Existence–uniqueness result for a nonlinear $n$-term fractional equation

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**A B S T R A C T**

We prove the existence–uniqueness of the solution to the nonlinear $n$-term time-fractional differential equation with constant coefficients in the Banach space $C([0, T])$,

$$b_0 D^{\beta_0} u(t) + \sum_{i=1}^{m-1} b_i D^{\beta_i} u(t) + \sum_{i=m}^{n-1} b_i D^{\alpha_i} u(t) + b_n D^{\alpha_n} u(t) = f(t, u(t)), \quad t \in (0, T), \quad x \in \mathbb{R},$$

$$u(0) = f(0), \quad u_t(0) = g(0),$$

$$0 < \beta_1 < \beta_2 < \cdots < \beta_{m-1} < 1 < \alpha_m < \alpha_{m+1} < \cdots < \alpha_n < 2$$

(respectively $0 < \beta_1 < \cdots < \beta_{m-1} < \alpha_m < \alpha_{m+1} < \cdots < \alpha_n < 1$,

$$1 < \beta_1 < \cdots < \beta_{m-1} < \alpha_m < \alpha_{m+1} < \cdots < \alpha_n < 2),$$

(1)

$f(t, u(t)) \in C([0, T] \times C([0, T]))$ is a given function meant to be composed with a real valued function $u$ and satisfies Assumption 1.

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**1. Introduction**

A fractional calculus appears in various fields of science covering many known classical fields, such as Abel integral equation and viscoelasticity, analysis of feedback amplifiers, capacitor theory, fractances, generalized voltage dividers, etc. Fractional differential equations draw a great application in nonlinear oscillations of earthquakes, many physical phenomena such as seepage flow in porous media (cf. [7]) and in fluid dynamical traffic model. Fractional derivatives can eliminate the deficiency of continuum traffic flow.

Basic approach to fractional calculus and its applications in mechanics, probability and statistics is given in [13] and references therein. The comprehensive treatment of the classical fractional equations techniques such as Laplace and Fourier transform method, method of Green's functions, Mellin transform and some numerical techniques are given in [23,34].

A survey of investigations of fractional calculus and its applications in mathematical analysis such as ODEs, PDEs, summation series, special functions, convolution integral equations, theory of generating equations, the theory of analytic functions, and so on, are the subject of [38]. In [10] and [9] are given qualitative properties of the solutions of such equations and remarks on numerical techniques for their solving.

Open problem in this field is finding some easy and effective methods for solving such equations. One attempt is given in [9]. New numerical approach based on the Laguerre polynomials are subject of [33]. Efficiency of this method is based on

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the distributional error estimate (cf. [33]). Theoretical foundation of this approach is given in [32]. Solving Cauchy problem for fractional PDEs of evolution type is given in [39] through the pseudo-differential operators.

Development of the theory of fractional calculus and its applications in physics, mechanics, chemistry, engineering, etc. causes the great interest for fractional differential equations (cf. [15,19–22,26,27,29,34]). Recently it found a great application in diffusion-wave phenomena [3–5,24,37], such as in the continuous random walk [14,16,17,40] or stochastic equations [8].

In classical approach linear initial fractional differential equations are solved by special functions [28,34,35]. In recent papers, for nonlinear problems, techniques of functional analysis such as fixed point theory, the Banach contraction principle, Leray–Schauder theory, etc. are applied for solving such kind of problems (cf. [1,10–12,41,42]). Dirichlet-type of problems for linear ordinary differential equations of fractional order are developed in [22] and [30]. Existence and multiplicity of solutions to nonlinear Dirichlet problem

\[ D^\alpha_{0+} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \quad u(0) = u(1) = 0, \quad 1 < \alpha < 2, \quad \alpha \in \mathbb{R}, \]

\[ D^\alpha_{0+} \] is the Riemann–Liouville differentiation, \( f : [0, 1] \times [0, \infty) \to [0, \infty) \) is continuous, are proved. By deriving Green's function, the existence and multiplicity of positive solutions are obtained by fixed point theorems (cf. [41,42]).

It is worth to mention two-term diffusion-wave problem for \( 0 < \alpha < 2 \) in [36], existence–uniqueness result obtained in Sobolev spaces, such as the regularity of the solution, and behavior when derivatives are non-entire values at interval (0, 2).

The recent results in the investigation of the existence and uniqueness of the solutions to nonlinear fractional order differential equations by fixed point theorems are given in [6,7]. This paper is devoted to nonlinear fractional differential equations due to the great interest of many researches and the application in various fields of science and engineering. The existence–uniqueness results are obtained for nonlinear two-term and consequently for the nonlinear \( n \)-term fractional differential equation integral form by Leray–Schauder fixed point theorem in a way of [6,7]. For the exact solution the proofs are given by the Banach contraction principle.

The existence of nonlinear fractional differential equations of one time fractional derivative, is considered in [6,20,23,34]. We follow precisely the approach of [6] in order to prove the existence of the solution given in integral form to the two-term fractional equation, for \( 0 < \beta < \alpha < 2 \). We use in general Assumption 1 for nonlinear term.

By the given technique using Green function from [34] for \( n \)-term, \( n = 2, 3, 4, \ldots \), fractional equation we prove the existence–uniqueness of the solution to \( n \)-term fractional equation, when nonlinear term satisfies Assumption 1.

The result can be generalized for arbitrary interval \( k − 1 < \beta < \alpha < k, k \in \mathbb{N} \), under appropriate conditions to nonlinear term.

2. Aims

The nonlinear Riemann–Liouville fractional order differential equation \( D^\alpha u(t) = f(t; u(t)) \) has, in the Banach space \( C[0; T] \), two well-known problems (see [22] and [33]) \( D^\alpha u(t) = f(t; u(t)); \quad t > 0 \) with \( 1−\alpha u(t)|_{t=0} = 0 \) and \( D^\alpha u(t) = f(t; u(t)); \quad t > 0 \) with \( u(0) = 0 \). The integer order values as the initial data can be determined by the measurements whereas non-integer order values have no clear physical meaning but they are appropriate for computation.

