BANACH ALGEBRAS AND BOTT PERIODICITY

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INTRODUCTION

RECENTLY Atiyah and Bott have given a new proof of the periodicity theorem for the infinite complex general linear group [1]. Their proof is given in the framework of vector bundles and K-theory. In the present work, the two principal ideas of [1], namely the approximation of Banach valued functions on the unit circle by polynomials and the deformation of polynomial maps into standard form are applied in the context of Banach algebras to obtain a single theorem for a real or complex Banach algebra $A$ with involution $\sigma$. By suitably choosing $A$ and $\sigma$ the theorems of Bott [3] can be deduced. The material is presented in the framework of homogeneous spaces. There is no $K$-theory. This omission is not by desire but rather by the lack, at the time of writing, of a workable description of the “one stage steps” in the periodicity theorem for the real $K$-theory.

The statements of the principal results are given in §1. Theorem 1.8 is the main theorem. The next two sections contain the proofs of the main results. The fourth section applies Theorem 1.8 to obtain the theorems of Bott [3].

I wish to thank my supervisor Dr. I. M. James for his helpful criticism of this work and I wish to express my gratitude to Professor M. F. Atiyah for suggesting the problem of proving the “real case” of the Bott periodicity theorem by polynomial methods and for pointing out that the right setting for the present approach is the concept of Banach algebras. I would also like to thank Professor J. F. Adams for his very helpful suggestions on both the subject matter and presentation of material.

§1. STATEMENT OF THE THEOREMS

We work with real or complex Banach algebras. They are associative and possess an identity but they are not necessarily commutative. The letter $A$ is reserved for a Banach algebra and $G$ for its group of invertible elements. For each positive integer $n$ the set $A_n$ of $n \times n$ matrices over $A$ is a Banach algebra with respect to the norm given by

$$\|a_{ij}\|^2 = \sum \|a_{ij}\|^2$$

where $a_{ij}$ is the $(i,j)$th element of the matrix $(a_{ij})$ in $A_n$. Suppose we have a way of associating to each algebra $A$ a set $S$ depending on $A$, then we write $S_n$ for the set corresponding to $A_n$. For example the group of invertible elements in $A_n$ is $G_n$. An important subset of $A$ consists of those elements whose square is the negative of the identity.
1.1. \textbf{Definition}. \( \mathcal{G} = \{ g \in A | g^2 = -1 \} \).

The base point of a group is always its identity element. There is no canonical choice for the base point of \( \mathcal{G} \). We reserve a symbol for it.

1.2. \textbf{Definition}. \( k \in \mathcal{G} \) is the base point of \( \mathcal{G} \).

The base point of \( \mathcal{G}_n \) is \( k \oplus \ldots \oplus k \) (direct sum of \( n \) copies of \( k \)).

The group \( G_n \) is identified with the subgroup of matrices \( M \oplus 1 \) in \( G_{n+1} \) where \( M \in G_n \). Similarly the set \( \mathcal{G}_n \) is identified with the set of matrices \( M \oplus k \) in \( \mathcal{G}_{n+1} \) where \( M \in \mathcal{G}_n \). These identifications are compatible with the choices of base points. We say that a sequence \( \{ Y_n \} \) of topological spaces \( Y_n, n = 1, 2, \ldots \) is increasing if \( Y_n \) is a subspace of \( Y_{n+1} \) for each \( n \).

1.3. \textbf{Definition}. The direct limit space \( \lim Y_n \) of an increasing sequence \( \{ Y_n \} \) has underlying set \( \bigcup Y_n \) and the weak topology.

The weak topology means that a subset of \( \lim Y_n \) is closed if and only if its intersection with each \( Y_n \) is closed in \( Y_n \). A sequence of maps \( f_n : Y_n \rightarrow Z_n \) compatible with the inclusions of two increasing sequences \( \{ Y_n \} \) and \( \{ Z_n \} \) induces a function \( \lim f_n : \lim Y_n \rightarrow \lim Z_n \) which is continuous. Each \( Y_n \) is a subspace of \( \lim Y_n \).

We write \( Y^X \) for the set of maps from \( X \) to \( Y \) and give \( Y^X \) the compact open topology. A map from \( Y \) to \( Z \) induces a function from \( Y^X \) to \( Z^X \) which is continuous. If \( Y \) is a subspace of \( Z \), then \( Y^X \) is a subspace of \( Z^X \). The following properties of function spaces will be used in the course of this work: if \( X \) is locally compact and Hausdorff, then a function from \( Z \) to \( Y^X \) is continuous if the adjoint function from \( Z \times X \) to \( Y \) is continuous; if \( X \) and \( Z \) are locally compact Hausdorff spaces, then \( (Y^X)^Z \) is naturally homeomorphic to \( Y^{X \times Z} \); if \( Y \) is a metric space with metric \( \rho \) and \( X \) is compact, then the compact open topology on \( Y^X \) is equivalent to the metric topology given by \( \rho(f, g) = \sup \rho(f(x), g(x)) \) where \( f, g \in Y^X \) and \( x \in X \). For the applications the exponent space \( X \) in the function space \( Y^X \) will be compact and Hausdorff.

If \( Y \) is a space with base point and \( I \) is the unit interval we write \( \Omega Y \) for the subspace of \( Y^I \) consisting of those functions which send the end points of \( I \) onto the base point of \( Y \). The loop space \( \Omega Y \) has a canonical base point namely the constant function at the base point of \( Y \).

There is an important map from \( \mathcal{G} \) to \( \Omega \mathcal{G} \). To define this map it is convenient to use the semicircle \( x = \cos \theta, y = \sin \theta, 0 \leq \theta \leq \pi \), as a model for the unit interval. The initial point of the interval is \((1, 0)\) and the end point is \((-1, 0)\).

1.4. \textbf{Definition}. \( \phi : \mathcal{G} \rightarrow \Omega \mathcal{G} \) is given by

\[
\phi(g)(x, y) = (x_1 + yg)(x_1 - yk),
\]

where \( g \in \mathcal{G}, x = \cos \theta, y = \sin \theta, 0 \leq \theta \leq \pi \).

The adjoint of \( \phi \) involves only the linear and algebraic operations in \( A \) which are continuous functions. Hence \( \phi \) is continuous. From the equations

\[
(x_1 + yg)(x_1 - yg) = (x^2 + y^2)k = 1
\]
we deduce that \( \phi(g)(x, y) \) is invertible. Also

\[
\phi(g)(1, 0) = \phi(g)(-1, 0) = \phi(k)(x, y) = 1.
\]

We deduce that \( \phi \) is well defined and base point preserving. One easily checks that the sequence \( \phi_n : \hat{G}_n \to \Omega G_n \) is compatible with the inclusions of the increasing sequences \( \{\hat{G}_n\} \) and \( \{\Omega G_n\} \).

**1.5. Definition.** A map \( f : Y \to Z \) is a weak homotopy equivalence if for every compact Hausdorff space \( X \) the induced map of homotopy classes \( f_* : [X, Y] \to [X, Z] \) is an isomorphism of sets.

**1.6. Theorem.** Provided \( k \) can be joined to \(-k\) in \( \hat{G} \), then \( \lim \phi_n : \lim \hat{G}_n \to \lim \Omega G_n \) is a weak homotopy equivalence.

