Column Reduced Rational Matrix Functions With Given Null-Pole Data in the Complex Plane

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ABSTRACT

Explicit formulas are given for rational matrix functions which have a prescribed null-pole structure in the complex plane and are column reduced at infinity. A full parametrization of such functions is obtained. The results are specified and developed further for matrix polynomials.

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0. INTRODUCTION

A rational $m \times m$ matrix function W is said to be *column reduced at infinity* if it admits a factorization

$$W(\lambda) = E(\lambda)\begin{pmatrix} \lambda^{\kappa_1} & & & \\ & \lambda^{\kappa_2} & & \\ & & \ddots & \\ & & & \lambda^{\kappa_m} \end{pmatrix}, \tag{0.1}$$

where E is a square rational matrix function which has no pole or zero at infinity (i.e., E is analytic and invertible at infinity). This concept was introduced into mathematical systems theory by [18] in the context of matrix polynomials. For a regular rational matrix function W with columns $w_1, \dots, w_m$ and column indices $\kappa_1 \geqslant \cdots \geqslant \kappa_m$, column reducedness at infinity is equivalent to the requirement (see [4]) that the vector functions $\lambda^{-\kappa_1}w_1(\lambda), \dots, \lambda^{-\kappa_k}w_k(\lambda)$ associated with positive columns indices $\kappa_1 \geqslant \cdots \geqslant \kappa_k > 0$ form a canonical set of right pole functions for W at infinity of orders $\kappa_1, \dots, \kappa_k$, respectively, and the row vector functions $\lambda^{\kappa_{m-l+1}}\tilde{w}_{m+1-l}(\lambda), \dots, \lambda^{\kappa_m}\tilde{w}_m(\lambda)$ associated with the last l rows $\tilde{w}_{m+1-l}(\lambda), \dots, \tilde{w}_m(\lambda)$ of $W(\lambda)^{-1}$, where l is the number of negative column indices, form a canonical set of left null functions for W at infinity of respective orders $\kappa_{m-l+1}, \dots, \kappa_m$. Intuitively, column reducedness means that pole-zero structure at infinity can be read off from looking at the columns separately; there are no poles or zeros arising from interactions among the columns.

The present paper concerns the problem of constructing a column reduced rational matrix function with prescribed null-pole data. More precisely, given an admissible Sylvester data set $\tau = (C, A_\pi; A_\zeta, B; \Gamma)$ (see Section 1 for the definition), the problem is to find a rational matrix function W such that

(i) W has τ as its $\mathbb{C}$-null-pole triple,
(ii) W is column reduced at infinity.

If we suppose that A_π is vacuous, then the problem is reduced to finding a matrix polynomial which has a prescribed (left) $\mathbb{C}$-null pair, and whose column indices at infinity coincide with the partial indices in its Smith-McMillan form at infinity.

The problem of construction of column reduced rational matrix functions with prescribed $\mathbb{C}$-null-pole triple arises naturally in the context of the problem of parametrizing rational matrix functions meeting a number of prescribed bitangential interpolation conditions which also have a prescribed McMillan degree; for details on this, see [3]. Specifically, a function W as in (i) and (ii) provides the coefficient matrix for the linear fractional map involved in the parametrization of the set of all such rational interpolants. The column-reducedness property of W is crucial in determining the McMillan degree of an interpolant in terms of the associated pair of rational functions used as the free parameters in the linear fractional map. We expect the results of this paper to lead to state-space formulas for the interpolants (see also [1] for an earlier state-space approach to this bitangential prescribed-McMillan-degree interpolation problem).

The problem of finding a column reduced rational matrix function with a prescribed $\mathbb{C}$-null-pole triple is also closely related to Wiener-Hopf factorization at infinity. Indeed, if G is a given $m \times m$ rational matrix function and W is constructed to be a column reduced rational matrix function as in (0.1) having the same $\mathbb{C}$-null-pole triple as does G, then $G(\lambda) = W(\lambda)F(\lambda)$, where F is a unimodular matrix polynomial, and hence (0.1) yields

$$G(\lambda) = E(\lambda)\begin{pmatrix} \lambda^{\kappa_1} & & & \\ & \lambda^{\kappa_2} & & \\ & & \ddots & \\ & & & \lambda^{\kappa_m} \end{pmatrix} F(\lambda),$$

which is a Wiener-Hopf factorization of G at infinity. State-space formulas for Wiener-Hopf factorization in turn are useful in solving singular integral, Wiener-Hopf, or Toeplitz equations with rational matrix symbols (see [6] and [10]).

In [11] a problem related to (i) and (ii) has been solved, namely the problem of finding a rational matrix function W satisfying (i) and

(ii$'$) W has the minimal possible McMillan degree.

A solution of our problem automatically yields a solution of the problem with (ii$'$) in place of (ii), but the converse is not necessarily true. For matrix polynomials condition (ii$'$) means that the solution has no zero at infinity. However, not every matrix polynomial which has no zero at infinity is column reduced at infinity. For example, the matrix polynomial

$$L(\lambda) = \begin{pmatrix} \lambda^2 & \lambda \\ 0 & 1 \end{pmatrix}$$

has no zeros at infinity, but is not column reduced. In fact, there does not exist an invertible constant matrix D for which $L(\lambda)D$ is column reduced at infinity.

In the present paper we construct a rational matrix function W satisfying (i) and (ii), and we give an explicit formula for such a solution W in realized form. We also parametrize the set of all W satisfying (i) and (ii). These results appear in Section 3. In Section 6 they are specified and developed further for matrix polynomials. Section 1 surveys the general theory of column reduced rational matrices. Preliminary material about null-pole data are collected together in the second section. Section 4 contains some auxiliary material about observable and controllable pairs which we need in Section 5 for the proof of the second main theorem of Section 3.

We close the introduction with a list of notation and terminology which will be used throughout the paper. An $n \times n$ matrix X (over the complex numbers) will often be identified with the linear transformation from $\mathbb{C}^m$ (complex m-tuples written as columns) into $\mathbb{C}^n$ associated with left multiplication. The symbol Ker X denotes the subspaces of $\mathbb{C}^m$ equal to the kernel of this linear transformation, while Im X denotes the subspace of $\mathbb{C}^n$ equal to its range of image. For X a square matrix, $\sigma(X)$ denotes the spectrum or set of eigenvalues (a finite subset of $\mathbb{C}$) of the associated linear transformation.

1. COLUMN REDUCEDNESS

Let $W(\lambda)$ be a regular $m \times m$ rational matrix function. For $j = 1, \dots, m$ let κ_j denote the highest power of λ in the jth column of $W(\lambda)$. Then $W(\lambda)$ is represented as

$$W(\lambda) = E(\lambda)\begin{pmatrix} \lambda^{\kappa_1} & & & \\ & \lambda^{\kappa_2} & & \\ & & \ddots & \\ & & & \lambda^{\kappa_m} \end{pmatrix}, \tag{1.1}$$

where

$$E(\lambda) = E_0 + \lambda^{-1}E_1 + \lambda^{-2}E_2 + \cdots. \tag{1.2}$$

The integer κ_j is called the jth *column index* of $W(\lambda)$, and $W(\lambda)$ is said to be *column reduced* at infinity if E_0 in (1.2) is invertible.

THEOREM 1.1. *A regular $m \times m$ rational matrix function W is column reduced at infinity if and only if the positive column indices of W coincide with the partial pole multiplicities and the negative column indices of W (taken with opposite sign) coincide with the partial zero multiplicities of W in the Smith-McMillan form of W at infinity.*

This theorem was first established [17] in 1979. It tells us that for a column reduced regular rational matrix function the poles and zeros at infinity may be read off from the columns separately.

In the next three theorems we state a number of important known properties of column reduced rational matrix functions.

THEOREM 1.2 (The predictable-degree property). *Let W be an $m \times m$ regular rational matrix function. Then W is column reduced at infinity if and only if for any $\mathbb{C}^m$-valued polynomial vector $p(\lambda)$ the column index of $W(\lambda)p(\lambda)$ (i.e., the highest power of λ in the column $W(\lambda)p(\lambda)$) is equal to*

$$\max_{i;\, p_i(\lambda)\neq 0} \{\deg p_i(\lambda) + \kappa_i\}, \tag{1.3}$$

where $p_i(\lambda)$ is the ith entry of $p(\lambda)$, and κ_i is the ith column index of $W(\lambda)$.

For matrix polynomials Theorem 1.2 was discovered by Forney [7]. For the proof of this theorem, see Theorem 6.3-13 in [14]. Although the latter theorem concerns only matrix polynomials, its proof may also be used for rational matrix functions.

THEOREM 1.3. *Let $W(\lambda)$ be a regular $m \times m$ rational matrix function. Then there exists a unimodular matrix polynomial $U(\lambda)$ such that*

$$\tilde{W}(\lambda) := W(\lambda)U(\lambda) \tag{1.4}$$

is column reduced at infinity.

Example 6.3.2 of [14] illustrates how one can find such a $U(\lambda)$. Note that if $W(\lambda)$ is not column reduced, then $\tilde{W}(\lambda)$ in (1.4) is obtained by applying elementary column operations to reduce the individual column indices until column reducedness is achieved. Theorem 1.3 may also be seen as a special case of Theorem I.2.1 in [6], which concerns Wiener-Hopf factorization of rational matrix functions relative to a contour. By taking a sufficiently large contour Theorem I.2.1 in [6] reduces to Theorem 1.3.

THEOREM 1.4. *Let $W(\lambda)$ and $\tilde{W}(\lambda)$ be $m \times m$ regular rational matrix functions which are column reduced at infinity. If*

$$\tilde{W}(\lambda) = W(\lambda)U(\lambda)$$

for a unimodular matrix polynomial $U(\lambda)$, then $W(\lambda)$ and $\tilde{W}(\lambda)$ have the same column indices at infinity and for the entries $U_{ij}(\lambda)$ of $U(\lambda)$ the following holds:

(a) $U_{ij}(\lambda) = 0$ if $\kappa_j < \kappa_i$
(b) $U_{ij}(\lambda)$ is a constant if $\kappa_j = \kappa_i$
(c) $U_{ij}(\lambda)$ has degree $\leqslant \kappa_j - \kappa_i$ if $\kappa_j > \kappa_i$

Here $\kappa_1, \dots, \kappa_m$ are the column indices of $W(\lambda)$ and $\tilde{W}(\lambda)$ at infinity.

For the proof of the equality of the indices, see Theorem 6.3-14 in [14] (the arguments given there are also valid for the rational case). The second part of Theorem 1.4 is an immediate corollary of the predictable-degree property of column reduced rational matrix functions (Theorem 1.2). Theorem 1.4 may also be viewed as a special case of Theorems I.1.1 and I.1.2 in [6], which concern the freedom one has in Wiener-Hopf factorization.

The column reducedness of a regular rational matrix function at a point $\lambda_0 \in \mathbb{C}$ can be defined in a similar way to column reducedness at infinity. Furthermore, we mention that in the papers [17] and [5] column reducedness at a point in $\mathbb{C}$ or at infinity is studied for rational matrix functions with full column rank but not necessarily regular.

2. NULL-POLE TRIPLES AND PROBLEM FORMULATION

In what follows W is a regular $m \times m$ rational matrix function. First we explain what is meant by the null-pole structure of W on $\mathbb{C}$. We begin with the poles. A pair of matrices (C, A), where A is $n \times n$ and C is $m \times n$, is called a *right pole pair* of W relative to $\mathbb{C}$ if

(P_1) $\bigcap_{j=1}^{n} \operatorname{Ker} CA^{j-1} = (0)$;
(P_2) there exists an $n \times m$ matrix $\tilde{B}$ such that $\operatorname{Im}(\tilde{B} \;\; A\tilde{B} \;\; \cdots \;\; A^{n-1}\tilde{B}) = \mathbb{C}^n$ and

$$W(\lambda) - C(\lambda - A)^{-1}\tilde{B}$$

is a polynomial in λ.

In this case $C(\lambda - A)^{-1}\tilde{B}$ is a minimal realization of the sum of the singular parts of the poles of W in $\mathbb{C}$. Pole pairs are unique up to similarity, and they may be constructed from the poles and the corresponding pole chains (see [4, Chapters 3, 4] for further details). For the null structure we employ $W(\cdot)^{-1}$. A pair of matrices (A, B), where A is $n \times n$ and B is $n \times m$, is called a *left null pair* of W relative to $\mathbb{C}$ if (A, B) is a left pole pair of W^{-1} relative to $\mathbb{C}$, that is,

(N_1) $\operatorname{Im}(B\ AB\ \cdots\ A^{n-1}B) = \mathbb{C}^n$;

(N_2) there exists an $m \times n$ matrix $\tilde{C}$ such that $\bigcap_{j=1}^{n} \operatorname{Ker} \tilde{C}A^{j-1} = (0)$ and

$$W(\lambda)^{-1} - \tilde{C}(\lambda - A)^{-1}B$$

is a polynomial in λ.

In this case $(\lambda - A)^{-1}BW(\lambda)p(\lambda)$ is a polynomial in λ for every $\mathbb{C}^m$-valued vector polynomial p such that Wp is also a polynomial. In fact, the latter property can be taken as the starting point for the definition of a left null pair.

