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Convex Polytopes and Enumeration

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This is an expository paper on connections between enumerative combinatorics and convex polytopes. It aims to give an essentially self-contained overview of five specific instances when enumerative combinatorics and convex polytopes arise jointly in problems whose initial formulation lies in only one of these two subjects. These examples constitute only a sample of such instances occurring in the work of several authors. On the enumerative side, they involved ordered graphical sequences, combinatorial statistics on the symmetric and hyperoctahedral groups, lattice paths, Baxter, André, and simsun permutations, q-Catalan and q-Schröder numbers. From the subject of polytopes, the examples involve the Ehrhart polynomial, the permutohedron, the associahedron, polytopes arising as intersections of cubes and simplices with half-spaces, and the cd-index of a polytope.

INTRODUCTION

A *convex polytope*, $P \subseteq \mathbf{R}^n$, is the convex hull of finitely many points in n-dimensional Euclidean space. That is, $P = \{\sum_{i=1}^{p} \lambda_i \mathbf{x}_i : \sum_{i=1}^{p} \lambda_i = 1, 0 \leq \lambda_i \leq 1 \text{ for } 0 \leq i \leq p\}$, where $\mathbf{x}_1, \ldots, \mathbf{x}_p$ are points in $\mathbf{R}^n$. Alternatively, a polytope is a *bounded* intersection of finitely many closed half spaces in $\mathbf{R}^n$. That is, it is a bounded set in $\mathbf{R}^n$ arising as the solution to a system of linear inequalities, $P = \{\mathbf{x} \in \mathbf{R}^n : A\mathbf{x} \leq \mathbf{b}\}$ for an $m \times n$ matrix A with real entries and a (column) vector $\mathbf{b} \in \mathbf{R}^m$. Unbounded convex sets arising from system of linear inequalities, *polyhedra*, are closely related and important objects, but they will not be featured here.

The subject of polytopes is exceptionally broad, lending itself to investigation from numerous points of view, including: linear optimization, geometry, computational geometry, extremal combinatorics. The techniques used in the study of polytopes have expanded in recent years to include tools from commutative algebra and algebraic geometry. The vast body of literature on polytopes, includes chapters in the "Handbook of Convex

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Geometry'' (e.g., [BayLe]), ''Handbook of Combinatorics'' (e.g., [KleeKlein]) and ''CRC Handbook of Discrete and Computational Geometry'' (e.g., [BiBj]). An extensive bibliography is included in [Zi].

The present paper is a slightly expanded version of the invited plenary talk given at the ''SIAM Discrete Mathematics Conference,'' Baltimore, June 17–20, 1996. The interest expressed by the audience, which prompted the writing of these notes, is acknowledged with appreciation. The focus of this paper is on connections between enumerative combinatorics and convex polytopes. More precisely, we aim to give an essentially self-contained overview of five specific instances when enumerative combinatorics and convex polytopes arise jointly in problems whose initial formulation lies in only one of these subjects. These examples constitute only a sample of such instances, intended to whet the appetite of the reader with expertise in one, the other, or neither of the two subjects.

The paper is organized as follows. Section 1 covers some basic concepts associated with polytopes, and Section 2 presents a few fundamental results concerning polytopes. The selection of topics in these two sections is limited to the background that is useful for the rest of the paper. For a thorough treatment, including proofs, the interested reader is referred to the list of references ([Zi] and its bibliography, in particular). For additional background on partially ordered sets and enumeration, the interested reader may consult [Ai] or [St1].

1. BASIC NOTIONS

The *dimension*, $\dim(P)$, of a polytope P is its affine dimension. Thus, the dimension is intrinsic to the polytope. The example in Fig. 1, a convex polygon, is a two-dimensional polytope—whether it is embedded in the plane or in a higher dimensional Euclidean space.

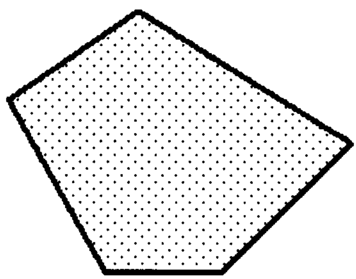


FIG. 1. A two-dimensional convex polytope.

Given a d-dimensional polytope $P \subseteq \mathbf{R}^n$, a *supporting hyperplane* of P is a hyperplane H in $\mathbf{R}^n$ such that P lies in one of the two closed half-spaces determined by H. Equivalently, there is an equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ for H, such that $a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq 0$ for all $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in P$. If H is a supporting hyperplane for P, then $P \cap H$ is a *face* of P. In particular, the empty set is a (n improper) face of every nonempty polytope, and the same face may arise from several supporting hyperplanes. It is easy to check (see, e.g., [Zi, p. 53]) that faces of polytopes are themselves polytopes. As such, the faces of a polytope may be counted according to dimension.

Let f_i denote the number of i-dimensional faces of a d-dimensional polytope P. In particular, $f_{-1} = 1$ counts the empty face, f_0 is the number of *vertices*, f_1 is the number of *edges*, f_{d-1} is the number of *facets* of P, and $f_d = 1$ counts P itself. Then $f(P) := (f_{-1}, f_0, f_1, \ldots, f_{d-1}, f_d)$ is the *f-vector* of P. For example, both polytopes shown in Fig. 2 have f-vector $f = (1, 4, 6, 4, 1)$. (Elsewhere in the literature, the f-vector may exclude one or both of the entries f_{-1} and f_d.)

With the natural relation of containment, the faces of a polytope form a partially ordered set (poset). If the empty face and the polytope itself are included as elements, then the resulting poset is a lattice, $\mathscr{L}(P)$, called the *face lattice* of P. The face lattice of a polytope is ranked, with rank function $\mathrm{rk}(F) = \dim(F) + 1$, and the f-vector of P gives the rank sizes of $\mathscr{L}(P)$ (the Whitney numbers of the second kind).

FACT 1.1 [St1, p. 122]. *The face lattice of a polytope is an Eulerian poset.*

An *Eulerian poset* is a ranked poset with a minimum and a maximum element (denoted $\hat{0}$ and $\hat{1}$, resp.), such that $\mu(x, y) = (-1)^{rk(y)-rk(x)}$ for all $x \leq y$ in the poset. Here, μ denotes the Möbius function of the poset. Equivalently, a poset is Eulerian if for every pair of elements x, y such

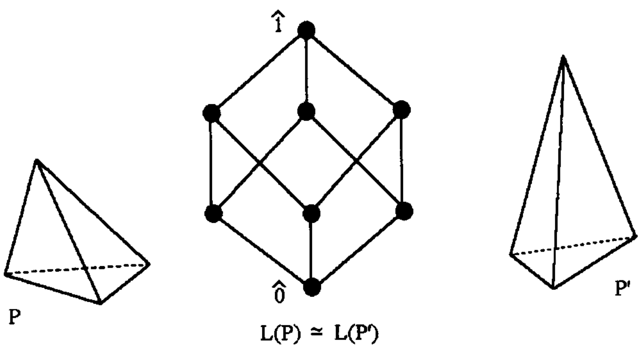


FIG. 2. Two combinatorially equivalent polytopes and their face lattice.

that $x < y$, the following condition holds: among the elements a such that $x \leq a \leq y$ there are as many of even rank as there are of odd rank. ([St2] offers an extensive discussion of Eulerian posets.)