We now go on to state a further theorem which can be regarded as a relative form of Theorem 1.6. It concerns a pair \((A, \sigma)\) where \( \sigma \) is an involution of \( A \) i.e. a continuous automorphism of period two (in the literature on Banach algebras involution usually means antiautomorphism). Given a subset \( S \) of \( A \), we write \( S^\sigma \) and \( S^{-\sigma} \) for the subsets of \( S \) pointwise fixed by \( \sigma \) and \(-\sigma \) respectively. The involution \( \sigma \) extends to an involution of \( A_n \) by defining \( \sigma(a_{ij}) = (\sigma a_{ij}) \). Of particular interest is the set \( \hat{G}^{-\sigma} \). It is a subspace of \( \hat{G} \) and we suppose that the base point \( k \) of Definition 1.2 is chosen in \( \hat{G}^{-\sigma} \) so that the conditions on \( k \) are now \( k^2 = -1 \) and \( \sigma k = -k \).

Given a space \( Y \) and subspace \( Z \) with base point, the relative loop space \( \Omega(Y, Z) \) is the subspace of \( Y^I \) consisting of those functions which send the initial point of \( I \) onto the base point of \( Z \) and the end point of \( I \) to some point in \( Z \).

There is an important map from \( \hat{G}^{-\sigma} \) to \( \Omega(G, G^\sigma) \). To define this map it is convenient to use the quarter circle \( x = \cos \theta, y = \sin \theta, 0 \leq \theta \leq \pi/2 \), as a model for the unit interval. The initial point is \((1, 0)\) and the end point \((0, 1)\).

**1.7. Definition.** \( \psi : \hat{G}^{-\sigma} \to \Omega(G, G^\sigma) \) is given by

\[
\psi(g)(x, y) = (x1 + yg)(x1 - yk),
\]

where \( g \in \hat{G}^{-\sigma}, x = \cos \theta, y = \sin \theta, 0 \leq \theta \leq \pi/2 \).

The formula for \( \psi \) is similar to the formula for \( \phi \) in Definition 1.6. To check that \( \psi \) is well defined we need only remark this time that \( \psi(g)(0, 1) = -gk \) which belongs to \( G^\sigma \) because \( g \) and \( k \) belong to \( G^{-\sigma} \). Hence the end point of the path \( \psi(g) \) lies in \( G^\sigma \) as it should.

**1.8. Theorem.** Provided \( k \) can be joined to \(-k\) in \( \hat{G}^{-\sigma} \) then

\[
\lim \psi_n : \lim \hat{G}_n^{-\sigma} \to \lim \Omega(G_n, G^\sigma_n)
\]

is a weak homotopy equivalence.

**1.9. Remark.** As Theorem 1.6 stands, it applies neither to the algebra of real numbers nor to the algebra of complex numbers because the hypothesis on \( \hat{G} \) in the statement of the theorem is not satisfied in these cases. In fact \( \hat{G} \) is empty when \( A = \mathbb{R} \) and has two points when \( A = \mathbb{C} \). However for a given algebra \( A \), the set \( \hat{G}_2 \) is nonempty because it contains the element
In \( \mathcal{G}_4 \) we have the element

\[
\ell = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

which can be joined to \(-k\) in \( \mathcal{G}_4 \) by the path

\[
\begin{pmatrix} \ell \cos \theta & \ell \sin \theta \\ \ell \sin \theta & -\ell \cos \theta \end{pmatrix}, \quad 0 \leq \theta \leq \pi.
\]

It follows that we can realise the hypothesis on \( \mathcal{G} \) if we replace our original algebra by the 4 \( \times \) 4 matrix algebra over it.

In Theorem 1.8 we have made the tacit assumption that \( G^{-\sigma} \) is non empty. This excludes the case in which \( \sigma \) is the identity involution. If \( g \in G^{-\sigma} \), then \( \mathcal{G}_5^{-\sigma} \) contains the element

\[
\ell = \begin{pmatrix} 0 & g \\ -g^{-1} & 0 \end{pmatrix}
\]

and as in the case of Theorem 1.6 there exists \( k \in \mathcal{G}_4^{-\sigma} \) which can be joined to \(-k\) in \( \mathcal{G}_4^{-\sigma} \). Hence if necessary we can replace \( A \) by \( A_4 \) to ensure the connectivity hypothesis on \( \mathcal{G}^{-\sigma} \) in the statement of Theorem 1.8.

1.10. Remark. The stages of the Bott periodicity theorem [3] are obtained from Theorem 1.8 by suitably choosing the algebra \( A \) and the involution \( \sigma \). Those stages which involve the loop space of a group are more readily obtained from Theorem 1.6. Some algebraic manipulation is required to throw \( \mathcal{G} \) and \( \mathcal{G}^{-\sigma} \) into the more recognisable form of homogeneous spaces. We take up these points in §4.

\section*{2. LINEARIZATION}

Theorems 1.6 and 1.8 are proved together. In the course of the proof we make several new definitions motivated by the form of the maps \( \phi \) and \( \psi \) in Definitions 1.4 and 1.7.

2.1. Definition. Let \( \mathcal{G} \) be the set of elements \( g \in G \) such that the spectrum of \( g \) contains no real numbers.

Now \( \mathcal{G} \) (Definition 1.1) is a subset of \( \mathcal{G} \) because the spectrum of an element in \( \mathcal{G} \) is contained in the set of two points \( \pm i \). The base point \( k \) of \( \mathcal{G} \) is chosen as the base point of \( \mathcal{G} \). If \( g \in \mathcal{G} \), then \( x1 + yg \) is invertible for each point \((x, y)\) on the unit circle. The formulae of Definitions 1.4 and 1.7 define extensions of \( \phi \) and \( \psi \) to the sets \( \mathcal{G} \) and \( \mathcal{G}^{-\sigma} \) respectively. At this point it is convenient to introduce some new notation to treat Theorems 1.6 and 1.8 simultaneously.

2.2. Definition. With reference to Theorem 1.6 let \( \Delta = \mathcal{G}, \Gamma = \mathcal{G}, \Omega = \Omega G \). With reference to Theorem 1.8 let \( \Delta = \mathcal{G}^{-\sigma}, \Gamma = \mathcal{G}^{-\sigma}, \Omega = \Omega(G, G^*) \). For the extension of \( \phi \) or \( \psi \) we write

\[ \chi : \Gamma \to \Omega \]
The proof of the main theorems breaks into two parts.

2.3. **Theorem.** \( \Delta \) is a deformation retract of \( \Gamma \).

2.4. **Theorem.** Provided \( k \) can be joined to \(-k\) in \( \Delta \) then
\[
\lim y_n : \lim \Gamma_n \to \lim \Omega_n
\]
is a weak homotopy equivalence.

The proof of Theorem 2.3 depends on some standard theory of Banach algebras and is held over until §3. In this section we prove Theorem 2.4 using methods due to Atiyah and Bott [1]. Before introducing these methods it is helpful for the later presentation of the proof to make some preliminary observations about the property of weak homotopy equivalence. We lead up to the main point which is made in Lemma 2.11.

2.5. **Lemma.** Let \( \{Y_n\} \) be an increasing sequence of topological spaces each of which satisfies the \( T_1 \) separation axiom. Then a compact subset of \( \lim Y_n \) lies in \( Y_m \) for some integer \( m \).