Additional information about the connections between null structure and pole structure of W is encoded in the null-pole subspace, which is defined as

$$W\mathscr{P}_{m\times 1} = \{Wp \mid p \in \mathscr{P}_{m\times 1}\},$$

where $\mathscr{P}_{m\times 1}$ denotes the set of all $\mathbb{C}^m$-valued vector polynomials. It turns out (see [4]) that given a left null pair (A_ζ, B) and a right pole pair (C, A_π) of W relative to $\mathbb{C}$, there exists a unique $n_\zeta \times n_\pi$ matrix Γ (where n_ζ is the order of A_ζ and n_π is the order of A_π) such that

$$\begin{aligned} W\mathscr{P}_{m\times 1} = \Big\{ & C(\lambda - A_\pi)^{-1}x + h(\lambda) \mid x \in \mathbb{C}^{n_\pi}, h \in \mathscr{P}_{m\times 1} \\ & \text{such that } \Gamma_x \text{ is the sum of all the residues} \\ & \text{of } (\lambda - A_\zeta)^{-1}Bh(\lambda) \text{ in } \mathbb{C}\Big\}. \end{aligned}$$

The quintet $(C, A_\pi; A_\zeta, B; \Gamma)$ is called a $\mathbb{C}$-*null-pole triple* for W, and one refers to Γ as the *coupling matrix* of the null-pole triple. The coupling matrix satisfies the following Sylvester equation:

$$\Gamma A_\pi - A_\zeta \Gamma = BC. \tag{2.1}$$

Hence in case $\sigma(A_\pi) \cap \sigma(A_\zeta) = \varnothing$, the matrix Γ is uniquely determined from the pole pair (C, A_π) and the null pair (A_ζ, B); when $\sigma(A_\pi)$ and $\sigma(A_\zeta)$ intersect, Γ adds the additional coupling information between pole data and zero data required to get a complete description of the polynomial module $W\mathscr{P}_{m\times 1}$; see [4, Chapter 4] for further details.

To state the main problem solved in this paper we need the notion of an admissible Sylvester data set. Let A_π and A_ζ be square matrices of order n_π

and n_ζ, respectively, let B and C be matrices of sizes $n_\zeta \times m$ and $m \times n_\pi$, respectively, and let Γ be an $n_\zeta \times n_\pi$ matrix. The quintet $(C, A_\pi; A_\zeta, B; \Gamma)$ is called an *admissible Sylvester data set* if

$$\bigcap_{j=1}^{n_\pi - 1} \operatorname{Ker} CA_\pi^{j-1} = \{0\}, \tag{2.2a}$$

$$\operatorname{Im}\begin{pmatrix} B & A_\zeta B & \cdots & A_\zeta^{n_\zeta - 1}B \end{pmatrix} = \mathbb{C}^{n_\zeta}, \tag{2.2b}$$

and Γ satisfies the Sylvester equation (2.1). The problem we shall deal with is the following. Given an admissible Sylvester data set $\tau = (C, A_\pi; A_\zeta, B; \Gamma)$, construct all regular rational matrix functions W such that

(i) W has τ as its $\mathbb{C}$-null pole triple,
(ii) W is column reduced at infinity.

The matrix-polynomial case is of special interest. Assume $W(\lambda) = L(\lambda)$ is an $m \times m$ regular matrix polynomial. Since a polynomial has no poles in $\mathbb{C}$, the matrix A_π in a right pole pair (C, A_π) of $L(\lambda)$ is necessarily vacuous. It follows that a $\mathbb{C}$-null-pole triple of $L(\lambda)$ is of the form $(0, 0; A_\zeta, B; 0)$, where (A_ζ, B) is a left null pair of $L(\lambda)$ and the coupling matrix maps $\mathbb{C}^{n_\zeta}$ into $\{0\}$. Thus for matrix polynomials our problem reduces to the following question. Given a pair of matrices (A, B), where A is $n \times n$, B is $n \times m$, and

$$\operatorname{Im}(B \;\; AB \;\; \cdots \;\; A^{n-1}B) = \mathbb{C}^n,$$

construct all regular $m \times m$ matrix polynomials which are column reduced at infinity and have (A, B) as left null pair.

3. MAIN THEOREMS

We first introduce the notation and linear transformations needed to state the main results. After the statements we present some perspective on the formulas and illustrate them with a simple example.

In what follows $\tau = (C, A_\pi; A_\zeta, B; \Gamma)$ is an admissible Sylvester data set and α is a complex number such that

$$\alpha \notin \sigma(A_\pi) \cup \sigma(A_\zeta) \cup \{0\}. \tag{3.1}$$

Our aim is to construct a column reduced rational matrix function which has τ as its $\mathbb{C}$-null-pole triple. For this purpose we need the following notation.

We write $\mathscr{X}_\pi$ for the space on which A_π acts, and $\mathscr{X}_\zeta$ for the space on which A_ζ acts. In particular, $\Gamma: \mathscr{X}_\pi \to \mathscr{X}_\zeta$. Let N be a complement of Ker Γ in $\mathscr{X}_\pi$, and let K be a complement of Im Γ in $\mathscr{X}_\zeta$. We choose $\Gamma^+: \mathscr{X}_\zeta \to \mathscr{X}_\pi$ to be the generalized inverse of Γ such that Im $\Gamma^+ = N$ and Ker $\Gamma^+ = K$. Let ρ_π be the projection of $\mathscr{X}_\pi$ onto Ker Γ along N, and ρ_ζ the projection of $\mathscr{X}_\zeta$ onto K along Im Γ. Thus $\rho_\pi = I - \Gamma^+\Gamma$ and $\rho_\zeta = I - \Gamma\Gamma^+$. We write η_π for the embedding of Ker Γ into $\mathscr{X}_\pi$, and η_ζ for the embedding of K into $\mathscr{X}_\zeta$.

We may choose bases $\{d_{jk}\}_{k=1,}^{\alpha_j}{}_{j=1}^{t}$ and $\{g_{jk}\}_{k=1,}^{\omega_j}{}_{j=1}^{s}$ in Ker Γ and K, respectively, such that the following hold:

(a$_1$) $\{d_{jk}\}_{k=2,}^{\alpha_j}{}_{j=1}^{t}$ is a basis of Ker $\Gamma \cap$ Ker C,
(a$_2$) $A_\pi d_{j,k+1} = d_{j,k}$, $k = 1, \dots, \alpha_j - 1$;
(b$_1$) $\{g_{j\omega_j}\}_{j=1}^{s}$ is a basis for a complement of Im Γ in Im Γ + Im B,
(b$_2$) $A_\zeta g_{j,k+1} - g_{jk} \in$ Im Γ + Im B, $k = 0, \dots, \omega_j - 1$, where $g_{j,0} := 0$.

Here we assume that $\alpha_1 \geqslant \cdots \geqslant \alpha_t$ and $\omega_1 \geqslant \cdots \geqslant \omega_s$. In the terminology of [11] (see also [4, p. 154]) the vectors $\{d_{jk}\}_{k=1,}^{\alpha_j}{}_{j=1}^{t}$ form an outgoing basis for τ at infinity, and $\{g_{jk}\}_{k=1,}^{\omega_j}{}_{j=1}^{s}$ an incoming basis for τ at infinity. With these bases we associate the following two operators.

$$S\colon \text{Ker}\,\Gamma \to \text{Ker}\,\Gamma, \qquad Sd_{jk} = d_{j,k+1} \quad \left(d_{j,\alpha_j+1} := 0\right), \tag{3.2}$$

$$T\colon K \to K, \qquad Tg_{jk} = g_{j,k+1} \quad \left(g_{j,\omega_j+1} := 0\right). \tag{3.3}$$

In the sequel Γ_1: $K \to$ Ker Γ is an arbitrary linear transformation which we may choose freely.

It is not difficult to show (see the proof of Theorem 3.1 below) that

$$\text{Im}\left(A_\zeta \eta_\zeta T - \eta_\zeta\right) \subset \text{Im}\left(\alpha - A_\zeta\right)\Gamma + \text{Im}\,B, \tag{3.4}$$

$$\text{Ker}\left(S\rho_\pi A_\pi - \rho_\pi\right) \supset \text{Ker}\,\Gamma\left(\alpha - A_\pi\right) \cap \text{Ker}\,C. \tag{3.5}$$

These inclusions allow us to choose operators

$$F\colon K \to \mathbb{C}^m, \qquad A_{12}\colon K \to \mathscr{X}_\pi, \tag{3.6}$$

$$H\colon \mathbb{C}^m \to \text{Ker}\,\Gamma, \qquad A_{21}\colon \mathscr{X}_\zeta \to \text{Ker}\,\Gamma \tag{3.7}$$

such that the following identities are fulfilled:

$$(\alpha - A_\zeta)\Gamma A_{12} = A_\zeta \eta_\zeta T - \eta_\zeta - BF, \qquad (3.8)$$

$$A_{21}\Gamma(\alpha - A_\pi) = S\rho_\pi A_\pi - \rho_\pi - HC, \qquad (3.9)$$

$$A_{21}\eta_\zeta(I - \alpha T) - (I - \alpha S)\rho_\pi A_{12} = \Gamma_1 T - S\Gamma_1 - HF. \qquad (3.10)$$

With the operators A_{12} and A_{21} defined by (3.8)–(3.10) we introduce

$$X := -\sum_{j=0}^{\omega_1 - 1} A_\pi^j A_{12} T^j \colon K \to \mathscr{X}_\pi, \qquad (3.11)$$

$$Y := -\sum_{j=0}^{\alpha_1 - 1} S^j A_{21} A_\zeta^j : \mathscr{X}_\zeta \to \operatorname{Ker} \Gamma. \qquad (3.12)$$

Finally, let

$$z_j = \alpha C(\alpha I - A_\pi)^{-1} d_{j\alpha_j}, \qquad j = 1, \dots, t, \qquad (3.13)$$

and choose vectors $y_j \in \mathbb{C}^m$, $j = 1, \dots, s$, such that

$$(A_\zeta - \alpha I)^{-1} B y_j - (I - \alpha T)^{-1} g_{j1} \in \operatorname{Im} \Gamma, \qquad j = 1, \dots, s. \quad (3.14)$$

We shall see later that the vectors $z_1, \dots, z_t, y_1, \dots, y_s$ are linearly independent vectors in $\mathbb{C}^m$, and hence one may choose vectors $z_{t+1}, \dots, z_{m-s}$ in $\mathbb{C}^m$ such that the following matrix is invertible:

$$E = (z_1 \ \cdots \ z_t \ z_{t+1} \ \cdots \ z_{m-s} \ y_1 \ \cdots \ y_s). \qquad (3.15)$$

We are now ready to state the main theorems.

THEOREM 3.1. *Let $\tau = (C, A_\pi; A_\zeta, B; \Gamma)$ be an admissible Sylvester data set, and let $\alpha \in \mathbb{C}$ be as in* (3.1). *Put*

$$\begin{aligned} W(\lambda) = E &- (\lambda - \alpha)C(\lambda - A_\pi)^{-1} \\ &\times\Big\{\big[\Gamma^+ + (\alpha - A_\pi)X\rho_\zeta - \Gamma_1\rho_\zeta\big](\alpha - A_\zeta)^{-1}B \\ &\qquad + \eta_\pi(\alpha S - I)^{-1}H\Big\}E \\ &+ (\lambda - \alpha)[CX(I - \alpha T) - F](I - \lambda T)^{-1}\rho_\zeta(\alpha - A_\zeta)^{-1}BE. \end{aligned}$$

Then W is a column reduced regular $m \times m$ rational matrix function which has τ as its $\mathbb{C}$-null-pole triple. The jth column index κ_j of W is given by

$$\kappa_j = \begin{cases} -\alpha_j, & 1 \leqslant j \leqslant t, \\ 0, & t + 1 \leqslant j \leqslant m - s, \\ \omega_{m-j+1}, & m - s + 1 \leqslant j \leqslant m, \end{cases}$$

where $\alpha_1 \geqslant \cdots \geqslant \alpha_t$ are the nonzero observability indices of the pair $(C|_{Ker\,\Gamma}, \rho_\pi A_\pi|_{Ker\,\Gamma})$, and $\omega_1 \geqslant \cdots \geqslant \omega_s$ are the nonzero controllability indices of the pair $(\rho_\zeta A_\zeta|_K, \rho_\zeta B)$. Moreover,

$$\begin{aligned} W(\lambda)^{-1} = E^{-1} &+ (\lambda - \alpha)E^{-1} \\ &\times\Big\{C(\alpha - A_\pi)^{-1}\big[\Gamma^+ + \eta_\pi Y(\alpha - A_\zeta) - \Gamma_1\rho_\zeta\big] \\ &\qquad + F(I - \alpha T)^{-1}\rho_\zeta\Big\}(\lambda - A_\zeta)^{-1}B \\ &+ (\lambda - \alpha)E^{-1}C(\alpha - A_\pi)^{-1}\eta_\pi(\lambda S - I)^{-1} \\ &\times[(I - \alpha S)YB + H]. \end{aligned}$$

THEOREM 3.2. *Let $\tau = (C, A_\pi; A_\zeta, B; \Gamma)$ be an admissible Sylvester data set. Every column reduced regular $m \times m$ rational matrix function W which has τ as its $\mathbb{C}$-null-pole triple is obtained via the method of Theorem* 3.1 *up to a certain constant invertible factor on the right.*

Description of column indices of column reduced rational matrix functions (or, equivalently, of factorization indices) in terms of observability and/or controllability indices as in Theorem 3.1 are known (see [13] and [16] for the case of matrix polynomials, [8] for the proper rational matrix case, and [10] for the general case). See also [3] and [15], where the column indices of a column reduced rational matrix function $W(\lambda)$ with prescribed $\mathbb{C}$-null-pole triple are given in terms of observability and controllability indices as in Theorem 3.1.