Note that upon choosing $x = \hat{0}$ and $y = \hat{1}$, the Eulerian condition yields

$$1 - f_0 + f_1 - \cdots + (-1)^d f_{d-1} + (-1)^{d+1} = 0,$$

which is equivalent to the formula for the Euler characteristic of the $(d-1)$-dimensional sphere, the boundary of a d-dimensional polytope.

It is easy to see that the class of Eulerian posets is strictly larger than that of face lattices of polytopes. A number of necessary conditions are known for a poset to be the face lattice of a polytope (see, e.g., [Zi, p. 57]), but there is no complete characterization for face lattices of polytopes.

The two polytopes shown in Fig. 2 have isomorphic face lattices. Such polytopes are said to have the same *combinatorial type*. The face lattice determines the dimension, f-vector, and incidence relation among the faces of a polytope, but not its geometry. In later sections, the discussion will pertain sometimes to polytopes up to combinatorial type, and other times to the geometry of the polytopes under consideration.

FACT 1.2. *The order dual of the face lattice of a polytope is also the face lattice of a polytope*. (*See e.g.*, [*Zi*, *p*. 64].)

By the *order dual* of a poset $X = X(S, \leq)$, we mean a poset X^* with the same underlying set S, and the "less than" relation replaced with "greater than." Given a polytope P, an explicit construction can be given for a polytope P^Δ, called the *polar* of P, such that $\mathscr{L}(P^\Delta) \simeq [\mathscr{L}(P)]^*$. Namely,

$$P^\Delta := \{x \in \mathbf{R}^n : \langle x, p \rangle \leq 1, \text{for all } p \in P\}.$$

The construction of the polar, the order reversing correspondence between the face lattices of P and P^Δ, and other properties of the polar are described in detail in [Zi, p. 59]. Here we note only that P should contain the origin in its interior in order to ensure that the polar is bounded, and we will refer to P^* as a *dual* of P if it has the combinatorial type given by $[\mathscr{L}(P)]^*$. It is obvious that if P and P^* are dual polytopes, and if P has f-vector $f(P) = (f_{-1}, f_0, f_1, \ldots, f_{d-1}, f_d)$, then P^* has f-vector $f(P^*) = (f_d, f_{d-1}, \ldots, f_0, f_{-1})$. Figure 3 shows several pairs of dual polytopes.

There are a number of classes of polytopes of special interest. For example:

Simplicial polytopes. These are polytopes all of whose (proper) faces are simplices (e.g., Fig. 3 (a), (c), (e), (f)). Thus, a d-dimensional simplicial

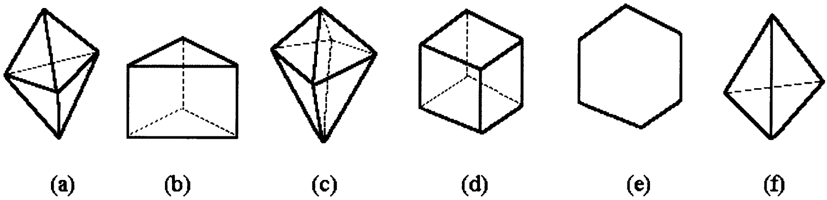


FIG. 3. Dual polytopes: (a) & (b), (c) & (d), (e) self-dual, (f) self-dual.

polytope can be described as one in which every facet has exactly d vertices (the minimum number of vertices required for a $(d-1)$-dimensional polytope).

Simple polytopes. These are duals of simplicial polytopes (e.g., Fig. 3 (b), (d), (e), (f)). Alternatively, a d-dimensional simple polytope is one for which every vertex is contained in exactly d facets (the minimum number of facets required to meet at a vertex for a d-dimensional polytope).

Centrally symmetric polytopes. These are polytopes which admit a center of symmetry. Thus, this is a geometric property. An example of a centrally symmetric polytope will arise later (the polytope $D(3)$ shown in Fig. 7 has center of symmetry with coordinates $(1, 1, 1)$).

Zonotopes. Zonotopes admit a number of equivalent descriptions (see [Zi, p. 198]), for instance:

—an affine projection of a cube (the hexagon can be viewed as a projection of the three-dimensional cube; see Figs. 3(d), (e);

—a d-dimensional polytope all of whose k-dimensional faces are centrally symmetric, for any particular k such that $2 \leq k \leq d-2$ (McMullen);

—a Minkowski sum of finitely many line segments.

The *Minkowski sum* of two convex polytopes P and Q in $\mathbf{R}^n$ is

$$P + Q := \{p + q : p \in P, q \in Q\},$$

using the usual (vector) addition in $\mathbf{R}^n$. Figure 4 shows a hexagon arising as the Minkowski sum of three line segments.

Cubical polytopes. These are polytopes all of whose (proper) faces are combinatorial cubes. Although cubical polytopes will not be discussed here, we mention them as a natural variation on the condition for simplicial polytopes, whose study is of current research interest. A counter-

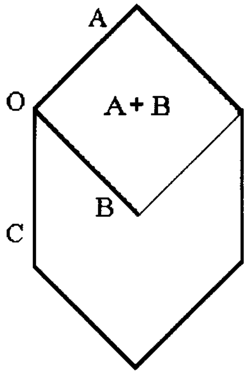


FIG. 4. The Minkowski sum of the line segments A, B, C.

part of the extensive theory for simplicial polytopes and complexes, is still being developed for cubical ones (see, e.g., [Ad; Cha; He2]).

In what follows, we will encounter simplicial and simple polytopes, and zonotopes.

Finally, we will use the notation $[n] := \{1, 2, \ldots, n\}$, and the usual interval notation $[i, k] := \{i, i+1, \ldots, k\}$.

2. BACKGROUND RESULTS

We turn now to a few fundamental results in the subject of polytopes, involving concepts that we will use later.

THEOREM 2.1 (Bruggesser and Mani, 1971). *The boundary of a polytope is shellable.*

This means that, given any d-dimensional polytope, there is an ordering of its facets $F_1, F_2, \ldots, F_t$, which is a *shelling order*; that is, having the property that for each $j = 2, 3, \ldots, t$ the intersection $F_j \cap (F_1 \cup F_2 \cup \cdots \cup F_{j-1})$ is (topologically) either a $(d-2)$-dimensional ball or a $(d-2)$-dimensional sphere.

For example, any ordering of the facets of a simplex is a shelling order. On the other hand, for the boundary of a three-dimensional cube, an ordering beginning with F_1 = the top facet and F_2 = the bottom facet is not a shelling order. Also, an ordering beginning with F_1 = the top facet, F_2 = left-hand side facet, F_3 = the bottom facet, and then setting F_4 = the right-hand side facet does not satisfy the shelling condition for $j = 4$. If, instead, we continue with F_4 = the back facet, either one of the two completions will be a shelling order.

The notion of shellability applies to a wider class of complexes, not only to boundaries of polytopes. For the boundary of a polytope, the Bruggesser–Mani proof of shellability establishes the existence of a shelling order using a most elegant idea. Consider supporting hyperplanes for the facets of the polytope and let L be a line which passes through an interior point, p, of the polytope, and which intersects the supporting hyperplanes of the facets one at a time. Now, start at p, travel along the line L in one direction and return from infinity along the line from the opposite direction. Index the facets in the order in which their supporting hyperplanes are encountered. This ordering of the facets turns out to provide a shelling, called a *line shelling* (see Fig. 5).