**Proof.** Let \( Y = \lim Y_n \) and let \( X \) be a compact subset of \( Y \). It is sufficient to show that an integer \( m \) exists such that \( X \cap Y_n = X \cap Y_{n+1} \) for all \( n \geq m \), for then \( X \subset Y_m \) as required. Choose if possible, for each integer \( n \), a point of \( X \cap Y_n - X \cap Y_{n+1} \) and let \( V \) be the set of points so chosen. The proof will be complete if we can show that \( V \) is a finite set. Since \( V \subset X \) and \( X \) is compact, it is sufficient to show that \( V \) has no limit point in \( X \). Given any point \( x \in X \), consider the set \( U = (Y - V) \cup x \). The intersection of \( U \) with \( Y_m \) is the complement in \( Y_m \) of a finite set. Hence by the \( T_1 \) axiom \( U \cap Y_m \) is open in \( Y_m \). By definition of the weak topology \( U \) is open in \( Y_m \). Hence \( U \cap X \) is a neighbourhood of \( x \in X \) which contains no point of \( V \) except possibly \( x \). Hence \( V \) has no limit point in \( X \) and the proof is complete.

The inclusions \( Y_n \to Y_{n+1} \) of an increasing sequence \( Y_n \) induce set functions \( [X, Y_n] \to [X, Y_{n+1}] \) where, as usual, \( [X, Y] \) is the set of homotopy classes of maps of a space \( X \) into a space \( Y \). The direct limit of sets \( \lim [X, Y_n] \) is defined in the usual way and there is an obvious map
\[
v : \lim [X, Y_n] \to [X, \lim Y_n].
\]

2.6. **Lemma.** If \( Y_n \) is an increasing sequence of \( T_1 \) spaces and \( X \) is compact, then \( v \) is an isomorphism of sets.

**Proof.** Let \( f : X \to \lim Y_n \) be a map. Then \( f(X) \) is a compact subset of \( \lim Y_n \). Hence by Lemma 2.5 \( f \) factors through \( Y_m \) for some integer \( m \). This shows that \( v \) is onto. By the same argument applied to \( X \times I \) instead of \( X \) a homotopy of maps into \( \lim Y_n \) factors through \( Y_m \) for some \( m \) which shows that \( v \) is one-one into.

2.7. **Corollary.** Let \( f_n : Y_n \to Z_n \) be a sequence of maps compatible with the inclusions of two increasing sequences \( \{Y_n\} \) and \( \{Z_n\} \) of \( T_1 \) spaces. Then the induced map
\[
\lim f_n : \lim Y_n \to \lim Z_n
\]
is a weak homotopy equivalence if and only if, for every compact Hausdorff space \( X \), the
induced function

\[ \lim[X, Y_n] \rightarrow \lim[X, Z_n] \]

is an isomorphism of sets.

**Proof.** By naturality the following diagram is commutative

\[ \begin{array}{c}
\lim[X, Y_n] \\
\downarrow \ \\
[X, \lim Y_n] \\
\end{array} \rightarrow \begin{array}{c}
\lim[X, Z_n] \\
\downarrow \\
[X, \lim Z_n]
\end{array} \]

where the horizontal maps are induced by the sequence \( \{f_n\} \) and the vertical maps are the isomorphisms of Lemma 2.6. The proof is now immediate from the definition of weak homotopy equivalence (Definition 1.5).

2.8. COROLLARY. Let \( f_n : Y_n \rightarrow Z_n \) be a sequence of weak homotopy equivalences compatible with the inclusions of two increasing sequences \( \{Y_n\} \) and \( \{Z_n\} \) of \( T_1 \) spaces. Then the induced map

\[ \lim f_n : \lim Y_n \rightarrow \lim Z_n \]

is also a weak homotopy equivalence.

**Proof.** For each integer \( n \) the function

\[ [X, Y_n] \rightarrow [X, Z_n] \]

induced by \( f_n \) is an isomorphism for any compact Hausdorff space \( X \). Hence

\[ \lim[X, Y_n] \rightarrow \lim[X, Z_n] \]

is an isomorphism and the proof follows by Corollary 2.7.

We can reformulate Corollary 2.7 as follows. Let \( \pi_0 \) \( Y \) be the set of path components of a space \( Y \). If \( X \) is compact and Hausdorff then \([X, Y]\) is naturally isomorphic to \( \pi_0 Y^X \). Corollary 2.7 now becomes

2.9.Lemma. Let \( f_n : Y_n \rightarrow Z_n \) be a sequence of maps compatible with the inclusions of two increasing sequences \( \{Y_n\} \) and \( \{Z_n\} \) of \( T_1 \) spaces. Then

\[ \lim f_n : \lim Y_n \rightarrow \lim Z_n \]

is a weak homotopy equivalence if and only if, for every compact Hausdorff space \( X \), the induced function

\[ \lim \pi_0 Y_n^X \rightarrow \lim \pi_0 Z_n^X \]

is an isomorphism of sets.

This formulation is particularly appropriate in certain questions involving Banach algebras for reasons we shall now explain. Given a Banach algebra \( A \) and a compact Hausdorff space \( X \), then \( A^X \) is canonically a Banach algebra with respect to the norm \( \|f\| = \sup \|f(x)\| \) where \( x \in X \) and \( f \in A^X \). Elements of \( A^X \) are added and multiplied by adding and multiplying their values in \( A \). An involution \( \sigma \) on \( A \) extends to an involution on \( A^X \) by \( (\sigma f)(x) = f(\sigma x) \) where \( x \in X \) and \( f \in A^X \). Let \( \Delta', \Gamma', \Omega' \) be the sets of Definition 2.2 associated with the algebra \( A^X \).
2.10. **Lemma.** The sets $\Delta'$, $\Gamma'$, $\Omega'$ are canonically homeomorphic to the functions spaces $\Delta^X$, $\Gamma^X$, $\Omega^X$ respectively.

**Proof.** This is a consequence of the way in which the algebraic operations in $A^X$ are defined and the fact that $(A^X)'$ is naturally homeomorphic to $(A')^X$ when $X$ is compact Hausdorff.

There is a natural choice for the base point of $\Delta'$ and $\Gamma'$ namely the constant function at the base point of $\Delta$. Referring to Theorem 2.4 we have

2.11. **Lemma.** $\lim n \chi_n : \lim \Gamma_n \to \lim \Omega_n$ is a weak homotopy equivalence for every Banach algebra if and only if the induced map

$$\lim \pi_0 \Gamma_n \to \lim \pi_0 \Omega_n$$

is an isomorphism of sets for every Banach algebra.

**Proof.** This is a consequence of Lemmas 2.9 and 2.10.

The preliminary observations on the property of weak homotopy equivalence are complete and we now proceed to the main ideas of this section.