Before commencing with the proof of Theorem 3.1 we give some background information to help the reader gain some perspective on the formulas in Theorem 3.1. We also present a simple example.

Let $\tau = (C, A_\pi; A_\zeta, B; \Gamma)$ be an admissible Sylvester data set and α a complex number as in (3.1). It is known (see, e.g., [4]) that there exists a rational matrix function W which is regular at infinity and which has τ as its $\mathbb{C}$-null-pole triple if and only if Γ is invertible, and then any such W has the form

$$W(\lambda) = D + C(\lambda - A_\pi)^{-1}\Gamma^{-1}BD$$

with inverse

$$W(\lambda)^{-1} = D^{-1} - D^{-1}C\Gamma^{-1}(\lambda - A_\zeta)^{-1}B,$$

where D is an arbitrary regular $m \times m$ matrix. If Γ is not invertible, one has to add nontrivial pole-zero structure at infinity. The latter may be described by an *admissible Sylvester data set over infinity*, that is, an admissible Sylvester data set

$$\tau_\infty = \left(C_\infty, A_{\pi\infty}; A_{\zeta\infty}, B_\infty; \Gamma_\infty\right)$$

with the additional property that the matrices $A_{\pi\infty}$ and $A_{\zeta\infty}$ are nilpotent. Such a quintet τ_∞ is called an $\{\infty\}$*-null-pole triple* for the rational matrix function W if in addition

$$\begin{aligned} W\Omega_{m\times 1} = \Big\{ & C_\infty(I - \lambda A_{\pi\infty})^{-1}x + h(\lambda) \mid x \in \mathbb{C}^{n_{\pi\infty}}, \\ & h \in \Omega_{m\times 1} \text{ such that } \Gamma_\infty x \text{ is equal to the} \\ & \text{residue of } (I - \lambda A_{\zeta\infty})^{-1}Bh(\lambda) \text{ at infinity}\Big\}, \end{aligned}$$

where $\Omega_{m\times 1}$ stands for the set of all strictly proper $\mathbb{C}^m$-valued rational functions.

Given now the admissible Sylvester data set τ, the complex number α, and an admissible Sylvester data set over infinity τ_∞, then it is known (see [9] or [4]) that there is a regular $m \times m$ rational matrix function W with $\mathbb{C}$-null-pole triple equal to τ and with $\{\infty\}$-null-pole triple equal to τ_∞ if and only if the matrix

$$\tilde{\Gamma} = \begin{pmatrix} \Gamma & \Gamma_{12} \\ \Gamma_{21} & \Gamma_\infty \end{pmatrix}$$

is invertible, where Γ_{12} and Γ_{21} are determined as the unique solution of the Stein equations

$$A_\zeta \Gamma_{12} A_{\pi\infty} - \Gamma_{12} = BC_\infty, \qquad A_{\zeta\infty}\Gamma_{21} A_\pi - \Gamma_{21} = B_\infty C.$$

In this case, any such W is given by

$$W(\lambda) = D + (\lambda - \alpha)(C \quad C_\infty)\begin{pmatrix} (\lambda - A_\pi)^{-1} & 0 \\ 0 & (I - \lambda A_{\pi\infty})^{-1} \end{pmatrix}$$

$$\times \begin{pmatrix} \Gamma & \Gamma_{12} \\ \Gamma_{21} & \Gamma_\infty \end{pmatrix}^{-1} \begin{pmatrix} (A_\zeta - \alpha)^{-1} BD \\ (I - \alpha A_\infty)^{-1} B_\infty D \end{pmatrix}$$

with inverse given by

$$W(\lambda)^{-1} = D^{-1} - (\lambda - \alpha)\left(D^{-1}C(\lambda - A_\pi)^{-1} \quad D^{-1}C_\infty(I - \lambda A_{\pi\infty})^{-1}\right)$$

$$\times \begin{pmatrix} \Gamma & \Gamma_{12} \\ \Gamma_{21} & \Gamma_\infty \end{pmatrix}^{-1} \begin{pmatrix} (A_\zeta - \lambda)^{-1} & 0 \\ 0 & (I - \lambda A_{\zeta\infty})^{-1} \end{pmatrix} \begin{pmatrix} B \\ B_\infty \end{pmatrix}.$$

The problem we are confronted with is to construct an admissible Sylvester data set over infinity τ_∞, which fits the given admissible Sylvester data set τ in the way described above, such that the resulting W is column reduced at infinity. This problem is a more refined version of the problem considered in [11], where W is only required to have minimal McMillan degree. One of the main points of the recipe in Theorem 3.1 is that in order to obtain a column reduced W one has to take

$$\tau_\infty = \big(CX(I - \alpha T) - F, T; S, (I - \alpha S)YB + H; \Gamma_\infty\big)$$

with

$$\Gamma_\infty = -Y(\alpha - A_\zeta)\Gamma(\alpha - A_\pi)X + \rho_\pi(\alpha - A_\pi)X + Y(\alpha - A_\zeta)\eta_\zeta - \Gamma_1.$$

It turns out that in this case $\tilde{\Gamma}^{-1}$ has the simple form

$$\tilde{\Gamma}^{-1} = \begin{pmatrix} \Gamma^+ & \eta_\pi \\ -\rho_\zeta & 0 \end{pmatrix},$$

and a straightforward computation shows that the formulas for $W(\cdot)$ and $W(\cdot)^{-1}$ in Theorem 3.1 result from those given in the previous paragraph by plugging this special choice of τ_∞ and with the matrix D chosen to be equal to

$$E = (z_1 \quad \cdots \quad z_t \quad z_{t+1} \quad \cdots \quad z_{m-s} \quad y_1 \quad \cdots \quad y_s).$$

This choice of D gives the appropriate basis for $\mathbb{C}^m$ with respect to which the resulting $W(\lambda)$ is column reduced with column indices in nondecreasing order.

The main point of Theorem 3.2 is that any $\{\infty\}$-null-pole triple of a column reduced W with $\mathbb{C}$-null-pole triple equal to τ must occur in this way.

An important distinguishing feature of the τ_∞ leading to a column reduced W is that the sizes of the nilpotent Jordan blocks of S and T must agree with the nonzero observability indices of the pair $(C \mid \operatorname{Ker} \Gamma, \rho_\pi A_\pi \mid \operatorname{Ker} \Gamma)$ and controllability indices of the pair $(\rho_\zeta A_\zeta \mid K, \rho_\zeta B)$. This is certainly a necessary condition, since the sizes of the Jordan blocks of T represent the partial pole multiplicities and those of S represent the partial zero multiplicities of W at infinity.

Unfortunately the results from [11] are not of an appropriate form to be directly applicable to the problem of this paper. This forces us to recall results from [10], where analogous problems were considered with spectral data added at a finite point rather than at infinity, and then use a Möbius transformation to map the finite point to infinity.

If Γ is invertible, then the spaces $\operatorname{Ker}\Gamma$ and K are trivial, the matrix E can be taken to be any $m \times m$ invertible matrix, and the formula for W in Theorem 3.1 collapses to

$$W(\lambda) = E - (\lambda - \alpha)C(\lambda - A_\pi)^{-1}\left\{\Gamma^{-1}(\alpha - A_\zeta)^{-1}B\right\}E,$$

which one may rewrite as

$$W(\lambda) = D + C(\lambda - A_\pi)^{-1}\Gamma^{-1}BD,$$

with D invertible and equal to $E - C\Gamma^{-1}(\alpha - A_\zeta)^{-1}BE$.

Let us now consider the following example. Take $m = 2$, $\eta_\pi = \eta_\zeta = 1$, and let

$$\tau = (C, A_\pi; A_\zeta, B, \Gamma) = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, 0; 0, (0 \quad 1); \gamma\right),$$

where γ is a real number. If $\gamma \neq 0$, we are in the situation considered in the previous paragraph, and in this case the solution is

$$W(\lambda) = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}(z - 0)^{-1}\gamma^{-1}(0 \quad 1)\right\}D = \begin{pmatrix} -1 & \gamma^{-1}z^{-1} \\ 0 & 1 \end{pmatrix}D,$$

where D is an arbitrary nonsingular 2×2 matrix. In this case the column indices are $\{0, 0\}$.

More interesting is the case $\gamma = 0$. Then the spaces are

$$\mathscr{X}_\pi = \mathbb{C}, \qquad \mathscr{X}_\zeta = \mathbb{C}, \qquad \operatorname{Ker}\Gamma = \mathbb{C}, \qquad N = \{0\},$$
$$\operatorname{Im}\Gamma = \{0\}, \qquad K = \mathbb{C},$$

the various mappings are

$$\rho_\pi = 1, \qquad \rho_\zeta = 1, \qquad \eta_\pi = 1, \qquad \eta_\zeta = 1$$

and $t = 1$, $\alpha_1 = 1$, $d_{11} = 1$, $s = 1$, $\omega_1 = 1$, $g_{11} = 1$. In this case we have

$$S = 0, \qquad T = 0,$$

both on $\mathbb{C}$, and $\Gamma_1 = \gamma_1\colon \mathbb{C}^1 \to \mathbb{C}^1$ is an arbitrary number. The linear transformations

$$F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \qquad H = (h_1 \quad h_2),$$

$A_{12} = a_{12}$ (a complex number), and $A_{21} = a_{21}$ are subject to

$$0 = 0 - 1 - (0 \quad 1)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \tag{3.8$'$}$$

$$0 = 0 - 1 - (h_1 \quad h_2)\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{3.9$'$}$$

and

$$a_{21} - a_{12} = -(h_1 \quad h_2)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}. \tag{3.10$'$}$$

From (3.8′) and (3.9′) we see that

$$f_2 = -1, \qquad h_1 = -1,$$

and then (3.10′) becomes

$$a_{21} - a_{12} = f_1 + h_2$$

with

$$F = \begin{pmatrix} f_1 \\ -1 \end{pmatrix}, \qquad H = (-1 \quad h_2).$$

Equations (3.11) and (3.12) give

$$X = -a_{12}, \qquad Y = -a_{21}.$$

According to (3.13) we should take

$$z_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

while (3.14) demands

$$y_1 = \begin{pmatrix} y_1^1 \\ -\alpha \end{pmatrix},$$

where y_1^1 is an arbitrary number. Thus the matrix E is given by

$$E = \begin{pmatrix} 1 & y_1^1 \\ 0 & -\alpha \end{pmatrix},$$

and E is invertible, since by assumption $\alpha \neq 0$. The formula for W in Theorem 3.1 works out to be

$$W(\lambda) = \begin{pmatrix} \alpha\lambda^{-1} & p_{-1}\lambda^{-1} + p_0 + p_1\lambda \\ 0 & -\lambda \end{pmatrix},$$

where

$$\begin{aligned} p_{-1} &= \alpha^2 a_{12} + \alpha\gamma_1 + \alpha y_1^1 + \alpha^2 h_2, \\ p_0 &= -2\alpha a_{12} - \gamma_1 - y_1^1 - \alpha(f_1 + h_2), \\ p_1 &= a_{12} + f_1. \end{aligned}$$

Here α is a fixed nonzero number, while a_{12}, y_1^1, h_2, γ_1, f_1 are free complex parameters. From the expressions for p_{-1}, p_0, p_1 we see that p_{-1}, p_0, p_1 are independently arbitrary parameters. The content of Theorems 3.1 and 3.2 is that the above $W(\lambda)$ describes the set of all column reduced rational matrix functions having

$$\tau = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, 0; 0, (0 \quad 1); 0 \right)$$

as $\mathbb{C}$-null-pole triple. Note that the column indices are $\{-1, 1\}$.

Once we know the answer, it is straightforward to verify directly that it has the desired properties for this example. In particular, to verify that τ as above indeed is a $\mathbb{C}$-null-pole triple for W as above, we need only to check that a rational vector function $\begin{pmatrix} r_1(\lambda) \\ r_2(\lambda) \end{pmatrix}$ has the form

$$\begin{pmatrix} r_1(\lambda) \\ r_2(\lambda) \end{pmatrix} = \begin{pmatrix} \alpha\lambda^{-1} & p_{-1}\lambda^{-1} + p_0 + p_1\lambda \\ 0 & -\lambda \end{pmatrix} \begin{pmatrix} h_1(\lambda) \\ h_2(\lambda) \end{pmatrix}$$

for some polynomials h_1 and h_2 if and only if there is a number x and two polynomials q_1 and q_2 such that

$$\begin{pmatrix} r_1(\lambda) \\ r_2(\lambda) \end{pmatrix} = \lambda^{-1}\begin{pmatrix} \alpha \\ 0 \end{pmatrix} + \begin{pmatrix} q_1(\lambda) \\ q_2(\lambda) \end{pmatrix}$$

subject to

$$q_2(0) = (0 \quad 1)\begin{pmatrix} q_1(0) \\ q_2(0) \end{pmatrix} = 0 \cdot x = 0.$$

Proof of Theorem 3.1. The proof is divided into eight parts. In the first four parts we justify the various choices made in the paragraph preceding the present theorem.

