Computers and Mathematics with Applications 64 (2012) 2735–2743

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Analysis and computation for a class of semilinear elliptic boundary value problems[☆]

Hai-long Zhu[a,c], Zhao-xiang Li[b,*], Zhong-hua Yang[b]

[a] Department of Mathematics, AnHui University of Finance & Economics, Bengbu 233030, China
[b] Department of Mathematics, Shanghai Normal University, Shanghai, 200234, China
[c] Institute of Applied Mathematics, AnHui University of Finance & Economics, Bengbu 233030, China

ARTICLE INFO

Article history:
Received 20 March 2012
Received in revised form 15 August 2012
Accepted 17 August 2012

Keywords:
Multiple solutions
Bifurcation
Sub-solution
Super-solution
Pseudo-arclength continuation

ABSTRACT

In this paper, with the help of super-solutions and sub-solutions, we set up a general framework and get a positive threshold Λ for solution existence and non-existence of a class of semilinear elliptic Dirichlet boundary value problems. Moreover, a result on multiplicity is obtained when λ is large enough. We also give a numerical method to solve and visualize the positive solutions of the problem. Theoretical results are illustrated by numerical simulation.

1. Introduction

In this paper, we consider the following semilinear elliptic boundary value problem (BVP)

$$\begin{cases} \Delta u(\boldsymbol{x}) + \lambda f(\boldsymbol{x}, u(\boldsymbol{x})) = 0, & \boldsymbol{x} \in \Omega, \\ u = 0, & \boldsymbol{x} \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a parameter, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ and $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfies the following hypotheses:

(H1) $f(0) = 0$,
(H2) $f'(0) = 0$,
(H3) $f''(0) < 0$,
(H4) There exists $\beta > 0$ such that $f(u) < 0$ for $u \in (0, \beta)$ and $f(u) > 0$ for $u > \beta$,
(H5) f is eventually increasing and $\lim_{u\to\infty} f(u)/u = 0$.

Let $F(u) = \int_0^u f(s)ds$, the energy functional $I_\lambda : H_0^1(\Omega) \to \mathbb{R}$ associated with problem (1.1) is defined by

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 d\boldsymbol{x} - \lambda \int_\Omega F(u) d\boldsymbol{x}. \tag{1.2}$$

[☆] This work was supported by the NSF of China (No. 10901106); the Shanghai Leading Academic Discipline Project (No. S30405); the Natural Science Foundation of Shanghai (No. 09ZR1423200); the Innovation Program of Shanghai Municipal Education Commission (No. 09YZ150); and the National Funds for Distinguished Young Scientists of Anhui Province (No. KJ2012B004).

* Corresponding author.
E-mail address: zxli@shnu.edu.cn (Z.-x. Li).

0898-1221/$ – see front matter
doi:10.1016/j.camwa.2012.08.004

The theoretical analysis and numerical results of multiple solutions to nonlinear equations of the form

$$\Delta u(\boldsymbol{x}) + f(\boldsymbol{x}, u(\boldsymbol{x})) = 0, \tag{1.3}$$

have been studied extensively by scientists.

In its theoretical aspect, Fowler [1] and Chandrasekhar [2] take the lead in studying the power type

$$\Delta u + u^p = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

which is the simplest, and yet the most basic form of nonlinearity. The recent works are [3,4], etc.

Henon [5] studied the stability of rotating stellar structures and proposed the equation

$$\Delta u + |\boldsymbol{x}|^l u^p = 0, \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

with $p > 1$, $l > 0$.

Lieb and Yau [6] considered Chandrasekhar's theory of stellar collapse. They showed that the Chandrasekhar equation for the white dwarf problem without the general relativistic effect is equivalent to the following equation

$$\Delta u + 4\pi(2u + u^2)^{3/2} = 0, \quad \text{in } B_R.$$

In view of the combined effects of superlinear and sublinear terms, Ambrosetti [7] considered the concave and convex nonlinearities

$$\Delta u(\boldsymbol{x}) + \lambda u(\boldsymbol{x})^q + u(\boldsymbol{x})^p = 0, \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

with $0 < q < 1 < p$.

In the computation aspect, for Eq. (1.3), there are 6 different numerical methods for computing such kinds of problems, as listed below:

(i) The Monotone Iterative Scheme (MIS) [8,9]: MIS is also called the barrier method or the method of super- and sub-solutions. It is a general, constructive method for finding stable solutions of semilinear elliptic equations.
(ii) The Mountain Pass Algorithm (MPA) [10]: MPA was proposed by Choi and McKenna to compute the solutions with the Morse Index (MI) 0 or 1.
(iii) The High Linking Algorithm (HLA) [11]: Ding, Costa and Chen established HLA for the sign-changing solution (MI $= 2$) of semilinear elliptic problems.
(iv) The Min–Max Algorithm (MMA) [12,13]: Li and Zhou designed MMA to find multiple saddle points with any Morse index which is more constructive than the traditional min–max theorem.
(v) The Search Extension Method (SEM) [14,15]: Chen and Xie proposed SEM, which searches the initial guess based on the linear combination of the eigenfunctions of the linearized problem and then gets a better initial guess by the continuation method for the discretized problem with the finite element method.
(vi) The Bifurcation Method (BM) [16–18]: BM was proposed by the authors. The method can compute not only a number of symmetric solutions of (1.3) as SEM does, but also various nonsymmetric solutions [16]. On the other hand, the difficulty in searching the initial guess in other methods can be solved effectively by the bifurcation method. Some publications also appeared in [19–22], which are concerned with the numerical computation of bifurcations and path following methods for nonlinear parameterized equations.

