JOURNAL OF ALGEBRA 39, 349–359 (1976)

Gorenstein Sequences and G_n Condition

MARIA GRAZIA MARINARI*

Department of Mathematics, Brandeis University, Waltham, Massachusetts 02154

Communicated by D. Buchsbaum

Received April 25, 1974; revised October 8, 1974

INTRODUCTION

Throughout this paper all rings are assumed to be commutative, noetherian, and with 1.

Among several equivalent definitions of Gorenstein rings (see [3, Theorem 6.3]), we focus our attention on the following one:

1. A ring A is Gorenstein iff every ideal generated by a regular sequence is unmixed and all of its primary components are irreducible.

This definition especially emphasizes that every Gorenstein ring is in particular a Macaulay ring whose regular sequences (in themselves closely connected to the Macaulay condition) have some special features, and suggested that we introduce "Gorenstein sequences" (**G**-sequences) of a ring A in the following way (see [6, Definition 2.1]):

2. An ordered sequence $\{x_1, \ldots, x_n\}$ of noninvertible elements of a ring A is called a Gorenstein sequence iff the following two conditions hold:

(i) $\{x_1, \ldots, x_n\}$ is a regular sequence,

(ii) for every $i \in \{1, \ldots, n\}$ the ideal $(x_1, \ldots, x_i)$ has irreducible minimal primary components.

The name Gorenstein sequence for $\{x_1, \ldots, x_n) \subset A$ satisfying (i) and (ii) above is due to the fact that for every $i \in \{1, \ldots, n\}$ and minimal $\mathfrak{p}^{(i)} \in \operatorname{Ass}(A/(x_1, \ldots, x_i))$, $A_{\mathfrak{p}^{(i)}}$ is clearly a local Gorenstein ring (in analogy to the well-known fact that $\{x_1, \ldots, x_n\} \subset A$ a regular sequence implies, for every $i \in \{1, \ldots, n\}$ and minimal $\mathfrak{p}^{(i)} \in \operatorname{Ass}(A/(x_1, \ldots, x_i))$, that $A_{\mathfrak{p}^{(i)}}$ is a local Macaulay ring).

By means of Gorenstein sequences, it was possible to give the following characterization of local Gorenstein rings (see [6, Proposition 4.2]):

* Supported by a scholarship of the Italian C.N.R.

3. A local ring $(A, \mathfrak{m})$ is Gorenstein iff $\mathfrak{m}$ contains a Gorenstein sequence of length $= \dim A$, which formally is the "translation" in terms of Gorenstein sequences of the usual characterization of local Macaulay rings in terms of regular sequences (see [1, Definition 3.3]).

In such a scheme of things, we proved for Gorenstein sequences some parallels of known properties of regular sequences (see [6, Sect. 2, 3]). Moreover, pushing further our analysis of the relationship between the Macaulay and Gorenstein conditions, we thought it natural to introduce the "$\mathbf{G}_n$ property" of a ring A (see [7, Theorem 2]) as follows:

4. A ring A is called $\mathbf{G}_n$ iff every $\mathfrak{p} \in \operatorname{Spec}(A)$ contains a Gorenstein sequence of length $\geqslant \min(n, \operatorname{ht} \mathfrak{p})$.

This is analogous to the well-known S_n property of Serre (a ring A is called S_n iff every $\mathfrak{p} \in \operatorname{Spec}(A)$ contains a regular sequence of length $\geqslant \min(n, \operatorname{ht} \mathfrak{p})$), and characterizes those rings A satisfying the following two conditions:

(i) A is an S_n ring;

(ii) for every $\mathfrak{p} \in \operatorname{Spec}(A)$ such that $\operatorname{ht} \mathfrak{p} \leqslant n$, $A_\mathfrak{p}$ is a local Gorenstein ring.

Later (in [8]), we examined several connections between $\mathbf{G}_n$ rings and other kinds of rings satisfying conditions weaker than the Gorenstein condition (precisely n-Gorenstein and n-Bass–Ishikawa rings[1] (see [2, 12] respectively)). Incidentally it was possible to prove (see [6, Corollary 2.8, 7, Corollary 3.1]:

5. In a Gorenstein (respectively, in a $\mathbf{G}_n$) ring A, every regular sequence (respectively, every regular sequence of length $\leqslant n$) is a Gorenstein sequence.

Here, we can observe that 1 gives two equivalent conditions, but for 5, converses are not true; actually it is possible to produce (see Sect. 3) examples of (local) rings A, all of whose regular sequences (respectively, all of whose regular sequences of length $\leqslant n$) are Gorenstein sequences, but that are not Gorenstein (respectively, not $\mathbf{G}_n$) rings.