In this paper we consider generalization of this problem in the two-term fractional differential equation \( D^\alpha u(t) + D^\beta u(t) = f(t; u(t)), \) where \( 0 < \beta < \alpha < 2 \), and its generalization to the \( n \)-term fractional time differential equations, with the initial data \( u(0) = f(0), \) and \( u(0) = g(0) \). In general, to \( n \)-term time fractional equations for \( 0 < \beta_0 < \beta_1 < \beta_2 < \cdots < \beta_{n−1} < 1 < \alpha_m < \alpha_{m+1} < \cdots < \alpha_n < 2 \) (respectively other combinations of derivatives at intervals (0, 1), (1, 2), or (0, 2), we have to solve Eq. (1).

We prove the existence and the uniqueness of the solution to the two-term fractional nonlinear equation with different combinations of the fractional derivatives \( 0 < \beta < \alpha < 2 \), using two forms of the solution: explicit solution in a form of the Mittag-Leffler functions, and integral form of the solution. As a techniques we employ the Banach contraction principle and the Leray–Schauder fixed point theorem, respectively.

Integral form is appropriate for the asymptotical expansion of the solution and examination of the main properties of the solution such as long time behaviour of the solution, i.e. boundedness as well as unboundedness when \( t \to \infty \). This form is more appropriate also in \( L^p \)-estimates of the solution when \( 1 \leq p < \infty \).

We prove the particular cases when fractional derivatives are in the intervals \( 0 < \beta < \alpha < 1 \), \( 1 < \beta < \alpha < 2 \), and \( 0 < \beta < 1 < \alpha < 2 \). Then, we generalize result to the \( n \)-term fractional equation. The solution have the form given in [39]. We give a generalization of that.

We apply the method of the solution to the existence–uniqueness results of important equations of physical science, mechanics and engineering (cf. [18,39]).

2.1. Basic notions

The Riemann–Liouville fractional integral of order \( \alpha > 0 \) is defined by

\[ I^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t − \tau)^{\alpha−1} f(\tau) d\tau, \quad \alpha > 0. \]
The Riemann–Liouville fractional derivative is defined as follows. Let $k$ be a positive integer and $\alpha \in (k-1, k)$, then $D^\alpha f(t) := D^k I^{k-\alpha} f(t)$. In particular, for $k = 1$,

$$
D^\alpha f(t) = \frac{d}{dt} \left[ \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^\alpha} d\tau \right], \quad \alpha \in (0, 1).
$$

The fractional derivative in the Caputo sense is defined as

$$
D^\beta f(t) = \left\{ \begin{array}{ll}
\frac{1}{\Gamma(k-\beta)} \int_0^t \frac{f^{(k)}(\tau)}{(t-\tau)^{\beta+1}} d\tau, & \beta \in (k-1, k), \\
\frac{d^k}{dt^k} f(t), & \beta = k,
\end{array} \right. \quad k \in \mathbb{N},
$$

(2)

The Caputo derivative is suitable for the initial value problems in physics and engineering when the initial conditions are integer-order derivatives.

For essentials of fractional calculus cf. [20,23,25,34].

As a tool in calculus the Laplace transform defined as

$$
\tilde{f}(p) = \mathcal{L}\{f(t); p\} = \int_0^\infty e^{-pt} f(t) dt, \quad p \in \mathbb{R}, \quad f^{(k)}(0^+) := \lim_{t \to 0^+} f^{(k)}(t),
$$

(3)

is used, as well as the inversion of the Laplace transform

$$
\int_0^\infty e^{-pt} t^{\beta\alpha-1} E_{\alpha, \beta}(\pm at^\alpha) dt = \frac{k! a^{-\beta}}{(s^\alpha + a)^{k+1}}, \quad \text{Res} > |a|^{1/\alpha} \quad (\text{cf. [34]}).
$$

(4)

Of particular importance for us is the result of applying the Laplace transform to the Caputo derivative that we will need for $m = 1$ and $m = 2$:

$$
\mathcal{L}\{D^\beta f(t); s\} = s^\beta \tilde{f}(s) - \sum_{k=0}^{m-1} s^{\beta-k-1} f^{(k)}(0^+), \quad m - 1 < \beta \leq m,
$$

where the Laplace transform is given with (3). The analogous formula for the Riemann–Liouville derivative needs derivatives of non-integer order at $t = 0$:

$$
\mathcal{L}\{D^\beta f(t); s\} = s^\beta \tilde{f}(s) - \sum_{k=0}^{m-1} [D^k f^{(m-\beta)}] f(0^+) s^{m-k-1}.
$$

(5)

When the limiting values $f^{(k)}(0^+)$ are finite and $m - 1 < \beta < m$ this formula simplifies to

$$
\mathcal{L}\{D^\beta f(t); s\} = s^\beta \tilde{f}(s).
$$

In particular, when $f^{(k)}(0^+) = 0$ for $k = 0, 1, \ldots, m - 1$, we recover identity between the two fractional derivatives consistently with Eq. (2).

In this paper we use Caputo derivative. Techniques for the Riemann–Liouville derivatives are the same but we use the different Laplace transform due to the different definitions of the derivatives.

Two-parametric Mittag-Leffler function is defined by

$$
E_{\alpha, \beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(k\alpha + \beta)}, \quad (\alpha > 0, \beta > 0).
$$

(6)

Analytic properties and asymptotical expansion of this function is given in [13,34].

In the sequel we shall use the following asymptotical expansion of the Mittag-Leffler function.

Lemma 1. (See [34, p. 34].) If $\alpha < 2$, $\beta$ is an arbitrary number, $\mu$ is such that $\frac{2\mu}{\pi} < \mu < \min(\pi, \pi \alpha)$ and $C_3$ is a real constant, then

$$
|E_{\alpha, \beta}(z)| \leq \frac{C_3}{(1 + |z|)},
$$

as $\mu \leq |\arg(z)| \leq \pi$, and $|z| > 0$.

Using formula for integration of the Mittag-Leffler function, term by term from [34],

$$
\int_0^z t^{\beta-1} E_{\alpha, \beta}(\lambda t^\alpha) dt = z^\beta E_{\alpha, \beta+1}(\lambda z^\alpha) \quad (\beta > 0),
$$

(7)

we establish the existence–uniqueness result.
In the sequel we use the following assumption on the nonlinear term.