In this paper, we consider (1.1) under hypotheses (H1)–(H5). Our main results are as follows.

Theorem 1.1. *Let* (H1)–(H5) *hold. Then there exists a* $\Lambda_1 \in \mathbb{R}$, $\Lambda_1 > 0$ *such that for all* $\lambda \in (0, \Lambda_1)$, (1.1) *has no positive solution.*

Theorem 1.2. *Let* (H1)–(H5) *hold. Then there exists a* $\Lambda \in \mathbb{R}$, $\Lambda > 0$ *such that for* $\lambda = \Lambda$, (1.1) *has at least one weak solution* $u \in H_0^1(\Omega)$.

Theorem 1.3. *Let* (H1)–(H5) *hold. Then there exists a* $\Lambda_2 \in \mathbb{R}$, $\Lambda_2 > 0$ *such that for all* $\lambda > \Lambda_2$, (1.1) *has at least two positive solutions.*

The rest of the paper is organized as follows. Section 2 is concerned with theoretical analysis, we will give the proofs of Theorems 1.1–1.3 in this section. Section 3 is about computation, we shall give the numerical method for solving (1.1) which satisfies (H1)–(H5) and visualize the numerical solutions with an example $f(u) = u^2(u-2)e^{-u^2}$. Finally, Section 4 gives some conclusions.

2. Proof of theorem

First of all, we will give the following lemmas which will be used to prove Theorems 1.1–1.3.

Lemma 2.1 ([23]). *Suppose there exists a sub-solution Ψ_1, a strict super-solution Φ_1, a strict sub-solution Ψ_2 and a super-solution Φ_2 for Eq. (1.1) such that $\Psi_1 < \Phi_1 < \Phi_2, \Psi_1 < \Psi_2 < \Phi_2$ and $\Psi_2 \not\leq \Phi_1$. Then Eq. (1.1) has at least three distinct solutions $u_s (s = 1, 2, 3)$ such that $\Psi_1 \leq u_1 < u_2 < u_3 \leq \Phi_2$.*

Lemma 2.2 ([24]). *Let g be a C^1 function such that $g(0) < 0, g(u) > 0$ and $g'(u) > 0$ for $u > \sigma$ for some $\sigma > 0$ and $\lim_{u\to\infty} g(u)/u = 0$. Then the boundary value problem $-\Delta u = \lambda g(u)$ in Ω, $u = 0$ on $\partial\Omega$ has a positive solution u_λ for λ large. Further $\|u_\lambda\|_\infty \to \infty$ as $\lambda \to \infty$.*

Lemma 2.3. *Let* (H1)–(H5) *hold. Then there exists a $\Lambda_2 \in \mathbb{R}$, $\Lambda_2 > 0$ such that for $\lambda > \Lambda_2$, (1.1) has a positive solution u_λ.*

Proof. Let e denote the unique positive solution of

$$-\Delta e = 1, \quad \text{in } \Omega, \qquad e = 0 \quad \text{on } \partial\Omega. \tag{2.1}$$

Then there exists $\Phi_2 = Me$ with $M > 0$ satisfying

$$M \geq \lambda f(M\|e\|_\infty),$$

because of $\lim_{u\to\infty} f(u)/u = 0$ and f is eventually increasing.

As a consequence, the function $\Phi_2 = Me$ is verified

$$-\Delta\Phi_2 = -\Delta Me = M \geq \lambda f(Me) = \lambda f(\Phi_2), \tag{2.2}$$

and hence Φ_2 is a super-solution of (1.1).

Next consider a C^1 function g as in Lemma 2.2 such that $g(u) < f(u), \ \forall u \geq 0$. This is clearly by the hypotheses (H1)–(H5). Let Ψ_2 be a positive solution for large λ described in Lemma 2.2 of the BVP:

$$-\Delta u = \lambda g(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{2.3}$$

Then

$$-\Delta\Psi_2 = \lambda g \quad (\Psi_2) < \lambda f(\Psi_2), \tag{2.4}$$

and hence Ψ_2 is a strict sub-solution of (1.1) for λ large enough.

Now choosing M large enough so that $\Psi_2 < Me = \Phi_2$, which is possible since $e(\boldsymbol{x}) > 0$ for $\boldsymbol{x} \in \Omega$ and $\partial e/\partial\boldsymbol{n} < 0$ for $\boldsymbol{x} \in \partial\Omega$ where $\boldsymbol{n}$ denotes the outward normal.