In fact, it is possible to prove the converses of 5 only under some particular hypotheses. What one can show (see [3, Theorem 6.3, Condition 3, 8. Theorem 2.1, Condition 10]) is that:

6. A Macaulay (respectively, an S_n) ring A in which every regular sequence (respectively, every regular sequence of length $\leqslant n$) is a Gorenstein sequence, is in effect a Gorenstein (respectively, a $\mathbf{G}_n$) ring.

[1] Recall that an S_n ring A is said to be: (a) n-Gorenstein iff $A_\mathfrak{p}$ is a local Gorenstein ring for every $\mathfrak{p} \in \operatorname{Spec}(A)$ such that $\operatorname{ht} \mathfrak{p} \leqslant n - 1$ (see [2, Sect. 1]). (b) n-Bass-Ishikawa iff $A_\mathfrak{p}$ is a local Gorenstein ring for every $\mathfrak{p} \in \operatorname{Spec}(A)$ such that $\operatorname{ht} \mathfrak{p} = n - 1$ (see [12, Sect. 3]).

This last result convinces us that Gorenstein sequences have some particular meaning in Macaulay and S_n rings: this is one reason why in Section 1 we examine several features of Macaulay and S_n rings from which follow some more information about the connections between n-Gorenstein and n-Bass-Ishikawa rings as well as some interesting properties of Gorenstein sequences.

In Section 2, we study the behavior of the $\mathbf{G}_n$ property under some change of rings (following up [7, Sect. 5]), proving, incidentally, in which sense the $\mathbf{G}_n$ property is preserved by $\otimes$.

Finally, in Section 3, we illustrate the above results by means of some examples.

1

In [8, Sect. 1], we examined several connections among n-Bass–Ishikawa, n-Gorenstein and $\mathbf{G}_n$ rings, which are in the following "hierarchy":

$$(n+1)\text{-Gorenstein} \Rightarrow \mathbf{G}_n \Rightarrow n\text{-Gorenstein} \Rightarrow n\text{-Bass–Ishikawa}$$

and showed, incidentally, that none of the implications can be reversed.

Now, we are going to prove that there is a complete equivalence between n-Gorenstein and n-Bass–Ishikawa conditions in rings satisfying suitable hypotheses on maximal ideals.

Recall that a local Macaulay ring satisfies the "maximal chain condition on prime ideals," precisely:

PROPOSITION 1.1. *Every local Macaulay ring* $(A, \mathfrak{m})$ *is biequidimensional* (cf. [4, Part I, Corollary 14.3.5]) *that is, for every* $\mathfrak{p} \in \operatorname{Spec}(A)$ *we have*:

$$\dim A = \dim A_{\mathfrak{p}} + \dim(A/\mathfrak{p}) \text{ (cf. [4, Part I, Corollary 16.5.11).}$$

We shall prove that it is possible to state a similar result about chains of prime ideals in an S_n local ring. For this, we need some preliminary remarks.

LEMMA 1.2. *Let* $(A, \mathfrak{m})$ *be an* S_n *local ring* (*n some nonnegative integer*). *Then, for every* $\mathfrak{p} \in \operatorname{Spec}(A)$ *with* $\operatorname{ht} \mathfrak{p} = \alpha \leqslant n-2$, *there exists a prime ideal* $\mathfrak{q} \supset \mathfrak{p}$ *such that* $\operatorname{ht} \mathfrak{q} = \alpha + 1$.

Proof. Let $\mathfrak{p} \in \operatorname{Spec}(A)$ be such that $\operatorname{ht} \mathfrak{p} = \alpha \leqslant n-2$. Then $\mathfrak{p}$ contains a regular sequence $\mathbf{x} = \{x_1, \dots, x_\alpha\}$ such that length $\mathbf{x} = \alpha = \operatorname{ht} \mathfrak{p}$ and moreover, there exist $\xi_1, \dots, \xi_{n-\alpha} \in \mathfrak{m}$ such that $\{\mathbf{x}, \xi_1, \dots, \xi_{n-\alpha}\}$ is a regular sequence of length $= n$ (A S_n implies depth $A \geqslant n$). In particular, for each $i \in \{1, \dots, n-\alpha\}$, $\{\mathbf{x}, \xi_i\}$ is a regular sequence of length $= \alpha + 1$. Consider the ideal $\mathfrak{a}_i = (\mathfrak{p}, \xi_i)$. Then, $\operatorname{grade}(A/\mathfrak{a}_i) = \alpha + 1$, namely, $\{\mathbf{x}, \xi_i\}$ is a regular sequence of length $= \alpha + 1$ contained in $\mathfrak{a}_i$ and it is maximal since for every element $a \in \mathfrak{a}_i$, $a = p + y\xi_i$, where $y \in A$ and $p \in \mathfrak{p}$ so $(\mathbf{x}, \xi_i)$: $a \neq (\mathbf{x}, \xi_i)$. Let, then, $\mathfrak{q} \in \operatorname{Ass}(A/\mathfrak{a}_i)$ be such that grade $(A/\mathfrak{q}) =$ grade $(A/\mathfrak{a}_i) = \alpha + 1$. Then, grade $(A/\mathfrak{q}) \leqslant n - 1$ $(\alpha \leqslant n - 2)$ and so, since A is an S_n ring, grade $(A/\mathfrak{q}) = \operatorname{ht} \mathfrak{q}$. Thus, $\operatorname{ht} \mathfrak{q} = \alpha + 1$ and $\mathfrak{q} \supset \mathfrak{p}$.