**Assumption 1.** (See [6].) Suppose that \( f \) is continuous on \( I \times \mathbb{R}, I \in [0, T], T > 0 \), and suppose that the following holds:
\[
|f(t, u(t)) - f(t, v(t))| \leq \lambda(t)s(r),
\]
where \( s(r) \) is continuous on \([0, \infty)\) and \( s(0) = 0, r = |u - v|, |I^1(\lambda(t))| < M, \) for \( t \in [0, T] \) (I is integral over \([0, t], t \leq T > 0\)).

For \( \lambda(t) = L > 0, s(|u - v|) = |u - v| \) the condition on \( f \) is Lipschitz's. Then, we have the following approximation for the nonlinear term
\[
|f(t, u(t))| \leq |f(t, 0)| + L|u(t)|,
\]
where \( L \) is the Lipschitz's constant.

For special choice of \( \lambda \) we have Osgood's condition.

**Remark 1.** Let \( s(r) \) be from \( C^1([0, T]) \) and \( s(0) = 0 \), we can use the first approximation for \( s \):
\[
s(r) = s(0) + s'(0)r = rs'(0).
\]
If \( \lambda(t) = L > 0 \), is a constant, then condition (8) becomes Osgood condition. If \( \lambda(t) = L > 0 \) and \( s(|u - v|) = |u - v| \) then condition (8) is the Lipschitz's condition.

In order to prove the existence–uniqueness result for the solution in integral form, we use Leray–Schauder fixed point theorem.

**Lemma 2.** (See [31].) If \( U \) is a closed bounded convex subset of a Banach space \( X \) and \( T : U \to U \), is completely continuous, then \( T \) has a fixed point in \( U \).

Let \( I = [0, T] \) and \( D = I \times C(I) \), where \( C(I) \) is the class of continuous functions defined on \( I \) supplied with the norm
\[
\|u\| = \max|u(t)|, \quad t \in I, \ u(t) \in C(I).
\]

2.2. Different form of the solution

We give integral form of the solution to Eq. (1) without setting the initial data. Then, we give the explicit solution to each problem separately in a form of the Mittag-Leffler function. We prove that these two forms are equivalent to Eq. (1). Using the explicit solution in the Mittag-Leffler form we prove the existence and uniqueness of the solution by the Banach contraction principle. For the proof of the existence–uniqueness solution for the integral form equation, we use Leray–Schauder fixed point theorem.

We use the following integral form for Eq. (1) from [34]. We have the following lemmas.

**Lemma 3.** Let \( u(t) \in C(I) \) and \( f(t, u(t)) \in C(D) \), if a solution to Eq. (1) where \( 0 < \beta < \alpha < 1 \) exists, it is given by
\[
u(t) = CG(t) + \int_0^t G(t - \tau)f(\tau, u(\tau))d\tau, \quad u(0) = f(0),
\]
where \( C = (D^{\beta-1}u(t) + D^{\alpha-1}u(t))|_{t=0}, G(t) = t^{\alpha-1}E_{\alpha-\beta,\alpha}(-t^{\alpha-\beta}), \) where \( E_{\alpha,\beta}(z) \) is the two-parameter function of the Mittag-Leffler type.

**Lemma 4.** Let \( u(t) \in C(I) \) and \( f(t, u(t)) \in C(D) \), if a solution to Eq. (1) when \( 1 < \beta < \alpha < 2 \) exists, it is given by
\[
u(t) = CG(t) + \int_0^t G(t - \tau)f(\tau, u(\tau))d\tau, \quad u(0) = f(0), \quad u_t(0) = g(0),
\]
where \( C = (D^{\beta-1}u(t) + D^{\alpha-1}u(t) + D^{\beta-2}u_t(t) + D^{\alpha-2}u_t(t))|_{t=0}, G(t) = t^{\alpha-1}E_{\alpha-\beta,\alpha}(-t^{\alpha-\beta}), \) where \( E_{\alpha,\beta}(z) \) is the two-parameter function of the Mittag-Leffler type.

**Lemma 5.** Let \( u(t) \in C(I) \) and \( f(t, u(t)) \in C(D) \), if a solution to Eq. (1) when \( 0 < \beta < 1 < \alpha < 2 \) exists, it is given by
\[
u(t) = CG(t) + \int_0^t G(t - \tau)f(\tau, u(\tau))d\tau,
\]
where \( C = (D^{\beta_0} - u(t) + D^{\alpha_1} - u(t) + D^{\alpha_2} - u(t)) \) if \( \alpha = 0 \), \( G(t) = t^{\alpha_1} E_{\alpha_1, \beta_1}(-t^{\alpha_1}) \), where \( E_{\alpha, \beta} \) is the two-parameter function of the Mittag-Leffler type.

**Lemma 6.** Let \( u(t) \in C(I) \) and \( f(t, u(t)) \in C(D) \), is a solution to Eq. (1) when \( 0 < \beta_0 < \beta_1 < \cdots < \beta_{m-1} < 1 < \alpha_m < \alpha_{m+1} < \cdots < \alpha_n < 2 \) exists, it is given by

\[
 u(t) = CG(t) + \int_0^t G(t - \tau) f(\tau, u(\tau)) d\tau,
\]

where \( C = (D^{\beta_0} - u(t) + D^{\alpha_1} - u(t) + \cdots + D^{\alpha_n} - u(t)) \) if \( \alpha = 0 \), \( G(t) = t^{\alpha_1} E_{\alpha_1, \beta_1}(-t^{\alpha_1}) \), where \( E_{\alpha, \beta} \) is the two-parameter function of the Mittag-Leffler type.

We give an explicit form for the solutions given in integral form in Lemmas 3–6, respectively. Due to simplicity we sometimes omit the weights.