The class of simplicial polytopes has received a substantial amount of attention, in particular in connection with properties of the associated f-vectors—relations that they satisfy, maximum and minimum values of their entries, characterizations (see, e.g., [BiBj]). The following is a classical result about the f-vector of triangulations of spheres and, hence, about simplicial polytopes.

THEOREM 2.2 (The Dehn–Sommerville equations; Dehn, 1905; Sommerville, 1927). *The f-vector of every triangulation of the $(d-1)$-dimensional sphere, in particular, the f-vector of every simplicial polytope, satisfies*

$$\sum_{j=i}^{d-1} (-1)^j \binom{j+1}{i+1} f_j = (-1)^{d-1} f_i \qquad \textit{for } -1 \le i \le d-1.$$

Note that the above equation for $i = -1$ yields the Euler characteristic of the $(d-1)$-sphere, the only linear relation satisfied by the f-vector of every d-dimensional polytope.

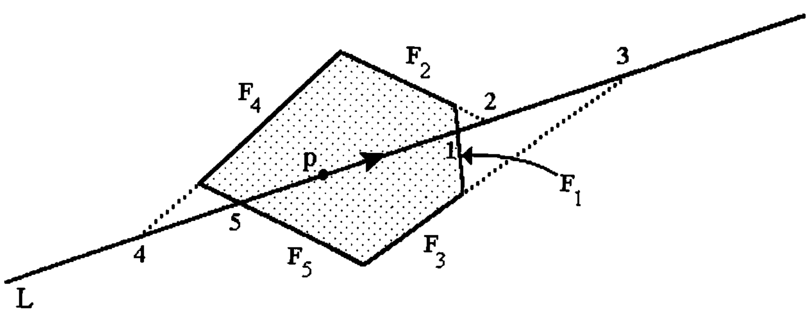


FIG. 5. A line shelling.

The possibility of an alternative, more compact form of the Dehn–Sommerville equations may be used as motivation for introducing the h-vector. Let $f = (f_{-1}, f_0, \ldots, f_{d-1}, f_d)$ be the f-vector of a simplicial polytope (more generally, a simplicial complex). Then the *h-vector*, $h = (h_0, h_1, \ldots, h_d)$, is defined via the polynomial identity

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-1} = \sum_{i=0}^{d} h_i x^{d-1}. \tag{1}$$

Thus, the f- and h-vectors contain equivalent information. Explicitly,

$$\begin{aligned} f_{i-1} &= \sum_{j=0}^{i} \binom{d-j}{i-j} h_j, \\ h_i &= \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{i-j} f_{j-1}. \end{aligned} \tag{2}$$

In terms of the h-vector, the Dehn–Sommerville equations assume a strikingly elegant form.

THEOREM 2.3 (Dehn–Sommerville equations in terms of the h-vector). *The h-vector of every triangulated $(d-1)$-dimensional sphere, in particular, the h-vector of every simplicial polytope, satisfies the relations*

$$h_i = h_{d-i} \qquad \textit{for } 0 \le i \le d.$$

It is interesting to observe that the symmetry of the h-vector combined with relation (2) implies that the first half of the entries in the f-vector determine the entire f-vector of a simplicial polytope, and, more generally, of a simplicial sphere.

Using the identity (1), one can compute the numbers h_i for any polytope. For simplicial polytopes (more generally, for shellable simplicial complexes), the h_i's computed from (1) turn out to be nonnegative and to have topological significance. But this is not the case for arbitrary polytopes (for the three-dimensional cube, for instance, relation (1) yields $h = (1, 5, -1, 1)$). The quest for the "right" general definition of the h-vector has led to the notion of generalized h-vectors [St4] and alternative definitions for the h-vector [Ad; Cha].

Note that Eq. (1) implies

$$f_{d-1} = \sum_{j=0}^{d} h_j; \tag{3}$$

that is, the h-vector refines the enumeration of the facets of a simplicial polytope (or shellable simplicial complex, more generally). This can be made more explicit. If $F_1, F_2, \ldots, F_t$ is a shelling order of the facets, let $R(F_i)$ be the unique smallest face of F_i which is not contained in $F_1 \cup F_2 \cup \cdots \cup F_{i-1}$. Since the facets are simplices, $R(F_i)$ is well defined (it is called the *restriction* of F_i with respect to the chosen shelling). Denote by r_i the number of facets whose restriction has i vertices. It is a counting exercise to check that $f_{i-1} = \sum_{j=0}^{i} \binom{d-j}{i-j} r_j$ for all i. Hence, comparing with (2), $h_j = r_j$ for all j. Note that this also shows that the sequence $(r_i)_i$ is independent of the choice of the shelling.

We now turn to a geometric property of polytopes whose vertices are integral points in $\mathbf{R}^n$. Let P be such a polytope, and let qP denote the *dilation* of P by a factor of q, $q \in \{1, 2, 3, \ldots\}$. That is, $qP := \{q\mathbf{x} : \mathbf{x} \in P\}$. Figure 6 shows an example.

The function $i(P; q) := |qP \cap \mathbf{Z}^n|$ was considered by Ehrhart beginning in the 1950s. He published his results later [Ehrh] and subsequently this function was further investigated by other authors (see, e.g., [Hi1; St1, p. 235; Hi2; Hi3; St3]). It turns out that $i(P; q)$ is a polynomial in q, called the *Ehrhart polynomial* of the integral polytope P. For example, the polytope shown in Fig. 6a has Ehrhart polynomial $i(P; q) = \binom{q+2}{2} = \frac{1}{2}q^2 + \frac{3}{2}q + 1$. This illustrates two general facts.

FACT 2.4 (Ehrhart). *The leading coefficient of the Ehrhart polynomial is the volume of the polytope.*

FACT 2.5 (Ehrhart). *The degree of the Ehrhart polynomial equals the dimension of the polytope.*

It is in fact the case that the generating function for the Ehrhart polynomial is a rational function of the form

$$1 + \sum_{q \geq 1} i(P; q) \lambda^q = \frac{\delta_0 + \delta_1 \lambda + \cdots + \delta_d \lambda^d}{(1 - \lambda)^{d+1}},$$

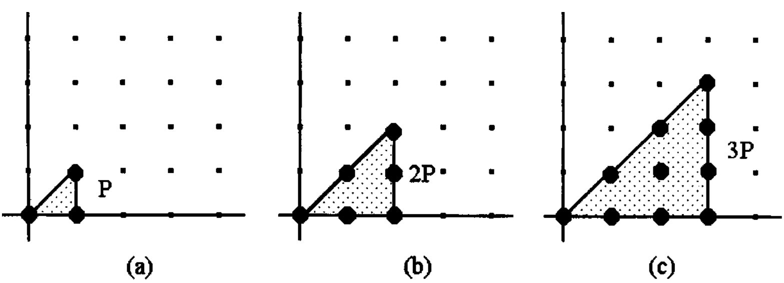


FIG. 6. An integral polytope P and its dilations $2P$ and $3P$.

where the δ_i's are nonnegative integers whose combinatorial properties are of interest (see, e.g., [Hi1] and its bibliography).

In concluding this section, we point out that Pick's classical theorem about the area of an integral convex polygon in the plane can be recovered from the general properties of the Ehrhart polynomial.