The result of Lemma 1.2 can be extended to "some" kinds of nonlocal S_n rings.

PROPOSITION 1.3. *Let A be an S_n ring. Suppose, further, that for every maximal ideal $\mathfrak{m} \subset A$, $\operatorname{ht} \mathfrak{m} \geqslant n$. Then, for every $\mathfrak{p} \in \operatorname{Spec}(A)$ such that $\operatorname{ht} \mathfrak{p} = \alpha \leqslant n - 2$, there exists a $\mathfrak{q} \in \operatorname{Spec}(A)$ such that $\mathfrak{q} \supset \mathfrak{p}$ and $\operatorname{ht} \mathfrak{q} = \alpha + 1$.*

Proof. Our hypothesis on maximal ideals implies that every $\mathfrak{p} \in \operatorname{Spec}(A)$ with $\operatorname{ht} \mathfrak{p} = \alpha < n - 1$ is contained in (at least) one maximal ideal $\tilde{\mathfrak{m}} \subset A$ ($\tilde{\mathfrak{m}}$ depending on $\mathfrak{p}$). Thus, localizing at $\tilde{\mathfrak{m}}$, we reproduce the same situation as in Lemma 1.2 and then can conclude likewise.

From Proposition 1.3, we get immediately the following:

COROLLARY 1.4. *Let A be an S_n equicodimensional ring (that is all maximal ideals $\mathfrak{m} \subset A$ have the same height* (cf. [4, Part I, Definition 14.2.1]). *Then, every $\mathfrak{p} \in \operatorname{Spec}(A)$ such that $\operatorname{ht} \mathfrak{p} = \alpha \leqslant n - 2$ is contained in some $\mathfrak{q} \in \operatorname{Spec}(A)$ such that $\operatorname{ht} \mathfrak{q} = \alpha + 1$.*

PROPOSITION 1.5. *Let A be an S_n ring all of whose maximal ideals have height $\geqslant n$. Then, for every $\mathfrak{p} \in \operatorname{Spec}(A)$: $n \leqslant \dim A_{\mathfrak{p}} + \dim(A/\mathfrak{p}) \leqslant \dim A$.*

Using previous results, we now can state the following:

PROPOSITION 1.6. *Every local n-Bass–Ishikawa ring $(A, \mathfrak{m})$ is an n-Gorenstein ring.*

(This fact also follows indirectly and independently from [2, Proposition 4.3].

Proof. A is by definition an S_n ring (see footnote 1), so it follows from Lemma 1.2 that for every $\mathfrak{p} \in \operatorname{Spec}(A)$ such that $\operatorname{ht} \mathfrak{p} < n - 1$, there exists a $\tilde{\mathfrak{q}} \in \operatorname{Spec}(A)$ such that $\operatorname{ht} \tilde{\mathfrak{q}} = n - 1$ and $\tilde{\mathfrak{q}} \supset \mathfrak{p}$. This in particular means that for every $\mathfrak{p} \in \operatorname{Spec}(A)$ such that $\operatorname{ht} \mathfrak{p} < n - 1$, $A_{\mathfrak{p}}$ is Gorenstein (namely, $A_{\mathfrak{p}} = (A_{\tilde{\mathfrak{q}}})_{\mathfrak{p}} A_{\tilde{\mathfrak{q}}}$ and $A_{\tilde{\mathfrak{q}}}$ is Gorenstein by hypothesis), i.e., A is n-Gorenstein.

The previous result can be extended to the nonlocal case in the following way:

COROLLARY 1.7. *Every n-Bass–Ishikawa ring, all of whose maximal ideals have height $\geqslant n$ (in particular every equicodimensional n-Bass–Ishikawa ring) is n-Gorenstein.*

We now state some further properties of S_n rings that, together with previous results about saturated chains of prime ideals, will allow us to prove some features of **G**-sequences of an S_n local ring like "truncation" and "unconditionality."