According to Definitions 1.4 and 1.7 an element of $\Omega$ is a function with values in $A$ defined on one half or one quarter of the unit circle. Such a function has a unique extension $f$ to the whole circle satisfying, in the case of Theorem 1.6 the two conditions

(i) $f(z) = f(-z), \quad |z| = 1,$
(ii) $f(1) = 1$

and, in the case of Theorem 1.8, the extra condition

(iii) $f(\bar{z}) = \sigma f(z),$

where, as usual, $\sigma$ is an involution of the algebra $A$, and $\bar{z}$ is the conjugate of $z$. Indeed any map defined on the unit circle with values in $G$ which satisfies conditions (i) and (ii) or (i), (ii) and (iii) in the case of an algebra with involution defines by restriction to the semicircle or quarter circle an element of $\Omega$. We may therefore regard elements of $\Omega$ as maps defined on the whole unit circle satisfying the invariance conditions above. Since we deal with real Banach algebras as well as complex Banach algebras it is sometimes more convenient to use the real variables $x = \cos \theta$, $y = \sin \theta$ as parameters for the unit circle rather than the complex variable $z = e^{i\theta}$.

2.12. **Definition.** Let $P$ be the subset of $\Omega$ consisting of polynomials in $x$ and $y$ with coefficients in $A$.

The conditions (i), (ii) and (iii) stated above translate into the following form for a polynomial $p(x, y)$

(i) $p(x, y) = p(-x, -y)$
(ii) $p(1, 0) = 1$
(iii) $p(x, -y) = \sigma p(x, y)$. 
2.13. Definition. Let $Q$ be the subset of $P$ consisting of quadratic polynomials.

The introduction of the sets $P$ and $Q$ is motivated by the forms of the maps $\phi$ and $\psi$ in Definitions 1.4 and 1.7. Indeed the map $\chi : \Gamma \rightarrow \Omega$ (Definition 2.2) factors through $Q$ and we have the sequence of maps

$$\Gamma \overset{\chi}{\rightarrow} Q \overset{i}{\rightarrow} P \overset{j}{\rightarrow} \Omega$$

where $i$ and $j$ are inclusions.

The proof of Theorem 2.4 breaks into three parts.

2.14. Proposition. The inclusion $j : P \rightarrow \Omega$ induces an isomorphism

$$\pi_0 P \rightarrow \pi_0 \Omega.$$ 

2.15. Proposition. The inclusions $i_n : Q_n \rightarrow P_n$ induce an isomorphism

$$\lim \pi_0 Q_n \rightarrow \lim \pi_0 P_n.$$ 

2.16. Proposition. The maps $\chi_n : \Gamma_n \rightarrow Q_n$ induce an isomorphism

$$\lim \pi_0 \Gamma_n \rightarrow \lim \pi_0 Q_n.$$ 

In combination with Lemma 2.11 the above propositions imply Theorem 2.4.

Proposition 2.14 is essentially a statement about polynomial approximation and is best proved by use of Fejér’s theorem of Fourier series which we state in the following form.

2.17. Fejér’s Theorem. Let $B$ denote a Banach space and $S$ the unit circle. Given a map $f : S \rightarrow B$ we define the Banach valued Fourier coefficients

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) \cos m\theta \, d\theta, \quad b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) \sin m\theta \, d\theta,$$

the partial Fourier sums

$$s_r(e^{i\theta}) = a_0/2 + \sum_{m=1}^{r} a_m \cos m\theta + b_m \sin m\theta$$

and the arithmetic means

$$t_n = \frac{1}{n} \sum_{r=0}^{n-1} s_r.$$ 

Then the sequence $\{t_n\}$ converges to $f$ in the norm topology of $B^S$.

The proof of this theorem as given in Titchmarsh [8, p. 414] for the special case of the real line carries over to the general Banach space if the modulus sign is replaced by the norm sign in the appropriate places. We refer to $t_n$ as the $n$th Fejér approximation to $f$. We can expand $t_n(e^{i\theta})$ as a polynomial in $x = \cos \theta$ and $y = \sin \theta$. Hence Fejér’s theorem gives explicit polynomial approximations to a Banach valued function on the unit circle.

2.18. Lemma. If $f : S \rightarrow A$ satisfies the condition $f(z) = f(-z)$ then so does a Fejér approximation to $f$. If $\sigma$ is an involution of $A$ and $f$ satisfies the condition $f(\bar{z}) = \sigma f(z)$ then so does a Fejér approximation to $f$. 

Proof. The Fourier coefficients of $f$ are given by

$$a_m + ib_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(e^{ie^\theta}e^{im\theta}) \, d\theta$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} f(e^{ie^\theta})e^{im\theta} \, d\theta + \frac{1}{\pi} \int_{0}^{\pi} f(e^{ie^\theta})e^{im\theta} \, d\theta$$

$$= \frac{1}{\pi} \int_{0}^{\pi} ((-1)^m f(-e^{i\theta}) + f(e^{i\theta})) e^{im\theta} \, d\theta$$

Hence, if $f(e^{i\theta}) = f(-e^{i\theta})$, then

$$a_m + ib_m = 0$$

when $m$ is odd. The vanishing of the odd Fourier coefficients implies that the partial Fourier sums $s_n$ in Fejér's Theorem 2.17 satisfy $s_n(e^{i\theta}) = s_n(-e^{i\theta})$ whence also $t_n(e^{i\theta}) = t_n(-e^{i\theta})$. This verifies the first statement of Lemma 2.18. For the second statement we have

$$\sigma(a_m + ib_m) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(e^{ie^\theta})e^{im\theta} \, d\theta$$

Hence, if $\sigma f(e^{i\theta}) = f(e^{-i\theta})$, then by substituting $\theta$ for $-\theta$ in the integral we obtain

$$\sigma(a_m + ib_m) = a_m - ib_m.$$}

Hence $\sigma a_m = a_m$ and $\sigma b_m = -b_m$ for each integer $m$.

It follows that $\sigma s_n(e^{i\theta}) = s_n(e^{-i\theta})$ and $\sigma t_n(e^{i\theta}) = t_n(e^{-i\theta})$ which verifies the second statement of Lemma 2.18.

2.19. Corollary. $P$ is dense in $\Omega$.

Proof. Let $t_n$ be the $n$th Fejér approximation to $f$. Define

$$f_n = t_n + 1 - t_n(1).$$

Then $f_n$ converges to $f$ because $t_n$ converges to $f$. Moreover, by Lemma 2.18, $f_n(e^{i\theta})$ satisfies the invariance conditions stated in the paragraph before Definition 2.12. Since the group of invertible elements of a Banach algebra is open it follows that $f_n$ belongs to $P$ for sufficiently large $n$ and the proof is complete.

Proof of Proposition 2.14. The group of invertible elements in a Banach algebra is open and therefore locally convex. Given $f \in \Omega$ choose $n$ sufficiently large so that the linear path $tf + (1-t)f_n$ lies in $\Omega$ for $0 \leq t \leq 1$, where $f_n$ is the approximation to $f$ defined in the proof of Corollary 2.19. The existence of a path joining an element of $\Omega$ to an element of $P$ shows that the inclusion $j: P \to \Omega$ induces an epimorphism

$$\pi_0 P \to \pi_0 \Omega.$$
Corollary 2.19. If the approximation is close enough, then the end points $p'(0)$ and $p'(1)$ of $p'$ can be joined linearly in $P$ to $p_0$ and $p_1$ respectively. Hence there is a path in $P$ joining $p_0$ to $p_1$ constructed in three stages by joining $p_0$ to $p'(0)$ linearly, $p'(0)$ to $p'(1)$ via $p'(t)$ and $p'(1)$ to $p_1$ linearly. This shows that the inclusion $j : P \to \Omega$ induces a monomorphism

$$\pi_0 P \to \pi_0 \Omega$$

and completes the proof of Proposition 2.14.

