UNIQUENESS AND STABILITY IN MULTIDIMENSIONAL HYPERBOLIC INVERSE PROBLEMS

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ABSTRACT. Under a weak regularity assumption, we prove the uniqueness in multidimensional hyperbolic inverse problems with a single measurement. Moreover we show that our uniqueness results yield the best possible Lipschitz stability in $L^2$-space in the inverse problems by means of the exact observability inequality.

1. Introduction and main results

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and its boundary $\partial \Omega$ be of class $C^2$. We consider two systems:

\begin{equation}
\begin{cases}
\frac{\partial^2 u}{\partial t^2}(x, t) = \Delta u(x, t) - q(x)u(x, t), & x \in \Omega, \ 0 < t < T \\
u(x, 0) = a(x), & x \in \Omega \\
u'(x, 0) = b(x), & x \in \Omega \\
\frac{\partial u}{\partial t}(x, t) = \xi(x, t), & x \in \partial \Omega, \ 0 < t < T.
\end{cases}
\end{equation}

(1.1)

\begin{equation}
\begin{cases}
\frac{\partial^2 y}{\partial t^2}(x, t) = \Delta y(x, t) - q(x)y(x, t) + f(x)R(x, t), & x \in \Omega, \ 0 < t < T \\
y(x, 0) = y'(x, 0) = 0, & x \in \Omega \\
y(x, t) = 0, & x \in \partial \Omega, \ 0 < t < T.
\end{cases}
\end{equation}

(1.2)

Here we set $u' = \frac{\partial u}{\partial t}$, $u'' = \frac{\partial^2 u}{\partial t^2}$, and $a \in H^1(\Omega)$, $h \in L^2(\Omega)$, $\xi \in L^2(\partial \Omega \times (0, T))$, $R$ are given suitably, and in (1.2) also $q$ is given, and $\nu(x) = (\nu_1(x), ..., \nu_n(x))$ denotes the unit outward normal vector to $\partial \Omega$ at $x$, $\frac{\partial y}{\partial \nu}$ the normal derivative: $\frac{\partial y}{\partial \nu}(x) = \sum_{i=1}^n \nu_i(x) \frac{\partial y}{\partial x_i}(x)$.

Throughout this paper, $H^2(\Omega)$, $H^2(\Omega \times (0, T))$, $W^{3,\infty}(\Omega \times (0, T))$, $H^1(\Omega)$ denote Sobolev spaces (e.g. Adams [1], Lions and Magenes [17]).

This paper treats two kinds of inverse problems for multidimensional hyperbolic equations.
Nonlinear inverse problem

Determine \( q(x) \), \( x \in \Omega \) from

\[ \frac{\partial u}{\partial \nu}|_{\partial \Omega \times (0,T)} \]

in (1.1).

Linear inverse problem

Determine \( f(x) \), \( x \in \Omega \) from

\[ \frac{\partial y}{\partial \nu}|_{\partial \Omega \times (0,T)} \]

in (1.2).

In our inverse problems, we are required to determine a coefficient of lower-order term or a right-hand side in hyperbolic equations from a single observation of Neumann boundary data.

First we are concerned with the uniqueness in the inverse problems:

Uniqueness in the nonlinear inverse problem

Let \( u = u(q) \) be a weak solution to (1.1). Does the normal derivative \( \frac{\partial u}{\partial \nu}|_{\partial \Omega \times (0,T)} \) determine \( q \) uniquely? In other words, does

\[ \frac{\partial u(q)}{\partial \nu}|_{\partial \Omega \times (0,T)} = \frac{\partial u(p)}{\partial \nu}|_{\partial \Omega \times (0,T)} \]

imply \( q(x) = p(x) \), \( x \in \Omega \) and \( u(q)(x,t) = u(p)(x,t) \), \( x \in \Omega \), \( 0 < t < T \)?

Uniqueness in the linear inverse problem

Let \( y = y(f) \) be a weak solution to (1.2). Does \( \frac{\partial y}{\partial \nu}|_{\partial \Omega \times (0,T)} \) determine \( f \) uniquely? More precisely, by taking into consideration the linearity in \( f \) of (1.2), does

\[ \frac{\partial y(f)}{\partial \nu}|_{\partial \Omega \times (0,T)} = 0 \]

imply \( f(x) = 0 \), \( x \in \Omega \) and \( y(f)(x,t) = 0 \), \( x \in \Omega \), \( 0 < t < T \)?

Second we consider the stability in these inverse problems. For example, by the stability in the linear inverse problem, we understand the following problem: Estimate \( f \) by

\[ \frac{\partial y(f)}{\partial \nu}|_{\partial \Omega \times (0,T)} \]

with suitable norms.
For the uniqueness in multidimensional inverse problems with a single observation, the paper by Bukhgeim and Klibanov [3] is epoch-making and gives a methodology on the basis of the Carleman estimate. After Bukhgeim and Klibanov [3], several papers by the Carleman estimate have been published concerning inverse problems (e.g. Isakov [5], [7], Khaidarov [9], Klibanov [10]). Furthermore we refer to Kubo [13].

On the other hand, for inverse hyperbolic problems by the Dirichlet-to-Neumann map, see Rakesh and Symes [20], Ramm and Rakesh [21], Ramm and Sjöstrand [22], Sun [24]. We note that the formulation by the Dirichlet-to-Neumann map requires repeat of observations, although we need not choose strictly positive data like (1.12).

For the Lipschitz stability in our formulation of the inverse problems, there are very few results. In particular, it is shown in Puel and Yamamoto [18], [19] that the uniqueness for unknown functions in $L^2(\Omega)$, implies the global Lipschitz stability with suitable choices of norms. However, in Puel and Yamamoto [18], [19], the assumption of $T$ for the stability is far from the best possible one, and we require extra regularity on $R$ by which the application of the result to the nonlinear inverse problem becomes more complicated. Our purpose of this paper is to refine the previous results in [18], [19] and establish the best possible stability for the nonlinear inverse problem.

We set

$$\rho = \min_{\eta \in \Omega} \max_{x \in \Omega} |x - \eta|.$$  

Without loss of generality we always assume that $0 \in \Omega$ and the minimum is attained at $\eta = 0$. That is,

$$\rho = \max_{x \in \Omega} |x|.$$  

Since $\Omega$ is a bounded domain, we note that $\rho < \infty$. As for the uniqueness, we state a result by Isakov [5] as follows, which is typical among published ones.

**Theorem A (Uniqueness in the linear inverse problem: [5]).** - We assume that

$$R \in W^{3,\infty}(\Omega \times (0,T))$$

and $R(x,0) \neq 0$, $x \in \overline{\Omega}$ and $T > \rho$. If $(y(f), f) \in H^2(\Omega \times (0,T)) \times L^2(\Omega)$ satisfies (1.2),

$$y(f)' \in H^2(\Omega \times (0,T))$$

and

$$\frac{\partial y(f)}{\partial \nu}(x,t) = 0, \ \ x \in \partial \Omega, \ 0 < t < T;$$

then $f(x) = 0$ and $y(f)(x,t) = 0$, $x \in \Omega$, $0 < t < T$.

Remark. - In Isakov [5], the system (1.2) is considered in $\Omega \times (-T,T)$. However, taking the even extension of $y$ in $t$ to $(-T,0)$ (e.g. the proof of Theorem 3.8 in Klibanov [10]), we can similarly prove the uniqueness in the case of $\Omega \times (0,T)$. Moreover, in [5], the regularity $R, R' \in C^2(\Omega \times [-T,T])$ is assumed, but we can easily weaken this assumption to ours (1.4).
Throughout this paper, we choose the $L^2(\Omega)$-norm for measuring $f$. Then it is a serious problem that we have to derive the regularity (1.5) from our assumption (1.4) and $f \in L^2(\Omega)$. In general, $f \in L^2(\Omega)$ guarantees $y(f) \in H^2(\Omega \times (0,T))$, but not (1.5), under the regularity (1.4) (e.g. Lions and Magenes [17]). Also in Khaidarov [9], Klibanov [10], Kubo [13], extra regularity conditions like (1.5) are assumed. Here we would like to assume only $f \in L^2(\Omega)$, not more, for the regularity of $f$, in order that our stability estimate may be as best as possible. Thus in the uniqueness like Theorem A, it is desirable to weaken the regularity assumption (1.5). Such the uniqueness under less regular assumptions is our first main result.

**Theorem 1.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and its boundary $\partial \Omega$ be of class $C^2$. We assume

\begin{equation}
R \in W^{3,\infty}(\Omega \times (0,T)),
\end{equation}

\begin{equation}
|R(x,0)| \geq r_0 > 0\quad \text{almost everywhere on } \overline{\Omega}
\end{equation}

with some constant $r_0 > 0$,

\begin{equation}
q \in L^\infty(\Omega)
\end{equation}

and

\begin{equation}
T > \rho.
\end{equation}

If for $f \in L^2(\Omega)$, the weak solution $y = y(f) \in L^2(\Omega \times (0,T))$ to (1.2) satisfies

\begin{equation}
\frac{\partial y(f)}{\partial \nu}(x,t) = 0, \quad x \in \partial \Omega, \quad 0 < t < T,
\end{equation}

then $f(x) = 0$ and $y(f)(x,t) = 0$, $x \in \Omega$, $0 < t < T$.

Next we give the answer to the nonlinear inverse problem, the determination of coefficient of lower order term.

**Theorem 2.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and its boundary $\partial \Omega$ be of class $C^2$, and we assume (1.8). Moreover we assume:

\begin{equation}
q, p \in L^\infty(\Omega).
\end{equation}

Let either of $u(q)$ and $u(p)$ satisfy

\begin{equation}
u \in W^{3,\infty}(\Omega \times (0,T)).
\end{equation}

Moreover let

\begin{equation} |a(x)| \geq a_0 > 0 \quad \text{almost everywhere on } \overline{\Omega}
\end{equation}

with some constant $a_0 > 0$. If $\frac{\partial u(q)}{\partial \nu}(x,t) = \frac{\partial u(p)}{\partial \nu}(x,t)$, $x \in \partial \Omega$, $0 < t < T$, then $q(x) - p(x)$, $u(q)(x,t) = u(p)(x,t)$, $x \in \Omega$, $0 < t < T$. 

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On the basis of Theorem 1, we refine the argument in Puel and Yamamoto [19] with an observability inequality, and for the linear inverse problem we can show

**Theorem 3.** We assume \((1.6) - (1.8)\). Then there exists a constant \(C = C(\Omega, T, q, R) > 0\) such that:

\[
C^{-1}\|f\|_{L^2(\Omega)} \leq \left\|\frac{\partial y(f)}{\partial \nu}\right\|_{H^1(0,T;L^2(\partial\Omega))} \leq C\|f\|_{L^2(\Omega)},
\]

for all \(f \in L^2(\Omega)\).

Here and henceforth we set

\[
\|\eta\|_{H^1(0,T;L^2(\partial\Omega))} = \left(\int_0^T \int_{\partial\Omega} |\eta(x,t)|^2 + |\eta'(x,t)|^2 dS_x dt\right)^{\frac{1}{2}}.
\]

This theorem means that our estimate is the best possible in the sense that it is an upper and lower estimate.

Finally we show an upper and lower estimate for the nonlinear inverse problem. Here assuming that \(q \in L^\infty(\Omega)\) is given, we are concerned with the stability around \(q\). In other words, \(q\) and \(u(q)\) are known, while \(p \in L^\infty(\Omega)\) is unknown.

**Theorem 4.** For unknown coefficients \(p\)'s, we define an admissible set \(U \subset L^\infty(\Omega)\) such that

\[
\text{the embedding } U \rightarrow L^\infty(\Omega) \text{ is compact.}
\]

We assume \((1.7), (1.8)\),

\[
u(x) \in W^{3,\infty}(\Omega \times (0, T))
\]

and

\[
|\alpha(x)| \geq \alpha_0 > 0 \text{ almost everywhere on } \overline{\Omega}
\]

with some constant \(\alpha_0 > 0\). Then there exists a constant \(C = C(\Omega, T, q, a, b, \xi, U) > 0\) such that

\[
C^{-1}\|q - p\|_{L^2(\Omega)} \leq \left\|\frac{\partial u(q)}{\partial \nu} - \frac{\partial u(p)}{\partial \nu}\right\|_{H^1(0,T;L^2(\partial\Omega))} \leq C\|q - p\|_{L^2(\Omega)},
\]

for all \(p \in U\).

The condition \((1.15)\) requires sufficient smoothness of \(q\) and compatibility conditions of sufficient order for \(a, b\) and \(\xi\) on \(\partial\Omega \times \{0\}\), which involve values and derivatives of the known \(q\) on \(\partial\Omega\). In particular, \(\xi(x, 0) \neq 0, x \in \partial\Omega\) must be satisfied by \((1.16)\).

The first inequality in \((1.17)\) shows the stability in our inverse problem and the second means that our estimate is the best possible. This best possible stability is new. For the formulation with the Dirichlet-to-Neumann map, we know only Hölder stability at most (Sun [24]).

**Remark.** If we can assume that \(\|q - p\|_{L^\infty(\Omega)}\) is sufficiently small, then we need not the compactness of \(U\), as is seen from the proof in §7. Also for the second inequality in \((1.17)\), it is sufficient to assume that \(U\) is bounded in \(L^\infty(\Omega)\).
The remainder of this paper is organized as follows:
§2. Carleman estimate,
§3. Proof of Theorem 1,
§4. Proof of Theorem 2,
§5. Observability inequality,
§6. Proof of Theorem 3,
§7. Proof of Theorem 4,

Our keys for the stability in the inverse problems are the observability inequality and the compactness-uniqueness argument, while the key for the uniqueness is modification of the argument by [3], [5], [10].

2. Carleman estimate

In this step, we will establish a Carleman estimate under a weaker regularity assumption. Let us set

\[ q(x) = \begin{cases} q(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^n \setminus \Omega \end{cases} \]

and

\[ (Pw)(x, t) = w''(x, t) - \Delta w(x, t) + \varphi(x)w(x, t), \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}. \]

We define a weight function \( \phi = \phi(x, t) \) by

\[ \phi(x, t) = |x|^2 - \beta |t|^2. \]

Here \( \beta \in (0, 1) \) is chosen later. Throughout this paper we use the notation:

\[ \phi_c = \{(x, t) ; x \in \Omega, \phi(x, t) > c^2 \} \]

for a constant \( c \geq 0 \). Moreover let \( H_0^1(\phi_c) = \{ v \in H^1(\phi_c) ; \phi |_{\partial \phi_c} = 0 \} \).