It follows that (1.1) has a solution

$$\Psi_2 \leq u_\lambda \leq \Phi_2 \tag{2.5}$$

for $\lambda > \Lambda_2$. □

Now we prove Theorems 1.1–1.3.

Proof of Theorem 1.1. Let the principal eigenvalue of

$$-\Delta\widetilde{v}(x) = \widetilde{\lambda}\widetilde{v}(x) \quad \text{in } \Omega, \qquad \widetilde{v}(x) = 0 \quad \text{on } \partial\Omega, \tag{2.6}$$

be denoted by $\widetilde{\lambda}$, $\widetilde{v}(x)$ denotes the corresponding eigenfunction satisfying $\widetilde{v}(x) > 0$ in Ω and $\|\widetilde{v}(x)\|_2 = 1$. Let $u(x)$ be a positive solution of (1.1). We multiply equality (2.5) by $u(x)$ and from the result obtained we subtract equality (1.1) multiplied by $\widetilde{v}(x)$. As a result, we get

$$\int_\Omega \{\lambda f(u) - \widetilde{\lambda}u\}\widetilde{v}(x)dx = 0, \tag{2.7}$$

which is easily deduced by applying Green's identity and the boundary conditions.

From $f(0) = 0, f'(0) = 0$ and $\lim_{u\to\infty} f(u)/u = 0$, there exists $K > 0$ such that $f(u) < Ku, \forall u \geq 0$. Let $\Lambda_1 \in \mathbb{R}$, $\Lambda_1 > 0$ be small enough, we will have $\widetilde{\lambda}/\lambda > K$ for $\lambda < \Lambda_1$, which conflicts with (2.6). Hence (1.1) has no positive solution for $\lambda < \Lambda_1$. This completes the proof. □

Proof of Theorem 1.2. A weak solution of problem (1.1), $u \in H_0^1(\Omega)$ satisfies the following variational equation

$$Q(u, v) \equiv (\nabla u, \nabla v) - \lambda(f(u), v) = 0, \quad \forall v \in H_0^1(\Omega). \tag{2.8}$$

From Theorem 1.1 and Lemma 2.3, we know that (2.7) has no positive solution for $\lambda < \Lambda_1$ and at least one positive solution for $\lambda > \Lambda_2$. Then there exists $\Lambda_1 < \Lambda < \Lambda_2$, (1.1) has at least one weak solution $u \in H_0^1(\Omega)$ since f is C^1. □

Proof of Theorem 1.3. Consider $\Phi_1(x) = \varepsilon\widetilde{v}(x)$, where $\widetilde{v}(x)$ is as defined in (2.5). Now $H(t) = \widetilde{\lambda}t - \lambda f(t) > 0$ for small positive t since $f'(0) = 0$ and $f''(0) < 0$. Thus

$$-\Delta\Phi_1 = \widetilde{\lambda}\varepsilon\widetilde{v}(x) > \lambda f(\varepsilon\widetilde{v}(x)), \tag{2.9}$$

if $\varepsilon > 0$ is small and hence $\Phi_1(x)$ is a strict super-solution of (1.1).

Clearly $\Psi_1 \equiv 0$ is a solution to (1.1). Now we have two sub-solutions Ψ_1, Ψ_2 and two super-solutions Φ_1, Φ_2 from the analysis above and Lemma 2.3. Further, we can obtain $\Psi_1 < \Psi_2 < \Phi_2$ from Lemma 2.3 and $\Psi_1 < \Phi_1 < \Phi_2$ while using the method similar to Lemma 2.3. Now applying Lemma 2.1, there exist at least two distinct positive solutions for λ large which easily follows. $\square$

3. Numerical algorithm

3.1. Numerical method

In this section, we will use BM to solve numerical solutions of (1.1) with $\Omega \subset \mathbb{R}^2$ under hypotheses (H1)–(H5). The method discussed in this section, which is used to find zeros of function f, is Newton's method with the following equation:

$$x^{n+1} = x^n - \frac{F(x^n)}{F'(x^n)}. \tag{3.1}$$

The following theorem is used to show sufficient conditions under which Newton's method converges to a solution.

Theorem 3.1 (*[25]*). *Let $D \subset \mathbb{R}^n$ be open and convex and let $F : D \to \mathbb{R}^n$ be continuously differentiable. Assume that for some norm $\|\cdot\|$ on $\mathbb{R}^n$ and $x_0 \in D$ the following condition holds:*

(a) *F satisfies*

$$\|F'(x) - F'(y)\| \le \gamma\|x - y\|$$

for all $x, y \in D$ and some constant $\gamma > 0$.

(b) *The Jacobian matrix $F'(x)$ is nonsingular for all $x \in D$, and there exists a constant $\beta > 0$, such that*

$$\|F'(x)^{-1}\| \le \beta, \quad x \in D.$$

(c) *For the constants*

$$\alpha := \|F'(x_0)^{-1}F(x_0)\| \quad \text{and} \quad q := \alpha\beta\gamma$$

the inequality

$$q < \frac{1}{2}$$

is satisfied.