Case 0 < \( \beta < \alpha < 1 \). Laplace transform gives

\[
 s^\beta \tilde{u}(s) - s^{\beta_1} \tilde{u}(0) + s^\alpha \tilde{u}(s) - s^{\alpha_1} \tilde{u}(0) = \tilde{f}(s, \tilde{u}(s)).
\]

From the inverse Laplace transform we have

\[
 u(t) = u(0) + t^{\alpha_1} E_{\alpha_1, \beta_1}(-t^{\alpha_1}) * f(t, u(t)).
\]

Case 1 < \( \beta < \alpha < 2 \). After applying Laplace transform, we obtain

\[
 s^\beta \tilde{u}(s) - s^{\beta_1} \tilde{u}(0) - s^{\beta_2} \tilde{u}_1(0) + s^\alpha \tilde{u}(s) - s^{\alpha_1} \tilde{u}(0) - s^{\alpha_2} \tilde{u}_1(0) = \tilde{f}(s, \tilde{u}(s)).
\]

The inverse Laplace transform gives

\[
 u(t) = u(0) + t u_1(0) + t^{\alpha_1} E_{\alpha_1, \beta_1}(-t^{\alpha_1}) * f(t, u(t)).
\]

Case 0 < \( \beta < 1 < \alpha < 2 \). Then,

\[
 s^\beta \tilde{u}(s) - s^{\beta_1} \tilde{u}(0) + s^\alpha \tilde{u}(s) - s^{\alpha_1} \tilde{u}(0) - s^{\alpha_2} u_1(t) = \tilde{f}(s, \tilde{u}(s)).
\]

Inverse Laplace transform gives the solution to this problem in a form of the Mittag-Leffler functions

\[
 u(t) = u(0) + t E_{\beta, \alpha}(-t^{\beta}) * f(t, u(t)).
\]

Case 0 < \( \beta_0 < \beta_1 < \beta_2 < \cdots < \beta_{m-1} < 1 < \alpha_m < \alpha_{m+1} < \cdots < \alpha_n < 2 \). We have to solve the equation

\[
 b_0 D^{\beta_0} u(t) + \sum_{i=1}^{m-1} b_i D^{\beta_i} u(t) + \sum_{i=m}^{n-1} b_i D^{\alpha_i} u(t) + b_n D^{\alpha_n} u(t) = f(t, u(t)),
\]

subject to the initial data \( u(0) = f(0), u_1(t) = g(0) \).

In this case the solution to the \( n \)-term equation is given with

\[
 u(t) = u(0) + \left( t^{-\alpha_{m+1}} + t^{-\alpha_m} + \cdots + s^{-\alpha_1} \right) \mathcal{L}^{-1} \left( \frac{1}{B(s)} \right) u_1(0) + \mathcal{L}^{-1} \left( \frac{1}{B(s)} \right) * f(t, u(t)).
\]

We have from [34] the explicit solution to the \( n \)-term Green function \( G_n(t) \). Recall it. Ordering the terms so that \( \alpha_n > \alpha_{n-1} > \cdots > \alpha_m > \beta_{m-1} > \beta_{m-2} > \cdots > \beta_1 > \beta_0 \), we have

\[
 \frac{1}{B(s)} = \frac{1}{b_n s^{\alpha_n} + b_n s^{\alpha_{n-1}} + \cdots + b_m s^{\alpha_m} + b_m s^{\beta_{m-1}} + \cdots + b_1 s^{\beta_1} + b_0 s^{\beta_0}}.
\]

From [34, (5.6)], the inverse Laplace transform reads as follows:

\[
 G_n(t) = \mathcal{L}^{-1} \left( \frac{1}{B(s)} \right) = \left( \frac{1}{b_n} \right) \sum_{m=0}^{\infty} \sum_{k_0 + k_1 + \cdots + k_{n-2} = m} \left( m; k_0, k_1, \ldots, k_{n-2} \right) \left( \prod_{i=0}^{n-2} \left( \frac{b_i}{b_n} \right)^{k_i} \right) \times \left( \prod_{i=0}^{\alpha_n-1} \left( \frac{b_i}{b_n} \right)^{k_i} \right)
\]

where \( \beta_n = \alpha_n, \ldots, \beta_m = \alpha_m, \ldots, \beta_{m-1} = \beta_m - 1, \ldots, \beta_0 = \beta_0 \).
Thus, the explicit solution to the integral form from Lemma 6 is given by

$$u(t) = u(0) + \left( t^{-\alpha_0+1} + t^{-\alpha_1} + \cdots + s^{-\alpha_0+1} \right) * L^{-1}\left( \frac{1}{B(s)} \right) u(t(0)) + L^{-1}\left( \frac{1}{B(s)} \right) * f(t, u(t)),$$

where $L^{-1}\left( \frac{1}{B(s)} \right)$ is given in [34] by (12).

Case $0 < \beta_0 < \beta_1 < \cdots < \beta_{m-1} < \alpha_m < \cdots < \alpha_n < 1$. Then, the solution is given by

$$u(t) = u(0) + L^{-1}\left( \frac{1}{B(s)} \right) * f(t, u(t)).$$

It is obvious that Banach contraction principle is applicable.

Case $1 < \beta_0 < \beta_1 < \cdots < \beta_{m-1} < \alpha_m < \cdots < \alpha_n < 2$. Then, the solution is given by

$$u(t) = u(0) + tu(t(0)) + L^{-1}\left( \frac{1}{B(s)} \right) * f(t, u(t)).$$

We give a proof for the first case $0 < \beta < \alpha < 1$. The other cases can be handled in the same way and proofs will be omitted.

**Lemma 7.** Solution to Eq. (1) is given with

$$u(t) = u(0) + t^{\alpha-1} E_{\alpha-\beta, \alpha}(-t^{\alpha-\beta}) * f(t, u(t)).$$

when $0 < \beta < \alpha < 1$.

