$

Applying Π in (104) to (114) and using (106), that $\bar\Pi^{\Theta,\Omega}_{\mathfrak{Obj}_{\mathcal{A}}}$ is an algebra morphism, and $\bar\Pi^{\Theta,\Omega}_{\mathfrak{Obj}_{\mathcal{A}}}(f') = f$, $\bar\Pi^{\Theta,\Omega}_{\mathfrak{Obj}_{\mathcal{A}}}(g') = g$, we find that

$$\Psi^\Omega(f) = \delta c^\alpha, \qquad \Psi^\Omega(g) = \epsilon c^\beta, \qquad \Psi^\Omega\bigl([f, g]\bigr) = \zeta c^{\alpha+\beta}, \tag{118}$$

where $\pi(\delta') = \delta$, $\pi(\epsilon') = \epsilon$, $\pi(\zeta') = \zeta$ and c^α, c^β, $c^{\alpha+\beta}$ are as in (107). Applying π to (115), (116) and using (117) then shows that $\zeta = \delta\epsilon\bar\chi(\alpha, \beta)$. Combining this with (107) and (118) now shows that

$$\Psi^\Omega\bigl([f, g]\bigr) = \bigl[\Psi^\Omega(f), \Psi^\Omega(g)\bigr],$$

where the bracket $[\,,]$ in $C^{\mathrm{ind}}(\mathcal{A}, \Omega, \frac{1}{2}\bar\chi)$ is defined using the form $\chi = \frac{1}{2}\bar\chi$, so that $\chi(\alpha, \beta) - \chi(\beta, \alpha)$ in (107) is $\bar\chi(\alpha, \beta)$ as $\bar\chi$ is antisymmetric. □

As in the previous cases, $\Psi^\Omega \circ \bar\Pi^{\Theta,\Omega}_{\mathfrak{Obj}_{\mathcal{A}}} : \mathrm{SF}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ or $\bar{\mathrm{SF}}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, \Upsilon, \Lambda^\circ) \to C^{\mathrm{ind}}(\mathcal{A}, \Omega, \frac{1}{2}\bar\chi)$ are also Lie algebra morphisms. We explain Theorem 6.12 using the following example. For simplicity, let $X, Y \in \mathcal{A}$ be indecomposable, with $[X] = \alpha$ and $[Y] = \beta$ in $C(\mathcal{A})$, and form $\bar\delta_{[X]}, \bar\delta_{[Y]}$ in $\bar{\mathrm{SF}}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}, \Theta, \Omega)$. Then $\Psi^\Omega(\bar\delta_{[X]}) = c^\alpha$ and $\Psi^\Omega(\bar\delta_{[Y]}) = c^\beta$. Consider the commutator $[\bar\delta_{[X]}, \bar\delta_{[Y]}]$.

We find $\Psi^\Omega(\bar\delta_{[X]} * \bar\delta_{[Y]}) = (\dim\operatorname{Ext}^1(Y, X) - \dim\operatorname{Hom}(Y, X))c^{\alpha+\beta} + c_{[\{1,2\},\kappa]}$, where $\kappa(1) = \alpha$ and $\kappa(2) = \beta$. This is because $\bar\delta_{[X]} * \bar\delta_{[Y]}$ is essentially the characteristic function of all Z in short exact sequences $0 \to X \to Z \to Y \to 0$. The effect of applying Ψ^Ω is to 'count' such sequences in a special way. The nontrivial extensions are parametrized by

$P(\mathrm{Ext}^1(Y,X))$ and contribute $\dim \mathrm{Ext}^1(Y,X)$ to the 'number' of such Z, and the trivial extension $0 \to X \to X \oplus Y \to Y \to 0$ has stabilizer group $(\mathrm{Aut}(X) \times \mathrm{Aut}(Y)) \ltimes \mathrm{Hom}(Y,X)$ and contributes $\dim \mathrm{Hom}(Y,X)$ to the number of 'virtual indecomposables' multiplying $c^{\alpha+\beta}$, and 1 to the number of 'virtual decomposables' multiplying $c_{[\{1,2\},\kappa]}$.

Exchanging X, Y and using (110) gives

$$\Psi^{\Omega}\big([\bar{\delta}_{[X]}, \bar{\delta}_{[Y]}]\big) = \bar{\chi}\big([X],[Y]\big)c^{\alpha+\beta}.$$

By (107), this is $[c^{\alpha}, c^{\beta}]$ in the Lie algebra $C^{\mathrm{ind}}(\mathcal{A}, \Omega, \frac{1}{2}\bar{\chi})$, so

$$\Psi^{\Omega}\big([\bar{\delta}_{[X]}, \bar{\delta}_{[Y]}]\big) = \big[\Psi^{\Omega}(\bar{\delta}_{[X]}), \Psi^{\Omega}(\bar{\delta}_{[Y]})\big],$$

as we want. Thus we see that Theorem 6.12 relies on (110) and a very particular way of 'counting' stack functions on $\mathfrak{Obj}_{\mathcal{A}}$, such that the 'number' of extensions of Y by X is $\dim \mathrm{Ext}^1(Y,X) - \dim \mathrm{Hom}(Y,X)$.

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