Part (a). In this part we show that one may choose operators F and A_{12} as in (3.6) such that (3.8) holds. From (3.3) and properties (b_1), (b_2) it follows that

$$(A_\zeta \eta_\zeta T - \eta_\zeta) g_{jk} \in \operatorname{Im} \Gamma + \operatorname{Im} B \tag{3.16}$$

for $k = 1, \dots, \omega_j$ and $j = 1, \dots, s$. Since τ is a Sylvester data set, we have $\Gamma A_\pi - A_\zeta \Gamma = BC$. The latter identity implies that

$$\operatorname{Im} \Gamma + \operatorname{Im} B = \operatorname{Im}(\alpha - A_\zeta)\Gamma + \operatorname{Im} B. \tag{3.17}$$

Equations (3.16) and (3.17) yield the inclusion (3.4), which we may rewrite as

$$\operatorname{Im}(A_\zeta \eta_\zeta T - \eta_\zeta) \subset \operatorname{Im}\big((\alpha - A_\zeta)\Gamma \quad B\big).$$

Hence we can find F and A_{12} as in (3.6) such that

$$(A_\zeta \eta_\zeta T - \eta_\zeta) = \big((\alpha - A_\zeta)\Gamma \quad B\big)\begin{pmatrix} A_{12} \\ F \end{pmatrix},$$

which yields (3.8). Note that (3.8) allows one to choose $\rho_\pi A_{12}$ as one wishes.

Part (b). We show that one may choose the operators H and A_{21} as in (3.7) such that (3.9) holds. From (3.2) and the property (a_2) one sees that

$$(S\rho_\pi A_\pi - \rho_\pi)d_{j,k+1} = 0, \qquad k = 1,\dots,\alpha_j - 1.$$

This together with (a_1) implies that

$$\operatorname{Ker}(S\rho_\pi A_\pi - \rho_\pi) \supset \operatorname{Ker}\Gamma \cap \operatorname{Ker} C. \tag{3.18}$$

Since $\Gamma A_\pi - A_\zeta \Gamma = BC$, we have

$$\operatorname{Ker}\Gamma \cap \operatorname{Ker} C = \operatorname{Ker}\Gamma(\alpha - A_\pi) \cap \operatorname{Ker} C. \tag{3.19}$$

Equations (3.18) and (3.19) yield the inclusion (3.5), which may be rewritten as

$$\operatorname{Ker}(S\rho_\pi A_\pi - \rho_\pi) \supset \operatorname{Ker}\begin{pmatrix} \Gamma(\alpha - A_\pi) \\ C \end{pmatrix}.$$

It follows that one can find operators H and A_{21} as in (3.7) such that

$$S\rho_\pi A_\pi - \rho_\pi = \begin{pmatrix} A_{21} & H \end{pmatrix}\begin{pmatrix} \Gamma(\alpha - A_\pi) \\ C \end{pmatrix},$$

which yield (3.9). Now note that (3.9) fixes A_{21} on $\operatorname{Im}\Gamma$. Hence one may choose $A_{21}\eta_\zeta$ as one wishes.

Since H and F are determined and Γ_1 is given, the right-hand side of (3.10) is now fixed. On the other hand $\rho_\pi A_{12}$ and $A_{21}\eta_\zeta$ are still free to be chosen. It follows that we can always choose A_{12} and A_{21} in such a way that (3.10) holds.

Part (c). This part concerns (3.14). Property (b_2) may be rewritten as

$$(A_\zeta - \alpha)g_{j\nu} + \alpha g_{j\nu} - g_{j,\nu-1} \in \operatorname{Im}\Gamma + \operatorname{Im} B, \qquad \nu = 1,\dots,\omega_j.$$

Here $g_{j0} = 0$. It follows that

$$\sum_{\nu=1}^{\omega_j} \alpha^{\nu-1}(A_\zeta - \alpha)g_{j\nu} + \alpha^{\omega_j} g_{j\omega_j} \in \operatorname{Im}\Gamma + \operatorname{Im} B.$$

Now use property (b_1), the definition of T in (3.3), and the identity (3.17) to conclude that

$$(A_\zeta - \alpha)(I - \alpha T)^{-1} g_{j1} \in \operatorname{Im}(A_\zeta - \alpha)\Gamma + \operatorname{Im} B, \qquad j = 1, \dots, s.$$

From the latter formula it is clear that we may choose $y_1, \dots, y_s$ in $\mathbb{C}^m$ such that (3.14) holds.

Part (d). In this part we show that the matrix E in (3.15) is well defined and invertible. First, we prove that the vectors $z_1, \dots, z_t$ are linearly independent. Assume $\sum_{j=1}^{t} \beta_j z_j = 0$. Then

$$(\alpha - A_\pi)^{-1} \sum_{j=1}^{t} \beta_j d_{j\alpha_j} \in \operatorname{Ker} C \cap \operatorname{Ker}\Gamma,$$

and hence

$$(\alpha - A_\pi)^{-1} \sum_{j=1}^{t} \beta_j d_{j\alpha_j} = \sum_{j=1}^{t} \sum_{k=2}^{\alpha_j} \gamma_{jk}\, d_{jk}$$

for some scalars γ_{jk}. It follows that

$$\sum_{j=1}^{t} \beta_j\, d_{j\alpha_j} = \sum_{j=1}^{t} \sum_{k=2}^{\alpha_j} \alpha\gamma_{jk}\, d_{jk} - \sum_{j=1}^{t} \sum_{k=2}^{\alpha_j} \gamma_{jk}\, d_{j,k-1}. \tag{3.20}$$

Since the vectors $\{d_{jk}\}_{k=1,j=1}^{\alpha_j\ \ t}$ are linearly independent, we see from (3.20) that $\gamma_{j2} = 0$ for $j = 1, \dots, t$. Thus (3.20) holds with 2 replaced by 3, which implies that $\gamma_{j3} = 0$ for $j = 1, \dots, t$. It follows that (3.20) remains true if 2 is replaced by 4. Proceeding in this way we find that all γ_{jk} are zero. But the left-hand side of (3.20) is zero, and we may conclude that $\beta_1 = \cdots = \beta_t = 0$. Thus $z_1, \dots, z_t$ are linearly independent.

Next, we prove that the vectors $(A_\zeta - \alpha)^{-1}By_j$, $j = 1, \ldots, s$, are linearly independent module $\operatorname{Im}\Gamma$. Indeed, if

$$\sum_{j=1}^{s} \lambda_j (A_\zeta - \alpha)^{-1} By_j \in \operatorname{Im}\Gamma, \tag{3.21}$$

then we see from (3.14) that

$$\sum_{j=1}^{s} \sum_{k=1}^{\omega_j} \lambda_j (-1)^{\omega_j} \alpha^{k-1} g_{jk} \in \operatorname{Im}\Gamma. \tag{3.22}$$

But $\{g_{jk}\}_{k=1,j=1}^{\omega_j \; s}$ is a basis of K, and $K \cap \operatorname{Im}\Gamma$ consists of the zero vector only. So the coefficient of g_{jk} in (3.22) is zero. In particular, $\lambda_1, \ldots, \lambda_s$ are zero. Thus (3.20) implies $\lambda_j = 0$ $(j = 1, \ldots, s)$.

By (3.13) and (2.1),

$$\begin{aligned}
(A_\zeta - \alpha)^{-1} Bz_j &= \alpha (A_\zeta - \alpha)^{-1} BC(\alpha - A_\pi)^{-1} d_{j\alpha_j} \\
&= \alpha (A_\zeta - \alpha)^{-1} \{\Gamma(A_\pi - \alpha) - (A_\zeta - \alpha)\Gamma\} \\
&\quad \times (\alpha - A_\pi)^{-1} d_{j\alpha_j} \\
&= \alpha(\alpha - A_\zeta)^{-1} \Gamma d_{j\alpha_j} - \alpha\Gamma(\alpha - A_\pi)^{-1} d_{j\alpha_j}.
\end{aligned}$$

Now, recall that $d_{j\alpha_j} \in \operatorname{Ker}\Gamma$. Thus

$$(A_\zeta - \alpha)^{-1} Bz_j \in \operatorname{Im}\Gamma, \qquad j = 1, \ldots, t.$$

By combining this with the result of the previous two paragraphs, we see that the vectors $z_1, \ldots, z_t$, $y_1, \ldots, y_s$ are linearly independent. Hence we can find vectors $z_{t+1}, \ldots, z_{m-s}$ with the property that the matrix E in (3.15) is invertible.

Part (e). In what follows we shall use an argument of the Möbius-transformation type (cf. [9, Section 3]) which will allow us to apply the main results from [10]. For this purpose we need the following matrices:

$$\hat{C} = \alpha C(A_\pi - \alpha)^{-1}, \qquad \hat{A}_\pi = -\tfrac{1}{2}(A_\pi + \alpha)(A_\pi - \alpha)^{-1},$$

$$\hat{A}_\zeta = -\tfrac{1}{2}(A_\zeta + \alpha)(A_\zeta - \alpha)^{-1}, \qquad \hat{B} = (A_\zeta - \alpha)^{-1}B,$$

$$\hat{T} = -\tfrac{1}{2}(I + \alpha T)(I - \alpha T)^{-1}, \qquad \hat{S} = -\tfrac{1}{2}(I + \alpha S)(I - \alpha S)^{-1},$$

$$\hat{H} = (I - \alpha S)^{-1}H, \qquad \hat{F} = \alpha F(I - \alpha T)^{-1},$$

$$\hat{X} = (\alpha - A_\pi)X, \qquad \hat{Y} = Y(\alpha - A_\zeta),$$

$$\hat{A}_{12} = -\alpha A_{12}(I - \alpha T)^{-1}, \qquad \hat{A}_{21} = -\alpha(I - \alpha S)^{-1}A_{21}.$$

In this part we rewrite the formulas (3.8)–(3.12) in terms of the above matrices.

Consider the Möbius transformation

$$\varphi(\lambda) = \alpha\frac{2\lambda - 1}{2\lambda + 1}, \qquad \varphi^{-1}(z) = -\frac{1}{2}\frac{z + \alpha}{z - \alpha}, \tag{3.23}$$

and set

$$\hat{\sigma} = \{\lambda \in \mathbb{C} \mid \varphi(\lambda) \in \sigma(A_\pi) \cup \sigma(A_\zeta) \cup \{0\}\}.$$

Note that $\hat{A}_\pi = \varphi^{-1}(A_\pi)$ and $\hat{A}_\zeta = \varphi^{-1}(A_\zeta)$. It follows (cf. [4, Theorem 5.1.3]) that the quintet

$$\hat{\tau} = \left(\hat{C}, \hat{A}_\pi; \hat{A}_\zeta, \hat{B}; \Gamma\right) \tag{3.24}$$

is a $\hat{\sigma}$-admissible Sylvester data set. From (3.8) and (3.9) it is straightforward to derive the following identities:

$$\hat{A}_\zeta \eta_\zeta - \eta_\zeta \hat{T} = \hat{B}\hat{F} + \Gamma \hat{A}_{12}, \tag{3.25}$$

$$\rho_\pi \hat{A}_\pi - \hat{S}\rho_\pi = \hat{H}\hat{C} + \hat{A}_{21}\Gamma, \tag{3.26}$$

which imply that

$$\rho_\zeta\left(\hat{A}_\zeta - \hat{B}\hat{F}\right)|_K = \hat{T}, \qquad \rho_\pi\left(\hat{A}_\pi - \hat{H}\hat{C}\right)|_{\operatorname{Ker}\Gamma} = \hat{S}, \tag{3.27}$$

$$(I - \rho_\pi)\hat{A}_{12} = \Gamma^+\left(\hat{A}_\zeta\eta_\zeta - \eta_\zeta\hat{T} - \hat{B}\hat{F}\right), \tag{3.28}$$

$$\hat{A}_{21}(I - \rho_\zeta) = \left(\rho_\pi\hat{A}_\pi - \hat{S}\rho_\pi - \hat{H}\hat{C}\right)\Gamma^+. \tag{3.29}$$

From (3.10) one obtains that

$$\hat{A}_{21}\eta_\zeta - \rho_\pi\hat{A}_{12} - \hat{H}\hat{F} = \Gamma_1\hat{T} - \hat{S}\Gamma_1. \tag{3.30}$$

The matrices X and Y defined by (3.11) and (3.12), respectively, are the unique solutions of the following Stein equations:

$$A_\pi XT - X = A_{12}, \qquad SYA_\zeta - Y = A_{21}. \tag{3.31}$$

From these identities one may derive that $\hat{X}$ and $\hat{Y}$ satisfy Sylvester equations, namely

$$\hat{A}_\pi\hat{X} - \hat{X}\hat{T} = \hat{A}_{12}, \qquad \hat{Y}\hat{A}_\zeta - \hat{S}\hat{Y} = \hat{A}_{21}. \tag{3.32}$$

To see this, let us check the first identity in (3.32). We have

$$\begin{aligned}
\hat{A}_\pi\hat{X} - \hat{X}\hat{T} &= \left(\hat{A}_\pi + \tfrac{1}{2}\right)\hat{X} - \hat{X}\left(\hat{T} + \tfrac{1}{2}\right)\\
&= -\alpha(A_\pi - \alpha)^{-1}(\alpha - A_\pi)X - (\alpha - A_\pi)\\
&\quad \times X\left\{-\alpha T(I - \alpha T)^{-1}\right\}\\
&= \left(\alpha X - \alpha^2 XT + \alpha^2 XT - \alpha A_\pi XT\right)(I - \alpha T)^{-1}\\
&= \alpha(X - A_\pi XT)(I - \alpha T)^{-1}\\
&= -\alpha A_{12}(I - \alpha T)^{-1} = \hat{A}_{12}.
\end{aligned}$$

The second identity in (3.32) is checked in a similar way.