One aspect of the above proof should be emphasised. The invariance conditions following Definition 2.12 imply that an element of $P$ can be written in the form

$$p(x, y) = 1 + c_1 y^2 + \cdots + c_m y^{2m} + xy(d_1 + d_2 y^2 + \cdots + d_m y^{2m-2})$$

(1)

where $2m$ is the formal degree of the polynomial (leading coefficients may be zero), $c_i$ and $d_i$ are elements of $A$ and in the case of an algebra with involution $\sigma c_i = c_i$ and $\sigma d_i = -d_i$. We have used the relation $x^2 + y^2 = 1$ to remove powers of $x$ higher than the first. In the course of the proof of Proposition 2.14 we showed that if two elements in $P$ could be joined by a path in $\Omega$ then they could be joined by a path of the form

$$p(t)(x, y) = 1 + c_1(t) y^2 + \cdots + c_m(t) y^{2m} + xy(d_1(t) + d_2(t) y^2 + \cdots + d_m(t) y^{2m-2})$$

(2)

where $0 \leq t \leq 1$. The degree $2m$ is independent of the parameter $t$.

We now proceed to the proof of Proposition 2.15. First we show that an element of $P$ of degree $2m$ can be joined in $P_m$ to a point of $Q_m$. Associate with $p \in P$, given explicitly by formula (1) above, the matrix

$$q(x, y) = \begin{pmatrix}
1 + c_1 y^2 + d_1 xy & c_2 y^2 + d_2 xy & \cdots & c_{m-1} y^2 + d_{m-1} xy & c_m y^2 + d_m xy \\
-y^2 & 1 & \cdots & 0 & 0 \\
0 & -y^2 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 1 & 0 \\
0 & 0 & \cdots & -y^2 & 1
\end{pmatrix}$$

By inspection $q$ belongs to $Q_m$. Using homotopies similar to those in [1] we can construct an explicit path joining $p$ to $q$ in $P_m$. Define a matrix $N_1$ by the formula

$$1 + N_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
y^2 & 1 & 0 & 0 & 0 \\
y^4 & y^2 & 1 & 0 & 0 \\
y^{2m-2} & y^{2m-4} & y^{2m-6} & y^2 & 1
\end{pmatrix}$$

Note that $1 + N_1 t$ belongs to $P_m$ for $0 \leq t \leq 1$. Multiplying $q(x, y)$ by $1 + N_1$ we obtain

$$q(x, y)(1 + N_1) = \begin{pmatrix}
p(x_1 y) & p_2(x_1 y) & \cdots & p_m(x, y) \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots \\
0 & 0 & \cdots & 1
\end{pmatrix}$$
where \( p_i(x, y) \), \( m \leq i \leq 3 \) are certain polynomials. Define a matrix \( N_2 \) by the formula
\[
1 + N_2 = \begin{pmatrix}
1 & p_2(x, y) & p_3(x, y) \\
0 & 1 & 0 \\
\vdots & \vdots & \vdots \\
0 & 1 & 1
\end{pmatrix}
\]
By construction \( 1 + N_2 \) belongs to \( P_m \) for \( 0 \leq t \leq 1 \) and \( (1 + N_2)^{-1} q(x, y)(1 + N_2 t) \)
defines a path joining \( p \) to \( q \) in \( P_m \).

Now suppose \( q_0 \) and \( q_1 \) are elements of \( Q \) which can be joined by a path in \( P \). According to the remarks made after the proof of Proposition 2.14 we may assume that the path \( p' \) joining \( q_0 \) to \( q_1 \) is given by formula (2). Associate to \( p' \) in \( P' \) the element \( q' \in Q_m \) defined by the canonical construction of the previous paragraph. By the nature of the construction the end points of \( q' \) are the same as the points obtained by applying the construction to \( q_0 \) and \( q_1 \). The end points are in fact
\[
q'(i)(x, y) = \begin{pmatrix}
1 + c_1(i)y^2 + d_1(i)xy & 0 & 0 \\
0 & 1 & 0 \\
0 & -y^2 & 0 \\
0 & 0 & 1 \\
0 & 0 & -y^2 & 1
\end{pmatrix}
\]
where \( i = 0, 1 \). Hence \( q'(i) \) can be joined to \( q_1 \) linearly and there is a path in \( Q_m \) joining \( q_0 \) to \( q_1 \) constructed in three stages by joining \( q_0 \) to \( q'(0) \) linearly, \( q'(0) \) to \( q'(1) \) via \( q'(t) \), and \( q'(1) \) to \( q_1 \) linearly.

To summarize, we have shown that, for any algebra \( A \), a point of \( P \) can be joined in \( P_m \) to a point of \( Q_m \) for some \( m \) and if two points of \( Q \) can be joined in \( P \) then they can be joined in \( Q_m \) for some \( m \). All this in effect shows that the sequence \( i_n : P_n \to Q_n \) induces an isomorphism
\[
\lim_\pi_0 P_n \to \lim_\pi_0 Q_n
\]
which completes the proof of Proposition 2.15.

We come finally to the proof of Proposition 2.16 which provides the last stage of the linearisation process by showing how to factorise a quadratic form into the product of two linear forms (recall that the image of \( \chi : \Gamma \to \Omega \) consists of factorised quadratic forms). An element \( q \in Q \) can be written in the form
\[
q(x, y) = 1x^2 + cy^2 + dxy
\]
where \( c \) and \( d \) belong to \( A \) and in the case of an algebra with involution \( \sigma c = c, \sigma d = -d \). Define an element \( r \in Q_2 \) by the matrix
\[
r(x, y) = \begin{pmatrix}
1x + dy & cky \\
k_y & 1x
\end{pmatrix} \begin{pmatrix}
1x & -ky \\
-k_y & 1x
\end{pmatrix}
\]
where as usual \( k \) is the base point of \( \Delta \) (Definition 2.2). Multiplying out we obtain
\[
r(x, y) = \begin{pmatrix}
q(x, y) & -(1x + dy)ky + ckyy \\
0 & 1
\end{pmatrix}
\]
Hence \( q \) can be joined linearly to \( r \) in \( Q_2 \). Unfortunately \( r \) is not in the image of \( \chi_2 : \Gamma_2 \rightarrow Q_2 \) because the matrix

\[
\begin{pmatrix}
0 & -k \\
-k & 0
\end{pmatrix}
\]

is the "wrong" base point for \( \Gamma_2 \) in the second linear factor of \( r(x, y) \). However we can construct a path joining \( r \) to a point in the image of \( \chi_2 \). For this it is sufficient to join the matrix

\[
\begin{pmatrix}
0 & -k \\
-k & 0
\end{pmatrix}
\]

to the "right" base point

\[
\begin{pmatrix}
k & 0 \\
0 & k
\end{pmatrix}
\]

by a path in \( \Delta_2 \). The path

\[
k \begin{pmatrix}
\cos \theta & \sin \theta \\
\sin \theta & -\cos \theta
\end{pmatrix}, \quad 0 \leq \theta \leq \pi/2,
\]

lies in \( \Delta_2 \) and joins

\[
\begin{pmatrix}
0 & -k \\
-k & 0
\end{pmatrix}
\]

to the matrix

\[
\begin{pmatrix}
k & 0 \\
0 & -k
\end{pmatrix}.
\]

At this point we use the connectivity hypothesis on \( \Delta \) in the statement of Theorem 2.4 namely that \( k \) can be joined to \(-k\) in \( \Delta \). Hence

\[
\begin{pmatrix}
k & 0 \\
0 & -k
\end{pmatrix}
\]

can be joined to

\[
\begin{pmatrix}
k & 0 \\
0 & k
\end{pmatrix}
\]

in \( \Delta_2 \) and this completes the demonstration.