Now we are ready to state a Carleman estimate.

**Proposition 1.** There exist constants \( M > 0 \) and \( \Lambda_0 > 0 \) depending on \( \beta \in (0, 1) \) such that we have a Carleman estimate:

\[ M \lambda \int_{\phi_c} (|\nabla v|^2 + |v'|^2) e^{2\lambda \phi} \, dx \, dt + M \lambda^3 \int_{\phi_c} |v|^2 e^{2\lambda \phi} \, dx \, dt \leq \int_{\phi_c} |g|^2 e^{2\lambda \phi} \, dx \, dt, \quad \lambda > \Lambda_0 \]

for all \( v \in H_0^1(\phi_c) \) satisfying

\[ (v, P \mu)_{L^2(\phi_c)} = (g, \mu)_{L^2(\phi_c)} \text{ for all } \mu \in H^2(\phi_c). \]

In (2.6), we note that \( g \) is uniquely determined by \( v \) if it exists.
Remark. – By identifying the dual of $L^2(\phi_c)$ with itself, we define $(H^2(\phi_c))'$: the dual of $H^2(\phi_c)$. Then $H^2(\phi_c) \subset L^2(\phi_c) \subset (H^2(\phi_c))'$. We denote the duality pairing between $(H^2(\phi_c))'$ and $H^2(\phi_c)$ by $(H^2(\phi_c))'(\cdot, \cdot)_{H^2(\phi_c)}$. For $v \in L^2(\phi_c)$, we define an element of $(H^2(\phi_c))'$, denoted by $< Pu >$, in the following manner:
\[
(H^2(\phi_c))'( < Pu >, \mu)_{H^2(\phi_c)} = (v, P\mu)_{L^2(\phi_c)}
\]
for all $\mu \in H^2(\phi_c)$. Then we can interpret (2.6) as
\[
(2.6') \quad < Pu > \in L^2(\phi_c).
\]
Henceforth for $v \in L^2(\phi_c)$, we define $\hat{v} \in L^2(\mathbb{R}^{n+1})$ by:
\[
(2.7) \quad \hat{v}(x,t) = \begin{cases} 
v(x,t), & (x,t) \in \phi_c, \\
0, & (x,t) \in \mathbb{R}^{n+1} \setminus \phi_c.
\end{cases}
\]
We can prove that the condition (2.6) is equivalent to $\sum_{i=1}^n \frac{\partial v_i}{\partial x_i} m_i = 0$ on $\partial \phi_c$, where $m = (m_1, m_2, \ldots, m_{n+1})$ is the unit outward normal vector to $\partial \phi_c$. Then from the proof of Proposition 1, we see
\[
< \hat{P} u > = P\hat{v} \quad \text{in} \quad \mathcal{D}(\mathbb{R}^{n+1})'.
\]

Remark. – In our Carleman estimate (2.5), we do not assume that $v \in H^2(\phi_c)$ like in usual Carleman estimates given in Isakov [5], [6]. In Tataru [26], the Carleman estimate is proved with non-zero boundary data for general partial differential operators under the assumption of pseudoconvexity, and if we realize and apply his estimate to our hyperbolic differential operator, then we can obtain the Carleman estimate within the same regularity condition as (2.6). However, we here choose a more direct way which is based on a version of Carleman estimate by Isakov [5], [6]. For direct derivation of Carleman estimates for the D'Alembertian, we can refer also to Chapter IV-§4 of Lavrent'ev, Romanov and Shishat'skii [15]. For a Carleman estimate under a much weaker regularity assumption, we can refer to Ruiz [23].

Proof of Proposition 1. – Let $c > 0$ and a sufficiently small $\delta > 0$ be given. Let $\Omega_\delta$ denote the $\delta$-neighbourhood of $\Omega$: $\Omega_\delta = \{x \in \Omega; \text{dist} (x, \Omega) < \delta\}$. We set
\[
\phi_{c, s, \delta} = \{(x, t); x \in \Omega_\delta, \phi(x, t) > (c - \delta)^2\}
\]
We first show:

Lemma 1. – There exist constants $M = M(c, \delta) > 0$ and $\Lambda_0 > 0$ depending on $\beta \in (0, 1)$ such that we have a Carleman estimate
\[
M \lambda \int_{\phi_{c, s, \delta}} (|\nabla v|^2 + |v|^2) e^{2\lambda \phi} \, dx \, dt + M \int_{\phi_{c, s, \delta}} |v|^2 e^{2\lambda \phi} \, dx \, dt
\leq \int_{\phi_{c, s, \delta}} |P v|^2 e^{2\lambda \phi} \, dx \, dt,
\]
\[
\lambda > \Lambda_0
\]
for all $v \in H^2(\phi_{c, s, \delta})$.

For the proof, see Isakov [5], [6].
The proof of Proposition 1 is based on Lemma 1 in view of the mollifier (e.g. Adams [1]). Take $J \in C_0^\infty(\mathbb{R}^{n+1})$, supp $J \subset \{(x, t); |x|^2 + t^2 < 1\}$, $J \geq 0$ such that
\[
\int_{\mathbb{R}^{n+1}} J(x,t) dx dt = 1.
\]
We set $J_\varepsilon(x,t) = \varepsilon^{-n-1} J\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right)$, $\varepsilon > 0$ and
\[
(J_\varepsilon * v)(x,t) = \int_{\mathbb{R}^{n+1}} J_\varepsilon(x-y,t-s)v(y,s) dy ds.
\]

Let $v \in H_0^1(\phi_\varepsilon)$ satisfy (2.6). By $< P\tilde{v} >$ we denote $g \in L^2(\phi_\varepsilon)$ in (2.6) which is uniquely determined by $v$. Since the domain $\phi_\varepsilon$ has a piecewise smooth boundary, we see
\[
(2.8) \quad \tilde{v} \in H^1(\mathbb{R}^{n+1}),
\]
(c.g. Lemma III-3.22 in Adams [1]).

Moreover we see
\[
(2.9) \quad < P\tilde{v} > = P\tilde{v} \quad \text{in } (D(\mathbb{R}^{n+1}))'.
\]

In fact, let $\mu \in C_0^\infty(\mathbb{R}^{n+1}) \equiv D(\mathbb{R}^{n+1})$. Then, in view of $< P\tilde{v} > \in L^1_{loc}(\mathbb{R}^{n+1})$, $\mu|_{\phi_\varepsilon} \in C_0^\infty(\phi_\varepsilon)$ and (2.6), we obtain
\[
(D(\mathbb{R}^{n+1}))(< P\tilde{v} >, \mu)_{D(\mathbb{R}^{n+1})} = \int_{\mathbb{R}^{n+1}} < P\tilde{v} > \mu dx dt = \int_{\phi_\varepsilon} < P\tilde{v} > \mu dx dt = (D(\mathbb{R}^{n+1}))(< P\tilde{v} >, \mu)_{D(\mathbb{R}^{n+1})}.
\]

Since $\mu \in C_0^\infty(\mathbb{R}^{n+1})$ is arbitrary, we see (2.9).

Let $\varepsilon_0 > 0$ be sufficiently small and fixed. Let $0 < \varepsilon < \varepsilon_0$. Then by (2.8) we see
\[
(2.10) \quad \tilde{v} * J_\varepsilon \in C_0^\infty(\mathbb{H}^{n+1}), \quad \text{supp } (\tilde{v} * J_\varepsilon) \subset \phi_\varepsilon - \delta, \delta,
\]
with some $\delta = \delta(\varepsilon_0) > 0$, and
\[
(2.11) \quad \|\tilde{v} * J_\varepsilon - \tilde{v}\|_{H^1(\phi_\varepsilon)} \longrightarrow 0 \quad \text{as } \varepsilon \downarrow 0
\]
(c.g. [1]). By (2.10) and $\tilde{v} * J_\varepsilon \in H_0^2(\phi_\varepsilon - \delta, \delta)$, we can apply Lemma 1 in $\phi_\varepsilon - \delta, \delta$ and obtain
\[
(2.12) \quad M \lambda \int_{\phi_\varepsilon - \delta, \delta} (|\nabla(\tilde{v} * J_\varepsilon)|^2 + |(\tilde{v} * J_\varepsilon)'|^2)e^{2\lambda \phi} dx dt + M \lambda^3 \int_{\phi_\varepsilon - \delta, \delta} |\tilde{v} * J_\varepsilon|^2 e^{2\lambda \phi} dx dt \leq \int_{\phi_\varepsilon - \delta, \delta} |P(\tilde{v} * J_\varepsilon)|^2 e^{2\lambda \phi} dx dt
\]
for all large $\lambda > 0$. Since $J_\varepsilon \in D(\mathbb{R}^{n+1})$ and $\tilde{v} \in L^2(\mathbb{R}^{n+1}) \subset (D(\mathbb{R}^{n+1}))^\prime$, we can regard $\tilde{v} * J_\varepsilon$ as the convolution between $\tilde{v} \in (D(\mathbb{R}^{n+1}))^\prime$ and $J_\varepsilon \in D(\mathbb{R}^{n+1})$, and so
\[
(2.13) \quad P(\tilde{v} * J_\varepsilon) = (P\tilde{v}) * J_\varepsilon \quad \text{in } (D(\mathbb{H}^{n+1}))^\prime
\]
(e.g. VI-3 of Yosida [28]). By (2.9),
\[
(2.14) \quad P(\tilde{v} * J_\varepsilon) = < P\tilde{v} > * J_\varepsilon \quad \text{in } (D(\mathbb{R}^{n+1}))^\prime.
\]
Since $\langle \tilde{P} \rangle \in L^2(\mathbb{R}^{n+1})$ by (2.6), it follows that:

\[(2.15) \quad P(\tilde{v} \ast J_e)(x,t) = \langle \tilde{P} \rangle \ast J_e(x,t) \quad \text{for almost all } (x,t) \in \mathbb{R}^{n+1}.
\]

Therefore, noting that $\phi_{c-\delta, \delta} \supset \phi_c$, we can rewrite (2.12) as

\[(2.16) \quad MA \int_{\phi_c} (|\nabla (\tilde{v} \ast J_e)|^2 + |(\tilde{v} \ast J_e')|^2) e^{2\lambda \phi} \, dx \, dt + MA^3 \int_{\phi_c} |\tilde{v} \ast J_e|^2 e^{2\lambda \phi} \, dx \, dt
\]
\[
\leq \int_{\phi_{c-\delta(\epsilon_0), \delta(\epsilon_0)}} |\langle \tilde{P} \rangle \ast J_e|^2 e^{2\lambda \phi} \, dx \, dt
\]
for all large $\lambda > 0$ and all $\epsilon \in (0, \epsilon_0)$. Here we note also that

\[
\text{supp} (\langle \tilde{P} \rangle \ast J_e) \subset \phi_{c-\delta(\epsilon_0), \delta(\epsilon_0)} \quad \text{(e.g. Adams [11]).}
\]

Since $||\langle \tilde{P} \rangle \ast J_e - \langle \tilde{P} \rangle||_{L^2(\mathbb{R}^{n+1})} \to 0$ as $\epsilon \downarrow 0$ by $\langle \tilde{P} \rangle \in L^2(\mathbb{R}^{n+1})$, noting that $\phi_{c-\delta(\epsilon_0), \delta(\epsilon_0)}$ is bounded and so $e^{2\lambda \phi} ||\tilde{P}||_{L^\infty(\phi_{c-\delta(\epsilon_0), \delta(\epsilon_0)})} < \infty$, we can make $\epsilon > 0$ going to 0 in (2.16), and by (2.11), we obtain:

\[
MA \int_{\phi_c} (|\nabla v|^2 + |v'|^2) e^{2\lambda \phi} \, dx \, dt + MA^3 \int_{\phi_c} |v|^2 e^{2\lambda \phi} \, dx \, dt
\]
\[
\leq \int_{\phi_{c-\delta(\epsilon_0), \delta(\epsilon_0)}} |\langle \tilde{P} \rangle \ast J_e|^2 e^{2\lambda \phi} \, dx \, dt = \int_{\phi_c} |\langle P \rangle \ast J_e|^2 e^{2\lambda \phi} \, dx \, dt.
\]

At the last equality and inequality, we use $\langle \tilde{P} \rangle \neq -0$ in $\mathbb{R}^{n+1} \setminus \phi_c$. Thus the proof of Proposition 1 is complete.

3. Proof of Theorem 1

The proof will be done by modifying the arguments in Isakov [5], Klibanov [10].

First Step. – By the assumption (1.8), we can take a constant $\beta \in (0,1)$ such that

\[(3.1) \quad T = \frac{\rho}{\sqrt{\beta}}.
\]

First we note

\[(3.2) \quad W^{1,1}(0,T;L^\infty(\Omega)) \subset C([0,T];L^\infty(\Omega)).
\]

Setting $y_1 = y'$, we have

\[(3.3) \quad \begin{cases}
   y_1''(x,t) = \Delta y_1(x,t) - q(x)y_1(x,t) + f(x)R(x,t), & x \in \Omega, 0 < t < T \\
   y_1(x,0) = 0, & y_1'(x,0) = f(x)R(x,0), \quad x \in \Omega \\
   y_1(x,t) = 0, & x \in \partial \Omega, 0 < t < T.
\end{cases}
\]

In view of (1.6) and (3.2), we can apply the regularity property of solutions to (1.2) and (3.3) (e.g. Lions [16], Lions and Magenes [17]), so that $y_1 = y' \in C([0,T];H^1_0(\Omega)) \cap C^1([0,T];L^2(\Omega))$ and $y \in C([0,T];H^1_0(\Omega))$, $y'' \in C([0,T];L^2(\Omega))$. 