Part (f). In this part we come to the matrix function $W(\cdot)$ appearing in the theorem. Note that $\sigma(\hat{S}) = \sigma(\hat{T}) = \{-\frac{1}{2}\}$ and $-\frac{1}{2} \notin \hat{\sigma}$. The two identities in (3.27) tell us (using the terminology introduced in Section 2 of [10]) that $(\hat{S}, \hat{H})$ is a zero correction pair for $\hat{\tau}$ and that $(\hat{F}, \hat{T})$ is a pole correction pair for $\hat{\tau}$. It follows that we can apply Theorem 2.1 in [10] to show that

$$\hat{\tau}_0 = \left(-\hat{C}\hat{X} - \hat{F}, \hat{T}; \hat{S}, -\hat{Y}\hat{B} + \hat{H}; \hat{Y}\Gamma\hat{X} - \hat{Y}\eta_\zeta - \rho_\pi \hat{X} + \Gamma_1\right) \quad (3.33)$$

is a minimal complement of $\hat{\tau}$. Let $\hat{W}(\cdot)$ be the $m \times m$ rational matrix function which is analytic at infinity, has the value I_m at infinity, and has $\hat{\tau} \oplus \hat{\tau}_0$ as its global null-pole triple. By Theorem 2.2 of [10] we have

$$\hat{W}(\lambda) = I_m + \hat{C}\left(\lambda - \hat{A}_\pi\right)^{-1}\left\{\left(\Gamma_{11}^{\times} - \eta_\pi \hat{Y}\right)\hat{B} + \eta_\pi \hat{H}\right\}$$

$$+(-\hat{C}\hat{X} - \hat{F})(\lambda - \hat{T})^{-1}\rho_\zeta \hat{B},$$

$$\hat{W}(\lambda)^{-1} = I_m - \left\{\left(\hat{C}\left(\Gamma_{11}^{\times} - \hat{X}\rho_\zeta\right) - \hat{F}\rho_\zeta\right\}\left(\lambda - \hat{A}_\zeta\right)^{-1}\hat{B}\right.$$

$$-\hat{C}\eta_\pi(\lambda - S)^{-1}(-\hat{Y}\hat{B} + \hat{H}),$$

where $\Gamma_{11}^{\times}: X_\zeta \to X_\pi$ is fixed by the identities

$$\Gamma\Gamma_{11}^{\times} = I - \eta_\zeta \rho_\zeta + \Gamma\hat{X}\rho_\zeta, \qquad \rho_\pi \Gamma_{11}^{\times} = \eta_\pi \hat{Y} + \rho_\pi \hat{X}\rho_\zeta - \Gamma_1 \rho_\zeta. \quad (3.34)$$

The identities in (3.34) imply that

$$\Gamma_{11}^{\times} = \rho_\pi \Gamma_{11}^{\times} + (I - \rho_\pi)\Gamma_{11}^{\times} = \rho_\pi \Gamma_{11}^{\times} + \Gamma^{+}\Gamma\Gamma_{11}^{\times}$$

$$= \eta_\pi \hat{Y} + \rho_\pi \hat{X}\rho_\zeta - \Gamma_1 \rho_\zeta + \Gamma^{+} - \Gamma^{+}\eta_\zeta \rho_\zeta + (I - \rho_\pi)\hat{X}\rho_\zeta$$

$$= \Gamma^{+} + \eta_\pi \hat{Y} + \hat{X}\rho_\zeta - \Gamma_1 \rho_\zeta$$

$$= \Gamma^{+} + \eta_\pi Y(\alpha - A_\zeta) + (\alpha - A_\pi)X\rho_\zeta - \Gamma_1 \rho_\zeta.$$

It is now straightforward to check that

$$\begin{aligned}\hat{W}(\varphi^{-1}(\lambda)) = I_m - (\lambda - \alpha)C(\lambda - A_\pi)^{-1} & \\ \times\Big\{\big[\Gamma^+ + (\alpha - A_\pi)X\rho_\zeta - \Gamma_1\rho_\zeta\big](\alpha - A_\zeta)^{-1}B & \\ + \eta_\pi(\alpha S - I)^{-1}H\Big\} & \\ + (\lambda - \alpha)\big[CX(I - \alpha T) - F\big] & \\ \times(I - \lambda T)^{-1}\rho_\zeta(\alpha - A_\zeta)^{-1}B, &\end{aligned}$$

and

$$\begin{aligned}\hat{W}(\varphi^{-1}(\lambda))^{-1} = I + (\lambda - \alpha)\Big\{C(\alpha - A_\pi)^{-1}\big[\Gamma^+ + \eta_\pi Y(\alpha - A_\zeta) - \Gamma_1\rho_\zeta\big] & \\ + F(I - \alpha T)^{-1}\rho_\zeta\Big\}(\lambda - A_\zeta)^{-1}B & \\ + (\lambda - \alpha)C(\alpha - A_\pi)^{-1}\eta_\pi(\lambda S - I)^{-1} & \\ \times\big[(I - \alpha S)YB + H\big]. &\end{aligned}$$

Now, put $W(\lambda) = \hat{W}(\varphi^{-1}(\lambda))E$. Then $W(\cdot)$ and $W(\cdot)^{-1}$ are given by the formulas appearing in the theorem. Furthermore, it is not difficult to show (cf. [4, Theorem 5.1.3]) that τ is a $\mathbb{C}$-null-pole triple for W.

Part (g). Put

$$\hat{d}_{jk} = \sum_{\nu=k}^{\alpha_j}\binom{\alpha_j - k}{\nu - k}(-\alpha)^{\nu-\alpha_j}d_{j\nu}, \qquad k = 1,\dots,\alpha_j, \quad j = 1,\dots,t; \tag{3.35}$$

$$\hat{g}_{jk} = (-1)^{k+1}\sum_{\nu=k}^{\omega_j}\binom{\nu - 1}{k - 1}\alpha^{\nu-1}g_{j\nu}, \qquad k = 1,\dots,\omega_j, \quad j = 1,\dots,s. \tag{3.36}$$

In this part we show that $\{\hat{d}_{jk}\}_{k=1,j=1}^{\alpha_j,\ t}$ is an outgoing basis for $\operatorname{Ker}\Gamma$ with respect to the pair $(\hat{C}, \hat{A}_\pi + \frac{1}{2})$ and that $\{\hat{g}_{jk}\}_{k=1,j=1}^{\omega_j,\ s}$ is an incoming basis for the complement K of $\operatorname{Im}\Gamma$ with respect to the pair $(\hat{A}_\zeta + \frac{1}{2}, \hat{B})$. (See Section 1 of [10] for the terminology.)

Obviously, $\{\hat{d}_{jk}\}_{k=1,j=1}^{\alpha_j \ \ t}$ is a basis of $\operatorname{Ker}\Gamma$. Fix $1 \leqslant k \leqslant \alpha_j - 1$. By property (a_2), we have $A_\pi \hat{d}_{j,k+1} - \alpha \hat{d}_{j,k+1} + \alpha \hat{d}_{jk} = 0$. Since $\hat{A}_\pi + \frac{1}{2} = \alpha(\alpha - A_\pi)^{-1}$, it follows that

$$\left(\hat{A}_\pi + \tfrac{1}{2}\right)\hat{d}_{jk} = \hat{d}_{j,k+1}, \qquad 1 \leq k \leqslant \alpha_j - 1.$$

Furthermore, because of property (a_1),

$$\hat{C}\hat{d}_{jk} = \alpha C(A_\pi - \alpha)^{-1}\hat{d}_{jk} = -C\hat{d}_{j,k+1} = 0, \qquad 1 \leqslant k \leqslant \alpha_j - 1.$$

Thus $\{\hat{d}_{jk}\}_{k=1,j=1}^{\alpha_j - 1 t}$ is a linearly independent set of vectors in $\operatorname{Ker}\Gamma \cap \operatorname{Ker}\hat{C}$. On the other hand

$$\hat{C}\hat{d}_{j\alpha_j} = \alpha C(A_\pi - \alpha)^{-1} d_{j\alpha_j} = -z_j.$$

Since the vectors $z_1, \dots, z_t$ are linearly independent [see part (d)], we may conclude that $\{\hat{d}_{jk}\}_{k=1,j=1}^{\alpha_j - 1 t}$ is a basis of $\operatorname{Ker}\hat{C} \cap \operatorname{Ker}\Gamma$, as desired.

Next, we turn to the vectors $\hat{g}_{jk}$ defined by (3.36). Obviously, $\{\hat{g}_{jk}\}_{k=1,j=1}^{\omega_j \ \ s}$ is a basis for K. We put $\hat{g}_{j,\omega_j+1} = 0$. One computes that

$$(\alpha - A_\zeta)\hat{g}_{j,k+1} - \alpha\hat{g}_{jk} \in \operatorname{Im}(\alpha - A_\zeta)\Gamma + \operatorname{Im} B, \qquad 1 \leqslant k \leqslant \omega_j - 1.$$

Since $\hat{A}_\zeta + \frac{1}{2} = \alpha(\alpha - A_\zeta)^{-1}$, we conclude that

$$\left(\hat{A}_\zeta + \tfrac{1}{2}\right)\hat{g}_{jk} - \hat{g}_{j,k+1} \in \operatorname{Im}\Gamma + \operatorname{Im}\hat{B}. \tag{3.37}$$

Note that $\hat{g}_{j,\omega_j} = (-1)^{\omega_j+1}\alpha^{\omega_j-1}g_{j,\omega_j} \in \operatorname{Im}\Gamma + \operatorname{Im} B$. It follows that (3.37) holds for $k = 1, \dots, \omega_j$. Since

$$\hat{g}_{j1} = \sum_{\nu=1}^{\omega_j} \alpha^{\nu-1} g_{j\nu} = \sum_{\nu=1}^{\omega_j} \alpha^{\nu-1} T^{\nu-1} g_{j1} = (I - \alpha T)^{-1} g_{j1}, \qquad j = 1, \dots, s.$$

we see from (3.14) that

$$\hat{g}_{j1} - \hat{B}y_j \in \operatorname{Im}\Gamma, \qquad j = 1, \dots, s. \tag{3.38}$$

We already know [see part (d)] that the vectors $\hat{B}y_1, \ldots, \hat{B}y_s$ are linearly independent modulo Im Γ. Thus (3.38) shows that $\hat{g}_{11}, \ldots, \hat{g}_{s1}$ are vectors in Im Γ + Im $\hat{B}$ which are linearly independent modulo Im Γ. On the other hand the vectors $\{\hat{g}_{jk}\}_{k=2,j=1}^{\omega_j\ \ s}$ are linearly independent modulo Im Γ + Im $\hat{B}$. Indeed, assume

$$\sum_{j=1}^{s}\sum_{k=2}^{\omega_j}\lambda_{jk}\hat{g}_{jk}\in \operatorname{Im}\Gamma+\operatorname{Im}\hat{B},$$

where λ_{jk} are complex numbers. Since $\hat{B}=(A_\zeta-\alpha)^{-1}B$ and $(\hat{A}_\zeta+\frac{1}{2})^{-1} = \alpha^{-1}(\alpha-A_\zeta)$, we see from (3.17) and (3.37) that

$$\sum_{j=1}^{s}\sum_{k=2}^{\omega_j}\lambda_{jk}\hat{g}_{j,k-1}\in \operatorname{Im}\Gamma+\operatorname{Im}B.$$

Recall that $\{g_{j\omega_j}\}_{j=1}^{s}$ is a basis of Im Γ + Im B modulo Im Γ. So there exist complex numbers $\mu_1, \ldots, \mu_s$ such that

$$\sum_{j=1}^{s}\left(\sum_{k=2}^{\omega_j}\lambda_{jk}\hat{g}_{j,k-1}-\mu_j g_{j\omega_j}\right)\in K\cap \operatorname{Im}\Gamma=\{0\}.$$

Now use (3.36) to conclude that the coefficient of g_{j1} in the left-hand side of the previous formula is equal to λ_{j2}, and hence $\lambda_{j2}=0$. Again using (3.36) we see that the coefficient of g_{j2} is equal to $\alpha\lambda_{j3}$, and so $\lambda_{j3}=0$. Proceeding in this way we obtain that all λ_{jk} are zero, and hence the vectors $\{\hat{g}_{jk}\}_{k=2,j=1}^{\omega_j\ \ s}$ are linearly independent modulo Im Γ + Im $\hat{B}$. Since $\{\hat{g}_{jk}\}_{k=1,j=1}^{\omega_j\ \ s}$ is a basis of K, we conclude that $\hat{g}_{11}, \ldots, \hat{g}_{s1}$ is a basis of Im Γ + Im $\hat{B}$ modulo Im Γ, as desired.