Suppose now we have two elements \( g_0 \) and \( g_1 \) in \( \Gamma \) whose images under \( \chi \) can be joined in \( Q \) by a path of the form

\[
g'(t)(x, y) = 1x^2 + c'(t)y^2 + d'(t)xy
\]

where \( c' \) and \( d' \) are elements of \( A^t \) and \( 0 \leq t \leq 1 \). We apply the factorisation process to show that \( g_0 \) and \( g_1 \) can be joined by a path in \( \Gamma_2 \). By assumption we have

\[
q'(0) = \chi(g_0) = (1x + yg_0)(1x - yk),
\]

\[
q'(1) = \chi(g_1) = (1x + yg_1)(1x - yk).
\]

Hence

\[
c'(0) = -g_0k, \quad d'(0) = g_0 - k
\]

\[
c'(1) = -g_1k, \quad d'(1) = g_1 - k.
\]
By the factorisation process of the previous paragraph we can associate with the path $q'$ a path

$$(1x + g'(t)y)(1x - k'y)$$

in the image of $\chi_2 : \Gamma_2 \to Q_2$, where

$$k' = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$
$$g'(i) = \begin{pmatrix} d'(i) & c'(i)k \\ k & 0 \end{pmatrix},$$

(compare with the definition of $r(x, y)$ in previous paragraph). We now show that $g_i$ can be joined to $f'(i), i = 0, 1$. The path

$$\begin{pmatrix} 1 & 1t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g_i - k & g_i \\ k & 0 \end{pmatrix} \begin{pmatrix} 1 & 1t \\ 0 & 1 \end{pmatrix}^{-1}$$

lies in $\Gamma_2$ and joins $g_i$ to the point

$$\begin{pmatrix} g_i \\ k - k \end{pmatrix}$$

which can be joined by a linear path in $\Gamma_2$ to the point

$$\begin{pmatrix} g_i \\ 0 \end{pmatrix}$$

Using the connectivity hypothesis on $\Delta$ in the statement of Theorem 2.4 there is a path joining

$$\begin{pmatrix} g_i \\ 0 \end{pmatrix}$$

to the point

$$\begin{pmatrix} g_i \\ 0 \end{pmatrix} = g'(i).$$

This completes the demonstration.

To summarise, we have shown that any point in $Q$ can be joined in $Q_m$ to a point in the image of $\chi_m : \Gamma_m \to Q_m$ for suitable $m$ and if the images under $\chi$ of two points in $\Gamma$ can be joined in $Q$ then these points can be joined in $\Gamma_m$ for some $m$. Hence the sequence $\chi_n : \Gamma_n \to Q_n$ induces an isomorphism

$$\lim \pi_0 \Gamma_n \to \lim \pi_0 Q_n$$

and completes the proof of Proposition 2.16.

We remark finally that by Corollary 2.8 and Theorem 2.3 the sequence of inclusions $\Delta_n \to \Gamma_n$ induces a weak homotopy equivalence

$$\lim \Delta_n \to \lim \Gamma_n.$$
§3. BANACH ALGEBRAS

This section is devoted to the proof of Theorem 2.3. Let us recall part of the statement of that theorem: the set of elements in a Banach algebra whose square is the negative of the identity is a deformation retract of the set of elements whose spectrum contains no real numbers. We shall write down an explicit formula for the deformation retraction in terms of certain standard notions from the analytic theory of Banach algebras. The necessary background material is recorded in the following four propositions. We write \( \text{spec}(g) \) for the spectrum of an element \( g \) in a Banach algebra.

3.1. Proposition. \( \text{spec}(g) \) is a compact subset of the plane [7, Chapter 1, p.6].

3.2. Proposition. If \( g \) belongs to a real Banach algebra, then \( \text{spec}(g) \) is self conjugate [7, Chapter 1, p.6].

3.3. Proposition. \( \text{spec}(g) \) is an upper semicontinuous function of \( g \) [4, pp.168, 171].

3.4. Proposition. Let \( g \) be an element of a complex Banach algebra \( A \) and \( D \) an open subset of the plane containing \( \text{spec}(g) \). Write \( A(D) \) for the algebra of complex valued functions locally holomorphic in \( D \). Let \( \gamma \) be a closed contour in \( D \) which bounds an open set containing \( \text{spec}(g) \). Then the function

\[
A(D) \to A
\]

which assigns to \( f(z) \) the element

\[
f(g) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z1 - g} \, dz
\]

in \( A \) is a homomorphism of algebras and sends the constant map \( f(z) = 1 \) onto the identity of \( A \) and the identity map \( f(z) = z \) onto the element \( g \). Moreover \( f(g) \) does not depend on the contour \( \gamma \). The spectrum of \( f(g) \) is the image under \( f \) of the spectrum of \( g \).

The above proposition is essentially an extension theorem for analytic functions. It is a basic result in the theory of Banach algebras and appears in one form or another in books on the subject. The version given above is a slightly modified form of that given by Lorch [5, p.105] but a more comprehensive account of the theorem appears in Hille and Phillips [4, pp.168, 171]. The last sentence of the proposition is the spectral mapping theorem [4, pp. 168, 171].

We can now give the proof of Theorem 2.5. If our Banach algebra is real we work in its complexification and check, where necessary, that our constructions are invariant under complex conjugation. In the case of an algebra with involution we must also check certain invariance conditions with respect to the involution. Referring to Theorem 2.5 let \( g \) be an element of \( \Gamma \). In Proposition 3.4 let \( D \) be the complement of the real axis. Then \( \text{spec}(g) \subset D \). Write \( D^+ \) and \( D^- \) for the parts of \( D \) above and below the real axis respectively. Define a function \( e \) on \( D \) to have value \( 1 \) on \( D^+ \) and \( 0 \) on \( D^- \). Then \( e \) is locally holomorphic in \( D \) and is an idempotent in the algebra \( A(D) \). Hence \( i(2e - 1) \) has square \( -1 \). The path

\[
f(z, t) = tz + (1 - t)(2e - 1), \quad 0 \leq t \leq 1,
\]