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Therefore $\Delta y = y'' + qy - fR \in C([0, T]; L^2(\Omega))$, so that $y \in C([0, T]; H^2(\Omega))$. Thus we obtain

\begin{equation}
\begin{aligned}
y \in & C([0, T]; H^2(\Omega) \cap H^1_0(\Omega)) \cap C^1([0, T]; H^1_0(\Omega)) \\
& \cap C^2([0, T]; L^2(\Omega)).
\end{aligned}
\end{equation}

**Second Step.** In view of (1.6) and (3.2), there exist small constants $\delta \in (0, T)$ and $r_0 > 0$ such that

\begin{equation}
|R(x, t)| \geq r_0, \quad x \in \Omega, \quad 0 \leq t \leq \delta.
\end{equation}

We extend $y = y(f)(\cdot, t), 0 \leq t < T$ to a function $y(x, t), |t| < T$ which is even in $t$:

\begin{equation}
y(x, t) = \begin{cases}
y(x, t), & x \in \Omega, \ t \geq 0 \\
y(x, -t), & x \in \Omega, \ t \leq 0.
\end{cases}
\end{equation}

Then since $y(x, 0) = y'(x, 0) = 0, x \in \Omega$, we see that

\begin{equation}
y \in C([-T, T]; H^2(\Omega) \cap H^1(\Omega)) \cap C^1([-T, T]; H^1_0(\Omega)) \cap C^2([-T, T]; L^2(\Omega)).
\end{equation}

We extend $R = R(x, t)$ to $t \in (-\delta, \delta)$ as an even function in $t$ and we denote the extension by the same notation $R$. Moreover we set

\begin{equation}
h(x, t) = \frac{R'(x, t)}{R(x, t)} , \quad x \in \Omega, \ |t| \leq \delta.
\end{equation}

Then by (1.6) we see

\begin{equation}
\begin{cases}
R \in W^{1, \infty}(\Omega \times (-\delta, \delta)), \\
h_{l(-\delta, 0)} \in W^{2, \infty}(\Omega \times (-\delta, 0)), \quad h_{l(0, \delta)} \in W^{2, \infty}(\Omega \times (0, \delta)).
\end{cases}
\end{equation}

Henceforth we set

\begin{equation}
(Nw)(x, t) = w'(x, t) - h(x, t)w(x, t), \quad x \in \Omega, \ |t| < \delta.
\end{equation}

Let us take a sufficiently small $\epsilon > 0$. We set

\begin{equation}
c(\epsilon) = (\rho^2 - \beta \delta^2 + \epsilon^2)^{\frac{1}{2}},
\end{equation}

where $\delta > 0$ is chosen such that (3.5) is true. Now we note that:

\begin{equation}
\phi_{c(\epsilon)} \subset \{(x, t); \sqrt{\rho^2 - \beta \delta^2} < |x| < \rho, \ |t| < \delta\}.
\end{equation}

Let $\chi \in C^\infty(\overline{\Omega} \times [-\delta, \delta])$ such that $0 \leq \chi(x, t) \leq 1, x \in \Omega, \ |t| \leq \delta, \chi(\cdot, t) = \chi(\cdot, -t)$ and

\begin{equation}
\chi(x, t) = \begin{cases}
1, & (x, t) \in \phi_{c(3\epsilon)} , \\
0, & (x, t) \in (\overline{\Omega} \times [-\delta, \delta]) \setminus \phi_{c(2\epsilon)}.
\end{cases}
\end{equation}
We set
\begin{equation}
(3.14) \quad v(x, t) = \chi(x, t) y(x, t), \quad x \in \Omega, \ |t| < \delta.
\end{equation}

We can easily see that \( N v \in H^1_0(\phi(\varepsilon)) \) by (1.2) and (3.7). In this step, we will prove that \( N v \) satisfies (2.6). Since \( C^\infty(\phi(\varepsilon)) \) is dense in \( H^2(\phi(\varepsilon)) \), it is sufficient to prove that there exists \( g \in L^2(\phi(\varepsilon)) \) such that:
\begin{equation}
(3.15) \quad (N v, P \mu)_{L^2(\phi(\varepsilon))} = (g, \mu)_{L^2(\phi(\varepsilon))}, \quad \mu \in C^\infty(\phi(\varepsilon)).
\end{equation}

We set \( \Sigma = (\partial \Omega \times (-T, T)) \cap \partial \phi(\varepsilon) \). Then
\begin{equation}
(3.16) \quad y, v \in H^2(\phi(\varepsilon)),
\end{equation}

\begin{equation}
(3.17) \quad \frac{\partial v}{\partial \nu} = v' = v = 0 \quad \text{on} \ \Sigma
\end{equation}
and
\begin{equation}
(3.18) \quad \frac{\partial v}{\partial x_1} = \cdots = \frac{\partial v}{\partial x_n} = v = v' = 0,
\end{equation}
on \( \partial \phi(\varepsilon) \setminus \Sigma \)

by (3.7), (3.13) and (1.9). Here we recall that \( \nu = \nu(x) = (\nu_1, \ldots, \nu_n) \) is the outward unit normal vector to \( \partial \Omega \) at \( x \) and \( \frac{\partial \nu}{\partial \nu} = \sum_{i=1}^{n} \nu_i \frac{\partial \nu_i}{\partial x_i} \) on \( \partial \Omega \times (-T, T) \). Henceforth we denote the outward unit normal vector to \( \partial \phi(\varepsilon) \) at \((x, t)\) by \( m(x, t) = (m_1(x, t), \ldots, m_n(x, t), m_{n+1}(x, t)) \in \mathbb{R}^{n+1} \). Then we easily see
\begin{equation}
(3.19) \quad m(x, t) = (\nu(x), 0), \quad (x, t) \in \Sigma.
\end{equation}

Now we calculate \( (N v, P \mu)_{L^2(\phi(\varepsilon))} \):
\begin{equation}
(3.20) \quad (N v, P \mu)_{L^2(\phi(\varepsilon))} = \int_{\phi(\varepsilon)} (v' - hv) P \mu dx dt
- \int_{\phi(\varepsilon)} v(P \mu)'dx dt + \int_{\partial \phi(\varepsilon)} v P \mu m_{n+1} d\sigma - \int_{\phi(\varepsilon)} (hv)(P \mu) dx dt
- \int_{\phi(\varepsilon)} v(P \mu)'dx dt - \int_{\phi(\varepsilon)} (hv)(P \mu) dx dt
\end{equation}
by integration by parts in \( t \), (3.17) and (3.18).

First for \( \int_{\phi(\varepsilon)} v(P \mu)'dx dt \), we can proceed as follows. Setting \( x_{n+1} = t \), by integration by parts, we obtain
\begin{equation}
(3.21) \quad \int_{D} v \frac{\partial^2 \mu}{\partial x_i^2} dx dt = \int_{D} \frac{\partial^2 v}{\partial x_i^2} \mu dx dt
+ \int_{D} \left( v \frac{\partial \mu}{\partial x_i} - \frac{\partial v}{\partial x_i} \right) m_i d\sigma, \quad 1 \leq i \leq n + 1,
\end{equation}
where $D \subset \mathbb{R}^{n+1}$ is a bounded domain and the boundary $\partial D$ is a finite sum of surfaces of class $C^2$. Therefore with $D = \phi_{c(\varepsilon)}$ in (3.21), we see:

$$\int_{\phi_{c(\varepsilon)}} v(P\mu')dxdt$$

$$= \int_{\phi_{c(\varepsilon)}} v\left(\frac{\partial^2}{\partial t^2} - \Delta\right)\mu' dxdt + \int_{\phi_{c(\varepsilon)}} vq\mu' dxdt$$

$$= \int_{\phi_{c(\varepsilon)}} (Pv)\mu' dxdt + \int_{\partial\phi_{c(\varepsilon)}} \left(\frac{\partial\mu'}{\partial x_{n+1}} - \mu' \frac{\partial v}{\partial x_{n+1}}\right) m_{n+1} d\sigma$$

$$- \sum_{i=1}^{n} \int_{\partial\phi_{c(\varepsilon)}} \left(\frac{\partial\mu'}{\partial x_i} - \mu' \frac{\partial v}{\partial x_i}\right) m_i d\sigma$$

$$= \int_{\phi_{c(\varepsilon)}} (Pv)\mu' dxdt + \sum_{i=1}^{n} \int_{\Sigma} \mu' \frac{\partial v}{\partial x_i} m_i d\sigma,$$

by (3.16) - (3.18). Moreover, noting (3.19) and $\sum_{i=1}^{n} \frac{\partial v}{\partial x_i} m_i = \frac{\partial v}{\partial t}$ on $\Sigma$, we obtain

(3.22)  $\int_{\phi_{c(\varepsilon)}} v(P\mu')dxdt = \int_{\phi_{c(\varepsilon)}} (Pv)\mu' dxdt$,

by (3.17). In view of (1.2), (3.7) and (3.14), in $L^2(\phi_{c(\varepsilon)})$, we can calculate:

(3.23)  $Pv = P(xy) = \chi P y + (\chi'' - \Delta \chi)y + 2\chi' y' - 2\nabla \chi \cdot \nabla y$

$$= \chi f R + (\chi'' - \Delta \chi)y + 2\chi' y' - 2\nabla \chi \cdot \nabla y \in C^1([-T, T]; L^2(\Omega)).$$

Here we note that $\Delta = \Delta_x = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, $\nabla = \nabla_x = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$, and $y' = \frac{\partial y}{\partial t}$.

Consequently by integration by parts in $t$, (3.17), (3.18) and (3.19), we obtain:

$$\int_{\phi_{c(\varepsilon)}} (Pv)\mu' dxdt$$

$$= -\int_{\phi_{c(\varepsilon)}} \frac{\partial}{\partial t}(\chi f R + (\chi'' - \Delta \chi)y + 2\chi' y' - 2\nabla \chi \cdot \nabla y)\mu dxdt$$

$$= -\int_{\phi_{c(\varepsilon)}} (\chi' f R + \chi f R')\mu dxdt + \int_{\phi_{c(\varepsilon)}} (P_1 y)\mu dxdt,$$

where we set

(3.24)  $P_1 y = \frac{\partial}{\partial t}((-\chi'' - \Delta \chi)y - 2\chi' y' + 2\nabla \chi \cdot \nabla y)$

which is a differential operator of second order in $(x, t)$. In view of (3.22), we see:

(3.25)  $\int_{\phi_{c(\varepsilon)}} v(P\mu')dxdt$

$$= -\int_{\phi_{c(\varepsilon)}} (\chi' f R + \chi f R')\mu dxdt + \int_{\phi_{c(\varepsilon)}} (P_1 y)\mu dxdt.$$

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Next we calculate \( \int_{\phi_{c(t)}} (hv)(P\mu)dxdt \). We set \( \phi^+_{c(t)} = \phi_{c(t)} \cap \{ t \geq 0 \} \) and \( \phi^-_{c(t)} = \phi_{c(t)} \cap \{ t \leq 0 \} \). Then \( \partial \phi^+_{c(t)} \), \( \partial \phi^-_{c(t)} \) \( \subset \partial \phi_{c(t)} \cup (\phi_{c(t)} \cap \{ t = 0 \}) \). We have

\[
\int_{\phi_{c(t)}} (hv)(P\mu)dxdt = \left( \int_{\phi^+_{c(t)}} + \int_{\phi^-_{c(t)}} \right) (hv)(P\mu)dxdt.
\]

In view of (3.9) and (3.16), we can apply (3.21):

\[
\int_{\phi^+_{c(t)}} (hv)(P\mu)dxdt = \int_{\phi^+_{c(t)}} P(hv)\mu dxdt
\]

\[
+ \int_{\partial \phi^+_{c(t)} \cap \{ t \geq 0 \}} (hv\mu' - \mu(hv)' + )m_{n+1}d\sigma
\]

\[
- \int_{\partial \phi^-_{c(t)} \cap \{ t \geq 0 \}} \sum_{i=1}^{n} \left( \frac{\partial \mu}{\partial x_i} - \mu \frac{\partial (hv)}{\partial x_i} \right) m_i d\sigma
\]

\[
+ \int_{\partial \phi^-_{c(t)} \cap \{ t = 0 \}} (hv\mu' - \mu(hv)' + )m_{n+1}d\sigma
\]

\[
- \int_{\partial \phi^-_{c(t)} \cap \{ t = 0 \}} \sum_{i=1}^{n} \left( \frac{\partial \mu}{\partial x_i} - \mu \frac{\partial (hv)}{\partial x_i} \right) m_i d\sigma
\]

\[
= \int_{\phi_{c(t)}} P(hv)\mu dxdt + I_1 + I_2 + I_3 + I_4,
\]

\[m = (m_1, \ldots, m_n, m_{n+1})\] being the outward unit normal vector to \( \partial \phi^+_{c(t)} \). Here and henceforth \( P \) acts on \( hv \) in the pointwise sense in \( \phi^+_{c(t)} \) (not in the sense of the distribution or \( (H^2(\phi_{c(t)}))^\prime \)); \( P(hv) \) is well-defined pointwise in \( \phi^+_{c(t)} \) by means of (3.9) and (3.16), and \( P(hv) \in L^2(\phi^+_{c(t)}) \). Similarly to the argument for obtaining (3.22), by (3.16) - (3.19) we see \( I_1 = I_2 = 0 \). Since \( y = y' = 0 \) in \( \Omega \times \{ t = 0 \} \) by (1.2), we see from (3.14) that \( \nu = \nu' = 0 \) in \( \phi_{c(t)} \cap \{ t = 0 \} \), so that \( I_3 = I_4 = 0 \) follows. Hence we have:

\[
\int_{\phi^+_{c(t)}} (hv)(P\mu)dxdt = \int_{\phi^+_{c(t)}} P(hv)\mu dxdt.
\]

Similarly we can obtain

\[
\int_{\phi^-_{c(t)}} (hv)(P\mu)dxdt = \int_{\phi^-_{c(t)}} P(hv)\mu dxdt,
\]

and so

\[
(3.26) \quad \int_{\phi_{c(t)}} (hv)(P\mu)dxdt = \int_{\phi_{c(t)}} P(hv)\mu dxdt.
\]

Moreover (3.26) implies that \( P(hv) \) taken in the pointwise sense in \( \phi_{c(t)} \) coincides with the one in sense of \( (H^2(\phi_{c(t)}))^\prime \). By (3.20), (3.25) and (3.26), we see

\[
(Nv, P\mu)_{L^2(\phi_{c(t)})}
\]

\[
= \int_{\phi_{c(t)}} (\chi' f R + \chi f R' - P_1 y + P(hv))\mu dxdt, \quad \mu \in C^\infty(\overline{\phi_{c(t)}}).
\]
Therefore (3.15) is seen with

(3.27) \[ g = \langle P(Nv) \rangle = \chi f R + \chi f R' - P_1 y - P(h v). \]

Thus we have proved that \( Nv \) satisfies (2.6).