Part (h). In this part we finish the proof. By using arguments similar to the ones employed in part (g) one shows that

$$(\hat{S}+\tfrac{1}{2})\hat{d}_{jk}=\hat{d}_{j,k+1},\qquad k=1,\ldots,\alpha_j,\quad \hat{d}_{j,\alpha_j+1}=0,\qquad(3.39)$$

$$(\hat{T}+\tfrac{1}{2})\hat{g}_{jk}=\hat{g}_{j,k+1},\qquad k=1,\ldots,\omega_j,\quad \hat{g}_{j,\omega_j+1}=0.\qquad(3.40)$$

In particular, the orders of the Jordan blocks of $\hat{S}$ are equal to the outgoing indices $\alpha_1, \ldots, \alpha_t$, and the orders of the Jordan blocks of $\hat{T}$ are equal to the incoming indices $\omega_1, \dots, \omega_s$. But then we can use the result of Section 4.3 in [10] (see also the end of Section 2 in [10]) to show that $\hat{W}(\cdot)$ factors as

$$\hat{W}(\lambda) = \hat{W}_-(\lambda)\hat{D}(\lambda), \tag{3.41}$$

where $\hat{W}_-(\cdot)$ is a regular $m \times m$ rational matrix function which does not have a pole or zero at $-\frac{1}{2}$ and

$$\hat{D}(\lambda)y = \begin{cases} \left(\dfrac{\lambda - \frac{1}{2}}{\lambda + \frac{1}{2}}\right)^{-\alpha_j} z_j, & y = z_j, \quad j = 1, \dots, t, \\ y, & y = z_j, \quad j = t+1, \dots, m-s, \\ \left(\dfrac{\lambda - \frac{1}{2}}{\lambda + \frac{1}{2}}\right)^{\omega_j} y_j, & y = y_j, \quad j = 1, \dots, s. \end{cases}$$

Note that

$$\hat{D}(\lambda)E = E\begin{pmatrix} \left(\dfrac{\lambda - \frac{1}{2}}{\lambda + \frac{1}{2}}\right)^{\kappa_1} & & \\ & \ddots & \\ & & \left(\dfrac{\lambda - \frac{1}{2}}{\lambda + \frac{1}{2}}\right)^{\kappa_m} \end{pmatrix}, \tag{3.42}$$

where $\kappa_1, \dots, \kappa_m$ are as in the theorem. Now take

$$W_-(\lambda) = \hat{W}_-(\varphi^{-1}(\lambda))ES,$$

where

$$S = \operatorname{diag}(\alpha^{\alpha_1}, \dots, \alpha^{\alpha_t}, 1, \dots, 1, \alpha^{-\omega_1}, \dots, \alpha^{-\omega_s}).$$

Then $W_-(\lambda)$ is a regular $m \times m$ rational matrix function which is analytic and invertible at infinity. Since $W(\lambda) = \hat{W}(\varphi^{-1}(\lambda))E$, we see from (3.41) and (3.42) that

$$W(\lambda) = W_-(\lambda)\begin{pmatrix} \lambda^{\kappa_1} & & \\ & \ddots & \\ & & \lambda^{\kappa_m} \end{pmatrix},$$

which completes the proof. ■

The proof of Theorem 3.2 will be given in Section 5. It requires some auxiliary results which we present in the next section.

4. AUXILIARY RESULTS ON OBSERVABLE PAIRS AND ON CONTROLLABLE PAIRS

Let (C, A) be a pair of matrices, where A is $n \times n$ and C is $m \times n$, such that (C, A) is observable. We write $\alpha_1 \geqslant \cdots \geqslant \alpha_t > 0$ for the nonzero observability indices of (C, A).

PROPOSITION 4.1. *Let (S, H) be a pair of matrices, where S is $n \times n$ and H is $n \times m$, such that*

(i) $A - HC = S$,
(ii) *S is nilpotent.*

and assume that the Jordan blocks in the Jordan normal form of S have orders $\alpha_1, \dots, \alpha_t$. Then there exists a basis $\{\tilde{e}_{jk}\}_{k=1, j=1}^{\alpha_j \;\; t}$ of $\mathbb{C}^n$ such that

(P_1) $A\tilde{e}_{jk} = \tilde{e}_{j,k+1}$, $k = 1, \dots, \alpha_j - 1$,
(P_2) $\{\tilde{e}_{jk}\}_{k=1, j=1}^{\alpha_j - 1, t}$ *is a basis for* $\operatorname{Ker} C$,
(P_3) $S\tilde{e}_{jk} = \tilde{e}_{j,k+1}$, $k = 1, \dots, \alpha_j$, *where* $\tilde{e}_{j,\alpha_j+1} = 0$.

For the proof of Proposition 4.1 we need a lemma. Consider the following 2×2 block matrix:

$$A = \begin{pmatrix} A_1 & M_{12} \\ 0 & A_2 \end{pmatrix}, \tag{4.1}$$

where A_1 is a nilpotent Jordan matrix of order $m_1 l$ which consists of m_1 Jordan blocks N of size $l \times l$, and $A_2 = N_1 \oplus \cdots \oplus N_{m_2}$, where for $\nu = 1, \dots, m_2$ the matrix N_ν is a nilpotent Jordan block of order $l_\nu < l$. Furthermore,

$$M_{12} = \begin{pmatrix} U_{11} & \cdots & U_{1m_2} \\ \vdots & & \vdots \\ U_{m_1 1} & \cdots & U_{m_1 m_2} \end{pmatrix}, \tag{4.2}$$

where for each i and j the entry U_{ij} is a matrix of size $l \times l_j$ which has the following form:

$$\begin{pmatrix} 0 & \cdots & 0 & * \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & * \end{pmatrix}. \tag{4.3}$$

Let $d_{11}, \dots, d_{1l}, d_{21}, \dots, d_{2l}, \dots, d_{m_1,1}, \dots, d_{m_1,l}$ be the first $m_1 l$ basis vectors of $\mathbb{C}^n$ (partitioned according to the partitioning of $A_1 = N \oplus \cdots \oplus N$).

LEMMA 4.2. *Let A in (4.1) have the properties described in the previous paragraph, and assume that $A^l = 0$. Then there exists a matrix F of size $m_1 l \times (l_1 + \cdots + l_{m_2})$ such that*

$$\begin{pmatrix} I & F \\ 0 & I \end{pmatrix}\begin{pmatrix} A_1 & M_{12} \\ 0 & A_2 \end{pmatrix}\begin{pmatrix} I & -F \\ 0 & I \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \tag{4.4}$$

and

$$\operatorname{Im}\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix} \subset \operatorname{span}\{d_{jk} \mid k = 1, \dots, l-1, j = 1, \dots, m_1\}. \tag{4.5}$$

Proof. The (1, 2)th block entry of A^p is equal to

$$\sum_{\nu=0}^{p-1} A_1^\nu M_{12} A_2^{p-1-\nu}. \tag{4.6}$$

Since $A^l = 0$, the matrix in (4.6) is zero for $p = l$. But then we can use

$$A_1 = N \oplus \cdots \oplus N, \qquad A_2 = N_1 \oplus \cdots \oplus N_{m_2} \tag{4.7}$$

and (4.2) to conclude that for $i = 1, \dots, m_1$ and $j = 1, \dots, m_2$ we have

$$\sum_{\nu=0}^{l-1} N^\nu U_{ij} N_j^{l-1-\nu} = 0. \tag{4.8}$$

Now fix i and j, and let u_k^{ij} be the kth entry in the last column of U_{ij}. We claim that

$$u_k^{ij} = 0, \qquad k = 1, \dots, l_j. \tag{4.9}$$

To see this, recall that $N_j^{l_j} = 0$ and $l_j < l$. Thus (4.8) may be rewritten as

$$\sum_{\nu=l-l_j}^{l-1} N^{\nu} U_{ij} N_j^{l-1-\nu} = 0. \tag{4.10}$$

Multiplying a matrix on the right by N_j removes the first column, moves all other columns one step to the left, and sets the last column equal to zero. Thus the first column of $U_{ij} N_j^{l_j-1}$ is precisely the last column of U_{ij}. For $\nu > l - l_j$ the first column of $U_{ij} N_j^{l-1-\nu}$ is equal to zero. Multiplying a matrix on the left by N removes the last row, moves all other rows one step down, and sets the first row equal to zero. It follows that the first column of the left-hand side of (4.10) is given by

$$\begin{pmatrix} 0 & \cdots & 0 & u_1^{ij} & \cdots & u_{l_2}^{ij} \end{pmatrix}^T,$$

where ^{T} denotes the transpose. Hence (4.10) implies (4.9).

Note that (4.4) is equivalent to

$$A_1 F - F A_2 = M_{12}. \tag{4.11}$$

Write $F = (F_{ij})_{i=1,j=1}^{m_1, m_2}$, where F_{ij} is a matrix of size $l \times l_j$. From the special form of A_1 and A_2 [see (4.7)] and (4.2) it follows that (4.11) may be rewritten as

$$N F_{ij} - F_{ij} N_j = U_{ij} \qquad (i = 1, \ldots, m_1, \quad j = 1, \ldots, m_2). \tag{4.12}$$

Now choose F_{ij} to be the following $l \times l_j$ Toeplitz matrix

$$F_{ij} = \begin{pmatrix} u_{l_j+1}^{ij} & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ \vdots & & & u_{l_j+1}^{ij} \\ u_l^{ij} & & & \vdots \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & & & u_l^{ij} \\ 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{4.13}$$

Then F_{ij} satisfies the equation (4.12), by virtue of (4.9), and thus (4.4) is fulfilled with $F = (F_{ij})_{i=1,j=1}^{m_1\ \ m_2}$, where F_{ij} is given by (4.13). Since the last row of F_{ij} is zero for each i and j, the inclusion (4.5) holds. ■

Proof of Proposition 4.1. The proof is divided into five parts. In parts (b)–(e) we take $A = S$, and in part (a) we justify this additional assumption.

Part (a). Since $A - HC = S$, properties (P_2) and (P_3) together imply (P_1). Furthermore, (C, S) is an observable pair, and the observability indices of the pair (C, S) coincide with those of (C, A). Therefore, in order to prove the proposition, we may without loss of generality assume that $A = S$.

Part (b). Since (C, A) is observable, we may (see [10]) choose a basis $\{e_{jk}\}_{k=1,j=1}^{\alpha_j\ \ t}$ of $\mathbb{C}^n$ such that (P_1) and (P_2) hold true for e_{jk}. Relative to the basis $\{e_{jk}\}_{k=1,j=1}^{\alpha_j\ \ t}$ the transformation A is represented by the block matrix

$$A = \begin{pmatrix} N_{11} & \cdots & N_{1t} \\ \vdots & & \vdots \\ N_{t1} & \cdots & N_{tt} \end{pmatrix},$$

where

$$N_{ii} = \begin{pmatrix} 0 & & 0 & * \\ 1 & & & * \\ & \ddots & & \vdots \\ & & 1 & * \end{pmatrix}, \qquad N_{ij} = \begin{pmatrix} 0 & \cdots & 0 & * \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & * \end{pmatrix} \quad (i \neq j).$$

For each i and j the matrix N_{ij} has size $\alpha_i \times \alpha_j$. The $*$'s denotes entries which we don't specify further, and the blank spots in N_{ii} stand for zero entries.

Let $l_1, \dots, l_r$ be the different numbers in the sequence $\alpha_1 \geqslant \cdots \geqslant \alpha_t > 0$, and let m_i be the number of times l_i occurs in the sequence $\alpha_1, \dots, \alpha_t$. Thus

$$\{\alpha_1, \dots, \alpha_t\} = \{\underbrace{l_1, \dots, l_1}_{m_1}, \underbrace{l_2, \dots, l_2}_{m_2}, \dots, \underbrace{l_r, \dots, l_r}_{m_r}\},$$

$$m_1 + \cdots + m_r = t.$$

Since $A = S$, we have $A^{l_1} = 0$, and therefore $Ae_{j\alpha_j} = 0$ for $j = 1, \ldots, m_1$. Thus

$$N_{ii} = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{pmatrix}, \qquad i = 1, \ldots, m_1,$$

$$N_{ij} = 0, \qquad i = 1, \ldots, m_1, \quad j = 1, \ldots, t, \quad j \neq i.$$

Put $J_1 = N_{11} \oplus \cdots \oplus N_{m_1, m_1}$. Then

$$A = \begin{pmatrix} J_1 & * \\ 0 & A_2 \end{pmatrix}, \tag{4.14}$$

where

$$A_2 = \begin{pmatrix} N_{m_1+1, m_1+1} & \cdots & N_{m_1+1, t} \\ \vdots & & \vdots \\ N_{t, m_1+1} & \cdots & N_{t,t} \end{pmatrix}.$$

Part (c). Since A is nilpotent, we see from (4.14) that the same holds for A_2. We claim that $A_2^{l_2} = 0$. To see this consider

$$A^{l_2} = \begin{pmatrix} J_1^{l_2} & B \\ 0 & A_2^{l_2} \end{pmatrix}.$$

By our assumption on the Jordan normal form of A we have rank $A^{l_2} = (l_1 - l_2)m_1$. Also, rank $A^{l_2} = (l_1 - l_2)m$. It follows that

$$\operatorname{Im}\begin{pmatrix} B \\ A_2^{l_2} \end{pmatrix} \subset \operatorname{Im} A^{l_2} = \operatorname{Im}\begin{pmatrix} J_1^{l_2} \\ 0 \end{pmatrix},$$

which shows that $A_2^{l_2} = 0$.