(d) *For $r := 2\alpha$ the closed ball $B[x_0, r] := \{x : \|x - x_0\| \le r\}$ is contained in D.*

Then F has a unique zero x^* in $B[x_0, r]$. Starting with x_0 Newton iteration

$$x^{n+1} = x^n - \frac{F(x^n)}{F'(x^n)}, \quad n = 0, 1, \ldots$$

is well-defined. The sequence $\{x^n\}$ converges to the zero x^* of F, and we have the error estimate

$$\|x^n - x^*\| \le 2\alpha q^{2^n - 1}, \quad n = 0, 1, \ldots.$$

We embed (1.1) into the nonlinear bifurcation problems with parameter $\tilde{\lambda}$ of the following form:

$$\begin{cases} \Delta u(\boldsymbol{x}) + \tilde{\lambda}u(\boldsymbol{x}) + \lambda f(u(\boldsymbol{x})) = 0, & \boldsymbol{x} \in \Omega, \\ u = 0, & \boldsymbol{x} \in \partial\Omega. \end{cases} \tag{3.2}$$

Considering the linearized equation of (3.2) at $u = 0$, we get

$$\begin{cases} \Delta\varphi(\boldsymbol{x}) + \tilde{\lambda}\varphi(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \Omega, \\ \varphi(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \partial\Omega, \end{cases} \tag{3.3}$$

because of $f(0) = 0$. It is well known that (3.3) always has a trivial solution. Furthermore, eigenpairs $\{\lambda_j, \varphi_j\}$ satisfying (3.3), where the eigenvalues are $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots \to \infty$, and the corresponding eigenfunctions $\{\varphi_j\}_1^\infty$ form a complete normalized orthogonal system S_∞ satisfying $\langle\varphi_i, \varphi_j\rangle = \delta_{ij}$, where δ_{ij} is a Kronecker's symbol.

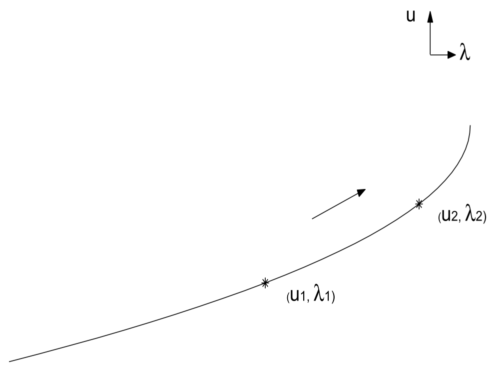

Fig. 1. Predictor–corrector method to calculate the untrivial solution branch with parameter λ.

To compute the positive solutions of (1.1), we consider the first single eigenpair $\{\lambda_1, \varphi_1\}$ here. Let $F = \Delta u + \tilde{\lambda} u + \lambda f(u), X = \{u|u \in C^2(\Omega), u|_{\partial\Omega} = 0\}, Y = \{u|u \in C^0(\Omega)\}, L = \Delta + \lambda_1$.

We define the inner product by

$$\langle u, v\rangle = \int_\Omega uv d\boldsymbol{x}.$$

L is a Fredholm self-adjoint operator with index zero, and

$$N(L^*) = N(L) = span\{\varphi_1\}, \tag{3.4}$$

where $N(L)$ and $N(L^*)$ are the null space of L and L^* respectively. Space X and Y have the decomposition

$$X = N(L) \oplus M, \qquad Y = N(L^*) \oplus R(L) = N(L) \oplus R(L),$$

where $M = R(L) \cap X$, $R(L)$ is the range of L.

Let P be the orthogonal projector from Y to $R(L)$

$$Pz = z - \langle z, \varphi_1\rangle\varphi_1, \quad z \in Y.$$

Eq. (3.2) is equivalent to

$$\begin{cases} PF(\tau\varphi_1 + \omega, \mu + \lambda_1) = 0, \quad \tau \in \mathbb{R}, \omega \in M, & \text{(a)} \\ \langle \varphi_1, F(\tau\varphi_1 + \omega, \mu + \lambda_1)\rangle = 0, & \text{(b)} \end{cases} \tag{3.5}$$

where $\mu = \tilde{\lambda} - \lambda_1, u = \tau\varphi_1 + \omega$. Since $PF_\omega(0, \lambda_1) = PF_u(0, \lambda_1) = PL = L$, and L restricted in M is regular, (3.5)(a) has a unique solution $\omega = \omega(\tau, \mu)$ which satisfies $\omega(0, 0) = 0$ by the implicit function theorem.