Third Step. Now we can apply Proposition 1 to \( Nv \), so that

(3.28) \[
M \lambda \int_{\phi_{\epsilon(t)}} \left( |\nabla (Nv)|^2 + |(Nv)'|^2 \right) e^{2\lambda t} dx \, dt + M \lambda^3 \int_{\phi_{\epsilon(t)}} |Nv|^2 e^{2\lambda t} dx \, dt \\
\leq \int_{\phi_{\epsilon(t)}} | \langle P(Nv) \rangle |^2 e^{2\lambda t} dx \, dt,
\]

for all large \( \lambda > 0 \).

Let us calculate \( \langle P(Nv) \rangle \) by (3.27). Henceforth we set

\[
h_1(\cdot,t) = \begin{cases} 
(h_{1|0,0})', & t \geq 0, \\
(h_{1|(-\delta,0)})', & t < 0,
\end{cases}
\]

and

\[
h_2(\cdot,t) = \begin{cases} 
(h_{1|0,0})'', & t \geq 0, \\
(h_{1|(-\delta,0)})'', & t < 0.
\end{cases}
\]

Then by (3.9) we see that \( h_1, h_2 \in L^\infty(\Omega \times (-\delta, \delta)) \). Noting \( y(\cdot,0) = y'(\cdot,0) = 0 \), (3.9) and (3.16), we can directly see that \( (hy)' = h_1 y + hy' \) and \( (hy)'' = hy'' + 2h_1 y' + h_2 y \) in \( (D(\Omega \times (-\delta, \delta)))' \). Therefore we obtain:

\[
P(h v) = (\chi h y)'' - \Delta (\chi h y) + q \chi h y \\
= \chi h(y'' + 2\chi'(hy') + \chi'' hy - \chi \Delta (hy)) - 2 \nabla \chi \cdot \nabla (hy) - hy \Delta \chi + q \chi h y \\
= \chi h(y'' - \Lambda y + qy) + \chi(2h_1 y' + h_2 y - 2 \nabla h \cdot \nabla y - y \Delta h) \\
+ 2 \chi'(h_1 y + hy') + \chi'' hy - 2 \nabla \chi \cdot \nabla (hy) - hy \Delta \chi \quad \text{in } (D(\Omega \times (-\delta, \delta)))'.
\]

On the other hand, noting that \( y \) and \( R \) are extended to \( (-\delta, \delta) \) as even functions in \( t \), by (1.2) and (3.8), we have

(3.29) \[ \chi h(y'' - \Delta y + qy) = \chi f R = \chi R' f \quad \text{almost everywhere in } \phi_{\epsilon(t)}. \]

Moreover by (3.14), we see \( \chi y' = v' - \chi' y \) and \( \chi \nabla y = \nabla v - y \nabla \chi \), so that

\[
\chi(2h_1 y' + h_2 y - 2 \nabla h \cdot \nabla y - y \Delta h) \\
= 2h_1(v' - \chi' y) + h_2 v - 2 \nabla h \cdot (\nabla v - y \nabla \chi) - v \Delta h \\
= 2h_1 v' - 2 \nabla h \cdot \nabla v + (h_2 - \Delta h) v + 2(\nabla h \cdot \nabla \chi - h_1 \chi') y \\
- Q_v + 2(\nabla h \cdot \nabla \chi - h_1 \chi') y,
\]

where we set

(3.30) \[ Q_v = 2h_1 v' - 2 \nabla h \cdot \nabla v + (h_2 - \Delta h) v. \]
Then we have

\begin{equation}
P(hv) = \chi R' f + (2\chi' y' - 2\nabla \chi \cdot \nabla y + (\chi'' - \Delta \chi)y)h + Qv
\end{equation}

\begin{equation}
= \chi R' f + Qv + P_2y \quad \text{in } (D(\phi_{c,(e)}))',
\end{equation}

where we set

\begin{equation}
P_2y = (2\chi' y' - 2\nabla \chi \cdot \nabla y + (\chi'' - \Delta \chi)y)h.
\end{equation}

Therefore it follows from (3.27) and (3.31) that

\begin{equation}
<P(Nv) > = \chi' f R - (P_1 + P_2)y - Qv \quad \text{almost everywhere in } \phi_{c,(e)}.
\end{equation}

By (3.13), (3.24) and (3.32), we see that

\begin{equation}
\chi' f R = (P_1 + P_2)y = 0 \quad \text{almost everywhere in } \phi_{c,(3e)}
\end{equation}

because the terms in $P_1$ and $P_2$ have derivatives of $\chi$ as factors. Furthermore by (3.30) and (3.9), we see

\begin{equation}
|Qv(x,t)| \leq M(|\nabla v|^2 + |v'|^2 + |v|^2)(x,t) \quad \text{almost everywhere in } \phi_{c,(e)}.
\end{equation}

Henceforth $M > 0$ denotes a generic constant which is independent of $\lambda > 0$.

Consequently (3.28) and (3.33) - (3.35) yield

\begin{equation}
M \lambda \int_{\phi_{c,(e)}} (|\nabla (Nv)|^2 + |(Nv)'|^2 + |Nv|^2) e^{2\lambda \phi} \, dx \, dt
\end{equation}

\begin{equation}
\leq \int_{\phi_{c,(e)}} \phi_{c,(3e)} |\chi' f R - (P_1 + P_2)y|^2 e^{2\lambda \phi} \, dx \, dt
\end{equation}

\begin{equation}
+ \int_{\phi_{c,(e)}} (|\nabla v|^2 + |v'|^2 + |v|^2) e^{2\lambda \phi} \, dx \, dt,
\end{equation}

for all large $\lambda > 0$.

On the other hand, for fixed $z \in \Omega$, since $v(\cdot, 0) = 0$, we can solve the ordinary differential equation $v'(x,t) - h(x,t)v(x,t) = (Nv)(x,t), |t| < \delta$, so that

\begin{equation}
v(x,t) = \int_0^t K(x,t,s)(Nv)(x,s) \, ds,
\end{equation}

where $K(x,t,s) = \frac{R(x,t)}{R(x,s)}$. Moreover the following inequality can be proved easily (e.g. Lemma 3.7 in Klibanov [10]):

\vspace{1em}

**Lemma 2.** - Let $\phi_c \subset \Omega \times (-\delta, \delta)$ and $c \geq 0$. Then

\begin{equation}
\int_{\phi_c} e^{2\lambda \phi} \left| \int_0^t |k(x,s)| \, ds \right|^2 \, dx \, dt < \delta^2 \int_{\phi_c} |k(x,t)|^2 e^{2\lambda \phi} \, dx \, dt
\end{equation}

for $k \in L^2(\phi_c)$. 

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Applying Lemma 2 to (3.37), we see

\[
\int_{\phi_{c(3)}} (|\nabla v|^2 + |v'|^2 + |v|^2)e^{2\lambda \phi} \, dx \, dt \\
\leq M \int_{\phi_{c(3)}} (|\nabla (Nv)|^2 + |(Nv)'|^2 + |Nv|^2)e^{2\lambda \phi} \, dx \, dt
\]

by (3.9) and (3.12). Consequently from (3.36) we obtain:

\[
(3.38) \quad M \lambda \int_{\phi_{c(3)}} (|\nabla (Nv)|^2 + |(Nv)'|^2 + |Nv|^2)e^{2\lambda \phi} \, dx \, dt \\
\quad \leq M \lambda \int_{\phi_{c(3)}} (\epsilon f R - (P_1 + P_2)y) |^2e^{2\lambda \phi} \, dx \, dt \\
\quad \leq e^{2\lambda (c(3)')^2} \int_{\phi_{c(3)}} (|\nabla (Nv)|^2 + |(Nv)'|^2 + |Nv|^2)e^{2\lambda \phi} \, dx \, dt
\]

for sufficiently large \( \lambda > 0 \). Here we note that the second term of the right hand side of (3.36) can be absorbed into the left hand side if we take large \( \lambda > 0 \).

On the other hand,

[the left hand side of (3.38)]

\[
\geq M \lambda \int_{\phi_{c(3)}} (|\nabla (Nv)|^2 + |(Nv)'|^2 + |Nv|^2)e^{2\lambda \phi} \, dx \, dt \\
\quad \geq M \lambda e^{2\lambda (c(3)')^2} \int_{\phi_{c(3)}} (|\nabla (Nv)|^2 + |(Nv)'|^2 + |Nv|^2)e^{2\lambda \phi} \, dx \, dt
\]

Therefore with (3.38), we obtain

\[
M \lambda e^{2\lambda (c(3)')^2} \int_{\phi_{c(3)}} (|\nabla (Nv)|^2 + |(Nv)'|^2 + |Nv|^2)e^{2\lambda \phi} \, dx \, dt \\
\quad \leq e^{2\lambda (c(3)')^2} \int_{\phi_{c(3)}} (|\nabla (Nv)|^2 + |(Nv)'|^2 + |Nv|^2)e^{2\lambda \phi} \, dx \, dt
\]

namely,

\[
\int_{\phi_{c(3)}} (|\nabla (Nv)|^2 + |(Nv)'|^2 + |Nv|^2)e^{2\lambda \phi} \, dx \, dt \\
\quad \leq \frac{1}{M \lambda} \int_{\phi_{c(3)}} (|\nabla (Nv)|^2 + |(Nv)'|^2 + |Nv|^2)e^{2\lambda \phi} \, dx \, dt
\]

for sufficiently large \( \lambda > 0 \). We make \( \lambda > 0 \) tend to \( \infty \), so that

\[
(Nv)(x, t) = v'(x, t) - h(x, t)v(x, t) = 0, \quad (x, t) \in \phi_{c(3)}
\]

because \( \chi' f R - (P_1 + P_2)y \in L^2(\Omega \times (-T, T)) \) is independent of \( \lambda \).

Moreover since \( v(\cdot, 0) = 0 \), the uniqueness in the initial value problem for the ordinary differential equation implies that \( v(x, t) = 0, \quad (x, t) \in \phi_{c(3)} \). By (3.13), we obtain \( y(x, t) = 0, \quad (x, t) \in \phi_{c(3)} \). Therefore \( (Fy)(x, t) = 0, \quad (x, t) \in \phi_{c(3)} \), so
that \( f(x)R(x,t) = 0 \), \((x,t) \in \phi_c(\epsilon)\). By the regularity property (1.6), we see that \( fR \in C([0,T];L^2(\Omega)) \). Therefore we obtain:

\[
f(x)R(x,0) = 0, \quad x \in \phi_c(\epsilon) \cap \{t = 0\} \subset \{x \in \Omega; \sqrt{\rho^2 - \beta \delta^2 + 16\epsilon^2} < |x| < \rho\}.
\]

By the condition (1.6), we see that \( f(x) = 0 \) for \( x \in \Omega \) satisfying \( \sqrt{\rho^2 - \beta \delta^2 + 16\epsilon^2} < |x| < \rho \). Since \( \epsilon > 0 \) is arbitrarily small, we see that

\[
f(x) = 0, \quad x \in \Omega, \sqrt{\rho^2 - \beta \delta^2} \leq |x| \leq \rho.
\]

**Fourth Step.** - By (3.39) we have

\[
(3.40) \quad \hat{y}(x, t) = 0, \quad x \in \Omega, \sqrt{\rho^2 - \beta \delta^2} < |x| < \rho, -T < t < T,
\]

\[
(3.41) \quad y(x, 0) = 0, \quad x \in \Omega,
\]

\[
(3.42) \quad y(x, t) = 0, \quad x \in \partial \Omega, -T < t < T
\]

and

\[
(3.43) \quad \frac{\partial y}{\partial \nu}(x, t) = 0, \quad x \in \partial \Omega, -T < t < T.
\]

In this step, we will prove

\[
(3.44) \quad f(x) = 0, \quad x \in \Omega, \sqrt{\rho^2 - 2\beta \delta^2} < |x| < \rho
\]

and

\[
(3.45) \quad y(x, t) = 0, \quad (x, t) \in \phi_{c_1(\epsilon)}
\]

for all sufficiently small \( \epsilon > 0 \). Here and henceforth we set

\[
(3.46) \quad c_1(\epsilon) = \sqrt{\rho^2 - 2\beta \delta^2 + \beta \epsilon^2}.
\]

We define a function \( \kappa = \kappa(t) \) such that:

\[
(3.47) \quad \kappa \in C_0^\infty(\mathbb{R}), \quad 0 \leq \kappa(t) \leq 1,
\]

\[
\kappa(t) = \begin{cases} 
1, & |t| \leq \sqrt{\delta^2 - \epsilon^2}, \\
0, & |t| > \delta
\end{cases}
\]

and we set

\[
(3.48) \quad \hat{R}(x, t) = R(x, 0) + \kappa(t)(R(x, t) - R(x, 0)), \quad (x, t) \in \phi_{c_1(\epsilon)}.
\]

Then by (3.47) we see

\[
(3.49) \quad \hat{R}(x, t) = \begin{cases} 
R(x, t), & |t| \leq \sqrt{\delta^2 - \epsilon^2}, \\
R(x, 0), & |t| > \delta.
\end{cases}
\]
Furthermore for $|t| \leq \delta$, we have $\hat{R}(x, t) - R(x, t) = (1 - \kappa(t))(R(x, 0) - R(x, t))$, so that $|\hat{R}(x, t) - R(x, t)| \leq \sup_{|t| \leq \delta} |R(x, 0) - R(x, t)| \leq (\|R\|_{L^\infty(\Omega \times (0, T))} + \|R\|_{L^\infty(\Omega \times (-T, 0))})\delta$.