**Proof.** If we apply the Laplace transform to Eq. (1), for the initial data $u(0) = g(0)$, $0 < \beta < \alpha < 1$, $s^\alpha \hat{u}(s) - s^\beta \hat{u}(s) - s^{\beta-1} \hat{u}(0) = \hat{f}(s, u(s))$. Solving (w.r.t) to $\hat{u}(s)$ we obtain (13). Then,

$$D^\alpha u(t) = D^\alpha u(t)|_{t=0} + D^\alpha (t^{\alpha-1} E_{\alpha-\beta, \alpha}(-t^{\alpha-\beta})) * f(t, u(t)),$$

$$D^\beta u(t) = D^\beta u(t)|_{t=0} + D^\beta (t^{\alpha-1} E_{\alpha-\beta, \alpha}(-t^{\alpha-\beta})) * f(t, u(t)).$$

Summing these two expressions, we obtain

$$D^\alpha u(t) + D^\beta u(t) = D^\alpha u(t)|_{t=0} + D^\beta u(t)|_{t=0} + \left[ D^\alpha (t^{\alpha-1} E_{\alpha-\beta, \alpha}(-t^{\alpha-\beta})) + D^\beta (t^{\alpha-1} E_{\alpha-\beta, \alpha}(-t^{\alpha-\beta})) \right] * f(t, u(t))$$

$$= D^\alpha u(t)|_{t=0} + D^\beta u(t)|_{t=0} + \left[ t^{\alpha-1} E_{\alpha-\beta, \alpha}(-t^{\alpha-\beta}) + t^{\alpha-1} E_{\alpha-\beta, \alpha}(-t^{\alpha-\beta}) \right] * f(t, u(t))$$

$$= D^\alpha u(t)|_{t=0} + D^\beta u(t)|_{t=0} + \Gamma^{-1} \sum_{n=0}^{\infty} \frac{(-1)^n t^{n \alpha / (\alpha - \beta)}}{\Gamma(n(\alpha - \beta))} + t^{\alpha-\beta} \sum_{n=0}^{\infty} \frac{(-1)^n t^{n / (\alpha - \beta)}}{\Gamma(n+1(\alpha - \beta))} * f(t, u(t))$$

$$= \frac{1}{\Gamma} \left( \sum_{n=0}^{\infty} \frac{(-1)^n t^{n \alpha / (\alpha - \beta)}}{\Gamma(n(\alpha - \beta))} + t^{\alpha-\beta} \sum_{n=0}^{\infty} \frac{(-1)^n t^{n / (\alpha - \beta)}}{\Gamma(n+1(\alpha - \beta))} \right) * f(t, u(t)).$$

We have

$$D^\alpha u(t) + D^\beta u(t) = \delta(t) * f(t, u(t)) = f(t, u(t)),$$

since

$$\lim_{t \to 0} \frac{1}{\Gamma} \sum_{n=0}^{\infty} \frac{(-1)^n t^{n \alpha / (\alpha - \beta)}}{\Gamma(n(\alpha - \beta))} = \delta(t).$$

The other cases for different combinations of the derivatives $\alpha, \beta,$ can be proved in a similar way. □

**Lemma 8.** The integral form (6) and Eq. (1) are equivalent.

**Proof.** We shall find

$$D^\alpha u(t) + D^\beta u(t) = f(t, u(t)).$$

We have

$$D^\alpha u(t) = CG^\alpha(t) + D^\alpha (G(t) * f(t, u(t))).$$
Similarly, 
\[ D^\beta u(t) = CG^\beta(t) + D^\beta(G(t) * f(t, u(t))). \]

Summing these two relations we obtain 
\[ D^\alpha u(t) + D^\beta u(t) = C(D^\alpha G(t) + D^\beta G(t)) + (D^\alpha G(t) + D^\beta G(t)) * f(t, u(t)) \]
\[ = (G^\alpha(t) + G^\beta(t)) * [C \delta(t) + \delta(t) * f(t, u(t))] = f(t, u(t)), \]
since \( C = 0 \) for \( t = 0 \). We shall give a proof that \( D^\alpha G(t) + D^\beta G(t) = \delta(t) \), where \( G(t) = t^{\alpha-1}E_{\alpha-\alpha, \alpha}(-t^{\alpha-\beta}). \)

We have by [34], by the differentiation of the fractional derivatives
\[ = t^{-1}E_{\alpha-\beta, 0}(-t^{\alpha-\beta}) + t^{\alpha-\beta-1}E_{\alpha-\beta, \alpha-\beta}(-t^{\alpha-\beta}) \]
\[ = \left( E_{\alpha-\beta, 0}(-t^{\alpha-\beta}) - E_{\alpha-\beta, 0}(-t^{\alpha-\beta}) - \frac{1}{t} \frac{1}{\Gamma(\alpha - \beta)} \right). \]

Since \( \lim_{t \to 0} \frac{1}{t} \frac{1}{\Gamma(\alpha - \beta)} = \delta(t) \), we obtain that \( D^\alpha G(t) + D^\beta G(t) = \delta(t). \)

2.3. Existence-uniqueness of the exact solutions

The proof for the existence–uniqueness of the solution to the problem (1) with the exact solution (13), with stated initial data is simpler. It can be solved by the Banach fixed point contraction principle. Let us see.

**Theorem 1.** Eq. (1) has a unique solution (13) in the space \( C([0, T]) \), \( T > 0 \), when \( 0 < \beta < \alpha < 1 \), and \( f \) is of the Lipschitz’s class.

**Proof.** We give a proof by the Banach fixed point theorem. Let be defined an operator
\[ (Au)(t) = u(0) + t^{\alpha-1}E_{\alpha-\beta, \alpha}(-t^{\alpha-\beta}) * f(t, u(t)), \] (14)
and be defined in the complete metric space
\[ \chi_c(T) = \left\{ u \in C(0, T; C) : \sup_{0 \leq t \leq T} |A(u(t)) - u(0)| \leq M \right\}, \] (15)
where \( T > 0 \), and the topology of \( \chi_c(T) \) is that induced by \( C((0, T); C([0, T])) \). We shall prove that setting \( T = T_c \) sufficiently small, that the map (14) is a contraction in \( \chi_c(T) \). When we prove that, then standard uniqueness argument [2] shows that this is in fact the only possible solution in \( C((0, T); C([0, T])) \). We have that \( Au \in C((0, T); C([0, T])) \) for all \( u \in \chi_c(T) \), \( T > 0 \). Then, we shall prove that for \( T_1 > 0 \) small enough, \( A(\chi_c(T_1)) \subset \chi_c(T_1) \).