THEOREM 2.6 (Pick, 1900). *Let P be a convex polygon in the plane, whose vertices have integral coordinates. Then*

$$\operatorname{area}(P) = |P \cap \mathbf{Z}^2| - 1 - \tfrac{1}{2}|\partial P \cap \mathbf{Z}^2|.$$

3. EXAMPLE 1—COUNTING DEGREE SEQUENCES

Let n be a positive integer and let $\mathscr{D}(n)$ denote the number of n-tuples $(d_1, d_2, \ldots, d_n) \in \mathbf{Z}^n$ such that there exists some simple labeled graph on n vertices in which vertex i has degree d_i, for each $i = 1, 2, \ldots, n$. Unlike other settings, where the degree sequence of a graph lists the degrees in nonincreasing order, here we are interested in all the permutations of the degrees. However, we count sequences such as $(2,2,2,2,2,2)$ and $(1,1,1,1)$ only once each, even though they are realizable by more than one isomorphism type or more than one labeled graph.

Thus, we have $\mathscr{D}(2) = 2$, counting the degree sequences $(0,0)$ and $(1,1)$, and $\mathscr{D}(3) = 8$, counting the degree sequences $(0,0,0)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$, $(2,1,1)$, $(1,2,1)$, $(1,1,2)$, $(2,2,2)$. It turns out that $\mathscr{D}(4) = 54$.

While the problem of enumerating degree sequences for unlabeled graphs (called graphical partitions) is not completely solved (see, e.g., [ErRi; RoAl]), the question for labeled graphs (finding $\mathscr{D}(n)$, or the number of graphical compositions) can be answered. The solution which we outline below is due to Stanley [St5] and relates to the theory of polytopes.

Consider the points $\tilde{e}_{ij} := (0,0,\ldots,0,1,0,\ldots,0,1,0,\ldots,0,1) \in \mathbf{R}^{n+1}$, for $1 \le i \le j \le n$, whose ith, jth, and $(n+1)$th coordinates are 1, and whose other coordinates are 0. Let $Z(n) = \sum_{1 \le i \le j \le n}[0, \tilde{e}_{ij}]$ be the Minkowski sum of the $\binom{n}{2}$ segments in $\mathbf{R}^{n+1}$ determined by the $\tilde{e}_{ij}$'s and the origin. Thus, $Z(n)$ is a zonotope. Figure 7 shows $Z(2)$ and its projection, $D(2)$, onto the hyperplane $x_3 = 0$, and the projection $D(3)$ of $Z(3)$ onto the hyperplane $x_4 = 0$.

If $d = (d_1, d_2, \ldots, d_n)$ is realizable as the degree sequence of a labeled graph, associate with it the point $\tilde{d} := (d_1, d_2, \ldots, d_n, \frac{1}{2}\sum_{v=1}^{n} d_v) \in \mathbf{R}^{n+1}$. It is clear that $\tilde{d} \in Z(n)$, since $\tilde{d}$ equals the sum of the $\tilde{e}_{ij}$ for the pairs of adjacent vertices i, j in any one labeled graph realizing the degree sequence d. Using work by Koren [Kor] and the Erdös−Gallai test for

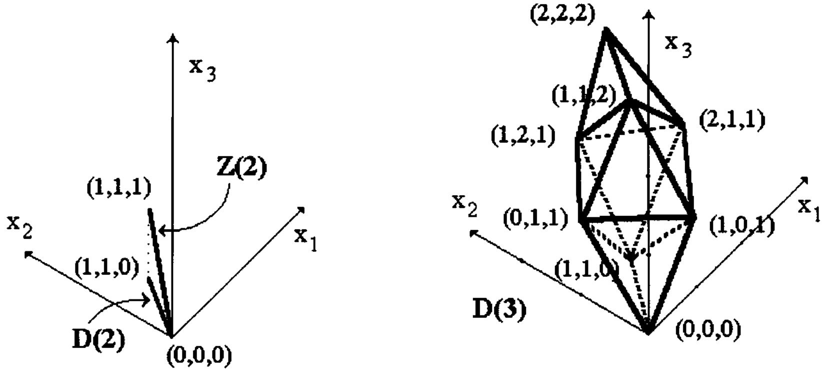


FIG. 7. The zonotopes $Z(2)$ and $D(2)$, and the projection $D(3)$ of $Z(3)$ onto the hyperplane $x_4 = 0$.

graphical partitions [ErGal], it turns out that every integral point in $Z(n)$ is of the form $\tilde{d}$ for some graphical sequence d. Thus, $\mathscr{D}(n) = |Z(n) \cap \mathbf{Z}^{n+1}| = i(Z(n), 1)$, the Ehrhart polynomial of $Z(n)$ evaluated at $q = 1$.

Before going on to the expression for the Ehrhart polynomial for $Z(n)$, let us remark that the polytope $D(n)$ obtained from taking the convex hull of the degree sequences $d = (d_1, \dots, d_n)$ has the same combinatorial type as $Z(n)$ but not the same Ehrhart polynomial. For example, $D(3)$ (shown in Fig. 7) contains the point $(1,1,1)$ (the midpoint of the segment joining $(0,0,0)$ and $(2,2,2)$), which is not graphical. At Micha Perles' suggestion, the polytope $D(n)$ was investigated by Peled and Srinivasan [PelSr].

Returning to $Z(n)$, the following theorem is applicable.

THEOREM 3.1 (see [St3; St5]). *Let $\beta_1, \beta_2, \dots, \beta_r \in \mathbf{Z}^m$ and consider the zonotope $Z = \sum_{i=1}^{r}[0, \beta_i]$. Then*

$$i(Z, q) = \sum_{s \geq 0} \left[\sum_{X, |X|=s} h(X) \right] q^s,$$

where X ranges over linearly independent subsets of $\{\beta_1, \beta_2, \dots, \beta_r\}$, and if $X = \{\beta_{i_1}, \beta_{i_2}, \dots, \beta_{i_s}\}$, then $h(X)$ is the greatest common divisor of the $s \times s$ minors of the $s \times m$ matrix whose rows are $\beta_{i_1}, \beta_{i_2}, \dots, \beta_{i_s}$.

Upon applying this theorem to the zonotope $Z(n)$ (the $\tilde{e}_{ij}$ play the role of the β_i's), the description of the Ehrhart polynomial can be reformulated much in the spirit of the matrix-tree theorem. The coefficient of q^s is a weighted sum of the spanning quasiforests of the complete labeled graph on n vertices, having s edges. A *quasiforest* is a graph each of whose connected components is either (i) a tree, or (ii) has one cycle and this cycle is of odd length.

THEOREM 3.2 (Stanley [St5]). *The Ehrhart polynomial of the zonotope* $Z(n)$,

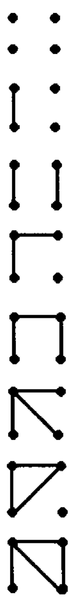
$$i(Z(n), q) = \sum_{s=0}^{n} c_{n,s} q^s,$$

has coefficients given by

$$c_{n,s} = \sum_{F \in Q\mathscr{F}(n,s)} \max\{1, 2^{c(F)-1}\},$$

where $Q\mathscr{F}(n,s)$ *is the set of labeled quasiforests with* n *vertices and* s *edges and* $c(F)$ *denotes the number of* (*odd-length*) *cycles in* F.