Part (d). Since $A_2^{l_2} = 0$, we may repeat the reasoning in the last paragraph of Part (b) with A_2 in place of A. It follows that

$$N_{ii} = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{pmatrix}, \qquad i = m_1 + 1, \ldots, m_1 + m_2,$$

$$N_{ij} = 0, \qquad i = m_1 + 1, \ldots, m_1 + m_2, \quad j = m_1 + 1, \ldots, t, \quad j \neq i.$$

Put

$$J_2 = N_{m_1+1,\,m_1+1} \oplus \cdots \oplus N_{m_1+m_2,\,m_1+m_2}.$$

Then

$$A = \begin{pmatrix} J_1 & * & * \\ 0 & J_2 & * \\ 0 & 0 & A_3 \end{pmatrix}. \tag{4.15}$$

From (4.15) it is clear that A_3 is nilpotent. We claim that $A_3^{l_3} = 0$. Indeed, consider

$$A^{l_3} = \begin{pmatrix} J_1^{l_3} & B_1 & B_2 \\ 0 & J_2^{l_3} & B_3 \\ 0 & 0 & A_3^{l_3} \end{pmatrix}.$$

By our conditions on the Jordan normal form of A we have

$$\operatorname{rank} A^{l_3} = (l_1 - l_3)m_1 + (l_2 - l_3)m_2.$$

On the other hand, $\operatorname{rank} J_1^{l_3} = (l_1 - l_3)m_1$, and $\operatorname{rank} J_2^{l_3} = (l_1 - l_3)m_2$. It follows that

$$\operatorname{Im}\begin{pmatrix} B_2 \\ B_3 \\ A_3^{l_3} \end{pmatrix} \subset \operatorname{Im} A^{l_3} = \operatorname{Im}\begin{pmatrix} J_1^{l_3} & B \\ 0 & J_2^{l_3} \\ 0 & 0 \end{pmatrix},$$

and therefore $A_3^{l_3} = 0$.

Part (e). Proceeding in this way, we see that

$$A = \begin{pmatrix} J_1 & M_{12} & \cdots & M_{1,r-1} & M_{1r} \\ 0 & J_2 & \cdots & M_{2,r-1} & M_{2r} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & J_{r-1} & M_{r-1,r} \\ 0 & 0 & \cdots & 0 & J_r \end{pmatrix}, \tag{4.16}$$

where J_i is a nilpotent Jordan matrix of order $m_i l_i$ which consists of m_i Jordan blocks of size $l_i \times l_i$ $(i = 1, \dots, r)$. Moreover, for

$$A_i = \begin{pmatrix} J_i & M_{i,i+1} & \cdots & M_{ir} \\ 0 & J_{i+1} & \cdots & M_{i+1,r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_r \end{pmatrix},$$

we have $A_i^{l_i} = 0$.

Now apply Lemma 4.2 to the block matrix

$$\begin{pmatrix} J_{r-1} & M_{r-1,r} \\ 0 & J_r \end{pmatrix}.$$

This allows us to find an invertible linear transformation V_1 on $\mathbb{C}^n$ such that relative to the basis $\{V_1 e_{jk}\}_{k=1,j=1}^{\alpha_j \;\; t}$ the matrix A has again the block form (4.16), but now with $M_{r-1,r} = 0$. The inclusion (4.5) shows that V_1 may be chosen in such a way that V_1 leaves $\operatorname{Ker} C$ invariant. In particular, $\{V_1 e_{jk}\}_{k=1,j=1}^{\alpha_j \;\; t}$ is again a basis of $\operatorname{Ker} C$. Pass to this new basis of $\mathbb{C}^n$, and apply Lemma 4.2 to the block matrix

$$\begin{pmatrix} J_{r-2} & M_{r-2,r-1} & M_{r-2,r} \\ 0 & J_{r-1} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

partitioned as a 2×2 block matrix in the indicated way. We find an invertible linear transformation V_2 on $\mathbb{C}^n$ that leaves $\operatorname{Ker} C$ invariant and is such that relative to the basis $\{V_2 e_{jk}\}_{k=1,j=1}^{\alpha_j \;\; t}$ the matrix A has again the block form (4.16), but now with

$$M_{r-1,r} = 0, \qquad M_{r-2,r-1} = 0, \qquad M_{r-2,r} = 0.$$

Proceeding in this way, we find an invertible linear transformation V on $\mathbb{C}^n$ such that

$$\tilde{e}_{jk} = V e_{jk}, \qquad k = 1, \dots, \alpha_j, \quad j = 1, \dots, t,$$

has the desired properties (P_1), (P_2), (P_3) for $A = S$. ■

Proposition 4.1 has a natural analogue for controllable pairs. Let (A, B) be a pair of matrices, where A is $n \times n$ and B is $n \times m$, such that (A, B) is controllable, and let $\omega_1 \geqslant \cdots \geqslant \omega_s > 0$ be the nonzero controllability indices of (A, B).

PROPOSITION 4.2. *Let (F, T) be a pair of matrices, where T is $n \times n$ and F is $m \times n$, such that*

(i) $A - BF = T$,
(ii) *T is nilpotent,*

and assume that the Jordan blocks in the Jordan normal form of T have orders $\omega_1, \dots, \omega_s$. Then there exists a basis $\{\tilde{f}_{jk}\}_{k=1,j=1}^{\omega_j \ \ s}$ of C^n such that

(Q_1) $A\tilde{f}_{jk} - \tilde{f}_{j,k+1} \in \operatorname{Im} B$, $k = 1, \dots, \omega_j$, *where* $\tilde{f}_{j,\omega_j+1} = 0$,
(Q_2) $\tilde{f}_{11}, \dots, \tilde{f}_{s,1}$ *is a basis of* $\operatorname{Im} B$,
(Q_3) $T\tilde{f}_{j,k} = \tilde{f}_{j,k+1}$, $k = 1, \dots, \omega_j$, *where* $\tilde{f}_{j,\omega_j+1} = 0$.

Proposition 4.2 may be derived from Proposition 4.1 by using a duality argument, or one may prove it by using the same type of arguments as were used in the proof of Proposition 4.1. We omit the details.

5. PROOF OF THEOREM 3.2

Throughout this section $\tau = (C, A_\pi; A_\zeta, B; \Gamma)$ is an admissible Sylvester data set, and $V(\cdot)$ is an arbitrary regular $m \times m$ rational matrix function which has τ as its $\mathbb{C}$-null-pole triple and which is column reduced at infinity. Our aim is to show that V may be constructed via the method of Theorem 3.1. The proof is divided into five parts.

Part (a). In this part we show that among all regular rational matrix functions having τ as $\mathbb{C}$-null-pole triple, those which are also column reduced at infinity have the minimal possible McMillan degree. This fact may be derived from Theorem 3.3 in [5], where a more general result for nonregular rational matrices is given; here we present a direct proof for the regular case.

We write $\deg V$ for the McMillan degree of V. Theorem 1.1 implies that

$$\deg V = \text{order } A_\pi + \sum_{\kappa_j > 0} \kappa_j = \text{order } A_\zeta + \sum_{\kappa_j < 0} - \kappa_j, \tag{5.1}$$

where $\kappa_1, \dots, \kappa_m$ are the column indices of V. Let W be the $m \times m$ regular rational matrix function defined in Theorem 3.1. Since V and W have the same $\mathbb{C}$-null-pole triple, there exists (cf. [3, Theorem 4.5.8]) a unimodular matrix polynomial $U(\lambda)$ such that $V(\lambda) = W(\lambda)U(\lambda)$. According to Theorem 1.4 this implies that up to a reordering V and W have the same column indices. So

$$\sum_{\kappa_j > 0} \kappa_j = \sum_{j=1}^{s} \omega_j = \dim K, \qquad \sum_{\kappa_j < 0} -\kappa_j = \sum_{j=1}^{t} \alpha_j = \dim \operatorname{Ker} \Gamma.$$

Here we use the notation introduced in the second and third paragraphs of Section 3. It follows that

$$\deg V = \dim X_\pi + \dim X_\zeta - \operatorname{rank} \Gamma.$$

Thus (see [3, Theorem 4.5.1]) the function $V(\cdot)$ is of minimal McMillan degree among all regular $m \times m$ rational matrix functions which have τ as $\mathbb{C}$-null-pole triple.

Part (b). Choose $\alpha \in \mathbb{C}$ such that (3.1) holds. Let φ be given by (3.23), and let $\hat{\tau}$ be defined by (3.24). Consider

$$\hat{V}(\lambda) = V(\varphi(\lambda))E^{-1}, \tag{5.2}$$

where $E = V(\alpha)$. Our choice of α implies that $V(\cdot)$ is analytic at α and $V(\alpha)$ is invertible. Thus $\hat{V}(\cdot)$ is well defined, $\hat{V}(\cdot)$ is analytic at infinity, and $V(\infty) = I_m$. Put

$$\hat{\sigma} = \{\lambda \in \mathbb{C} \mid \varphi(\lambda) \in \sigma(A_\pi) \cup \sigma(A_\zeta) \cup \{0\}\}.$$

By applying [3], Theorem 5.1.3, and similarity transformations in the spaces $\mathscr{X}_\pi$ and $\mathscr{X}_\zeta$ we see that $\hat{\tau}$ is a $\hat{\sigma}$-null-pole triple for $\hat{V}(\cdot)$. Since $\varphi(-\frac{1}{2}) = \infty$, we may choose a $\{-\frac{1}{2}\}$-admissible Sylvester data set $\hat{\tau}_0$ such that $\hat{\tau} \oplus \hat{\tau}_0$ is a global null-pole triple for $\hat{V}$. By the result of part (a) the function $\hat{V}$ is of minimal McMillan degree among all regular $m \times m$ rational matrix functions which have $\hat{\tau}$ as $\hat{\sigma}$-null-pole triple. Thus $\hat{\tau}_0$ is a minimal complement (see [9, Section 2]) of $\hat{\tau}$.

Part (c). By the second part of Theorem 2.1 in [10], every minimal complement of $\hat{\tau}$ is similar to a minimal complement of $\hat{\tau}$ obtained via the construction described in the first part of Theorem 2.1 in [10]. So without loss of generality we may assume that

$$\hat{\tau}_0 = \left(-\hat{C}\hat{X} - \hat{F}, \hat{T}; \hat{S}, -\hat{Y}\hat{B} + \hat{H}; \Gamma_0\right).$$

Here $(\hat{S}, \hat{H})$ is a zero correction pair for $\hat{\tau}$ and $(\hat{F}, \hat{T})$ is a pole correction pair of $\hat{\tau}$, such that $\sigma(\hat{S}) = \sigma(\hat{T}) = \{-\frac{1}{2}\}$. The linear transformations $\hat{X}$: $K \to \mathscr{X}_\pi$ and $\hat{Y}$: $\mathscr{X}_\zeta \to \operatorname{Ker}\Gamma$ are the unique solutions of the Lyapunov equations

$$\hat{A}_\pi \hat{X} - \hat{X}\hat{T} = \hat{A}_{12}, \qquad \hat{Y}\hat{A}_\zeta - \hat{S}\hat{Y} = \hat{A}_{21},$$

where $\hat{A}_{12}$: $K \to \mathscr{X}_\pi$ and $\hat{A}_{21}$: $\mathscr{X}_\zeta \to \operatorname{Ker}\Gamma$ are linear transformations which satisfy the identities (3.28), (3.29), and (3.30) for some linear operator Γ_1: $K \to \operatorname{Ker}\Gamma$. Finally,

$$\Gamma_0 = \hat{Y}\Gamma\hat{X} - \hat{Y}\eta_\zeta - \rho_\pi \hat{X} + \Gamma_1. \tag{5.3}$$

Since V is column reduced at infinity, $\hat{V}$ is column reduced at $-\frac{1}{2}$. Hence the matrices $\hat{S} + \frac{1}{2}$ and $\hat{T} + \frac{1}{2}$ are nilpotent, the orders of the Jordan blocks in the Jordan normal form of $\hat{S} + \frac{1}{2}$ coincide with the absolute values of the negative column indices of V, and the sizes of the Jordan blocks of $\hat{T} + \frac{1}{2}$ are equal to the positive column indices of V. Now recall [see part (a)] that up to a reordering V and W have the same column indices. Furthermore, we may use the description of the column indices of W given in Theorem 3.1. So $\hat{S} + \frac{1}{2}$ is nilpotent, and the orders of the Jordan blocks in its Jordan normal form are equal to the observability indices of the pair $(\hat{C}|_{\operatorname{Ker}\Gamma}, \rho_\pi \hat{A}_\pi|_{\operatorname{Ker}\Gamma})$. Similarly, $\hat{T} + \frac{1}{2}$ is nilpotent and the orders of the Jordan blocks in its Jordan normal form are equal to the controllability indices of the pair $(\rho_\zeta \hat{A}_\zeta|_K, \rho_\zeta \hat{B})$.