We have
\[ \| (Au)(t) - u(0) \| \leq t^{\alpha-1} \| E_{\alpha-\beta, \alpha}(-t^{\alpha-\beta}) \| |f(t, u(t))|, \] (16)
By adding \( \pm u(0) \) into (16) and by (8) we obtain
\[ \| (Au)(t) - u(0) \| \leq t^{\alpha-1} \| E_{\alpha-\beta, \alpha}(-t^{\alpha-\beta}) \| \left( f(t, u(t)) - u(0) + u(0) \right) \]
\[ \leq M \| t^{\alpha} \| \| E_{\alpha-\beta, \alpha-1}(-t^{\alpha-\beta}) \| + \| u(0) \| \| t^{\alpha} \| \| E_{\alpha-\beta, \alpha-1}(-t^{\alpha-\beta}) \|. \]

since (15) holds. In order to prove that \( t^{\alpha}E_{\alpha-\beta, \alpha-1}(-t^{\alpha-\beta}) \) is bounded for any \( t < T_0, T_0 > 0 \), we shall use the formula for the asymptotic expansion for negative argument given in Theorem 1.6 from [34]; here it is Lemma 1. Applying Lemma 1 to the above Mittag-Leffler function, we obtain
\[ \frac{t^{\alpha}}{1 + t^{\alpha-\beta}} \sim t^{\alpha} \rightarrow 0, \text{ when } t \rightarrow 0. \]
In the opposite case
\[ \frac{t^{\alpha}}{1 + t^{\alpha-\beta}} \sim \frac{T_0^\beta}{M}, \text{ when } t < T_0, \text{ since } 0 < \beta < 1. \]
So, we can choose \( T_1 > 0 \) such that the right-hand side of (16) is less than or equal to \( M \).

Thus, \( \| (Au)(t) - u(0) \| \leq M \), and for \( T_1 > 0 \) small enough, \( A(\chi_c(T_1)) \subset \chi_c(T_1) \).

Finally, we show that there exists \( T_2 < T_1 \) such that \( A \) is a contraction in \( \chi_c(T_2) \). For \( 0 \leq t < T_1 \) one has
\[ \| Au(t) - Av(t) \| \leq \| t^{\alpha-1}E_{\alpha-\beta, \alpha}(-t^{\alpha-\beta}) \| \left( f(t, u(t)) - f(t, v(t)) \right) \]
\[ \leq C |u(t) - v(t)|, \]
where \( L \) is the Lipschitz constant and \( C < T_2^{\max(\alpha, \beta)} \) in any case. Thus, there exists \( T_2 < 1 \) such that operator \( A \) has a unique fixed point in \( \chi_c(T_2) \) which satisfies Eq. (1), i.e. which is the solution to Eq. (1). \( \Box \)
2.4. Existence–uniqueness in the integral form

According to [34], the exact solution to problem (1) when $0 < \beta < \alpha < 1$, is given with a help of the Laplace transform. We have

$$u(t) = CG(t) + \int_0^t G(t - \tau)f(\tau, u(\tau))d\tau,$$

where $C = (D^{\beta-1}u(t) + D^{\alpha-1}u(t))|_{t=0}$, $G(t) = t^{\alpha-1}E_{\alpha-\beta, \alpha}(-t^{\alpha-\beta})$, where $E_{\alpha, \beta}(z)$ is the two-parameter function of the Mittag-Leffler type.

We use the following estimation for integral of the Mittag-Leffler function.

**Lemma 9.** The following inequality holds:

$$\int_0^t (t - \tau)^{\alpha-1}E_{\alpha-\beta, \alpha}(- (t - \tau)^{\alpha-\beta})d\tau \leq C \begin{cases} t^\alpha & \text{when } |t| \leq 1, \\ t^\beta & \text{when } |t| \geq 1. \end{cases}$$

**Proof.** Applying formula (7)

$$\int_0^t (t - \tau)^{\alpha-1}E_{\alpha-\beta, \alpha}(- (t - \tau)^{\alpha-\beta})d\tau = t^{\alpha}E_{\alpha-\beta, \alpha+1}(-t^{\alpha-\beta})$$

and by the asymptotic formula from Lemma 1 we obtain

$$\int_0^t (t - \tau)^{\alpha-1}E_{\alpha-\beta, \alpha}(- (t - \tau)^{\alpha-\beta})d\tau \leq C \begin{cases} t^\alpha & \text{when } |t| \leq 1, \\ t^\beta & \text{when } |t| \geq 1. \end{cases}. \quad \square$$

We prove the following theorem.

**Theorem 2.** Let the conditions for nonlinear term of Assumption 1 are fulfilled. Then, there exists a continuous solution $u(t) \in C([0, \bar{\mu}))$ to Eq. (1) for suitable $\bar{\mu} < T$.

**Proof.** Following the notation from [6] we have: $I_{\mu} = [0, \mu], \tilde{\eta} > 0$ for fixed $\eta > 0$, $\| f \| = \max |f(t, u)|, t \in I, |u| \leq \eta$, where

$$0 \leq \bar{\mu} \leq T, \quad 0 \leq \tilde{\eta} \leq \eta, \quad \| f \| = \begin{cases} t^\alpha & \text{when } |t| \leq 1, \\ t^\beta & \text{when } |t| \geq 1 \leq C\bar{\eta} \leq \tilde{\eta}. \end{cases}$$

We denote by $A = A(\bar{\mu}, \tilde{\eta})$ the closed bounded and convex set of functions $\phi$ in $C(I_{\mu}, \mathbb{R})$ where $\phi(0) = 0$ and $|\phi(t)| \leq \tilde{\eta}$, $\forall t \in I_{\mu}$.

Define the operator $G$, such that $\forall \phi \in A$ with

$$G\phi(t) = CG(t) + \int_0^t G(t - \tau)f(\tau, \phi(\tau))d\tau,$$

where $C = (D^{\beta-1}f(\phi(t)) + D^{\alpha-1}f(\phi(t)))|_{t=0}$, $G(t) = t^{\alpha-1}E_{\alpha-\beta, \alpha}(-t^{\alpha-\beta})$. We apply Leray–Schauder fixed point theorem to prove the existence of a fixed point $G$ in $A$ what will be the solution to the equation in $A$. Such as in [6] for any $\phi$ in $A$, the operator $G$ is well defined since the norm $\| f \|$ of $f(\tau, \phi(\tau))$ is bounded for $\phi$ in $A$, $G\phi(0) = \phi(0) = 0$, and $G\phi(t)$ is continuous for $t \in I, \bar{\mu} \in I$. We have

$$|G\phi(t)| \leq |G(t - \tau)||f(\tau, \phi(\tau))|d\tau \leq \int_0^t |(t - \tau)^{\alpha-1}E_{\alpha-\beta, \alpha}(-(t - \tau)^{\alpha-\beta})||f(\tau, \phi(\tau))||d\tau. $$

Integrate (w.r.) to $\tau$, by (7), we obtain

$$|G\phi(t)| \leq \| f \| \int_0^t \frac{-t^\alpha}{(1 + | -t^{\alpha-\beta}|)} dt \leq \| f \| \begin{cases} t^\alpha & \text{when } |t| \leq 1, \\ t^\beta & \text{when } |t| \geq 1 \leq C\tilde{\eta}. \end{cases}$$

since $t \in I_{\tilde{\eta}}$. We used here asymptotical expansion of the Mittag-Leffler function from Lemma 1. Follows, $G : A \rightarrow A$. 