Therefore in view of (1.6), we can previously choose sufficiently small $\delta > 0$ satisfying
\[(3.50) \quad |\hat{R}(x, t)| \geq \frac{r_0}{4}, \quad x \in \Omega, \quad |t| \leq \delta\]
and
\[(3.51) \quad \delta = \frac{T}{\sqrt{N}} \quad \text{for some large } N \in \mathbb{N} \]
as well as (3.5). Hence by (3.49) we obtain
\[(3.52) \quad |\hat{R}(x, t)| \geq \frac{r_0}{4}, \quad (x, t) \in \phi(x, t)\).

Moreover we can see
\[(3.53) \quad \hat{R}(x, t)f(x) = \begin{cases} 0, & (x, t) \in \phi(x, t) \cap \{(x, t); |x| \leq \rho \}\; \text{or} \; |t| \leq \sqrt{\beta^2 - \beta^2}
R(x, t)f(x), & (x, t) \in \phi(x, t) \cap \{(x, t); |x| > \sqrt{\beta^2 - \beta^2}\} \end{cases} \]

In fact, as direct calculations show, if $(x, t) \in \phi(x, t) \cap \{(x, t); |x| \leq \sqrt{\beta^2 - \beta^2}\}$, then $|t| \leq \sqrt{\beta^2 - \beta^2}$, and so (3.49) implies that $\hat{R}(x, t)f(x) = R(x, t)f(x)$ for $(x, t) \in \phi(x, t) \cap \{(x, t); |x| \leq \sqrt{\beta^2 - \beta^2}\}$. Moreover from (3.39), we can directly see that $\hat{R}(x, t)f(x) = 0$ if $(x, t) \in \phi(x, t) \cap \{(x, t); \sqrt{\beta^2 - \beta^2} < |x| \leq \rho\}$.

Therefore (3.40) yields
\[(3.54) \quad (Py)(x, t) = \hat{R}(x, t)f(x), \quad (x, t) \in \phi(x, t).\]

Here since $\phi(x, t) \subset \Omega \times (-\sqrt{2\delta}, \sqrt{2\delta})$, we have
\[\partial \phi(x, t) \cap (\partial \Omega \times (-T, T)) \subset \partial \Omega \times (-\sqrt{2\delta}, \sqrt{2\delta}),\]
so that
\[(3.55) \quad y(x, t) = \frac{\partial y}{\partial t}(x, t) = 0, \quad (x, t) \in \partial \phi(x, t) \cap (\partial \Omega \times (-T, T)),\]
because $\delta > 0$ is sufficiently small. Hence in view of (3.52), we can repeat the argument in Third Step to the system (3.54) with (3.41) - (3.43), so that we obtain (3.44) and (3.45).

Fifth Step. - We will complete the proof of Theorem 1. Repeating $m$-times the argument in Fourth Step, we see that
\[(3.56) \quad f(x) = 0, \quad x \in \Omega, \quad \sqrt{\rho^2 - m\beta^2} \leq |x| \leq \rho\]
and
\[(3.57) \quad y(x, t) = 0 \quad \text{in } \phi \sqrt{\rho^2 - m\beta^2},\]
where $\phi \sqrt{\rho^2 - m\beta^2} = \{(x, t) \in \Omega \times (-T, T); |x| \leq \sqrt{\beta^2 - \beta^2} > \rho^2 - m\beta^2\}$. We have $\phi \sqrt{\rho^2 - m\beta^2} \subset \Omega \times (-\sqrt{m\beta}, \sqrt{m\beta})$ and so we can actually repeat the argument until $m \in \mathbb{N}$ satisfies $\delta \sqrt{m} \leq T < \delta \sqrt{m + 1}$, namely, $m = N$ by (3.51). Then $\rho^2 - N\beta^2 = \rho^2 - \beta T^2 = 0$ by (3.1). Therefore we see $f(x) = 0$, $x \in \Omega$ and by the uniqueness of solution to the problem (1.2) with $f = 0$, it follows that $g(x, t) = g(f)(x, t) = 0$, $x \in \Omega$, $0 < t < T$. Thus the proof of Theorem 1 is complete.

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4. Proof of Theorem 2

Without loss of generality, we may assume that \( u(p) \in W^{3,\infty}(\Omega \times (0, T)) \). Setting \( y(x, t) = u(q)(x, t) - u(p)(x, t), \ R(x, t) = u(p)(x, t) \) and \( f(x) = p(x) - q(x) \), \( x \in \Omega \), \( 0 < t < T \), we obtain (1.2). Moreover, by [17] for example, we see that \( y \in H^2(\Omega \times (0, T)) \).
Since \( R(x, 0) = u(p)(x, 0) = a(x) \), \( x \in \Omega \), the conditions (1.11) and (1.12) imply (1.6). Since
\[
\frac{\partial u(q)(x, t)}{\partial \nu} = \frac{\partial u(p)}{\partial \nu}(x, t), \quad x \in \partial \Omega, \ 0 < t < T
\]
means that \( \frac{\partial y}{\partial \nu}(x, t) = 0, \ x \in \partial \Omega, \ 0 < t < T \), the conclusion \( q(x) - p(x) = 0, \ x \in \Omega \) follows from Theorem 1.

5. Observability inequality

In this section, for the proofs of Theorems 3 and 4, we will establish an observability inequality. We consider an initial value problem
\[
\begin{cases}
\phi''(x, t) = \Delta \phi(x, t) - q(x)\phi(x, t), & x \in \Omega, \ 0 < t < T \\
\phi(x, 0) = \phi_0(x), \quad \phi'(x, 0) = \phi_1(x), & x \in \Omega \\
\phi(x, t) = 0, & x \in \partial \Omega, \ 0 < t < T.
\end{cases}
\]
(5.1)

For \( \phi_0 \in H^1_0(\Omega) \) and \( \phi_1 \in L^2(\Omega) \), there exists a unique solution \( \phi = \phi(\phi_0, \phi_1) \in C([0, T], H^1_0(\Omega)) \cap C^1([0, T]; L^2(\Omega)) \) (e.g. Lions [16], Lions and Magenes [17]) and
\[
\left\| \frac{\partial \phi(\phi_0, \phi_1)}{\partial \nu} \right\|_{L^2(\partial \Omega \times (0, T))} \leq C(\|\phi_0\|_{H^1_0(\Omega)} + \|\phi_1\|_{L^2(\Omega)})
\]
(5.2)

with a constant \( C > 0 \) independent of \( \phi_0 \) and \( \phi_1 \) (e.g. Komornik [11], [12], Lions [16]). Furthermore we can derive the reverse inequality called an observability inequality:

**PROPOSITION B** (Komornik [11], [12]). - Let \( q \in L^\infty(\Omega) \) and
\[
T > 2\rho;
\]
(5.3)
then there exists a constant \( C = C(\Omega, T, q) > 0 \) such that
\[
\|\phi_0\|_{H^1_0(\Omega)} + \|\phi_1\|_{L^2(\Omega)} \leq C \left\| \frac{\partial \phi(\phi_0, \phi_1)}{\partial \nu} \right\|_{L^2(\partial \Omega \times (0, T))}
\]
(5.4)
for all \( \phi_0 \in H^1_0(\Omega) \) and \( \phi_1 \in L^2(\Omega) \).

**Remark.** - For \( q = 0 \), the inequality (5.4) is proved in Ho [4], Lions [16]. Furthermore this kind of inequalities are proved by the microlocal analysis (Bardos, Lebeau and Rauch [2]) and the Carleman estimate (Kazemi and Klibanov [8], Tataru [25]).

The condition (5.3) on \( T \) should be noticed, which requires that \( T \) must be larger twice than the critical value in (1.8) for the uniqueness. The condition (5.3) is necessary for estimating two functions \( \phi_0 \) and \( \phi_1 \), and too much for determining either of \( \phi_0 \) and \( \phi_1 \). In fact, we can prove another observability inequality.
PROPOSITION 2. - Let \( q \in L^\infty(\Omega) \) and
\[
T > \rho;
\]
then there exists a constant \( C = C(\Omega, T, q) > 0 \) such that
\[
C^{-1} \| \phi_1 \|_{L^2(\Omega)} \leq \left\| \frac{\partial \phi(0, \phi_1)}{\partial \nu} \right\|_{L^2(\partial\Omega \times (0, T))} \leq C \| \phi_1 \|_{L^2(\Omega)}
\]
for all \( \phi_1 \in L^2(\Omega) \).

The second inequality in (5.6) is straightforward from (5.2). For the first inequality in (5.6), we notice that since we assume that \( a = 0 \), we can reduce the critical time length (compare (5.5) with (5.3)). The rest of this section is devoted to the proof of the first inequality. The proposition is related with contrôlabilité exacte élargie (Chapitre I, §9 in Lions [16]) and for completeness we prove it here. Our proof is done along Komornik [12], except for using \( \phi_0 = 0 \).

Proof of Proposition 2. - By means of the estimate (5.2), it is sufficient to prove the conclusion (5.6) for \( \phi(0, \phi_1) \in C^2(\overline{\Omega} \times [0, T]) \).

First Step. - We will establish:

LEMMA 3. - Let \( q = 0 \) in \( \Omega \). Then under the assumption (5.5), the conclusion of Proposition 2 is true.

Proof of Lemma 3. - By (5.5) and the definition (1.3) of \( \rho \), for a sufficiently small \( \epsilon > 0 \), we can choose \( x_0 \in \Omega \) such that
\[
\sup_{x \in \Omega} |x - x_0| < \rho + \epsilon < T.
\]
Henceforth we fix such \( \epsilon > 0 \) and \( x_0 \in \Omega \), and we set
\[
m(x) = x - x_0, \quad x \in \mathbb{R}^n.
\]
We denote the scalar product of vectors \( m, \nu \in \mathbb{R}^n \) by \( m \cdot \nu \). Then

LEMMA 4. - Let \( \phi \in C^2(\overline{\Omega} \times [0, T]) \) satisfy
\[
\phi''(x, t) = \Delta \phi(x, t), \quad x \in \Omega, 0 < t < T;
\]
then
\[
\int_0^T \int_{\partial\Omega} 2 \frac{\partial \phi}{\partial \nu} m \cdot \nabla \phi + (m \cdot \nu)(|\phi'|^2 - |\nabla \phi|^2) dS dt
= \left| \int_{\Omega} 2 \phi (m \cdot \nabla \phi) dx \right|_{t=T}^{t=0} + \int_0^T \int_{\Omega} n|\phi'|^2 + (2 - n)|\nabla \phi|^2 dx dt.
\]
The lemma is proved in Komornik [12] as Lemma 2.2.3. In fact, the proof is finished by multiplication of (5.9) by \( 2m \cdot \nabla \phi \) and integration by parts.
Let \( \phi = \phi(0, \phi_1) \in C^2(\Omega \times [0, T]) \) satisfy (5.1). Then by \( \phi = 0 \) on \( \partial\Omega \times (0, T) \), we have \( \phi' = 0 \) and \( \nabla \phi = (\frac{\partial \phi}{\partial n}) \nu \) on \( \partial\Omega \times (0, T) \), so that we obtain:

\[
\int_0^T \int_{\partial\Omega} (m \cdot \nu) \left| \frac{\partial \phi}{\partial \nu} \right|^2 dS dt = \left[ \int_{\Omega} 2\phi'(m \cdot \nabla \phi) dx \right]_{t=0}^{t=T} + \int_0^T \int_{\Omega} n|\phi'|^2 + (2-n)|\nabla \phi|^2 dx dt
\]

by Lemma 4.

On the other hand, multiplying the first equation in (5.1) by \( (n-1)\phi \) and integrating by parts, we have:

\[
\left[ \int_{\Omega} (n-1)\phi \phi' dx \right]_{t=0}^{t=T} + \int_0^T \int_{\Omega} (1-n)|\phi'|^2 + (n-1)|\nabla \phi|^2 dx dt = 0,
\]

by noting that \( \phi = 0 \) on \( \partial\Omega \times (0, T) \). Addition of this and (5.10) yields

\[
\int_0^T \int_{\partial\Omega} (m \cdot \nu) \left| \frac{\partial \phi}{\partial \nu} \right|^2 dS dt = \left[ \int_{\Omega} \phi' M \phi dx \right]_{t=0}^{t=T} + \int_0^T \int_{\Omega} |\phi'|^2 + |\nabla \phi|^2 dx dt,
\]

where we set

\[
M \phi = 2m \cdot \nabla \phi + (n-1)\phi.
\]

By the conservation of energy, we have \( \int_{\Omega} |\phi'(x, t)|^2 + |\nabla \phi(x, t)|^2 dx = ||\phi_1||_{L^2(\Omega)}^2 \) for all \( t \in [0, T] \) (e.g. Theorem 1.1.1 in [12]). Moreover we assume that \( \phi_0 = 0 \), so that

\[
\int_{\Omega} \phi'(x, 0)(M \phi)(x, 0) dx = 0.
\]

(This makes the critical value of \( T \) half, i.e., (5.5).)

Consequently we obtain

\[
\int_{\Omega} \phi'(x, T)(M \phi)(x, T) dx + T||\phi_1||_{L^2(\Omega)}^2 = \int_0^T \int_{\partial\Omega} (m \cdot \nu) \left| \frac{\partial \phi}{\partial \nu} \right|^2 dS dt.
\]

Next we show:

**LEMMA 5.** - We have:

\[
\left| \int_{\Omega} \phi'(x, t)(M \phi)(x, t) dx \right| \leq \sup_{x \in \Omega} \{m(x)\} ||\phi_1||_{L^2(\Omega)}^2, \quad t \geq 0.
\]

For the proof, see pp. 38-39 in [12].