Substituting $\omega(\tau, \mu)$ into (3.5)(b) yields

$$g(\tau, \mu) = \langle \varphi_1, F(\tau\varphi_1 + \omega(\tau, \mu), \mu + \lambda_1)\rangle = 0. \tag{3.6}$$

Then we get

$$F(\omega, \tau) = F(\tau\varphi_1 + \omega, \mu + \lambda_1) = \Delta\omega + \lambda_1\omega + h(\tau, \mu), \tag{3.7}$$

where $h(\tau, \mu) = \mu(\tau\varphi_1 + \omega) + \lambda f(\tau\varphi_1 + \omega), \omega = \omega(\tau, \mu)$. Eq. (3.2) is transformed into

$$\begin{cases} \Delta\omega + (\mu + \lambda_1)\omega + \mu\tau\varphi_1 + \lambda f(\tau\varphi_1 + \omega) = 0, & \boldsymbol{x} \in \Omega, \\ \omega = 0, & \boldsymbol{x} \in \partial\Omega, \\ \langle \varphi_1, \omega\rangle = 0. \end{cases} \tag{3.8}$$

Newton's method is used to solve the ω and τ, then we get $u = \tau\varphi_1 + \omega$. In order to ensure the numerical solution's truthfulness and validity, our safeguard here is that we take different spacing h to discretize the region Ω with a uniform mesh. Here we compute (3.8) with different h, which satisfying $\frac{1}{h}$ is a rational number or any irrational number. If those output solutions are all close to each other, we accept them as authentic, otherwise, we reject them.

As mentioned above, we've got the first positive solution of (1.1). The next is to show the second positive solution with the same λ, in order to emphasize the dependence on λ, (1.1) is regarded as $(1.1)_\lambda$ here. Let $(u_1, \lambda_1) = (u_{\lambda_1}, \lambda_1)$ be a solution on the solution branch of $F(u_\lambda, \lambda) = 0$, where $F(u_\lambda, \lambda) = \Delta u + \lambda f(u)$. To get the solution branch, the most obvious parameter is the control variable λ. Let (u_1, λ_1) be the predictor point of $F(u_2, \lambda_2) = 0$, we can get the solution (u_2, λ_2) of $(1.1)_\lambda$ (see Fig. 1). While the parameter λ has the advantage of having practical significance, it encounters difficulties at the turning point, where the pulling direction is normal to the branch (see (u_3, λ_3) in Fig. 2).

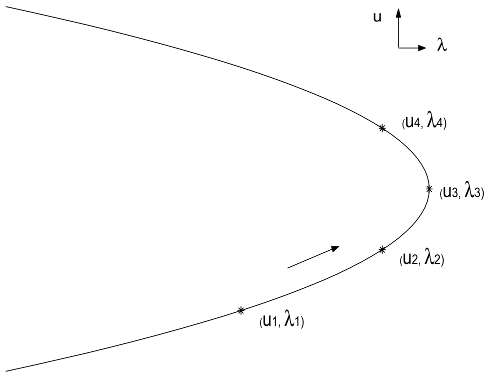

Fig. 2. Pseudo arclength method to calculate the untrivial solution branch with curve parameter s to cross the turning point (u_3, λ_3).

The solution branch of $F(u_\lambda, \lambda) = 0$ can be parameterized by the curve parameter. A general curve parameter is called s. A parameterization by s means that the solution of $F(u_\lambda, \lambda) = 0$ depends on s:

$$u = u(\boldsymbol{x}; s), \qquad \lambda = \lambda(s).$$

This means pulling the imaginary particle in the direction tangent to the branch, the turning point does not pose problems. We adjoint to problem (1.1) the arclength normalization equation [26,27]

$$N(u, \lambda, s) = \dot{u}^{*T}(u - u^*) + \dot{\lambda}^*(\lambda - \lambda^*) - (s - s^*) = 0, \tag{3.9}$$

where (u^*, λ^*) is the solution previously calculated, and $\dot{u}^{*T}$, $\dot{\lambda}^*$ can be calculated by

$$\dot{u}^{*T} = \beta v, \quad \beta \in \mathbb{R},$$
$$\dot{\lambda}^* = \beta,$$

where v, β satisfy

$$F_u^* v = -F_\lambda^*,$$
$$\beta = \frac{\pm 1}{\sqrt{1 + \|v\|^2}}.$$

From equation $F(u_\lambda, \lambda) = 0$ and Eq. (3.9), we get the system:

$$G(u, \lambda, s) = \begin{pmatrix} F(u(\boldsymbol{x}; s), \lambda(s)) \\ N(u(\boldsymbol{x}; s), \lambda(s), s) \end{pmatrix} = 0. \tag{3.10}$$

We can solve the solution branch by Newton's method. By this way, we can compute the second positive solution of $(1.1)_\lambda$ which confirms Theorem 1.3.

3.2. Numerical example

In this subsection, we consider an example $\tilde{f}(u) = u^2(u-2)e^{-u^2}$ which satisfies hypotheses (H1)–(H5). Besides, we have $\tilde{F}(u) = \frac{1}{2}(1 - u^2)e^{-u^2} = \int_0^u \tilde{f}(s)ds$.