PROPOSITION 1.8. *Let* $(A, \mathfrak{m})$ *be a local ring and* $x \in \mathfrak{m}$ *a regular element. If* $A' = A/(x)$ *is* S_n, *then so is* A.

Proof. In [2, Proposition 3], it was proved that if A' is S_n, then every $\mathfrak{q} \in \operatorname{Spec}(A)$ such that $\operatorname{ht} \mathfrak{q} \geqslant n$ contains a regular sequence of length $= n$, so, to prove our contention, it is enough to prove that $A_\mathfrak{p}$ is a local Macaulay ring for every $\mathfrak{q} \in \operatorname{Spec}(A)$ such that $\operatorname{ht} \mathfrak{q} < n$. Here, the proof runs exactly as in [2] loco citato replacing the word Gorenstein with the word Macaulay.

From Proposition 1.8, it is possible to deduce the following:

COROLLARY 1.9. *Let* A *be* S_n. *Then, so is* $A[[X]]$ (*and conversely*).

Proof. The proof can easily be deduced from that of [2, Corollary to Proposition 3].

COROLLARY 1.10. *Let* $(A, \mathfrak{m})$ *be a local ring and let* $\mathbf{x} = \{x_1, \ldots, x_n\} \subset \mathfrak{m}$ *be a regular sequence. If* $A_n = A/(x_1, \ldots, x_n)$ *is an* S_1 *ring, then so is* $A_i = A/(x_1, \ldots, x_i)$ *for every* $i \in \{1, \ldots, n-1\}$.

Remark. The meaning of Corollary 1.10 is essentially this: If $(A, \mathfrak{m})$ is any local ring and $\mathbf{x} = \{x_1, \ldots, x_n\} \subset \mathfrak{m}$ is any regular sequence that generates an unmixed ideal, then all the ideals $(\mathbf{x}_i)$ generated by regular sequences gotten by cutting the given one, have to be unmixed.

PROPOSITION 1.11. *Let* $(A, \mathfrak{m})$ *be a local ring such that* depth $A \geqslant n - 1$. *If, for every regular sequence* $\mathbf{x} \subset \mathfrak{m}$ *such that* length $\mathbf{x} = n - 1$, $A/(\mathbf{x})$ *is an* S_1 *ring, then* A *is* S_n.

Proof. The Proposition will be proven by showing that every regular sequence $\mathbf{y} \subset \mathfrak{m}$ such that length $\mathbf{y} < n$ generates an unmixed ideal $(\mathbf{y})$ (this, by [12, Theorem 2.2] is equivalent to showing that A is S_n). Let $\mathbf{y} \subset \mathfrak{m}$ be a regular sequence of length $= \alpha < n - 1$ (nothing to prove for regular sequences $\mathbf{y}$ such that length $\mathbf{y} = n - 1$). Since $\alpha < n - 1$, there exist $\eta_{\alpha+1}, \ldots, \eta_{n-1} \in \mathfrak{m}$ such that $\bar{\mathbf{y}} = \{\mathbf{y}, \eta_{\alpha+1}, \ldots, \eta_{n-1}\}$ is a regular sequence of length $= n - 1$. The ideal $(\bar{\mathbf{y}})$ is unmixed by hypothesis, so, in particular, $(\mathbf{y})$ is unmixed (cf. Corollary 1.10).

From Proposition 1.11 we get:

PROPOSITION 1.12. *Let* A *be a ring such that* depth $A_\mathfrak{m} \geqslant n - 1$ *for each* *maximal ideal* $\mathfrak{m} \subset A$. *If every regular sequence* $\mathbf{x} \subset A$ *such that length* $\mathbf{x} = n - 1$ *generates a* **G**-*ideal*[2] (*or equivalently if every weak* **G**-*sequence*[3] $\mathbf{x} \subset A$ *of length* $= n - 1$ *generates an unmixed ideal*). *Then* A *is n-Bass–Ishikawa as well as n-Gorenstein.*

Proof. Under given hypothesis, for every maximal ideal $\mathfrak{m} \subset A$, $A_{\mathfrak{m}}$ is an S_n local ring in which every ideal $(\tilde{\mathbf{x}})$ generated by a regular sequence $\tilde{\mathbf{x}}$ of length $= n - 1$ is a **G**-ideal, that is, it is n-Bass–Ishikawa (cf. [12, Theorem 3.2]) as it is even n-Gorenstein (cf. Proposition 1.6). Since n-Bass–Ishikawa and n-Gorenstein are both local properties, our proof is then complete.

We now can show a proposition that is the analog of Corollary 1.10.