Application of Lemma 5 in (5.13) yields

\[
(T - \sup_{x \in \Omega} \{m(x)\}) ||\phi_1||_{L^2(\Omega)}^2 \leq \int_0^T \int_{\partial\Omega} (m \cdot \nu) \left| \frac{\partial \phi}{\partial \nu} \right|^2 dS dt.
\]
Here by (5.7) we see \( T - \sup_{x \in \Omega} |m(x)| > T - \rho - \epsilon > 0 \) and
\[
\int_0^T \int_{\Omega} (m \cdot \nu) \frac{\partial \psi}{\partial \nu}^2 \, dS \, dt \leq \int_0^T \int_{(m \cdot \nu) > 0} (m \cdot \nu) \frac{\partial \psi}{\partial \nu}^2 \, dS \, dt
\]
\[
\leq \sup_{x \in \Omega} |m(x)| \int_0^T \int_{(m \cdot \nu) > 0} \frac{\partial \psi}{\partial \nu}^2 \, dS \, dt \leq C \int_0^T \int_{\partial \Omega} \frac{\partial \psi}{\partial \nu}^2 \, dS \, dt.
\]
Thus we obtain
\[
\| \psi_1 \|_{L^2(\Omega)}^2 \leq \frac{C}{T - (\rho + \epsilon)} \int_0^T \int_{\partial \Omega} \frac{\partial \psi}{\partial \nu}^2 \, dS \, dt,
\]
and the proof of Lemma 3 is complete.

**Second Step.** On the basis of Lemma 3, we complete the proof of Proposition 2 for general \( q \in L^\infty(\Omega) \). For this, we can apply the method of norm inequalities due to Komornik [11], [12], which has been originally used for the proof of proposition B. His method is useful in the sense that we need only the uniqueness for the corresponding elliptic system, not for the original hyperbolic system. However, here for technical convenience, we apply another way, the compactness-uniqueness argument, which is actually used also for the proof of Theorem 3.

Now we show:

**Lemma 6.** Let us consider
\[
\begin{cases}
\psi''(x, t) = \Delta \psi(x, t) - q(x) \psi(x, t) + F(x, t), & x \in \Omega, \ 0 < t < T \\
\psi(x, 0) = \psi_0(x), & x \in \Omega \\
\psi'(x, 0) = \psi_1(x), & x \in \Omega \\
\psi(x, t) = 0, & x \in \partial \Omega, \ 0 < t < T.
\end{cases}
\]
(5.14)

where \( q \in L^\infty(\Omega) \). Then there exists a constant \( C = C(\Omega, T, q) > 0 \) such that
\[
\left\| \frac{\partial \psi}{\partial \nu} \right\|_{L^2(\partial \Omega \times (0, T))} \leq C(\| F \|_{L^1(0, T; L^2(\Omega))} + \| \psi_0 \|_{H^1(\Omega)} + \| \psi_1 \|_{L^2(\Omega)})
\]
for \( F \in L^1(0, T; L^2(\Omega)) \), \( \psi_0 \in H^1_0(\Omega) \) and \( \psi_1 \in L^2(\Omega) \).

**Lemma 7.** Let us consider
\[
\begin{cases}
z''(x, t) = \Delta z(x, t) - q(x) z(x, t) + F(x, t), & x \in \Omega, \ 0 < t < T \\
z(x, 0) = z_0(x), & x \in \Omega \\
z'(x, 0) = z_1(x), & x \in \Omega \\
z(x, t) = \eta(x, t), & x \in \partial \Omega, \ 0 < t < T.
\end{cases}
\]
(5.15)

where \( q \in L^\infty(\Omega) \). Then for every \( z_0 \in L^2(\Omega) \), \( z_1 \in H^{-1}(\Omega) \), \( F \in L^2(0, T; L^2(\Omega)) \) and \( \eta \in L^2(\partial \Omega \times (0, T)) \), there exists a unique solution \( z \in C((0, T]; H^1(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega)) \) and we can take a constant \( C = C(\Omega, T, q) > 0 \) such that:
\[
\| z \|_{L^\infty(0, T; L^2(\Omega))} \leq C(\| F \|_{L^2(0, T; L^2(\Omega))} + \| z_0 \|_{L^2(\Omega)} + \| z_1 \|_{H^{-1}(\Omega)} + \| \eta \|_{L^2(\partial \Omega \times (0, T))}).
\]

Here \( H^{-1}(\Omega) \) denotes the dual of \( H^1_0(\Omega) \) (e.g. [1], [17]) and the weak solution to (5.15) is defined by the transposition method (e.g. p. 23 in Komornik [12]).
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Proof of Lemma 6. - We can apply Théorème 1.4.1 in [16] (or Lasiecka, Lions and Triggiani [14]), regarding \(-q(x)\psi(x, t) + F(x, t)\) as non-homogeneous term, so that

\begin{equation}
\left\| \frac{\partial \psi}{\partial t} \right\|_{L^2(\partial \Omega \times (0, T))} \leq C \left( \left\| \psi_0 \right\|_{H^1_0(\Omega)} + \left\| \psi_1 \right\|_{L^2(\Omega)} \right) + \left\| F \right\|_{L^1(0,T;L^2(\Omega))} + \left\| q \right\|_{L^1(0,T;L^2(\Omega))}.
\end{equation}

Moreover by a usual a-priori estimate (e.g. Theorem 3.8.2 in [17]), we see that

\begin{equation}
\left\| \psi_t \right\|_{L^\infty(0,T;L^2(\Omega))} \leq C \left( \left\| \psi_0 \right\|_{H^1_0(\Omega)} + \left\| \psi_1 \right\|_{L^2(\Omega)} + \left\| F \right\|_{L^1(0,T;L^2(\Omega))} \right).
\end{equation}

Noting that \(q \in L^\infty(\Omega)\), we substitute this into (5.16) and the proof of Lemma 6 is complete.

Proof of Lemma 7. - For \(F = 0\), we can refer to the proof of Theorem 2.2.5 in [12], for example. For \(z_0 = 0\), \(z_1 = 0\) and \(\eta = 0\), the estimate is straightforward from Theorem 3.8.2 in [17] for example.

Now we proceed to completion of the proof of Proposition 2. We introduce

\begin{equation}
\left\{ \begin{array}{l}
\psi''(x, t) = \Delta \psi(x, t), \quad x \in \Omega, \ 0 < t < T \\
\psi(x, 0) = 0, \quad \psi'(x, 0) = \phi_1(x), \quad x \in \Omega \\
\psi(x, t) = 0, \quad x \in \partial \Omega, \ 0 < t < T.
\end{array} \right.
\end{equation}

and

\begin{equation}
\left\{ \begin{array}{l}
z''(x, t) = \Delta z(x, t) - q(x)\phi(x, t), \quad x \in \Omega, \ 0 < t < T \\
z(x, 0) = z'(x, 0) = 0, \quad x \in \Omega \\
z(x, t) = 0, \quad x \in \partial \Omega, \ 0 < t < T,
\end{array} \right.
\end{equation}

where \(\phi = \phi(0, \phi_1)\) is the weak solution to (5.1); then:

\begin{equation}
\phi(x, t) = \phi(0, \phi_1)(x, t) = \psi(x, t) + z(x, t), \quad x \in \Omega, \ 0 < t < T.
\end{equation}

By Lemma 3, we have

\begin{equation}
\left\| \phi_1 \right\|_{L^2(\Omega)} \leq C \left\| \frac{\partial \psi}{\partial t} \right\|_{L^2(\partial \Omega \times (0,T))}
\end{equation}

under the assumption (5.5).

On the other hand, applying Lemma 6 to (5.18), we obtain

\begin{equation}
\left\| \frac{\partial z}{\partial t} \right\|_{L^2(\partial \Omega \times (0,T))} \leq C \left\| q \phi \right\|_{L^1(0,T;L^2(\Omega))}.
\end{equation}

Next application of Lemma 7 to (5.1) with \(\phi_0 = 0\), yields \(\left\| \phi \right\|_{L^\infty(0,T;L^2(\Omega))} \leq C \left\| \phi_1 \right\|_{H^{-1}(\Omega)}\). Therefore by \(q \in L^\infty(\Omega)\), the estimate (5.21) implies

\begin{equation}
\left\| \frac{\partial z}{\partial t} \right\|_{L^2(\partial \Omega \times (0,T))} \leq C \left\| \phi_1 \right\|_{H^{-1}(\Omega)}.
\end{equation}
Combining (5.19), (5.20) with (5.22), we obtain
\[
\|\phi_1\|_{L^2(\Omega)} \leq C \left\| \frac{\partial \phi}{\partial \nu} - \frac{\partial z}{\partial \nu} \right\|_{L^2(\partial\Omega \times (0,T))} \leq C \left( \frac{\partial \phi(0,\phi_1)}{\partial \nu} \right)_{L^2(\partial\Omega \times (0,T))} + C\|\phi_1\|_{H^{-1}(\Omega)}.
\]
(5.23)

We have to take away the term \(\|\phi_1\|_{H^{-1}(\Omega)}\) in (5.23) for the completion of the proof of Proposition 2. It is sufficient to prove
\[
\inf_{\|\phi_1\|_{L^2(\Omega)} = 1} \left\| \frac{\partial \phi(0,\phi_1)}{\partial \nu} \right\|_{L^2(\partial\Omega \times (0,T))} > 0,
\]
by \(\phi(0,\alpha \phi_1) = \alpha \phi(0, \phi_1)\) for \(\alpha \in \mathbb{R}\). Assume contrarily that (5.24) is not true. Then there exist \(\phi^n_1 \in L^2(\Omega), n \geq 1\) such that
\[
\|\phi^n_1\|_{L^2(\Omega)} = 1, \quad n \geq 1
\]
and
\[
\lim_{n \to \infty} \left\| \frac{\partial \phi(0,\phi^n_1)}{\partial \nu} \right\|_{L^2(\partial\Omega \times (0,T))} = 0.
\]
(5.26)

By (5.25) we can extract a subsequence, denoted again by the same notation, such that
\[
\lim_{m,n \to \infty} \|\phi^n_1 - \phi^m_1\|_{H^{-1}(\Omega)} = 0,
\]
(5.27)

because the embedding \(L^2(\Omega) \hookrightarrow H^{-1}(\Omega)\) is compact (e.g. [17]). Then from (5.23) we have
\[
\|\phi^n_1 - \phi^m_1\|_{L^2(\Omega)} \leq C \left\| \frac{\partial \phi(0,\phi^n_1)}{\partial \nu} - \frac{\partial \phi(0,\phi^m_1)}{\partial \nu} \right\|_{L^2(\partial\Omega \times (0,T))} + C\|\phi^n_1 - \phi^m_1\|_{H^{-1}(\Omega)}.
\]
Consequently (5.26) and (5.27) imply that \(\lim_{m,n \to \infty} \|\phi^n_1 - \phi^m_1\|_{L^2(\Omega)} = 0\), so that there exists \(\phi_0^1 \in L^2(\Omega)\) such that \(\lim_{n \to \infty} \|\phi^n_1 - \phi_0^1\|_{L^2(\Omega)} = 0\). By (5.25) and (5.26), we see
\[
\|\phi_0^1\|_{L^2(\Omega)} = 1
\]
and
\[
\frac{\partial \phi(0,\phi_0^1)}{\partial \nu}(x,t) = 0, \quad x \in \partial\Omega, \quad 0 < t < T.
\]
(5.29)

We recall that \(\phi = \phi(0,\phi_1^0) \in C([0,T]; H^1_0(\Omega)) \cap C^1([0,T]; L^2(\Omega))\) is the weak solution to
\[
\begin{cases}
\phi''(x,t) - \Delta \phi(x,t) - g(x)\phi(x,t), & x \in \Omega, \quad 0 < t < T \\
\phi(x,0) = 0, & x \in \partial\Omega \\
\phi(x,t) = 0, & x \in \partial\Omega, \quad 0 < t < T.
\end{cases}
\]
(5.30)
UNIQUENESS AND STABILITY IN INVERSE PROBLEMS

Setting
\[
\Phi(x,t) = \begin{cases} 
\phi(0,\phi^0_1)(x,t), & x \in \Omega, \ 0 < t < T, \\
0, & x \in \mathbb{R}^n \setminus \Omega, \ 0 < t < T,
\end{cases}
\]
\[
\Phi(x,t) = \Phi(x,-t), \quad x \in \mathbb{R}^n, \ -T < t < 0
\]
and
\[
\tilde{q}(x) = \begin{cases} 
q(x), & x \in \Omega \\
0, & x \in \mathbb{R}^n \setminus \Omega,
\end{cases}
\]
we can directly show that \( \Phi \) is a weak solution to \( \Phi''(x,t) - \Delta \Phi(x,t) - \tilde{q}(x)\Phi(x,t), x \in \mathbb{R}^n, -T < t < T \) and \( \Phi \in H^1([-T,T];L^2(\mathbb{R}^n)) \cap C([-T,T];H_0^2(\mathbb{R}^n)) \) by (5.29), and \( \phi(x,0) = 0, x \in \Omega \) and \( \phi(x,t) = 0, x \in \partial \Omega, 0 < t < T \) (see (5.30)). Moreover \( \Phi \) vanishes outside \( \{x \in \mathbb{R}^n; |x-x_0| \leq \rho + \epsilon \} \times (-T,T) \). By means of (5.7), we can apply the unique continuation theorem by Ruiz [23], so that \( \Phi(x,t) = 0, x \in \mathbb{R}^n, -T < t < T \) follows, which implies that \( \phi^0_1(x) = 0, x \in \Omega \) by \( \Phi \in C([-T,T];H_0^2(\mathbb{R}^n)) \). This contradicts (5.28).
Thus the proof of Proposition 2 is complete.

6. Proof of Theorem 3

First we show:

**Lemma 8.** - Let us consider

\[
\begin{cases} 
\nu''(x,t) = \Delta \nu(x,t) - q(x)\nu(x,t) + F(x,t), & x \in \Omega, \ 0 < t < T \\
\nu(x,0) = \nu_0(x), \ \nu'(x,0) = \nu_1(x), & x \in \Omega \\
\nu(x,t) = 0, & x \in \partial \Omega, \ 0 < t < T,
\end{cases}
\]

then there exists a constant \( C = C(\Omega,T,q) > 0 \) such that

\[
\left\| \frac{\partial \nu}{\partial t} \right\|_{H^1([0,T];L^2(\Omega))} \leq C(\|F\|_{W^{1,1}(0,T;L^2(\Omega))} + \|\nu_0\|_{H_0^2(\Omega)} + \|\nu_1\|_{H_0^1(\Omega)}),
\]

for all \( F \in W^{1,1}(0,T;L^2(\Omega)) \), \( \nu_0 \in H^2(\Omega) \cap H_0^1(\Omega) \) and \( \nu_1 \in H_0^1(\Omega) \).