In this subsection, we will chose three types of domains for the computation of numerical solutions:

(a) Square $\Omega = \Omega_1 = [0, 1] \times [0, 1]$;
(b) Disk $\Omega = \Omega_2 = \{\boldsymbol{x} \in \mathbb{R}^2 | |\boldsymbol{x} - (0, 0)| \le 1\}$;
(c) L-shaped region $\Omega = \Omega_3 = S_1/S_2$, where $S_1 = [-1, 1] \times [-1, 1]$, $S_2 = [-1, 0] \times [-1, 0]$.

The rationale for choosing (a)–(c) is based on the special geometrical and topological features offered by each type of domains.

For the square, type (a), (3.3) have analytic solutions $\varphi_{n,m} = \sin(n\pi x_1)\sin(m\pi x_2)$ while $\lambda_{n,m} = (n^2 + m^2)\pi^2$, so many pure and applied mathematicians and physicists are interested in this domain [28].

The disk, type (b), has the strongest symmetry, whereupon the analytic information about nonlinear elliptic equations is relatively easy.

L-shaped region, type (c): an L-shaped region formed from three rectangles is interesting for several reasons. It is one of the simplest geometries for which solutions of (3.3) cannot be expressed analytically, so numerical computation is necessary. Furthermore, the 270° nonconvex corner causes a singularity in the solution. Mathematically, the gradient of the first eigenfunction is unbounded near the corner. Physically, a membrane stretched over such a region would rip at the corner.

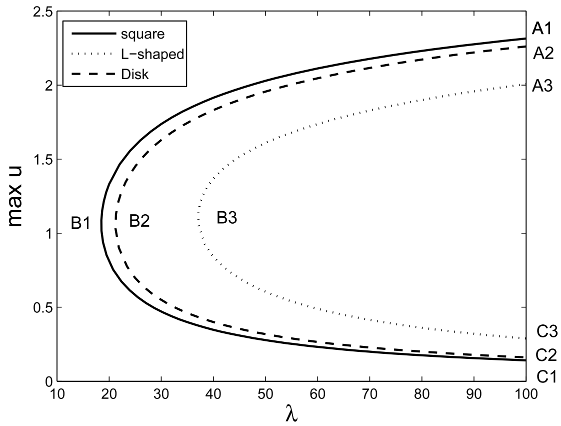

Fig. 3. Solution branch of (1.1) while $f(u) = u^2(u-2)e^{-u^2}$ with different λ on three types of domains.

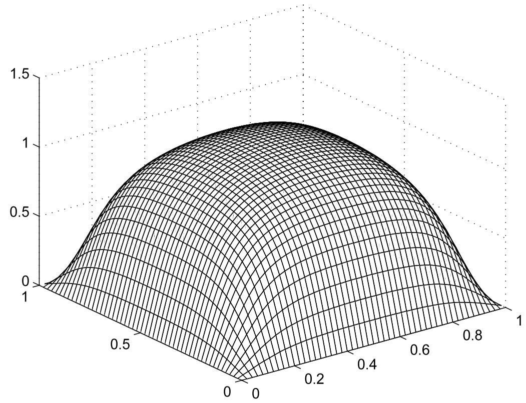

Fig. 4. The positive solution u of (1.1) on Ω_1 with $\lambda = 90$ on the branch A1 to B1.

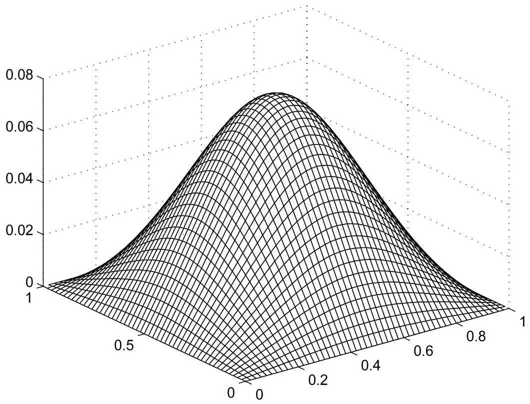

Fig. 5. The positive solution u of (1.1) on Ω_1 with $\lambda = 90$ on the branch B1 to C1.

This singularity limits the accuracy of finite difference methods with uniform grids. MathWorks has adopted a surface plot of the first eigenfunction of the L-shaped region of (3.3) as the company logo.

In Fig. 3, the solution branches on three different types of domains: square, disk and L-shaped domains are respectively drawn by the solid line, the dashed line and the dotted line.

Case (a) *Square.* Problem (1.1) has no positive solution with $\lambda < 18.5151$ and at least two positive solutions with $\lambda > 18.5151$. Two different solutions are displayed in Figs. 4 and 5 while $\lambda = 90$. Fig. 4 denotes the solution u of (1.1) on the branch A1 to B1, with $\max u = 1.1231, J = -13.7031, \tilde{\varepsilon}_n = 10^{-5}$, Fig. 5 denotes the solution u of (1.1) on the branch B1 to C1, with $\max u = 0.0713, J = -43.1095, \tilde{\varepsilon}_n = 10^{-5}$, where $J = J(u)$ is the energy value of the solution and $\tilde{\varepsilon}_n$ denotes the relative convergence error from $\tilde{\varepsilon}_n = \|u_{n+1} - u_n\| < \varepsilon$.