Figure 8 illustrates the computation of the Ehrhart polynomial of $Z(4)$ via this theorem. In particular, one obtains $\mathscr{D}(4) = i(Z(4), 1) = 54$.



# edges	Isomorphism type	# labeled quasiforests	c(F)	Contribution to i(Z(4), q)
s = 0		1	0	1
s = 1		6	0	6 q
s = 2		3	0	3 q^2
		12	0	12 q^2
s = 3		12	0	12 q^3
		4	0	4 q^3
		4	1	4 q^3
s = 4		12	1	12 q^4

i(Z(4), q) = 1 + 6 q + 15 q + 20 q^3 + 12 q^4

FIG. 8. Illustration of Theorem 3.2, computing the Ehrhart polynomial of $Z(4)$.

We note the remarkable generating function of the numbers $\mathscr{D}(n)$ of degree sequences.

THEOREM 3.3 [St5].

$$\sum_{n\geq 0}\mathscr{D}(n)\frac{x^n}{n!} = \frac{1}{2}\left[\sqrt{1+2\sum_{n\geq 1}n^n\frac{x^n}{n!}}\cdot\left(1-\sum_{n\geq 1}(n-1)^{n-1}\frac{x^n}{n!}\right)+1\right]$$
$$\cdot\exp\left[\sum_{n\geq 1}n^{n-2}\frac{x^n}{n!}\right].$$

Theorem 3.2 establishes that the degree sequences counted by $\mathscr{D}(n)$ are equinumerous with the labeled quasiforests on n vertices, weighted in a certain way. Is there a direct bijective proof of this fact? This question was posed in 1991 by Stanley and it remains unsolved.

A similar Ehrhart polynomial approach can be used to count the score sequences for tournaments with n players (the polytope involved is a translation of the permutohedron). The answer turns out to be the number of labeled forests on n vertices. In this case, a direct bijective proof was given by Kleitman and Winston [KleitWi].

4. EXAMPLE 2—THE PERMUTOHEDRON

The permutohedron is an example of a polytope associated with a family of combinatorial objects, namely, permutations. In fact, it is not the only polytope defined from permutations (another example is the Birkhoff polytope, i.e., the $(n-1)^2$-dimensional polytope in $\mathbf{R}^{n^2}$ of $n\times n$ doubly stochastic matrices).

To each permutation $\sigma\in S_n$, the symmetric group on n letters, associate the point $p_\sigma := (\sigma(1),\sigma(2),\ldots,\sigma(n))$ in $\mathbf{R}^n$. The *permutohedron* is the convex hull of $\{p_\sigma : \sigma\in S_n\}$. Our discussion of the permutohedron draws on its long history going back to the start of this century and which includes work by Schoute (1905), Rado (1952), Gaiha and Gupta (1977), Emelichev *et al*. (1984), and generalizations by Billera and Sarangarajan (1995). The last reference, [BiSa], includes a broad overview, generalizations, and a substantial bibliographic list.

It is immediately clear that for each p_σ we have (i) the sum of all the coordinates is equal to $\binom{n+1}{2}$, and (ii) the sum of any s coordinates, $1\leq s\leq n-1$, is at least $1+2+\cdots+s=\binom{s+1}{2}$. Consequently, every point in P_n satisfies these conditions and it turns out that this is indeed a description of P_n in terms of linear inequalities:

$$P_n = \left\{ \mathbf{x} \in \mathbf{R}^n : \sum_{i=1}^{n} x_i = \binom{n+1}{2} \text{ and } \sum_{i \in I} x_i \geq \binom{|I|+1}{2}, \text{ for all } I \subset [n] \right\}.$$

The first condition shows that $\dim(P_n) \leq n - 1$. It is easy to find $n - 1$ affinely independent p_σ's and, thus, $\dim(P_n) = n - 1$. Figure 9 shows P_2, P_3, and, taking advantage of the dimension being only 3, also P_4.

A *composition* of the set $[n]$ is an ordered partition of it. That is, an ordered collection $B_1/B_2/ \cdots /B_k$ of nonempty, pairwise disjoint sets called *blocks*, whose union is $[n]$. Set partitions and compositions are well-studied combinatorial objects of interest from the point of view of enumeration and partially ordered sets. The *refinement* order on set compositions is defined as follows: if $\pi = B_1/B_2/ \cdots /B_k$ and $\rho = C_1/C_2/ \cdots /C_j$ are two compositions of $[n]$, we say that π *covers* ρ (that is, $\pi \geq \rho$ and there is no composition τ such that $\pi > \tau > \rho$) if $k + 1 = j$ and if there is an index i such that $B_1 = C_1, B_2 = C_2, \ldots, B_{i-1} = C_{i-1}$ and $B_{i+1} = C_{i+2}, B_{i+2} = C_{i+3}, \ldots, B_k = C_{k+1}$, and $B_i = C_i \cup C_{i+1}$. If a minimum element is added to the compositions of $[n]$ ordered by refinement, the resulting poset is a ranked lattice of rank n (the rank function is given by $\mathrm{rk}(\pi) = n + 1 - \#$ blocks of π).

FACT 4.1. *The face lattice of the permutohedron P_n is isomorphic with the lattice of compositions of $[n]$ ordered by refinement. Consequently, for $i = 0, 1, \ldots, n - 2$,*

$$f_i(P_n) = (n - i)!S(n, n - i),$$

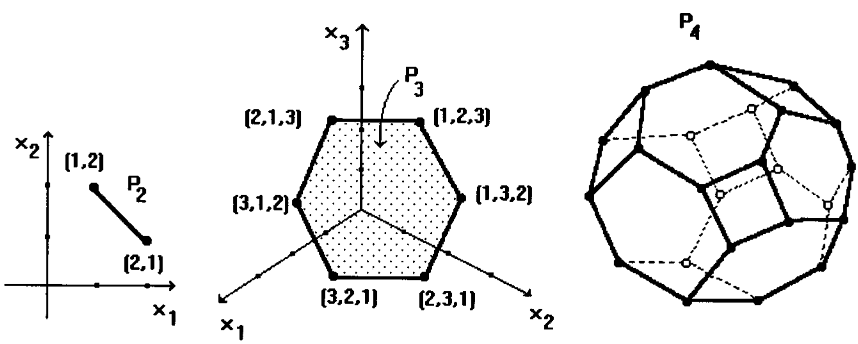


FIG. 9. The permutohedra P_2, P_3, P_4.

where $S(n,k)$ is the Stirling number of the second kind (counting the partitions of $[n]$ into k blocks).

The vertex $p_\sigma = (\sigma(1),\dots,\sigma(n))$ of P_n corresponds to the composition $\{\sigma(1)\}/\{\sigma(2)\}/\cdots/\{\sigma(n)\}$. Higher dimensional faces of P_n correspond to coarser compositions. If $\pi = B_1/B_2/\cdots/B_k$ is a composition of $[n]$, then the corresponding face of P_n has a vertex p_σ for each permutation σ whose values on $1,2,\dots,|B_1 \cup B_2 \cup \cdots \cup B_i|$ are the elements of the union $B_1 \cup B_2 \cup \cdots \cup B_i$, for every $1 \le i \le k$.