PROPOSITION 1.13. *Let* $(A, \mathfrak{m})$ *be a local ring and let* $\mathbf{x} \subset \mathfrak{m}$ *be any regular sequence that generates a* **G**-*ideal* (*or, equivalently, any weak*-**G**-*sequence that generates an unmixed ideal*). *Then,* $\mathbf{x}$ *is a* **G**-*sequence.*

Proof. It suffices to observe that if $\mathbf{x} \subset A$ is a regular sequence generating a **G**-ideal $(\mathbf{x})$, then $A/(\mathbf{x})$ and $A/(\mathbf{y})$ ($\mathbf{y}$ any regular sequence truncated from $\mathbf{x}$) are 1-Gorenstein rings (cf. footnote 1) and this in particular means that $\mathbf{x}$ is a **G**-sequence.

Finally, we prove two results about unconditionality of (weak)-**G**-sequences in local rings.

PROPOSITION 1.14. *Let* $(A, \mathfrak{m})$ *be a local ring and let* $\mathbf{x} = \{x_1, ..., x_n\} \subset \mathfrak{m}$ *be any regular sequence that generates a* **G**-*ideal* (*or, equivalently, any weak*-**G**-*sequence that generates an unmixed ideal*). *Then, for every permutation* $i \mapsto j(i)$ *of the index set* $\{1, ..., n\}$, $\mathbf{x}^* = \{x_{j(1)}, ..., x_{j(n)}\}$ *is a* **G**-*sequence.*

Proof. The assertion follows immediately from Proposition 1.13 since one easily can remark that in a given situation, if $A/(\mathbf{x})$ is a 1-Gorenstein ring, then so is $A/(\mathbf{x}^*)$.

COROLLARY 1.15. *In an* S_n *local ring* $(A, \mathfrak{m})$, *every* (*weak*)-**G**-*sequence of length* $= \alpha < n$ *is unconditioned.*

Proof. It is enough to observe that in an S_n local ring every regular sequence of length $= \alpha < n$ generates an unmixed ideal (cf. [12, Theorem 2.2]) and then apply the Proposition 1.14.

[2] Recall that an ideal $\mathfrak{a} \subseteq A$ is called a **G**-ideal iff it is unmixed and all of its primary components are irreducible (cf. [3, Definition 5]).

[3] Recall that a regular sequence $\mathbf{x} = \{x_1, ..., x_n\} \subset A$ is called a weak-**G**-sequence iff the ideal $(x_1, ..., x_n)$ has irreducible minimal primary components (cf. [6, Definition 2.1(ii)]).

2

In [6, Theorem 5] we proved the $\mathbf{G}_n$ property ascends (respectively, descends) by flat ring homomorphisms $\varphi: A \to B$ provided all fibres of φ are $\mathbf{G}_n$ (respectively, φ is faithfully flat). We are now going to deduce just from this result the ascent of $\mathbf{G}_n$ property from a ring A to the group ring $A[F]$ (where F is an abelian finite group). Moreover, avoiding the above condition on fibres, we are going to give a direct proof for the ascent of $\mathbf{G}_n$ property from a ring A to the formal power series ring $A[[X]]$, provided $A[[X]]$ satisfies suitable hypotheses of biequidimensionality.

PROPOSITION 2.1. *Let F be a finite abelian group. If A is a $\mathbf{G}_n$ ring, then so is the group ring $A[F]$.*

Proof. $A[F]$ is a flat A-algebra (cf. [[11, Theorem 17]) and also all fibres of $A \to A[F]$ are Gorenstein rings since for every $\mathfrak{p} \in \operatorname{Spec}(A)$ we have $k(\mathfrak{p}) \otimes_A A[F] = \bigoplus_{|F|} (k(\mathfrak{p}) \otimes_A A) = \bigoplus_{|F|} k(\mathfrak{p})$. Then, our contention follows from [6, Corollary 5.2].

To prove the ascent of $\mathbf{G}_n$ property from A to $A[[X]]$, we will use the following fact proven in [8, Sect. 3(v)].

LEMMA 2.2. *Let $(A, \mathfrak{m})$ be a biequidimensional local ring and $x \in \mathfrak{m}$ a regular element. If $A' = A/(x)$ is a $\mathbf{G}_n$ ring, then so is A.*

PROPOSITION 2.3. *Let A be any ring and let $A[[X]]$ be its formal power series ring. Then:*

(i) *$A[[X]]$ $\mathbf{G}_n$ implies $A\mathbf{G}_n$.*

Moreover, if for every maximal ideal $\mathfrak{M} \subset A[[X]]$ the local ring $A[[X]]_{\mathfrak{M}}$ is biequidimensional, it is still true that:

(ii) *A $\mathbf{G}_n$ implies $A[[X]]$ $\mathbf{G}_n$.*

Proof. (i) follows immediately from [6, Theorem 5.1] since $A[[X]]$ is a faithfully flat A-algebra.