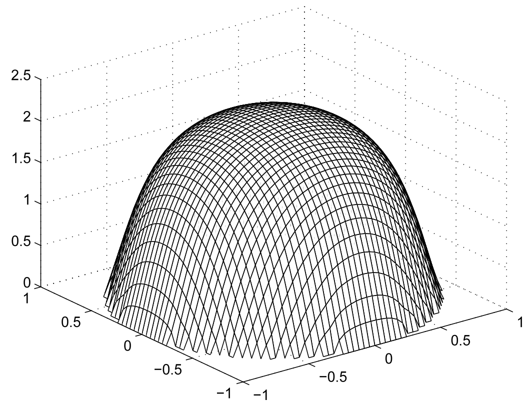

Fig. 6. The positive solution u of (1.1) on Ω_2 with $\lambda = 90$ on the branch A2 to B2.

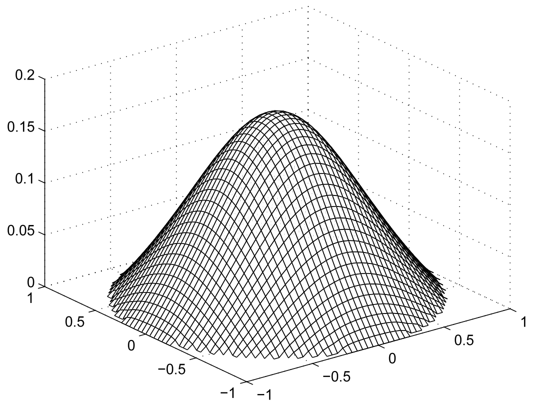

Fig. 7. The positive solution u of (1.1) on Ω_2 with $\lambda = 90$ on the branch B2 to C2.

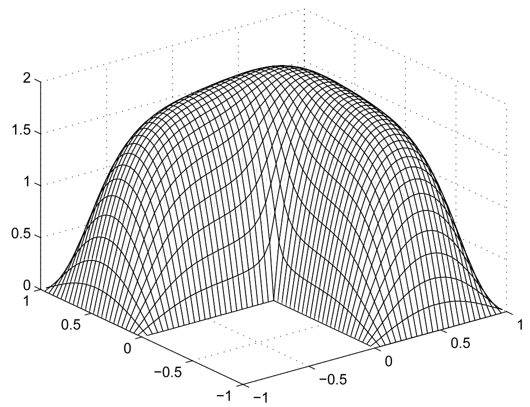

Fig. 8. The positive solution u of (1.1) on Ω_3 with $\lambda = 90$ on the branch A3 to B3.

Case (b) *Disk.* Problem (1.1) has no positive solution with $\lambda < 21.23015$ and at least two positive solutions with $\lambda > 21.23015$. Two different solutions are displayed in Figs. 6 and 7 while $\lambda = 90$. Fig. 6 denotes the solution u of (1.1) on the branch A2 to B2, with $\max u = 2.0784$, $J = -2.9339$, $\tilde{\varepsilon}_n = 10^{-6}$, Fig. 7 denotes the solution u of (1.1) on the branch B2 to C2, with $\max u = 0.1501$, $J = -42.7872$, $\tilde{\varepsilon}_n = 10^{-6}$.

Case (c) L-*shaped domain.* Problem (1.1) has no positive solution with $\lambda < 37.1125$ and at least two positive solutions with $\lambda > 37.1125$. Two different solutions are displayed in Figs. 8 and 9 while $\lambda = 90$. Fig. 8 denotes the solution u of (1.1) on the branch A3 to B3, with $\max u = 1.8714$, $J = -4.7302$, $\tilde{\varepsilon}_n = 10^{-5}$, Fig. 9 denotes the solution u of (1.1) on the branch B3 to C3, with $\max u = 0.3102$, $J = -31.0397$, $\tilde{\varepsilon}_n = 10^{-5}$.

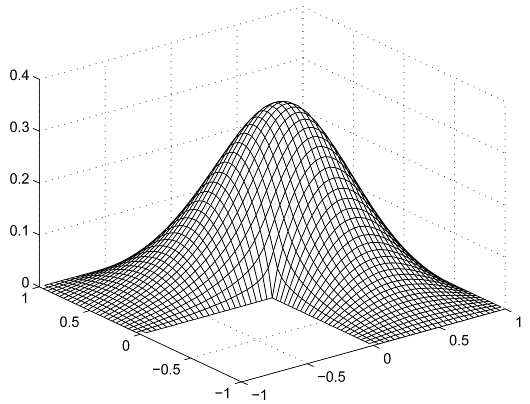

Fig. 9. The positive solution u of (1.1) on Ω_3 with $\lambda = 90$ on the branch B3 to C3.

4. Conclusion

In this paper, our main work has two aspects:

(i) With the help of super-solutions and sub-solutions, we set up a general framework and get a positive threshold Λ for solution existence and non-existence. Moreover a result on multiplicity is obtained when λ is large enough.