For example, the permutahedron P_4 has 14 facets, falling into two types: 8 facets are hexagonal, corresponding to the 8 compositions of $\{1,2,3,4\}$ into one block of three elements and one block of one element, and 6 facets are quadrilaterals, corresponding to the 6 compositions of $\{1,2,3,4\}$ into two blocks, each of two elements.

It is easy to see that P_n is a simple polytope. Each vertex p_σ lies in precisely $n-1$ facets, namely those facets corresponding to compositions (into two blocks) whose first block is $B_1 = \{\sigma(1),\sigma(2),\dots,\sigma(m)\}$ for some $m \in \{1,2,\dots,n-1\}$. In fact, the 1-skeleton of P_n is isomorphic to the Cayley graph of S_n with the set of Weyl transpositions, $\{(1,2),(2,3),\dots,(n-1,n)\}$ as the set of generators.

Considering a dual, P_n^*, of P_n, we have a simplicial polytope with f-vector

$$f(P_n^*) = (1!S(n,1), 2!S(n,2),\dots,n!S(n,n),1).$$

By definition (recall relation (2)), the entries of the h-vector are

$$h_i(P_n^*) = \sum_{j=0}^{i} (-1)^{i-j}\binom{n-1-j}{i-j}(j+1)!S(n,j+1).$$

A classical inclusion−exclusion argument yields

$$h_i(P_n^*) = A(n,i) \qquad \text{for } i = 0,1,\dots,n-1,$$

where the $A(n,i)$'s are the *Eulerian numbers*. These count permutations according to their number of descents. A permutation $\alpha \in S_n$ has a *descent* at i, $1 \le i \le n-1$, if $\alpha(i) > \alpha(i+1)$. For example, the permutation 6 2 1 3 5 4 $\in S_6$ has three descents (for $i = 1,2,5$). The number of descents is a classical combinatorial statistic for permutations. It has generalizations to other objects of combinatorial interest (e.g., elements of other reflection groups and tableaux).

Thus, the face structure of P_n (and P_n^*) corresponds to set compositions —objects of independent combinatorial interest. Furthermore, the natural refinement of the facet count provided by the h-vector of P_n^* (relation (3)) corresponds with an enumeratively natural combinatorial statistic on permutations.

Finally, it turns out that the boundary complex of P_n^* is the order complex of the boolean lattice on n elements, and $h_i(P_n^*) = A(n, i)$ can also be derived from the general theory of EL-shellable posets (an exposition on order complexes and (pure) poset shellability can be found in [BjGarSt; St1]; a natural extension to nonpure posets appears in [Bj; BjWac]; a recent general combinatorial approach to shellability is given in [Koz]).

5. EXAMPLE 3—SLICING SIMPLICES AND CUBES

Consider the simplex $\Delta^n := \{\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n : 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq 1\}$. Suppose that a point $\mathbf{x}$ is selected uniformly at random from Δ^n. What is the probability that the average of $0, 1$, and the coordinates of $\mathbf{x}$ falls between the $(i-1)$th and ith coordinates of $\mathbf{x}$? The geometric probability question of finding

$$\Pr\left[x_{i-1} < \frac{0 + x_1 + x_2 + \cdots + x_n + 1}{n + 2} \leq x_i \right]$$

can be rephrased as follows: consider the hyperplanes $H_i \subset \mathbf{R}^n$ with equation

$$x_1 + x_2 + \cdots + x_n + 1 = (n + 2) x_i,$$

for $i = 1, 2, \dots, n$. These form a pencil of hyperlanes through the point $(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}) \in \mathbf{R}^n$, and they determine $n + 1$ regions, $R_{n,0}, R_{n,1}, \dots, R_{n,n}$ in the simplex Δ^n:

$$R_{n,i} := \left\{ \mathbf{x} \in \Delta^n : \sum_{k=1}^{n} x_k + 1 \geq (n+2) x_j \text{ for } j \leq i, \right.$$
$$\left. \sum_{k=1}^{n} x_k + 1 \leq (n+2) x_j \text{ for } j > i \right\}.$$

In the low-dimensional cases shown in Fig. 10, the volumes of the regions are proportional to the Eulerian numbers:

	$\text{vol}(R_{n,0})$	$\text{vol}(R_{n,1})$	$\text{vol}(R_{n,2})$	$\text{vol}(R_{n,3})$
$n = 1$	$\frac{1}{2}$	$\frac{1}{2}$		
$n = 2$	$\frac{1}{12}$	$\frac{4}{12}$	$\frac{1}{12}$	
$n = 3$	$\frac{1}{144}$	$\frac{11}{144}$	$\frac{11}{144}$	$\frac{1}{144}$

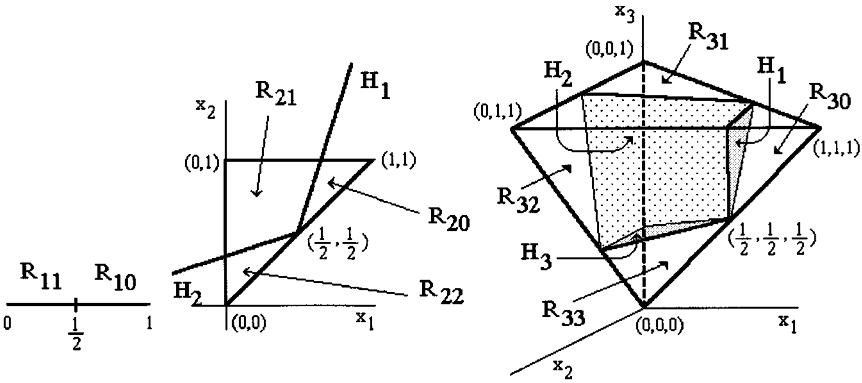


FIG. 10. The regions and sections determined in Δ^n, for $n = 1, 2, 3$, by the hyperplanes $H_i : x_1 + x_2 + \cdots + x_n + 1 = (n+2)x_i$, $1 \le i \le n$.

THEOREM 5.1 (Schmidt-Simion, 1996). *Let $A(m,i)$ denote the number of permutations in S_m having exactly i descents (the Eulerian numbers). Then, for $0 \le i \le n$, the volumes of the regions $R_{n,i}$ are given by*

$$\mathrm{vol}(R_{n,i}) = \frac{A(n+1,i)}{n!(n+1)!},$$

and the $(n-1)$-dimensional volumes of the sections $S_{n,i} := \Delta^n \cap H_i$ for $1 \le i \le n$ are given by

$$\mathrm{vol}(S_{n,i}) = c(n) \cdot \frac{A(n,i)}{(n-1)!n!},$$

where $c(n) = \sqrt{n^2 + 3n}/(n+1)$.

Consequently, each of these sequences of volumes is symmetric and unimodal (in fact, logarithmically concave).

We describe three further problems of a similar nature, whose three-dimensional case is illustrated in Fig. 11. Again, the results (Fact 5.2)–(Fact 5.4) involve well-known enumerative quantities: binomial coefficients, Eulerian numbers, and the analogue of Eulerian numbers for the hyperoctahedral group. The first problem is a simple variation on the previous one and can be readily solved.