Part (d). In this part we apply Propositions 4.1 and 4.2. We know that $(\hat{S}, \hat{H})$ is a zero correction pair for $\hat{\tau}$. Hence

$$\rho_\pi\left(\hat{A}_\pi + \tfrac{1}{2}\right)|_{\operatorname{Ker}\Gamma} - \left(\hat{H}\hat{C}|_{\operatorname{Ker}\Gamma}\right) = \hat{S} + \tfrac{1}{2}.$$

The result of the previous part for $\hat{S} + \frac{1}{2}$ allows us to apply Proposition 4.1. So $\operatorname{Ker}\Gamma$ has a basis $\{\hat{d}_{jk}\}_{k=1,j=1}^{\alpha_j\ \ t}$ such that $\{\hat{d}_{jk}\}_{k=1,j=1}^{\alpha_j-1\, t}$ is a basis of $\operatorname{Ker}\hat{C} \cap \operatorname{Ker}\Gamma$ and

$$\rho_\pi\left(\hat{A}_\pi + \tfrac{1}{2}\right)\hat{d}_{jk} = \hat{d}_{j,k+1}, \qquad k = 1, \dots, \alpha_j - 1, \tag{5.4}$$

$$\left(\hat{S} + \tfrac{1}{2}\right)\hat{d}_{jk} = \hat{d}_{j,k+1}, \qquad k = 1, \dots, \alpha_j, \tag{5.5}$$

where $\hat{d}_{j,\alpha_j+1} = 0$. Since $\Gamma\hat{A}_\pi - \hat{A}_\zeta\Gamma = \hat{B}\hat{C}$, the operator $\hat{A}_\pi$ maps $\operatorname{Ker}\hat{C} \cap \operatorname{Ker}\Gamma$ into $\operatorname{Ker}\Gamma$. Thus (5.3) remains true if ρ_π in (5.4) is deleted. In other words, $\{\hat{d}_{jk}\}_{k=1,j=1}^{\alpha_j\ \ t}$ is an outgoing basis for $\operatorname{Ker}\Gamma$ with respect to the pair $(\hat{C}, \hat{A}_\pi + \frac{1}{2})$.

Next, we use that $(\hat{F}, \hat{T})$ is a pole correction pair of $\hat{\tau}$. So

$$\rho_\zeta\left(\hat{A}_\zeta + \tfrac{1}{2}\right)|_K - \rho_\zeta\hat{B}\hat{F} = \hat{T} + \tfrac{1}{2}.$$

The result of the previous part for $\hat{T} + \frac{1}{2}$ allows us to apply Proposition 4.2. So K has a basis $\{\hat{g}_{jk}\}_{k=1,j=1}^{\omega_j\ \ s}$ such that $\hat{g}_{11}, \dots, \hat{g}_{s1}$ is a basis of $\operatorname{Im}\rho_\zeta\hat{B}$ and

$$\rho_\zeta\left(\hat{A}_\zeta + \tfrac{1}{2}\right)\hat{g}_{jk} - \hat{g}_{j,k+1} \in \operatorname{Im}\rho_\zeta\hat{B}, \qquad k = 1, \dots, \omega_j, \tag{5.6}$$

$$\left(\hat{T}_\pi + \tfrac{1}{2}\right)\hat{g}_{j,k} = \hat{g}_{j,k+1}, \qquad k = 1, \dots, \omega_j, \tag{5.7}$$

where $\hat{g}_{j,\omega_j+1} = 0$. Recall that ρ_ζ is the projection of $\mathscr{X}_\zeta$ onto K along $\operatorname{Im}\Gamma$. Thus $\hat{g}_{11}, \dots, \hat{g}_{s1}$ is a basis of $\operatorname{Im}\Gamma + \operatorname{Im}\hat{B}$ modulo $\operatorname{Im}\Gamma$ and, because of (5.6),

$$\left(\hat{A}_\zeta + \tfrac{1}{2}\right)\hat{g}_{jk} - \hat{g}_{j,k+1} \in \operatorname{Im}\Gamma + \operatorname{Im}\hat{B}, \qquad k = 1, \dots, \omega_j.$$

In other words $\{\hat{g}_{jk}\}_{k=1,j=1}^{\omega_j\ \ s}$ is an incoming basis for K with respect to $(\hat{A}_\zeta + \frac{1}{2}, \hat{B})$.

Part (e). Now put

$$d_{jk} = \alpha^{\alpha_j - k}\sum_{\nu=k}^{\alpha_j}\binom{\alpha_j - k}{\nu - k}(-1)^{\alpha_j-\nu}\hat{d}_{j\nu}, \qquad k = 1, \dots, \alpha_j, \quad j = 1, \dots, t; \tag{5.8}$$

$$g_{jk} = \left(-\frac{1}{\alpha}\right)^{k-1}\sum_{\nu=k}^{\omega_j}\binom{\nu - 1}{k - 1}\hat{g}_{j\nu}, \qquad k = 1, \dots, \omega_j, \quad j = 1, \dots, s. \tag{5.9}$$

It is straightforward to check that $\{d_{jk}\}_{k=1,j=1}^{\alpha_j\ \ t}$ and $\{g_{jk}\}_{k=1,j=1}^{\omega_j\ \ s}$ are bases of $\operatorname{Ker}\Gamma$ and K, respectively, such that properties (a_1), (a_2), (b_1), and (b_2) hold (see the third paragraph of Section 3). Furthermore, (3.35) and (3.36) are valid. Next, define S and T as in (3.2) and (3.3), and put

$$H = (I - \alpha S)\hat{H}, \qquad F = \alpha^{-1}\hat{F}(I - \alpha T),$$

$$A_{12} = -\alpha^{-1}\hat{A}_{12}(I - \alpha T), \qquad A_{21} = -\alpha^{-1}(I - \alpha S)\hat{A}_{21},$$

$$X = (\alpha - A_\pi)^{-1}\hat{X}, \qquad Y = \hat{Y}(\alpha - A_\zeta)^{-1}.$$

Then (3.8), (3.9), and (3.10) are satisfied, and the operators X and Y are also given by (3.11) and (3.12), respectively. Let $V_0(\cdot)$ be the regular $m \times m$ rational matrix function given in Theorem 3.1, where the operators appearing in the paragraphs preceding Theorem 3.1 are defined as above. The arguments given in the proof of Theorem 3.1 show that

$$\hat{V}_0(\lambda) := V_0(\varphi(\lambda))V_0(\alpha)^{-1}$$

is a regular rational matrix function which is analytic at ∞, has the value I_m at ∞, and has $\hat{\tau} \oplus \hat{\tau}_0$ as its global null-pole triple. It follows that $\hat{V}_0(\lambda) = \hat{V}(\lambda)$, and therefore

$$V(\lambda) := V_0(\lambda)V_0(\alpha)^{-1}V(\alpha),$$

which completes the proof. ∎

6. COLUMN REDUCED MATRIX POLYNOMIALS

In this section the results of the previous section are specified and developed further for matrix polynomials. Let (A, B) be a pair of matrices, where A is $n \times n$ and B is $n \times m$, such that

$$\operatorname{Im}(B \quad AB \quad \cdots \quad A^{n-1} \quad B) = \mathbb{C}^n. \tag{6.1}$$

Our aim is to parametrize all regular $m \times m$ matrix polynomials $L(\lambda)$ for which

(j) $L(\lambda)$ has (A, B) as its left null pair;
(jj) $L(\lambda)$ is column reduced at infinity.

A regular matrix polynomial satisfying (j) and (jj) has been constructed in the papers [2, 16] by studying the controllability matrix in the left-hand side of (6.1). In these papers a desired matrix polynomial is constructed entrywise. In the present paper all regular matrix polynomials satisfying (j) and (jj) are obtained explicitly in realized form.

Let $\omega_1 \geqslant \cdots \geqslant \omega_s$ be the positive controllability indices of the pair (A, B). From the Brunowski canonical form of (A, B) (see [14]) it follows that we may choose a basis $\{g_{jk}\}_{k=1,j=1}^{\omega_j\ \ s}$ in $\mathbb{C}^n$ such that

(c_1) $\{g_{j\omega_j}\}_{j=1}^{s}$ is a basis of Im B;
(c_2) $Ag_{j,k+1} - g_{jk} \in \operatorname{Im} B$, $k = 0, \dots, \omega_j - 1$, where $g_{j0} := 0$.

With the basis $\{g_{jk}\}_{k=1,j=1}^{\omega_j\ \ s}$ we associate the operator $T: \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$Tg_{jk} = \begin{cases} g_{j,k+1}, & k = 1, \dots, \omega_j - 1, \\ 0, & k = \omega_j. \end{cases} \tag{6.2}$$

Note that $\operatorname{Im}(AT - I) \subset \operatorname{Im} B$, and hence there exists an operator F: $\mathbb{C}^n \to \mathbb{C}^m$ such that

$$AT + BF = I. \tag{6.3}$$

Next, choose a complex number α that is not an eigenvalue of A. From the definition of T in (6.2) it follows that

$$(I - \alpha T)\left(\sum_{\nu=1}^{\omega_j} \alpha^{\nu-1} g_{j\nu} \right) = g_{j1}, \qquad j = 1, \dots, s.$$

Since T is nilpotent, $I - \alpha T$ is invertible and thus

$$(I - \alpha T)^{-1} g_{j1} = \sum_{\nu=1}^{\omega_j} \alpha^{\nu-1} g_{j\nu}, \qquad j = 1, \dots, s. \tag{6.4}$$

The latter equality implies that

$$(\alpha - A)(I - \alpha T)^{-1} g_{j1} \in \operatorname{Im} B, \qquad j = 1, \dots, s. \tag{6.5}$$

Indeed, substituting (6.4), we get

$$(\alpha - A)(I - \alpha T)^{-1} g_{j1} = (\alpha - A)\left(\sum_{\nu=1}^{\omega_j} \alpha^{\nu-1} g_{j\nu} \right)$$

$$= \alpha^{\omega_j} g_{j\omega_j} + \sum_{\nu=1}^{\omega_j} \alpha^{\nu-1} A g_{j,\nu-1} - \sum_{\nu=1}^{\alpha_j} \alpha^{\nu-1} A g_{j\nu}$$

$$= \alpha^{\omega_j} g_{j\omega_j} - \sum_{\nu=1}^{\omega_j} \alpha^{\nu-1} (A g_{j\nu} - g_{j,\nu-1}),$$

where $g_{j0} = 0$. By properties (c_1) and (c_2) of the basis $\{g_{jk}\}_{k=1,j=1}^{\omega_j \ \ s}$ we obtain the desired formula (6.5). From (6.5) we can get vectors $y_1, \dots, y_s$ in $\mathbb{C}^m$ such that

$$By_j = (\alpha - A)(I - \alpha T)^{-1} g_{j1}, \qquad j = 1, \dots, s. \tag{6.6}$$

Since the vectors $g_{11}, \dots, g_{s1}$ are linearly independent, the same must be true for the vectors $y_1, \dots, y_s$ (because $\alpha - A$ and $I - \alpha T$ are invertible). We may extend $y_1, \dots, y_s$ to a basis of $\mathbb{C}^m$ by choosing $y_{s+1}, \dots, y_m$ so that $\{y_{s+1}, \dots, y_m\}$ is a basis of Ker B. Finally, let the operator V: $\mathbb{C}^m \to \mathbb{C}^m$ be defined by

$$Ve_j = y_j, \qquad j = 1, \dots, m, \tag{6.7}$$

where the vectors $e_1, \dots, e_m$ form the standard basis of $\mathbb{C}^m$.

We are now ready to construct a regular matrix polynomial satisfying (j) and (jj).

THEOREM 6.1. *Let* (A, B) *be a pair of matrices such that* (6.1) *holds. Choose a basis* $\{g_{jk}\}_{k=1,j=1}^{\omega_j \ \ s}$ *such that conditions* (c_1) *and* (c_2) *are fulfilled, and let* T *be the linear transformation defined by* (6.2), *choose* F *as in* (6.3), *define* V *by* (6.7), *and let* α *be the complex number used in the definition of* V. *Then*

$$L(\lambda) = V + (\lambda - \alpha) \sum_{j=0}^{\omega_1 - 1} \lambda^j F T^j (\alpha - A)^{-1} BV \tag{6.8}$$

is a regular $m \times m$ matrix polynomial satisfying (j) *and* (jj). *Moreover, the nonzero column indices of* $L(\lambda)$ *are* $\omega_1, \dots, \omega_s$.

Theorem 6.1 may be derived from Theorem 3.1 by applying Theorem 3.1 to the quintet $(0, 0; A, B; 0)$, which is an admissible Sylvester data set with $\mathscr{X}_\pi = \{0\}$. We omit the details.

The next theorem asserts that the construction carried out in Theorem 4.1 yields all regular $m \times m$ regular matrix polynomials satisfying (j) and (jj) up to a certain invertible constant factor on the right.

THEOREM 6.2. *Let* $L(\lambda)$ *be a regular* $m \times m$ *matrix polynomial satisfying* (j) *and* (jj). *then* $L(\lambda)$ *may be represented as*

$$L(\lambda) = D + (\lambda - \alpha) F(I - \lambda T)^{-1}(\alpha I - A)^{-1} BD, \tag{6.9}$$

with α *a complex number which is not an eigenvalue of* A *and with matrices* T *and* F *of sizes* $n \times n$ *and* $m \times n$, *respectively, for which*

$$AT + BF = I, \tag{6.10}$$

$$T \text{ is nilpotent.} \tag{6.11}$$

Also, there exists a basis $\{g_{jk}\}_{k=1,j=1}^{\omega_j\ \ s}$ *of* $\mathbb{C}^n$ *such that conditions* (c$_1$), (c$_2$) *are fulfilled*, (6.2) *holds, and*

$$D = VR, \tag{6.12}$$

where V *is given by* (6.7) *and* R *is a constant invertible matrix whose* (i, j)*th entry is zero whenever* $\omega_j < \omega_i$.

Theorem 6.2 may be proved in the same way as Theorem 3.2. Details are omitted. Note that the statement about the entries of R is an immediate consequence of Theorem 1.4.

Theorems 6.1 and 6.2 develop further the results in Section 2 of [12]. In particular, Theorem 6.1 shows that the matrix polynomial constructed in Theorem 2.1 of [12] is column reduced up to an invertible constant on the right.

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