**Proof of Lemma 8.** - Setting \( \nu_1 = \nu' \) and noting

\[
W^{1,1}(0,T;L^2(\Omega)) \subset C([0,T];L^2(\Omega))
\]
(e.g. [17]), we see that \( \nu_1 \) satisfies

\[
\begin{cases} 
\nu_1''(x,t) = \Delta \nu_1(x,t) - q(x)\nu_1(x,t) + F'(x,t), & x \in \Omega, \ 0 < t < T \\
\nu_1(x,0) = \nu_1(x), & x \in \Omega \\
\nu_1'(x,0) = \Delta \nu_0(x) - q(x)\nu_0(x) + F(x,0), & x \in \Omega \\
\nu_1(x,t) = 0, & x \in \partial \Omega, \ 0 < t < T.
\end{cases}
\]

Therefore Lemma 6 and (6.2) imply

\[
\left\| \frac{\partial \nu_1}{\partial t} \right\|_{L^2(\partial \Omega \times (0,T))} \leq C(\|F\|_{L^1(0,T;L^2(\Omega))} + \|\nu_1\|_{H_0^1(\Omega)} + \|\Delta \nu_0 - q\nu_0 + F(\cdot,0)\|_{L^2(\Omega)})
\]
\[
\leq C(\|F\|_{W^{1,1}(0,T;L^2(\Omega))} + \|\nu_0\|_{H^2(\Omega)} + \|\nu_1\|_{H_0^1(\Omega)}),
\]

which completes the proof of Lemma 8.
Proof of the second inequality in (1.13). Henceforth $C > 0$ denotes a generic constant which is independent of $f \in L^2(\Omega)$. Apply Lemma 8 to (1.2) and we obtain

$$\left\| \frac{\partial y(f)}{\partial u} \right\|_{H^1(0,T; L^2(\Omega))} \leq C\|f\|_{H^1(0,T; L^2(\Omega))} \leq C\|f\|_{L^2(\Omega)}$$

in view of (1.6). Therefore we see the second inequality in (1.13).

Proof of the first inequality in (1.13). Setting $y_1 = y(f)'$, we have

$$\begin{cases}
y''(x,t) = \Delta y_1(x,t) - q(x)y_1(x,t) + f(x)R'(x,t), \quad x \in \Omega, \ 0 < t < T \\
y_1(x,0) = 0, \quad \psi'(x,0) = f(x)R(x,0), \quad x \in \Omega \\
y_1(x,t) = 0, \quad x \in \partial \Omega, \ 0 < t < T.
\end{cases}$$

Moreover in relation with (6.4), we introduce

$$\begin{cases}
\phi''(x,t) = \Delta \phi(x,t) - q(x)\psi(x,t), \quad x \in \Omega, \ 0 < t < T \\
\phi(x,0) = 0, \quad \phi'(x,0) = f(x)R(x,0), \quad x \in \Omega \\
\phi(x,t) = 0, \quad x \in \partial \Omega, \ 0 < t < T.
\end{cases}$$

and

$$\begin{cases}
\psi''(x,t) = \Delta \psi(x,t) - q(x)\psi(x,t) + f(x)R'(x,t), \quad x \in \Omega, \ 0 < t < T \\
\psi(x,0) = \psi'(x,0) = 0, \quad x \in \Omega \\
\psi(x,t) = 0, \quad x \in \partial \Omega, \ 0 < t < T
\end{cases}.$$

then

$$y_1 = \phi + \psi.$$

Setting $\mu(x,t) = \psi'(x,t), \ x \in \Omega, \ 0 < t < T$, we have:

$$\begin{cases}
\mu''(x,t) = \Delta \mu(x,t) - q(x)\mu(x,t) + f(x)R''(x,t), \quad x \in \Omega, \ 0 < t < T \\
\mu(x,0) = 0, \quad \mu'(x,0) = f(x)R'(x,0), \quad x \in \Omega \\
\mu(x,t) = 0, \quad x \in \partial \Omega, \ 0 < t < T.
\end{cases}$$

By (1.6), we apply the regularity property (e.g. [16], [17]) to (6.6) and (6.8), so that

$$\begin{cases}
\|\mu\|_{L^\infty(0,T; L^2(\Omega))} = \|\psi'\|_{L^\infty(0,T; L^2(\Omega))} < C\|f\|_{L^2(\Omega)}, \\
\|\psi\|_{L^\infty(0,T; H^1_0(\Omega))} \leq C\|f\|_{L^2(\Omega)}
\end{cases}$$

and

$$\|\mu\|_{L^\infty(0,T; H^1_0(\Omega))} \leq C\|f\|_{L^2(\Omega)}.$$

From (6.6) we have $\Delta \psi(x,t) = \psi''(x,t) + q(x)\psi(x,t) - f(x)R'(x,t), \ x \in \Omega, \ 0 < t < T$, so that $\|\Delta \psi\|_{L^\infty(0,T; L^2(\Omega))} \leq C\|f\|_{L^2(\Omega)}$ by (6.9) and $q \in L^\infty(\Omega)$. Combining this with
\[ \| \psi \|_{L^\infty(0,T;L^2(\Omega))} \leq C \| f \|_{L^2(\Omega)}, \]

we obtain \[ \| \psi \|_{L^\infty(0,T;H^1(\Omega))} \leq C \| f \|_{L^2(\Omega)}. \]

Therefore, by the trace theorem, we see

\[ \| \frac{\partial \psi}{\partial \nu} \|_{L^\infty(0,T;H^{1/2}(\partial \Omega))} \leq C \| f \|_{L^2(\Omega)}. \]  

On the other hand, application of Lemma 8 to (6.6), yields

\[ \| \frac{(\partial \psi}{\partial \nu})' \|_{L^2(0,T;L^2(\partial \Omega))} \leq C \| f R' \|_{H^1(0,T;L^2(\Omega))} \leq C \| f \|_{L^2(\Omega)} \]

by means of (1.6).

Defining an operator \( K : L^2(\Omega) \rightarrow L^2(\partial \Omega \times (0,T)) \) by:

\[ (K f)(x,t) = \frac{\partial \psi}{\partial \nu}(x,t), \quad x \in \partial \Omega, \ 0 < t < T, \]

we see by (6.11) and (6.12) (e.g. Theorem III.2.1 in Temam [27]) that

\[ K \text{ is a compact operator,} \]

because the embedding \( H^{1/2}(\partial \Omega) \rightarrow L^2(\partial \Omega) \) is compact.

In view of \( T > \rho \), we apply Proposition 2 to (6.5), so that

\[ \| f R(\cdot,0) \|_{L^2(\Omega)} \leq C \left\| \frac{\partial \phi}{\partial \nu} \right\|_{L^2(\partial \Omega \times (0,T))}. \]

Therefore since \( |R(x,0)| \geq r_0 > 0 \) for almost all \( x \in \overline{\Omega} \) by (1.6), we obtain

\[ \| f \|_{L^2(\Omega)} \leq C \left\| \frac{\partial \phi}{\partial \nu} \right\|_{L^2(\partial \Omega \times (0,T))} = C \left\| \frac{\partial}{\partial \nu} (y_1 - \psi) \right\|_{L^2(\partial \Omega \times (0,T))} \]

by means of (6.7). Consequently by the triangle inequality and \( y_1 = y(f)' \), we have

\[ \| f \|_{L^2(\Omega)} \leq C \left\| \frac{\partial y(f)}{\partial \nu} \right\|_{H^1(0,T;L^2(\partial \Omega))} + C \| K f \|_{L^2(\partial \Omega \times (0,T))}. \]

When we take away the second term in (6.15), we can complete the proof of the first inequality in (1.13). For this, let us apply the compactness-uniqueness argument as in Second Step in §5. Contrarily assume that the first inequality in (1.13) does not hold. Then there exist \( f_n \in L^2(\Omega), \ n \geq 1 \) such that

\[ \| f_n \|_{L^2(\Omega)} = 1, \quad n \geq 1 \]

and

\[ \lim_{n \to \infty} \left\| \frac{\partial y(f_n)}{\partial \nu} \right\|_{H^1(0,T;L^2(\partial \Omega))} = 0. \]
By (6.16), we can extract a subsequence, denoted again by \( \{f_n\}_{n \geq 1} \), such that \( f_n \), \( n \geq 1 \) converge to some element \( f_0 \in L^2(\Omega) \) weakly in \( L^2(\Omega) \). Then (6.14) yields that

\[
(6.18) \quad \lim_{m,n \to \infty} \|Kf_n - Kf_m\|_{L^2(\partial\Omega \times (0,T))} = 0.
\]

On the other hand, it follows from (6.15) that

\[
\|f_n - f_m\|_{L^2(\Omega)} \leq C \left( \frac{\partial y(f_n)}{\partial \nu} - \frac{\partial y(f_m)}{\partial \nu} \right)_{H^1(0,T;L^2(\partial\Omega))} + C\|Kf_n - Kf_m\|_{L^2(\partial\Omega \times (0,T))}
\]

\[
\leq C \left( \frac{\partial y(f_n)}{\partial \nu} \right)_{H^1(0,T;L^2(\partial\Omega))} + C \left( \frac{\partial y(f_m)}{\partial \nu} \right)_{H^1(0,T;L^2(\partial\Omega))}
\]

\[
+ C\|Kf_n - Kf_m\|_{L^2(\partial\Omega \times (0,T))}, \quad m, n \geq 1.
\]

Therefore by (6.17) and (6.18), we see that \( \lim_{m,n \to \infty} \|f_n - f_m\|_{L^2(\Omega)} = 0 \), namely, \( \lim_{n \to \infty} \|f_n - f_0\|_{L^2(\Omega)} = 0 \). By (6.16) we obtain

\[
(6.19) \quad \|f_0\|_{L^2(\Omega)} = 1.
\]

Moreover by the second inequality in (1.13) which has been already proved, we have

\[
\lim_{n \to \infty} \left( \frac{\partial y(f_n)}{\partial \nu} - \frac{\partial y(f_0)}{\partial \nu} \right)_{H^1(0,T;L^2(\partial\Omega))} \leq C \lim_{n \to \infty} \|f_n - f_0\|_{L^2(\Omega)} = 0,
\]

with which we combine (6.17), so that

\[
(6.20) \quad \frac{\partial y(f_0)}{\partial \nu}(x,t) = 0, \quad x \in \partial\Omega, \ 0 < t < T.
\]

In view of (1.8), we apply Theorem 1 and \( f_0 = 0 \) follows. This contradicts (6.19). Thus the proof of Theorem 3 is complete.

7. Proof of Theorem 4

Proof of the second inequality in (1.17). Henceforth by \( C \) we denote a generic positive constant which is dependent on \( \Omega, T, q, a, b, \xi \) and \( \mathcal{U} \), but independent of \( p \). Setting \( \psi(x,t) = u(p)(x,t) - u(q)(x,t) \), we have:

\[
(7.1) \quad \begin{cases}
\psi''(x,t) = \Delta \psi(x,t) - p(x)\psi(x,t) + (q - p)(x)u(q)(x,t), \\
x \in \Omega, \ 0 < t < T \\
\psi(x,0) = \psi'(x,0) = 0, \quad x \in \Omega \\
\psi(x,t) = 0, \quad x \in \partial\Omega, \ 0 < t < T.
\end{cases}
\]

By (1.14), we choose \( C > 0 \) such that

\[
(7.2) \quad \|p\|_{L^\infty(\Omega)} \leq C, \quad p \in \mathcal{U}.
\]
Multiplying the both sides of the first equation in (7.1) by $\psi'$ and integrating by parts in $x$, from $\psi|_{\partial \Omega} = 0$, we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\psi'(x,t)|^2 + |\nabla \psi(x,t)|^2 dx = - \int_\Omega p(x)\psi(x,t)\psi'(x,t) dx + \int_\Omega (q - p)(x)u(q)(x,t)\psi'(x,t) dx, \quad t > 0.
\]

For simplicity, we define the energy $E(t)$ by
\[
E(t) = \frac{1}{2} \int_\Omega |\psi'(x,t)|^2 + |\nabla \psi(x,t)|^2 dx, \quad t \geq 0.
\]

Then by (7.2), (1.15) and Schwarz's inequality, Poincaré's inequality, we have
\[
E'(t) \leq \frac{1}{2} \int_\Omega |p(x)\psi'(x,t)|^2 dx + \frac{1}{2} \int_\Omega |\psi(x,t)|^2 dx + \frac{1}{2} \int_\Omega |p - q(x)|^2 dx \leq CE(t) + C\|p - q\|^2_{L^2(\Omega)}, \quad 0 \leq t \leq T.
\]

Therefore by $E(0) = 0$ and Gronwall's inequality, we see
\[
E(t) \leq CT\|p - q\|^2_{L^2(\Omega)}e^{CT}, \quad 0 \leq t \leq T,
\]

namely,
\[
\|\psi\|_{L^\infty(0,T; H^1_0(\Omega))} + \|\psi'\|_{L^\infty(0,T; L^2(\Omega))} \leq C\|p - q\|_{L^2(\Omega)}, \quad 0 \leq t \leq T.
\]

In view of (1.15), (7.2) and (7.4), we apply Lemma 8 to (7.1), so that
\[
\left\| \frac{\partial \psi}{\partial \nu} \right\|_{H^1(0,T; L^2(\partial \Omega))} \leq C\left( \|p\|_{H^1(0,T; L^2(\Omega))} + \|(q - p)u(q)\|_{H^1(0,T; L^2(\Omega))} \right)
\]
\[
\leq C\|p - q\|_{L^2(\Omega)}.
\]

Thus we finish the proof of the second inequality in (1.17).

Proof of the first inequality in (1.17).