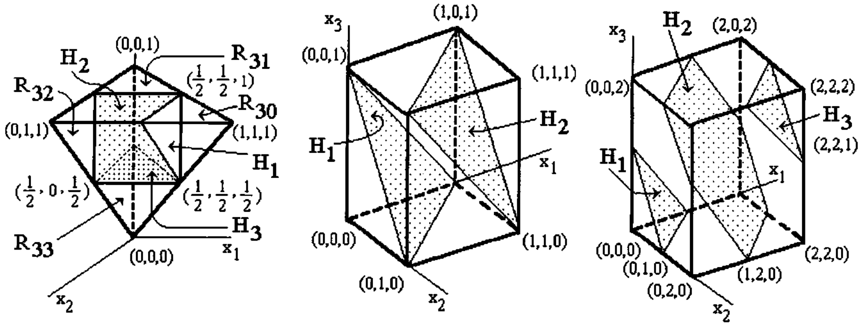


FIG. 11. Illustration of Facts 5.2, 5.3, 5.4 for $n = 3$.

FACT 5.2. *Consider the simplex Δ^n. The hyperplanes $H_i : x_i = \frac{1}{2}$, $1 \le i \le n$, partition Δ^n into $n + 1$ regions whose volumes are given by*

$$\operatorname{vol}(R_{n,i}) = \binom{n}{i} \Big/ n!2^n.$$

The next problem goes back to Laplace and appears in work of Foata and Stanley in the 1980's. The Eulerian numbers arise again.

FACT 5.3. *Consider the n regions determined in the unit cube $[0,1]^n$ by the hyperplanes $H_i : x_1 + x_2 + \cdots + x_n = i$, $1 \le i \le n - 1$. The volumes of the regions (in increasing—or else decreasing—order of their distance to the origin) are: $A(n,0)/n!, A(n,1)/n!, \ldots, A(n, \ n-1)/n!$. The $(n-1)$-dimensional volumes of the sections are also proportional to Eulerian numbers.*

FACT 5.4. *Consider now the cube with side-length 2, $[0,2]^n$, and the hyperplanes $H_i : x_1 + x_2 + \cdots + x_n = 2i - 1$, for $1 \le i \le n$. The volumes of the resulting $n + 1$ regions and n sections were investigated by Chakerian and Logothetti (1991), who established recurrence relations satisfied by the sequence of volumes. It turns out that the recurrences imply that*

$$\operatorname{vol}(R_{n,i}) = \frac{\overline{A}(n,i)}{n!}, \qquad 0 \le i \le n,$$

where $R_{n,i} := \{\underline{\mathbf{x}} \in [0,2]^n : \sum_{k=1}^n x_k \le 2j - 1 \text{ for } j < i, \sum_{k=1}^n x_k \ge 2j - 1 \text{ for } j \ge i\}$, and the $\overline{A}(n,i)$'s are the Eulerian numbers for signed permutations (the hyperoctahedral group).

More precisely, $\overline{A}(n,i)$ denotes the number of *signed permutations* on n letters, having exactly i *descents*. A *signed permutation* on n letters is a permutation of $\{1, 2, \ldots, n\}$ in which each letter may bear a minus sign. Thus, there are $2^n n!$ signed permutations on n letters. For notational convenience, we will write $\overline{m}$ instead of $-m$. The notion of *descent* for signed permutations is based on the linear ordering $1 < 2 < \cdots < n < \overline{n} < \overline{n-1} < \cdots < \overline{2} < \overline{1}$ of the symbols, together with the fact that if the last letter in the permutation is negative ("barred"), then the last position contributes a descent. For example, the signed permutation $\overline{2}\ 4\ 5\ 1\ 6\ \overline{3}$ has three descents (occurring in positions 1, 3, and 6). For $n = 2$, we have $\overline{A}(2,0) = 1$ (counting the signed permutation 1 2), $\overline{A}(2,1) = 6$ (counting the signed permutations $\overline{1}\ 2$, $1\ \overline{2}$, $2\ 1$, $\overline{2}\ 1$, $2\ \overline{1}$, and $\overline{2}\ \overline{1}$), and $\overline{A}(2,2) = 1$ (counting the signed permutation $\overline{1}\ \overline{2}$).

It is well known that the unit cube, $[0,1]^n$, is dissected by the hyperplanes $x_i = x_j$, $1 \le i < j \le n$, into $n!$ simplices, each having volumes $1/n!$, and each corresponding naturally to a permutation in the symmetric group S_n. In response to a question asked by Foata, Stanley showed how the interiors of these $n!$ simplices can be mapped to the cube in a measure preserving fashion, so that the region $R_{n,i}$ (of Fact 5.3) is partitioned (up to a set of measure zero) by the images of (the interior of) precisely those simplices which correspond to permutations in S_n having i descents. A similar map for the situation in Fact 5.4 is described in [SchmSi].

We conclude this section with two problems.

The symmetric group and the signed permutations are, in fact, the elements of the Weyl groups for the root systems A_{n-1} and B_n, respectively. In both cases, the notion of a descent is motivated by the geometric context of reflection groups. Is there a unified approach to the results stated in this section, in the framework of Coxeter systems?

What other combinatorial sequences arise naturally as volumes of regions and sections?

6. EXAMPLE 4—THE ASSOCIAHEDRON

In this section, Catalan and Schröder numbers and two types of combinatorial objects—lattice paths and Baxter permutations—arise in connection with a polytope called the associahedron.

The Catalan sequence: $1, 1, 2, 5, 14, 42, 132, \ldots,$ and the Schröder sequence: $1, 2, 6, 22, 90, 394, \ldots,$ enumerate remarkably many and varied types of objects of enumerative and structural interest in combinatorics (see, e.g., [Co, pp. 52, 56; St1; St6; BoShaSi] for several examples). Here it will suffice to mention one combinatorial interpretation of these numbers, in terms of certain lattice paths.

Consider lattice paths in the plane, which begin at the origin, end at the point (N, N), and do not run above the line $x = y$. If the allowable steps are $(1,0)$ (East) and $(0,1)$ (North), the number of paths is the Nth Catalan number, $C_N = (1/(N+1))\binom{2N}{N}$, and we call such paths *Catalan paths*. If diagonal steps $(1,1)$ are allowed as well, the number of paths is the Nth Schröder number, and we call such paths *Schröder paths*.

By counting the Catalan paths ending at (N, N) according to their number of East–North corners, we obtain a q-analogue of the Nth Catalan number:

$$C_N(q) := \sum_{i=1}^{N} \frac{1}{N}\binom{N}{i}\binom{N}{i-1} q^i.$$

The coefficients of this polynomial are the Narayana numbers, also well known in combinatorial enumeration. Similarly, by counting Schröder paths according to their number of diagonal steps, we obtain a polynomial $\mathrm{Sch}_N(q)$ which is a q-analogue of the Nth Schröder number. For example, from Fig. 12, we see that $C_3(q) = q + 3q^2 + q^3$ and $\mathrm{Sch}_3(q) = 5 + 10q + 6q^2 + q^3$. Of course, the constant term in $\mathrm{Sch}_N(q)$ is $\mathrm{Sch}_N(0) = C_N$.