joins the identity function \( f(z) = z \) to an element of square \(-1\) in \( A(D) \). We claim that
the extended function
\[ f(g, t) = tg + (1 - t)i(2e(g) - 1) \]
given by Proposition 3.4 defines a deformation retraction of \( \Gamma \) onto \( \Delta \). To check continuity of \( f(g, t) \) as a function of two variables it is sufficient to check that \( e(g) \) is continuous as a function of \( g \in \Gamma \). By Propositions 3.3 and 3.1 we can choose a contour \( \gamma \) in \( D \) which encloses the spectra of all elements in a suitably small neighbourhood \( N \) of \( g \). Integration over \( \gamma \) is continuous as a function of the integrand. Hence \( e(g) \) is a continuous function in the neighbourhood \( N \) and so continuous in \( \Gamma \). Next observe that \( \text{spec} f(g, t) \) contains no real numbers because, by the last sentence of Proposition 3.4 \( \text{spec} f(g, t) \) is the set of points
\[ tz + (1 - t)i(2e(z) - 1), \quad z \in \text{spec} \, g, \quad 0 \leq t \leq 1, \]
which is in \( D \). Using Proposition 3.4 we see that \( f(g, 0) \) has square \(-1\). By inspection \( f(g, 1) = g \). Finally, when \( g^2 = -1 \), we have
\[
e(g) = \frac{1}{2\pi i} \int_{\gamma} \frac{e(z)}{z1 - g} \, dz
\]
\[
= \frac{1}{2\pi i} \int_{\gamma} \frac{e(z)(z1 + g)}{1 + z^21} \, dz
\]
\[
= \frac{1}{2\pi i} \int_{\gamma} \frac{ze(z)}{1 + z^21} \, dz + \frac{g}{2\pi i} \int_{\gamma} \frac{e(z)}{1 + z^21} \, dz
\]
\[
= \frac{1}{2} - i\frac{g}{2}
\]
Hence \( f(g, t) = g \) when \( g \in \Delta \). We have now verified that \( f(g, t) \) defines a deformation retraction \( \Gamma \rightarrow \Delta \) at least in the case of a complex Banach algebra. For the case of real Banach algebras, suppose \( g \) is self conjugate, then
\[
\overline{e(g)} = -\frac{1}{2\pi i} \int_{\gamma} \frac{\overline{e(z)}}{z1 - \overline{g}} \, dz
\]
Now \( e(z) = 1 - e(\overline{z}) \) and by choosing \( \gamma \) to be symmetric about the real axis we obtain
\[
\overline{e(g)} = \frac{1}{2\pi i} \int_{\gamma} \frac{1 - e(z)}{z1 - g} \, dz
\]
\[
= 1 - e(g)
\]
It follows that \( f(g, t) \) is invariant under complex conjugation. Finally in the case of an algebra with involution \( \sigma \), suppose \( \sigma g = -g \). Choose the contour \( \gamma \) to be radially symmetric about the origin then
\[
\sigma e(g) = \frac{1}{2\pi i} \int_{\gamma} \frac{e(z)}{z1 - \sigma(g)} \, dz
\]
\[
= \frac{1}{2\pi i} \int_{\gamma} \frac{e(z)}{z1 + g} \, dz
\]
This implies that \( \sigma f(g, t) = -f(g, t) \). Hence in all cases \( f(g, t) \) defines a deformation retraction \( \Gamma \rightarrow \Delta \) and this completes the proof of Theorem 2.3.

§4. THE THEOREMS OF BOTT

The main Theorem 1.8 is applied to Clifford algebras to obtain the Bott periodicity theorems for the infinite real and complex general linear groups. The various stages of the Bott theorem [3] can be obtained by identifying Clifford algebras in terms of matrix algebras or more directly by applying Theorem 1.8 to suitably chosen matrix algebras and involutions.

The first task is to bring Theorems 1.6 and 1.8 into a form suitable for the applications. The base point \( k \) of \( \hat{G} \) (Definition 1.1) acts by conjugation on the algebra \( A \) and defines an involution which we call \( \tau \). The map \( G \rightarrow \hat{G} \) which sends \( g \in G \) onto \( g^{-1}kg \in \hat{G} \) has kernel \( G' \) (the fixed point set of \( \tau \)). Hence there is an induced map of the homogeneous space of left cosets \( G/G' \) into \( G \). This map is not in general onto. If \( X \) is a topological space with base point we write \( X_\ast \) for the base point component. The base point of a homogeneous space is the coset containing the identity.

4.1. Lemma. The map \((G/G')_\ast \rightarrow \hat{G}_\ast\) induced by \( g \rightarrow g^{-1}kg \) is one-one and onto.

In the case of an algebra with involution \( \sigma \) recall that \( k \) lies in \( G^{-\sigma} \). Hence \( \tau \) and \( \sigma \) commute.

4.2. Lemma. The map \((G^\sigma/(G^\sigma)\tau)_\ast \rightarrow \hat{G}^{-\sigma}_\ast\) induced by \( g \rightarrow g^{-1}kg \) is one-one and onto.

We shall give a proof of this lemma. The proof of Lemma 4.1 is obtained by suppressing the appropriate statements involving \( \sigma \).

Proof. It is sufficient to show that \( G^\sigma_\ast \) maps onto \( \hat{G}^{-\sigma} \). For this we demonstrate that the image of \( G^\sigma_\ast \) is both open and closed in \( \hat{G}^{-\sigma} \). The following proof of this fact was suggested to me by J. F. Adams.

Define an equivalence relation \( \sim \) on \( \hat{G}^{-\sigma} \) by \( u \sim v \) if \( u = g^{-1}vg \) where \( u, v \in \hat{G}^{-\sigma} \) and \( g \in G^\sigma_\ast \). It will be shown that each equivalence class is open and therefore closed. The equivalence class containing \( k \) is precisely the image of \( G^\sigma_\ast \). To show that each equivalence class is open consider elements \( u, u(1 + \delta) \in \hat{G}^{-\sigma} \) where \( \delta \) is small. We have the relations

\[
\sigma \delta = \delta u^{-1}(1 + \delta)u = (1 + \delta)^{-1}.
\]

Now \( \delta \) generates a cyclic algebra whose closure contains the elements \((1 + \delta)^\pm\) and \((1 + \delta)^{-\pm}\) defined uniquely for small \( \delta \) by the binomial expansion. Using the formulae above we see that \( \sigma \) and conjugation by \( u \) are automorphisms of the closure of the algebra.
generated by $\delta$ and we have the relations

\[ \sigma(1 + \delta)^*= (1 + \delta)^*, \quad u^{-1}(1 + \delta)u = (1 + \delta)^{-1} \]

Indeed $\sigma(1 + t\delta)^* = (1 + t\delta)^*$, $0 \leq t \leq 1$, so that $(1 + \delta)^*$ lies in $G_*$. The equation

\[ (1 + \delta)^*_u(1 + \delta)(1 + \delta)^{-1} = u \]

shows that $u(1 + \delta) \sim u$. Hence all elements in a sufficiently small neighbourhood of $u$ are equivalent to $u$ and this completes the proof.

It need not concern us whether the maps of Lemma 4.1 and 4.2 are homeomorphisms because a one-one onto map of Hausdorff spaces is certainly a weak homotopy equivalence and this is sufficient for our purpose.