(ii) With the help of the bifurcation method, we solve and visualize the multiple positive solutions of (1.1) under hypotheses (H1)–(H5). The algorithm can compute more than one positive solution of (1.1), and it can visualize the positive solution branch of (1.1). Theoretical results are illustrated by numerical simulation.

References

[1] R. Fowler, Further studies of Emden's and similar differential equations, Q. J. Math. 2 (1931) 259–288.
[2] S. Chandrasekhar, An Introduction to the Study of Stellar Structure, University of Chicago Press, 1939.
[3] R. Kajikiya, Non-radial solutions with group invariance for the sublinear Emden–Fowler equation, Nonlinear Anal. 47 (2001) 3759–3770.
[4] I. Flores, A resonance phenomenon for ground states of an elliptic equation of Emden–Fowler type, J. Differential Equations 198 (2004) 1–15.
[5] M. Henon, Numerical experiments on the stability of spherical stellar systems, Astronm. Astrophys. 24 (1973) 229–238.
[6] E. Lieb, H.T. Yau, The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics, Comm. Math. Phys. 112 (1987) 147–174.
[7] A. Ambrosetti, Combined effects of concave and convex nonlinearities in some elliptic problems, J. Funct. Anal. 122 (1994) 519–543.
[8] C.V. Pao, Block monotone iterative methods for numerical solutions of nonlinear elliptic equations, Numer. Math. 72 (1995) 239–262.
[9] Y. Deng, G. Chen, W.M. Ni, et al., Boundary element monotone iteration scheme for semilinear elliptic partial differential equations, Math. Comp. 65 (1996) 943–982.
[10] Y.S. Choi, P.J. McKenna, A mountain pass method for the numerical solutions of semilinear elliptic problems, Nonlinear Anal. 20 (1993) 417–437.
[11] Z.H. Ding, D. Costa, G. Chen, A high-linking algorithm for sign-changing solutions of semilinear elliptic equations, Nonlinear Anal. 38 (1999) 151–172.
[12] Y. Li, J.X. Zhou, A minimax method for finding multiple critical points and its applications to semilinear PDEs, SIAM J. Sci. Comput. 23 (2002) 840–865.
[13] X.D. Yao, J.X. Zhou, A minimax method for finding multiple critical points in Banach spaces and its application to quasi-linear elliptic PDE, SIAM J. Sci. Comput. 26 (2005) 1796–1809.
[14] C.M. Chen, Z.Q. Xie, Search-extension method for multiple solutions of nonlinear problem, Comput. Math. Appl. 47 (2004) 327–343.
[15] C.M. Chen, Z.Q. Xie, Analysis of search-extension method for finding multiple solutions of nonlinear problem, Sci. China, Ser. A Math. 51 (2008) 42–54.
[16] Z.H. Yang, Z.X. Li, H.L. Zhu, Bifurcation method for solving multiple positive solutions to Henon equation, Sci. China, Ser. A Math. 51 (2008) 2330–2342.
[17] Z.X. Li, H.L. Zhu, Z.H. Yang, Bifurcation method for solving multiple positive solutions to Henon equation on the unit cube, Commun. Nonlinear Sci. Numer. Simul. 16 (2011) 3673–3683.
[18] Z.X. Li, Z.H. Yang, H.L. Zhu, Bifurcation method for solving multiple positive solutions to boundary value problem of Henon equation on unit disk, Comput. Math. Appl. 62 (2011) 3775–3784.
[19] H.B. Keller, On Numerical Methods in Bifurcation Problems, Springer, Berlin, 1987.
[20] Z.H. Yang, H.B. Keller, A direct method for computing higher order folds, SIAM J. Sci. Stat. Comput. 7 (1986) 351–361.
[21] Eugene L. Allgower, Kurt Georg, Continuation and path following, Acta Numer. 1–64 (1992).
[22] Eugene Allgower, Klaus Bohmer, Zhen Mei, Branch switching at a corank-4 bifurcation point of semi-linear elliptic problems with symmetry, IMA J. Numer. Anal. 14 (1994) 161–182.
[23] R. Shivaji, A remark on the existence of three solutions via sub-super solutions, in: V. Lakshmikantham (Ed.), in: Lecture Notes in Pure and Applied Mathematics, vol. 109, Spring, Berlin, 1987, pp. 561–566.
[24] A. Castro, J. Garner, R. Shivaji, Existence results for classes of sublinear semipositone problems, Results Math. 23 (1993) 214–220.
[25] R. Kress, Numerical Analysis, Springer-Verlag, Berlin, 2003.
[26] Z, H. Yang, Non-linear Bifurcation: Theory and Computation, Science Press, Beijing, 2007 (in Chinese).
[27] R. Seydel, Practical Bifurcation and Stability Analysis, Springer-Verlag, Berlin, 1994.
[28] M. Zhen, W.S. Chih, Forced symmetry-breaking via boundary conditions, Nonlinear Anal. 44 (2001) 509–526.