(ii) We will prove that for every maximal ideal $\mathfrak{M} \subset A[[X]]$, $A[[X]]_{\mathfrak{M}}$ is a $\mathbf{G}_n$ ring (then, so is $A[[X]]$ since to be a $\mathbf{G}_n$ ring is a local property). By the way, X is in the Jacobson radical of $A[[X]]$ so, putting $\mathfrak{m} = \mathfrak{M} \cap A$, we get $A[[X]]_{\mathfrak{M}}/XA[[X]]_{\mathfrak{M}} \simeq A_{\mathfrak{m}}$. Thus, our thesis becomes an immediate consequence of Lemma 2.2.

Remark. We derived the above proof from [2, Corollary to Proposition 3], where the ascent of n-Gorenstein property from A to $A[[X]]$ has been shown with no hypotheses on $A[[X]]$. It might be that it is possible to do the same for $\mathbf{G}_n$ property, or picking out another approach to the question, or proving that Lemma 2.2 holds also for local rings that are not biequidimensional (which we were not able to do).

Lemma 2.2 states a lifting of $\mathbf{G}_n$ property from $A' = A/(x)$ to A, in case A is a local biequidimensional ring and $x \in \mathfrak{m}$ is a regular element. We will now see how $\mathbf{G}_n$ property "transforms" passing from a ring A to some of its residue rings.

PROPOSITION 2.4. *Let A be a ring and $x \in A$ be a regular element. If A is $\mathbf{G}_n$, n a nonzero positive integer, then $A' = A/(x)$ is $\mathbf{G}_{n-1}$.*

Proof. It is known from [1, Proposition 3.1 and Theorem 3.1] that with our given hypothesis, A' is an S_{n-1} ring, so, for showing our contention, it is enough to prove that for every $\mathfrak{p}' \in \operatorname{Spec}(A')$ such that $\operatorname{ht} \mathfrak{p}' \leqslant n - 1$, $A'_{\mathfrak{p}'}$ is Gorenstein. In this connection observe that for every $\mathfrak{p}' \in \operatorname{Spec}(A')$, there exists $\mathfrak{p} \in \operatorname{Spec}(A)$ such that $A'_{\mathfrak{p}'} = A_\mathfrak{p}/xA_\mathfrak{p}$ and $\operatorname{ht} \mathfrak{p} = \operatorname{ht} \mathfrak{p}' + 1$ so, if $\operatorname{ht} \mathfrak{p}' \leqslant n - 1$, $A'_{\mathfrak{p}'}$ is Gorenstein since $A_\mathfrak{p}$ is so by hypothesis (cf. [3, Corollary 6.6]).

COROLLARY 2.5. *Let A be a ring, let n, h be two nonnegative integers such that $h \leqslant n$, and let $\{x_1, \ldots, x_h\} \subset A$ be a regular sequence. If A is G_n, then $A' = A/(x_1, \ldots, x_h)$ is $\mathbf{G}_{n-h}$.*

PROPOSITION 2.6. *Let A be a ring, let n, h be any two nonnegative integers, and let $\{x_1, \ldots, x_h\} \subset A$ be a weak-$\mathbf{G}$-sequence. If A is $\mathbf{G}_n$, then $A' = A/(x_1, \ldots, x_h)$ is $\mathbf{G}_{\max(0, n-h)}$.*

Proof. If $\max(0, n - h) = n - h$, the assertion becomes a particular case of Corollary 2.5 since, clearly, $\{x_1, \ldots, x_h\}$ a weak-$\mathbf{G}$-sequence means in particular $\{x_1, \ldots, x_h\}$ a regular sequence. On the other hand, if $\max(0, n - h) = 0$, then the assertion follows immediately from the definition of weak-$\mathbf{G}$-sequence.

To conclude, we are going to examine in which sense the $\mathbf{G}_n$ property is preserved by $\otimes$. To do this, we need a few definitions and facts about some (we always assume locally noetherian) schemes and relative morphisms.

DEFINITION 2.7. A scheme $X = (X, \mathcal{O}_X)$ is said to be $\mathbf{G}_n$, if for every $x \in X$, the local ring O_x is $\mathbf{G}_n$.

DEFINITION 2.8. A morphism of schemes $f: X \to Y$ is said to be $\mathbf{G}_n$ if it is flat and each fibre $f^{-1}(y) = X \times_Y (\operatorname{Spec}(k(y))$ *is* $\mathbf{G}_n$.