We shall prove that operator $G$ is continuous on $A$. Suppose that $\phi_n$ for $n = 0, 1, 2, \ldots$ is convergent sequence with $\lim_{n \to \infty} \phi_n \to \phi \in A$, and $|\phi_n - \phi| \to 0$, as $n \to \infty$.

Choosing $t \in I$, $\mu \in I$, we obtain

$$
|G\phi_n - G\phi| = \int_0^t \left| G(t - \tau)f(\tau, \phi_n(\tau)) - G(t - \tau)f(\tau, \phi(\tau)) \right| d\tau
$$

By Lemma 9

$$
\leq C \int_0^t \left| (t - \tau)^{\alpha - 1} E_{\alpha - \beta, \alpha} \left( - (t - \tau)^{\alpha - \beta} \right) \right| d\tau \leq C \left\{ \begin{array}{ll}
\frac{t^\alpha}{\Gamma(\alpha)} & \text{when } |t| \leq 1 \\
\frac{t^\beta}{\Gamma(\beta)} & \text{when } |t| \geq 1
\end{array} \right.
$$

since $s(t, r)$ is continuous on $[0, \infty)$ and $s(t, 0) = 0$ it follows $s(t, r) \to 0$, as $r \to 0$. In our case $r = |\phi_n(t) - \phi(t)| \to 0$, as $n \to \infty$, and $|f(\lambda)(t)| < M$. Thus, the mapping $G : A \to A$ is continuous.

We shall prove that $G\phi$ is an equicontinuous on a set of $\mathbb{C}([\beta, R])$ and that it is uniformly bounded. Let $\phi \in A$ and let $t, \tau \in I$, and suppose $t \leq \tau, \mu \in I$. Then, we have

$$
|G\phi - G\phi| = \int_0^t \left| G(t - \tau)f(\tau, \phi(\tau)) - G(t - \tau)f(\tau, \phi(\tau)) \right| d\tau
$$

Using formula for integration of the Mittag-Leffler function, term by term given with (7) and Lemma 1 for asymptotical expansion of the Mittag-Leffler function, we obtain

$$
\leq \|f\| \left[ \frac{\tau^\alpha}{1 + |\tau - \tau^\alpha|} - \frac{\tau^\alpha}{1 + |\tau - \tau^\alpha|} + \frac{(\tau - t)^\alpha}{1 + |\tau - \tau^\alpha|} \right]
$$

After canceling the hardest parts, we obtain

$$
\leq \|f\| \left[ \frac{\tau^\alpha}{1 + |\tau - \tau^\alpha|} - \frac{\tau^\alpha}{1 + |\tau - \tau^\alpha|} \right] \leq \left[ \frac{\tau^\alpha - \tau^\alpha}{1 + |\tau - \tau^\alpha|} \right].
$$

Using formula $\tau^\alpha - t^\alpha = (\tau - t)/(\tau^\alpha + 1 + \tau^\alpha - 1)$, we obtain as common factor $(\tau - t)$. Small changes of functions $\phi$ cause small changes of $G\phi$. Thus, the operator $G\phi$ is equicontinuous. In Lipschitz’s case we use the first approximation (9) which simplifies the proof. Continuing (17) we obtain

$$
\leq C \int_0^t \left[ \left| \left( t - \tau \right)^{\alpha - 1} E_{\alpha - \beta, \alpha} \left( - (t - \tau)^{\alpha - \beta} \right) \phi_n(\tau) - \phi(\tau) \right| \right] d\tau \leq C \left\{ \begin{array}{ll}
\frac{t^\alpha}{\Gamma(\alpha)} & \text{when } |t| \leq 1 \\
\frac{t^\beta}{\Gamma(\beta)} & \text{when } |t| \geq 1
\end{array} \right.
$$

Since $|\phi_n - \phi| \to 0$, the same holds for $|G\phi_n - G\phi|$. We applied Lebesgue theorem on dominant convergence. □
Remark 2. The same calculation can be done when $\alpha \in (1, 2)$ and $\beta \in (0, 1)$, and these two points can be distributed in different intervals $k < \alpha < k + 1$, $k = 1, 2, 3, \ldots$, and $k - 1 < \beta < k$, $k = 1, 2, 3, \ldots$.

We can formulate the following theorem.

**Theorem 3.** Let the conditions for nonlinear term of Assumption 1 are fulfilled. Then, there exists a continuous solution $u(t) \in C([0, \tilde{\mu}])$ to Eq. (1) for suitable $\tilde{\mu} < T$, when $\alpha \in (1, 2)$ and $\beta \in (0, 1)$.

The same holds for the different combinations of $\alpha, \beta$ given in Section 2.2. The proofs are similar and will be omitted.

**2.5. Uniqueness of the solution**

We prove the uniqueness of the solution given in integral form to the nonlinear two-term fractional equation (1) following precisely fixed point approach of [7] given for one-time fractional equation. Then, we give a generalization to the $n$-term time fractional equation.