First Step. - We show:

**Lemma 9.** Let $z = z(f)(x,t)$ be the weak solution to (1.2) with $R(x,t) = u(q)(x,t)$,
\[
\begin{cases}
z''(x,t) = \Delta z(x,t) - q(x)z(x,t) + f(x)u(q)(x,t), & x \in \Omega, \quad 0 < t < T \\
z(x,0) = z'(x,0) = 0, & x \in \Omega \\
z(x,t) = 0, & x \in \partial \Omega, \quad 0 < t < T.
\end{cases}
\]

Then there exists a constant $C = C(\Omega, T, q, a, b, \xi, \mathcal{U}) > 0$ such that
\[
\left\| \frac{\partial u(p)}{\partial \nu} - \frac{\partial u(q)}{\partial \nu} - \frac{\partial z(q - p)}{\partial \nu} \right\|_{H^1(0,T; L^2(\partial \Omega))} \leq C\|p - q\|_{L^\infty(\Omega)}\|p - q\|_{L^2(\Omega)}
\]
for all $p \in \mathcal{U}$.
Proof of Lemma 9. — By Lemma 8 and \( u(q) \in W^{3,\infty}(\Omega \times (0, T)) \subset H^1(0, T; L^2(\Omega)) \), we note \( \frac{\partial z(p)}{\partial \nu} \in H^1(0, T; L^2(\partial \Omega)) \). By a well-known a-priori estimate (e.g. Lions [16], Lions and Magenes [17]), we have

\[
\|z(f)\|_{L^\infty(0, T; H^1_0(\Omega))} + \|z(f)'\|_{L^\infty(0, T; L^2(\Omega))} \leq C\|f u(q)\|_{L^1(0, T; L^2(\Omega))} \leq C\|f\|_{L^2(\Omega)}.
\]

We set:

\[ d(x, t) = u(p)(x, t) - u(q)(x, t) - z(q - p)(x, t), \quad x \in \Omega, \quad 0 < t < T; \]

then \( d \) satisfies

\[
\begin{cases}
  d''(x, t) = \Delta d(x, t) - p(x)d(x, t) + (q - p)(x)z(q - p)(x, t), \quad x \in \Omega, \quad 0 < t < T \\
  d(x, 0) = d'(x, 0) = 0, \quad x \in \Omega \\
  d(x, t) = 0, \quad x \in \partial \Omega, \quad 0 < t < T.
\end{cases}
\]

By Lemma 8, (7.2) and (7.8) with \( f = q - p \), we can repeat the energy estimate for the proof of the second inequality in (1.17), so that

\[
\|d\|_{L^\infty(0, T; H^1_0(\Omega))} + \|d'\|_{L^\infty(0, T; L^2(\Omega))} \leq C\|q - p\|_{L^\infty(\Omega)} \|z(q - p)\|_{L^2(0, T; L^2(\Omega))}
\]

and

\[
\left\| \frac{\partial d}{\partial \nu} \right\|_{H^1(0, T; L^2(\partial \Omega))} \leq C\left( \|p d\|_{H^1(0, T; L^2(\Omega))} + \|p - q\|_{L^2(\Omega)} \right).
\]

Therefore

\[
\left\| \frac{\partial d}{\partial \nu} \right\|_{H^1(0, T; L^2(\partial \Omega))} \leq C\left( \|p\|_{L^\infty(\Omega)} \|p - q\|_{L^2(\Omega)} \right)
\]

by (7.2) and (7.8). Thus the proof of Lemma 9 is complete.

Second Step. — In this step, we will prove the first inequality in (1.17) provided that \( \|p - q\|_{L^\infty(\Omega)} \) is sufficiently small. By the assumptions (1.15) and (1.16), application of Theorem 3 to (7.6) yields

\[
\|p - q\|_{L^2(\Omega)} \leq C\left\| \frac{\partial z(q - p)}{\partial \nu} \right\|_{H^1(0, T; L^2(\partial \Omega))}.
\]

Consequently by (7.7) we have

\[
\begin{align*}
\|p - q\|_{L^2(\Omega)} & \leq C\left\| \frac{\partial u(p)}{\partial \nu} - \frac{\partial u(q)}{\partial \nu} - \frac{\partial z(q - p)}{\partial \nu} \right\|_{H^1(0, T; L^2(\partial \Omega))} \\
& \leq C\left( \|\frac{\partial u(p)}{\partial \nu} - \frac{\partial u(q)}{\partial \nu}\|_{H^1(0, T; L^2(\partial \Omega))} + \|\frac{\partial u(p)}{\partial \nu} - \frac{\partial u(q)}{\partial \nu}\|_{H^1(0, T; L^2(\partial \Omega))} \right)
\end{align*}
\]

Consequently by (7.8) we have

\[
\begin{align*}
\|p - q\|_{L^2(\Omega)} & \leq C\left( \|\frac{\partial u(p)}{\partial \nu} - \frac{\partial u(q)}{\partial \nu}\|_{H^1(0, T; L^2(\partial \Omega))} + \|\frac{\partial u(p)}{\partial \nu} - \frac{\partial u(q)}{\partial \nu}\|_{H^1(0, T; L^2(\partial \Omega))} \right)
\end{align*}
\]
Therefore
\[(1 - C\|p - q\|_{L^\infty(\Omega)})\|p - q\|_{L^2(\Omega)} \leq C\left\| \frac{\partial u(p)}{\partial \nu} - \frac{\partial u(q)}{\partial \nu} \right\|_{H^1(0, T; L^2(\partial \Omega))},\]

which implies that the first inequality in (1.17) holds true provided that \(\|p - q\|_{L^\infty(\Omega)} \leq \delta, \delta > 0\) being a small constant which is dependent on \(\Omega, T, q, a, b, \xi\) and \(\mathcal{U}\).

**Third Step.** Finally we will prove the first inequality in (1.17) for general \(p \in \mathcal{U}\). Only in this step, we use the compactness (1.14) of \(\mathcal{U}\) in \(L^\infty(\Omega)\). For this, it is sufficient to verify
\[(7.9) \quad \inf_{\|p - q\|_{L^\infty(\Omega)} \geq \delta, p \in \mathcal{U}} \frac{\left\| \frac{\partial u(p)}{\partial \nu} - \frac{\partial u(q)}{\partial \nu} \right\|_{H^1(0, T; L^2(\partial \Omega))}}{\|p - q\|_{L^2(\Omega)}} > 0,
\]

where \(\delta > 0\) is the sufficiently small constant chosen in Second Step. Contrarily assume that (7.9) is not true; then there exist
\[(7.10) \quad p_n \in \mathcal{U}, \quad \|p_n - q\|_{L^\infty(\Omega)} \geq \delta, \quad n \geq 1
\]
such that
\[(7.11) \quad \lim_{n \to \infty} \frac{\left\| \frac{\partial u(p_n)}{\partial \nu} - \frac{\partial u(q)}{\partial \nu} \right\|_{H^1(0, T; L^2(\partial \Omega))}}{\|p_n - q\|_{L^2(\Omega)}} = 0.
\]

By the assumption (1.14), we can choose \(p_0 \in L^\infty(\Omega)\) and a subsequence, denoted again by \(p_n, n \geq 1\), such that
\[(7.12) \quad \lim_{n \to \infty} \|p_n - p_0\|_{L^\infty(\Omega)} = 0.
\]

Then (7.10) and (7.12) imply
\[(7.13) \quad \|p_0 - q\|_{L^\infty(\Omega)} \geq \delta.
\]

Since \(\sup_{n \geq 1} \|p_n - q\|_{L^2(\Omega)} < \infty\) by \(p_n \in \mathcal{U}\), the condition (7.11) yields
\[(7.14) \quad \lim_{n \to \infty} \left\| \frac{\partial u(p_n)}{\partial \nu} - \frac{\partial u(q)}{\partial \nu} \right\|_{H^1(0, T; L^2(\partial \Omega))} = 0.
\]

On the other hand, we can prove
\[(7.15) \quad \left\| \frac{\partial u(p_n)}{\partial \nu} - \frac{\partial u(p_0)}{\partial \nu} \right\|_{L^2(\partial \Omega \times (0, T))} \leq C\|p_n - p_0\|_{L^\infty(\Omega)}, \quad n \geq 1,
\]

where \(C > 0\) is independent of \(n \geq 1\).

**Proof of (7.15).** - Set \(u_n(x, t) = u(p_n)(x, t) - u(p_0)(x, t), x \in \Omega, 0 < t < T, n \geq 1\). Then \(u_n\) satisfies
\[
\begin{aligned}
\left\{ \begin{array}{l}
u_n''(x, t) = \Delta u_n(x, t) - p_n(x)u_n(x, t) + (p_0 - p_n)(x)u(p_0)(x, t), \\
x \in \Omega, \quad 0 < t < T \\
u_n(x, 0) = u_n'(x, 0) = 0, \\
u_n(x, t) = 0, \\quad x \in \partial \Omega, \quad 0 < t < T.
\end{array} \right.
\end{aligned}
\]
Noting (7.2) and Poincaré's inequality, multiplying the both sides of the first equation in (7.16) by \( u'_n \), integrating by parts, we can obtain:

\[
\begin{align*}
\frac{d}{dt} \left( \int_{\Omega} |u'_n(x,t)|^2 + |\nabla u_n(x,t)|^2 \, dx \right) \\
\leq \int_{\Omega} |p_n(x)u_n(x,t)|^2 \, dx + \int_{\Omega} |u'_n(x,t)|^2 \, dx \\
+ \int_{\Omega} |(p_0 - p_n)(x)|^2 |u(p_0)(x,t)|^2 \, dx + \int_{\Omega} |u'_n(x,t)|^2 \, dx \\
\leq C \int_{\Omega} |u'_n(x,t)|^2 + |\nabla u_n(x,t)|^2 \, dx + \|p_0 - p_n\|_{L^\infty(\Omega)}^2 \|u(p_0)(\cdot,t)\|_{L^2(\Omega)}^2
\end{align*}
\]

in a way similar to (7.3). Here and henceforth \( C > 0 \) is independent of \( n \geq 1 \). Therefore Gronwall's inequality yields

\[
\int_{\Omega} |u'_n(x,t)|^2 + |\nabla u_n(x,t)|^2 \, dx \leq C\|p_0 - p_n\|_{L^\infty(\Omega)}^2 \|u(p_0)\|_{L^\infty(0,T;L^2(\Omega))}^2, \quad n \geq 1, \quad 0 \leq t \leq T,
\]

by \( u_n(x,0) = u'_n(x,0) = 0, \quad x \in \Omega \). Applying a priori estimation (e.g. Théorème 1.4.2 in [16]) and Lemma 7 to \( u(p_0) \), we see

\[
\|u(p_0)\|_{L^\infty(0,T;L^2(\Omega))} \leq C(\|a\|_{L^2(\Omega)} + \|b\|_{H^{-1}(\Omega)} + \|\xi\|_{H^2(\partial\Omega \times (0,T))}) \equiv C.
\]

Therefore (7.17) and (7.18) imply

\[
\|u_n\|_{L^\infty(0,T;L^2(\Omega))} \leq C\|p_n - p_0\|_{L^\infty(\Omega)}, \quad n \geq 1.
\]

Now we apply Lemma 6 to (7.16) in view of (7.2), (7.18) and (7.19), so that the proof of (7.15) is complete.

**Remark.** In Theorem 2.2.5 in [12], \( p_0 \) is fixed. However since \( p_0 \) is in a bounded set in \( L^\infty(\Omega) \), we can apply the transposition method in [12] and prove (7.18) with \( C > 0 \) independent of \( p_0 \).

Now we proceed to the completion of the proof of (1.17). In view of (7.12) and (7.15), we see that

\[
\lim_{n \to \infty} \left\| \frac{\partial u(p_n)}{\partial \nu} - \frac{\partial u(p_0)}{\partial \nu} \right\|_{L^1(\partial \Omega \setminus (0,T))} = 0,
\]

which implies

\[
\frac{\partial u(q)}{\partial \nu}(x,t) = \frac{\partial u(p_0)}{\partial \nu}(x,t), \quad x \in \partial \Omega, \quad 0 < t < T
\]

by (7.14). In view of (1.15) and (1.16), it follows from Theorem 2 that \( p_0(x) = q(x) \), \( x \in \Omega \), which contradicts (7.13). Hence by this contradiction, (7.9) must be true. Thus the proof of Theorem 4 is complete.
8. Concluding Remarks

I. For the proof of Theorem 1 where the uniqueness is established within less regular solutions, the key is a modification of a well-known Carleman estimate (Proposition 1). The weaker regularity assumption for the uniqueness in Theorem 1 is essential for the global Lipschitz stability in our inverse problems.

II. For showing the Lipschitz stability, our method is based on the Carleman estimate and the observability inequality. Therefore we can prove Theorems 3 and 4 for other equations such as a wave equation with damping term, a plate equation, Maxwell’s equations, an isotropic Lamé system for which we can establish Carleman estimates and observability inequalities.

Remark added in the revision. – Throughout this paper, we take the whole boundary \( \partial \Omega \) where the normal derivative is given. The argument concerning the observability is valid for a suitable subboundary. More precisely, correspondingly to Proposition 2 by the multiplier method, we can prove:

**Proposition 3.** Let \( x_0 \in \mathbb{R}^n \) be arbitrarily fixed and let us set

\[
\Gamma(x_0) = \{ x \in \partial \Omega ; (x - x_0) \cdot \nu(x) > 0 \}.
\]

If \( T > \sup_{x \in \Omega} |x - x_0| \), then there exists a constant \( C = C(\Omega, T, q, x_0) > 0 \) such that

\[
\left\| \phi_1 \right\|_{L^2(\Omega)} \leq C \left\| \frac{\partial \phi(0, \phi_1)}{\partial \nu} \right\|_{L^1(\Gamma(x_0) \times (0, T))}
\]

for all \( \phi_1 \in L^2(\Omega) \).

Therefore we can replace (1.9) of Theorem 1 by:

(1.9') \[
\frac{\partial y(f)}{\partial \nu}(x, t) = 0, \quad x \in \Gamma(x_0), \ 0 < t < T,
\]

for the uniqueness, then our argument for the global Lipschitz stability can work. The author has found a paper by M. Kubo “Uniqueness in inverse hyperbolic problems – Carleman estimate for boundary value problems–” (to appear in Journal of Mathematics of Kyoto University, 1998) after I had submitted the present paper. M. Kubo’s paper leads us to the uniqueness under the condition (1.9'), so that we can establish the global Lipschitz stability in determining \( f \) in (1.2) from Neumann data on the subboundary \( \Gamma(x_0) \).

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