LEMMA 2.9 Basic field finite type extension). *Let k be a field and let A be a k-algebra. If k' is a finitely generated field extension of k and A is a $\mathbf{G}_n$ ring, then $A' = A \otimes_k k'$ is $\mathbf{G}_n$ (and conversely).*

Proof. We will do it essentially as in [10, Proposition 4]. Observe that the ring homomorphism $\varphi: A \to A'$ is clearly flat and since for every $\mathfrak{p} \in \operatorname{Spec}(A)$ $k(\mathfrak{p}) \otimes_A A' = k(\mathfrak{p}) \otimes_A (A \otimes_k k') = k(\mathfrak{p}) \otimes_k k'$, it has Gorenstein fibres (cf. [13, Corollary 2]. Then, A' is $\mathbf{G}_n$ by [6, Corollary 5.2] (the converse again follows from [6, Theorem 5.1] since $\varphi: A \to A'$ is a faithfully flat ring homomorphism).

COROLLARY 2.10. *Let $f: X \to Y$ and $g: Y' \to Y$ be morphisms of schemes, respectively, $\mathbf{G}_n$ and of finite type. Then, the morphism $f' = f_{(Y')}: X \times_Y Y' \to Y'$ is $\mathbf{G}_n$.*

Proof. It follows from Lemma 2.9 and [4, Part 2, Lemma 7.3.7].

We now can state the following:

PROPOSITION 2.11. *Let B, C be A-algebras such that $\varphi: A \to B$ is flat and C is finitely generated over A. Then, if B, C, and all fibres of φ are $\mathbf{G}_n$, so is $B \otimes_A C$.*

Proof. Our hypotheses mean that the morphism $f: \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is $\mathbf{G}_n$ and thus, (by Corollary 2.10) so is the morphism $f': \operatorname{Spec}(B \otimes_A C) \to \operatorname{Spec}(C)$. The assertion follows, therefore, from [6, Theorem 5.1] since we have a flat ring homomorphism $\varphi': C \to B \otimes_A C$ such that C and all its fibres are $\mathbf{G}_n$, which actually implies that $B \otimes_A C$ is $\mathbf{G}_n$.

3

In this section, we collect some examples and counterexamples about the results of the previous sections.

First, we want to explain how it is possible to produce examples of (local) rings A all of whose regular sequences (respectively, regular sequences of length $\leqslant n$) are $\mathbf{G}$-sequences and that are not Gorenstein (respectively, not $\mathbf{G}_n$) rings.

From 5 of the Introduction, it results that to do this we need (local) rings that are not Macaulay (respectively, not S_n). Actually, it is enough to choose any local $\mathbf{G}_i$ ring $(A, \mathfrak{m})$ such that $\dim A > \operatorname{depth} A = i$ (for both cases). By means of the following two examples we will better explain how such a kind of ring satisfies the required conditions.

(a) Let $B = k[X, Y, Z]/(X(X, Y, Z)) = k[x, y, z]$ $(x^2 = xy = xz = 0)$, $k = \bar{k}$. Then, $A = B_{(x,y,z)}$ is a $\mathbf{G}_0$ local ring such that depth $A = 0$ (cf. [6, Example 4]), that is, every regular sequence of A is a $\mathbf{G}$-sequence (really the only regular sequence of A is the empty set, and the zero ideal that it generates has its minimal primary component irreducible), i.e., in particular, every regular sequence of length $\leqslant 1$ is a $\mathbf{G}$-sequence. Nevertheless, A is not Gorenstein (respectively, not $\mathbf{G}_1$) since clearly, it is not S_1.

(b) Let $B = k[X, Y, Z, T]/(X, Y)(Z, T) = k[x, y, z, t]$ $(xt = xz = yz = yt = 0)$, $k = \bar{k}$. Then, $A = B_{(x,y,z,t)}$ is a $\mathbf{G}_1$ ring of dimension $= 2$ and depth $= 1$, that is, again, every regular sequence (in particular, every regular sequence of length $\leqslant 2$) of A is a $\mathbf{G}$-sequence, but A is not Gorenstein (respectively, not $\mathbf{G}_2$) since it is not S_2 (cf. [5, Example 3.4.1]).

Recall now that in Lemma 1.2 and Proposition 1.5, two characterizations of saturated chains of prime ideals are given in some (local) S_n rings. The next examples will show that we cannot improve such characterizations. In fact, the first one produces a local S_n ring $(A, \mathfrak{m})$ having a prime ideal of height $= n - 1$ that is not contained in any $\mathfrak{q} \in \operatorname{Spec}(A)$ such that ht $\mathfrak{q} = n$, and the second one produces a Macaulay ring for which a very short maximal chain of prime ideals exists.

(c) Let $B = k[X, Y, Z]/(X(Y, Z)) = k[x, y, z]$ $(xy = xz = 0)$. Then, $A = B_{(x,y,z)}$ is an S_1 local ring of dimension 2, but $(y, z)A$ (minimal prime divisor of zero) is not contained in any $\mathfrak{p} \in \operatorname{Spec}(A)$ such that ht $\mathfrak{p} = 1$.