Let $u(t)$ and $v(t)$ be two solutions to Eq. (1). Denote their difference with $w(t)$. Then, we solve the equation

$$D^\alpha w(t) + D^\beta w(t) = f(t, u(t)) - f(t, v(t)).$$

The exact solution is according to [34] given with

$$w(t) = CG(t) + \int_0^t G(t - \tau) \left(f(\tau, u(\tau)) - f(\tau, v(\tau))\right) d\tau,$$

where

$$C = \left(D^{-1}(\bar{w}(t)) + D^{\alpha-1}(w(t))\right)|_{z=0}, G(t) = \frac{t^{\alpha-1}E_{\alpha-\alpha}(t^{\alpha-\beta})}{\Gamma(\alpha)}.$$

(a) Suppose that Assumption 1 is fulfilled. We have

$$w(t) = \int_0^t \left[\left(t - \tau\right)^{\alpha-1}E_{\alpha-\alpha}\left(-\left(t - \tau\right)^{\alpha-\beta}\right)(f(\tau, u(\tau)) - f(\tau, v(\tau)))\right] d\tau.$$

By Assumption 1 and asymptotical expansion of the Mittag-Leffler function, we obtain

$$|w(t)| \leq C \int_0^t \left|\left(t - \tau\right)^{\alpha-1}E_{\alpha-\alpha}\left(-\left(t - \tau\right)^{\alpha-\beta}\right)s(w(\tau))\lambda(\tau)\right| d\tau.$$

From the first approximation (cf. Remark 1)

$$|w(t)| \leq C\int_0^t \left|\left(t - \tau\right)^{\alpha-1}E_{\alpha-\alpha}\left(-\left(t - \tau\right)^{\alpha-\beta}\right)w(\tau)\lambda(\tau)\right| d\tau.$$

By Gronwall inequality and Lemma 9

$$|w(t)| \leq Cw(0) \begin{cases} e^{t_\alpha \alpha} \int_0^t \lambda(\tau) d\tau, & \text{when } |t| \leq 1, \\ e^{t_\alpha \beta} \int_0^t \lambda(\tau) d\tau, & \text{when } |t| \geq 1. \end{cases}$$

Since by Assumption 1, $\int_0^t \lambda(\tau) d\tau \leq M, M > 0$, then,

$$|w(t)| \leq MCw(0) \quad \text{for any } t.$$

We know that $w(0) = 0$. It follows $|w(t)| = 0$, i.e. the solution is unique. Thus, $u(t) = v(t)$.

(b) We give a proof for Lipschitz’s case $|f(t, u(t)) - f(t, v(t))| \leq Lw(t)$ (cf. Assumption 1), where $L$ is the Lipschitz’s constant. Then,

$$|w(t)| \leq L \int_0^t \left|\left(t - \tau\right)^{\alpha-1}E_{\alpha-\alpha}\left(-\left(t - \tau\right)^{\alpha-\beta}\right)w(\tau)\right| d\tau.$$

Applying Gronwall inequality, since $w(0) = 0$, we obtain

$$\sup_t w(t) \leq Cw(0) \begin{cases} e^{t_\alpha \alpha} \int_0^t \lambda(\tau) d\tau, & \text{when } |t| \leq 1, \\ e^{t_\alpha \beta} \int_0^t \lambda(\tau) d\tau, & \text{when } |t| \geq 1. \end{cases}$$

Then, $w(t) = 0$, since $w(0) = 0$ and $I^1(\lambda)(t) \leq M$. Thus, the solution is unique.
Remark 3. If $\lambda(t) = t^{-\gamma}$, $\gamma \in (0, 1)$ in Assumption 1, applying the formula (7) we solve the integral
\[
\int_0^t (t - \tau)^{\alpha-1-\gamma} E_{\alpha-\beta, \alpha} (-(t - \tau)^{\alpha-\beta}) d\tau = t^{\alpha-\gamma} E_{\alpha-\beta, \alpha+1} (-t^{\alpha-\beta}) = \begin{cases} 
t^\beta - t \gamma, & |t| \leq 1, \\
t^{\alpha-\gamma} - t^\beta - \gamma, & |t| \geq 1.
\end{cases}
\]
Then, we repeat the proofs of existence and uniqueness of the solution with these changes.

3. Application

3.1. The three-term equation

Setting $d = 0$ in [34] in four-time fractional equation on p. 156, we obtain the three-terms time fractional equation with constant coefficients
\[aD_t^{\gamma} u(t) + bD_t^{\mu} u(t) + cD_t^{\beta} u(t) = f(t, u(t)), \quad a, b, c \text{ are constants},\]
where the derivatives are the Riemann–Liouville fractional derivatives, $1 > \gamma > \alpha > \beta > 0$. By the Laplace transform and its inversion the Green function
\[G_3(t) = \frac{1}{\alpha} \sum_{k=0}^{\infty} \left(\frac{-1}{k!}\right)^k t^{(\gamma-\beta)(k+1)-1} \frac{1}{E_{\gamma-\alpha, \gamma-\beta+\alpha-\beta}(t^{\gamma-\alpha})} \left(\frac{b}{\alpha} t^{\gamma-\alpha}\right),\]
gives the solution
\[u(t) = C_0 G_3(t) + G_3(t) * f(t, u(t)) = C_0 G_3(t) + \int_0^t G_3(t - \tau) f(\tau, u(\tau)) d\tau,
\]
where $C_0$ depends on the initial data and the order of the fractional derivatives like in previous sections. But, here we use the Laplace transform formula for the Riemann–Liouville fractional derivatives given with (5). A proof of the existence–uniqueness of the solution can be obtained following the same lines of the proof of Theorem 2 with changes caused by the difference between the two forms of the fractional derivatives.

3.2. General n-term equation

The existence–uniqueness result from Theorem 2 can be generalized to the $n$-term fractional order differential equation with constant coefficients (1) (cf. [34]). Assume that $1 > \alpha_n > \alpha_{n-1} > \cdots > \alpha_1 > \beta_1 > \beta_0 > 0$, respectively $2 > \alpha_n > \alpha_{n-1} > \cdots > \alpha_1 > \beta_1 > \beta_0 > 0$. After finding the Laplace transform we obtain (11). To prove the existence–uniqueness result to the $n$-term equation under the appropriate conditions on nonlinear term. The techniques are the same as in the previous sections and will be omitted. Moreover, it follows by Theorem 2.

Conclusion 1. The similar technique for the existence–uniqueness result can be used between two consecutive integers $k - 1 < \beta < \alpha < k, k = 1, 2, \ldots$. All other cases of ($\cdot$)-term time fractional equation can be solved using the Green function from [34] and applying Theorem 2 to the integral form of the solution. The technique is general.

References