Our next observations concern the relative loop space $\Omega(G, G^\sigma)$. The projection map $G \to G/G^\sigma$ has a local cross-section because $G$ is the group of invertible elements in a Banach algebra and $G^\sigma$ is a closed subgroup [6, p.571]. Hence the map $G \to G/G^\sigma$ is a fibration. It follows that the obvious map $\Omega(G, G^\sigma) \to \Omega G/G^\sigma$ is a Serre fibration. The fibre is the space of paths in $G^\sigma$ which begin at the identity. Since this space is contractible we have

4.3. LEMMA. The map $\Omega(G, G^\sigma) \to \Omega G/G^\sigma$ is a weak homotopy equivalence.

There are one or two comments to be made about weak homotopy equivalence and base point components. First of all note that in the spaces with which we are working components and path components are the same thing because the group of invertible elements in a Banach algebra is locally pathwise connected. We shall use $Z_*$ for the path component containing the base point of $Z$. For any space $X$ the set $[X, Z_\ast]$ is a subset of $[X, Z]$. It follows that a weak homotopy equivalence which is base point preserving induces a weak homotopy equivalence of base point path components. Using Corollary 2.7 one easily checks that if a sequence $f_n : Y_n \to Z_n$ of base point preserving maps of $T_1$ spaces induces a weak homotopy equivalence $\lim Y_n \to \lim Z_n$ then it also induced a weak homotopy equivalence $\lim(Y_n)_\ast \to \lim(Z_n)_\ast$. We are now ready to reformulate Theorems 1.6 and 1.8.

4.4. THEOREM. Let $\tau$ be the involution defined by conjugation by $k \in G$. Suppose $k$ can be joined to $-k$ in $G$. Then there is a weak homotopy equivalence

\[ \lim(G_n/G^\sigma_n)_\ast \to \lim(\Omega G_n)_\ast \]

4.5. THEOREM. Let $\tau$ be the involution defined by conjugation by $k \in \hat{G}^{-\sigma}$. Suppose $k$ can be joined to $-k$ in $\hat{G}^{-\sigma}$. Then there is a weak homotopy equivalence

\[ \lim(G_n/(G^\sigma_n))_\ast \to \lim(\Omega G_n/G^\sigma_n)_\ast \]

Proofs. The maps $(G_n/G^\sigma_n)_\ast \to (\hat{G}_n)_\ast$ of Lemma 4.1 are compatible with the usual inclusions and each is a weak homotopy equivalence. Hence

\[ \lim(G_n/G^\sigma_n)_\ast \to \lim(\hat{G}_n)_\ast \]

is a weak homotopy equivalence. From Theorem 1.6 and remarks made above there is a
weak homotopy equivalence

$$\lim (G_n^\sigma) \rightarrow \lim (\Omega G_n)$$

This proves Theorem 4.4. With the addition of Lemma 4.3 the proof of Theorem 4.5 is similar.

4.6. EXAMPLE. Let $C'$ be the real or complex Clifford algebra on $r$ generators $e_1, \ldots, e'$ and relations

$$e_i^2 = -1, \quad e_i e_j + e_j e_i = 0, \quad i \neq j.$$

Let $\sigma$ be the main involution of $C'$ defined on generators by $\sigma e_i = -e_i$. Now $(C', \sigma)$ is generated by

$$e_1 e_r, \ldots, e_{r-1} e,$$

and can therefore be identified with $C'^{-1}$; moreover if we choose $e_r$ as the base point $k$ of the general theory then the involution $\tau$ on $(C')^\sigma$ becomes the main involution on $C'^{-1}$. Hence we can identify $(C'^{-1})^\tau$ with $C'^{-2}$. Theorem 4.5 applies to the pair $(C', \sigma)$ for $r \geq 2$. Hence we have

4.7. PROPOSITION. Let $G'$ be the group of invertible elements in the Clifford algebra $C'$. There is a weak homotopy equivalence

$$\lim (G'_n/G'_{n-1}) \rightarrow \lim (\Omega G'_n/G'_n)$$

when $r \geq 1$.

The following table gives the Clifford algebras in terms of matrix algebras [2, p.12]:

<table>
<thead>
<tr>
<th>$r$</th>
<th>real</th>
<th>complex</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$C'$</td>
<td>$C'$</td>
</tr>
<tr>
<td>2</td>
<td>$H$</td>
<td>$C \oplus C$</td>
</tr>
<tr>
<td>3</td>
<td>$H \oplus H$</td>
<td>$C_3$</td>
</tr>
<tr>
<td>4</td>
<td>$H_3$</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$C_4$</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$R_8$</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>$R_8 \oplus R_8$</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>$R_{16}$</td>
<td>$C'^{\ast 2}$</td>
</tr>
</tbody>
</table>

Using Proposition 4.7 the various stages of the Bott periodicity theorem can now be written down, for example choosing $r = 5$ we have a weak homotopy equivalence

$$\lim (GL(4n, C)/GL(2n, H)) \rightarrow \lim (GL(8n, R)/GL(4n, C))$$

To get the periodicity theorems for the real and complex general linear groups observe that, in the real case, $G^2_n/G^6_n$ can be identified with $GL(8n; R)$ and $G^1_n/G^4_n$ can be identified with $GL(16.8n; R)$; in the complex case $G^2_n/G^4_n$ can be identified with $GL(2n, C)$ and $G^3_n/G^6_n$ can be identified with $GL(4n, C)$. Hence by iterating Proposition 4.7 the appropriate number of times we obtain weak homotopy equivalences

$$\lim GL(8n, R) \rightarrow \lim (\Omega^8 GL(16.8n, R))$$
\[
\lim GL(2n, C) \to \lim(\Omega^2 GL(4n, C))_.
\]

4.8. \textbf{Remark.} If we use the standard inclusions \(GL(r) \to GL(r + 1)\) then \(\lim GL(r)\) and \(\lim GL(mr)\), where \(m\) is a fixed integer, are essentially the same thing. Since the limit sign and loop sign commute up to weak homotopy equivalence we can write simply

\[
GL(\infty, R)_* \cong \Omega^8 GL(\infty, R)
\]

\[
GL(\infty, C)_* \cong \Omega^2 GL(\infty, C)
\]

where \(\cong\) denotes weak homotopy equivalence and \(GL(\infty) = \lim_{r \to \infty} GL(r)\). We say that \(GL(\infty, R)\) is periodic with period eight and \(GL(\infty, C)\) is periodic with period two.

4.9. \textbf{Remark.} What we have said about the algebra \(C'\) applies with appropriate modifications to the algebra \(A \otimes C'\) where \(A\) is any Banach algebra. In particular the periodicity theorem applies in the sense that \(\lim G_*\) is periodic with period eight in the real case and period two in the complex case, where as usual \(G\) is the group of invertible elements in \(A\).

4.10. \textbf{Remark.} As remarked at the beginning of this section the theorems of Bott can be obtained without appealing to the structure theorems of Clifford algebras by applying Theorems 4.4 and 4.5 directly to matrix algebras. Let us give one illustration. Choose for \(A\) the algebra \(C_2\) and for \(k\) the matrix

\[
\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}
\]

Then \(G_* = GL(2n, C)\) and \(G' = GL(n, C) \times GL(n, C)\). By Theorem 4.4 there is a weak homotopy equivalence

\[
\lim(GL(2n, C)/GL(n, C) \times GL(n, C))_* \to \lim(\Omega GL(2n, C))_*.
\]

As a matter of fact Theorem 4.4 can be deduced from Theorem 4.5 by using the direct sum of \(A\) with itself and the involution which switches the factors.

\section*{REFERENCES}


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