(d) Let $A_1 = k[X, Y]/(X^2, XY, Y^2) = k[x, y]$ $(x^2 = xy = y^2 = 0)$, $A_2 = k[z, t]$, and $A = A_1 \oplus A_2$. Then, A is a two-dimensional Macaulay (i.e., S_n for each positive integer n) ring, but the prime ideal $(x, y) \oplus A_2 = \mathfrak{p}$ is such that ht $\mathfrak{p} = 0$ and it is not contained in any $\mathfrak{q} \in \operatorname{Spec} A$ with ht $\mathfrak{q} > 0$ (in fact, $\mathfrak{p}$ is maximal), so, in particular, we have: $0 = \dim A_{\mathfrak{p}} + \dim(A/\mathfrak{p}) < 2$.

In Section 1, Corollary 1.10, we showed how unmixedness of an ideal generated by a regular sequence in a local ring is preserved for the ideals generated by regular sequences gotten by cutting the given one. As expected, this fact is true only for local rings. In fact:

(e) Let $A_1 = k[X, Y]$, $A_2 = k[Z, T, U]/(Z^2, ZT)$, and $A = A_1 \oplus A_2$. Let $a = (X, U)$ and $b = (Y, 1)$, $\{a, b\} \subset A$ is clearly a regular sequence and $A/(a, b) \simeq k$ is an S_1 ring, but $A/(a) \simeq k[Y] \oplus k[Z, T]/(Z^2, ZT)$ is not S_1, that is, the ideal (a, b) is unmixed, but the ideal (a) is not so.

Finally, recall that in Section 2, we examined the $\mathbf{G}_n$ property behavior passing from a ring A to some of its quotients by regular sequences. Now we want to show by means of two examples that in Proposition 2.4 and Corollary 2.5, respectively, the hypotheses $n \geqslant 1$ and $n \geqslant h$ are necessary.

(f) Let $A = k[X, Y, Z]/(XY, YZ, ZX) = k[x, y, z]$ $(xy = xz = zy = 0)$. A is a one-dimensional Macaulay ring that is $\mathbf{G}_0$. However, $A' = A/(\xi)$ where $\xi \in A$, $\xi = x + y + z$ is not a $\mathbf{G}_0$ ring (cf. [6, oss.].

(g) *Let* $A = k[X, Y, Z, T]/(XT - YZ, Y^2 - XZ, Z^2 - YT) = k[x, y, z, t]$ $(xt - yz = y^2 - xz = z^2 - yt = 0)$ and $\mathfrak{m} = (x, y, z, t)$. $A_\mathfrak{m}$ is a two-dimensional Macaulay ring that is $\mathbf{G}_1$, $x, t \in \mathfrak{m}A_\mathfrak{m}$ form a parameter system of $A_\mathfrak{m}$ (and so a regular sequence), but $A' = A_\mathfrak{m}/(x, t) \simeq k[Y, Z]/(Y^2, YZ, Z^2)$ is not a $\mathbf{G}_0$ ring (cf. [3, Example 6.11]).

References

1. J. Dieudonné, Topics in local algebra, *Notre Dame Math. Lect.* **10** (1967).
2. R. Fossum and I. Reiten, Commutative n-Gorenstein rings, *Math. Scand.* **31** (1972).
3. S. Greco, Anelli di Gorenstein, Sem. Ist. Mat., Genova, 1968.
4. A. Grothendieck, Éléments de Géométrique Algébrique IV. I, II, *Inst. Hautes Études Sci. Publ. Math.* **20** (1966), **24** (1965).
5. R. Hartshorne, Complete intersections and connectedness, *Amer. J. Math.* **84** (1962).
6. M. G. Marinari, Successioni di Gorenstein e proprietà G_n, *Rend. Sem. Mat. Univ. Padova* **48** (1972).
7. M. G. Marinari, Alcune caratterizzazioni degli anelli G_n, *Matematiche (Catania)* **28** (1973).
8. M. G. Marinari, Su alcune classi di anelli c.n., to appear.
9. T. Millevoi, Sulle estensioni degli anelli di Gorenstein, *Rend. Sem. Mat. Univ. Padova* **41** (1968).
10. M. Paugam, La condition (G_a) de Ischekeck, *C. R. Acad. Sci. Paris* **276** (1973).
11. P. Ribenboim, Rings and modules, *Intersci. Publ.* **24** (1969).
12. R. Y. Sharp, Cousin complex characterization, *J. London Math. Soc.* **3** (1971), No. 2.
13. T. Watanabe *et al.*, On tensor product of Gorenstein rings, *J. Math. Kyoto Univ.* **9** (1969), No. 3.