Since, for a given N, the Schröder paths can be obtained from the Catalan paths by replacing the East–North corners with diagonal steps independently, we have $\mathrm{Sch}_N(q) = C_N(1+q)$. This relation can be rewritten as

$$\mathrm{Sch}_N(q-1) = C_N(q),$$

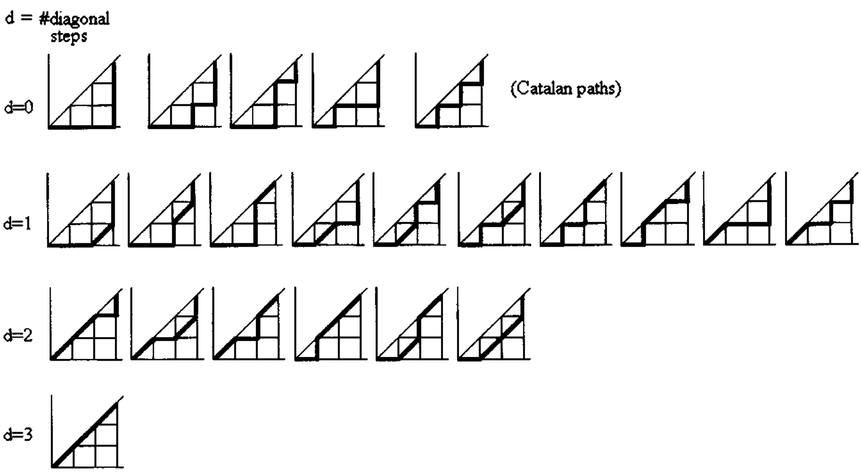


FIG. 12. The 22 Schröder paths for $N = 3$.

which is reminiscent of the relation between the f- and h-vectors of a polytope (Eq. (1)). This is not merely a coincidence.

FACT 6.1 (Lee, 1989; Bonin, Shapiro and Simion, 1993). *The* $(N-1)$-*dimensional associahedron*, Q_N, *has h-vector and f-vector given by*

$$\frac{1}{q}C_N(q) = \sum_{i=0}^{N-1} h_i(Q_N)q^{N-1-i}$$

$$\frac{1}{1+q}\mathrm{Sch}_N(q) = \sum_{i=0}^{N-1} f_{i-1}(Q_N)q^{N-1-i}.$$

The history of the associahedron turns out to span more than 30 years, going back to work by Stasheff in the 1960s. Later, independently, Perles posed the following problem: given a convex $(N+2)$-gon, its sets of pairwise noncrossing diagonals form an abstract simplicial complex of dimension $N-2$; is there an $(N-1)$-dimensional simplicial polytope having this simplicial complex as its boundary complex? This question was settled in the affirmative, and the *associahedron* is the desired polytope (up to combinatorial type). Haiman (unpublished manuscript) gave linear inequalities defining a polytope as desired. Independently, Lee [Le] constructed the associahedron via a sequence of stellar subdivisions of the $(N-1)$-dimensional simplex. The same simplicial complex considered by Perles appears in the work of Penner and Waterman [PenWat], who established that it is topologically a sphere (a necessary condition for being the boundary of a polytope). Their work is motivated by connections of this simplicial complex with a(n idealized) model for RNA secondary structure in molecular biology.

For example, the associahedra Q_2, Q_3, and Q_4 are shown in Fig. 13. The facets correspond to maximal sets of pairwise noncrossing diagonals of the $(N+2)$-gon, that is, triangulations of the polygon. When $N=2$, the convex polygon is a quadrilateral; its two diagonals contribute the two vertices of Q_2, the boundary of a one-dimensional polytope. When $N=3$, there are five diagonals which form five pairs of noncrossing diagonals; the simplicial complex can be realized geometrically (coincidentally) as a convex pentagon. When $N=4$, it is possible to have three mutually noncrossing diagonals in a convex hexagon; the figure showing Q_4 is based on Lee's construction [Le].

We now turn to the enumeration of Baxter permutations, investigated by many authors, including Baxter (1967), Chung *et al.* (1978), Cori, Dulucq, and Viennot (1986), Gire (1993), Guibert (1995) (references are given in [DuSi]).

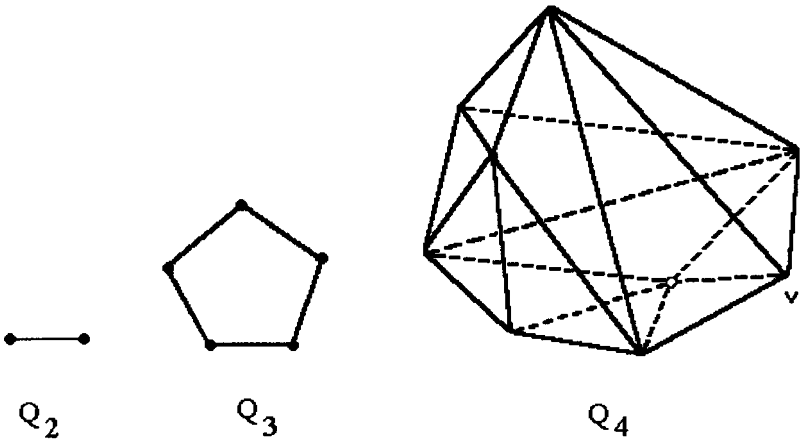


FIG. 13. The associahedron Q_N for $N = 2, 3, 4$.

A permutation $\sigma \in S_N$ is a *Baxter permutation* if it contains no 2413 and no 3142 pattern in which the roles of 2 and 3 are played by consecutive values. More explicitly, there are no indices $1 \leq i < j < k < m \leq N$ such that $\sigma(k) < \sigma(i) = \sigma(m) - 1 < \sigma(j)$ or $\sigma(j) < \sigma(i) = \sigma(m) + 1 < \sigma(k)$. Clearly, all permutations in S_N for $N \leq 3$ satisfy the Baxter condition; in S_4, the permutations 3142 and 2413 fail the Baxter condition.

THEOREM 6.2 (Cori, Dulucq, and Viennot, 1986). *The number of alternating Baxter permutations in S_N is either the square of a Catalan number or else the product of two consecutive Catalan numbers, namely,*

$$C_n^2 \qquad \text{if } N = 2n,$$

or

$$C_n C_{n+1} \qquad \text{if } N = 2n + 1.$$

By an *alternating permutation* we mean $\sigma \in S_N$ such that $\sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > \cdots$ (i.e., σ has descents in positions $2, 4, 6, 8, \ldots$). As an illustration for Theorem 6.2, consider $N = 4$. There are four alternating Baxter permutations in S_4, namely: 1324, 1423, 2314, 2413 and indeed $4 = (C_2)^2$.

In [DuSi], in the context of refining the enumeration of certain permutations (alternating permutations of genus 0), the following statistic was introduced for alternating permutations,

$$\widehat{\mathrm{des}}(\sigma) := \mathrm{des}(\sigma(1)\sigma(3)\sigma(5)\cdots) + \mathrm{des}(\sigma(2)\sigma(4)\sigma(6)\cdots),$$

the number of descents among the local minima plus the number of descents among the local maxima of the alternating permutation. For instance, $\widehat{\mathrm{des}}(3\ 5\ 4\ 7\ 2\ 8\ 1\ 6) = \mathrm{des}(3\ 4\ 2\ 1) + \mathrm{des}(5\ 7\ 8\ 6) = 2 + 1 = 3$.

It turns out that this statistic, called the *alternating descents* statistic, yields a q-analogue of Theorem 6.2. That is, there is a refinement of the enumeration of alternating Baxter permutations expressed in terms of polynomials in the variable q, from which Theorem 6.2 can be recovered by setting $q = 1$.

THEOREM 6.3 (Dulucq and Simion, 1995). *The distribution of alternating descents over the class of alternating Baxter permutations is*

$$\frac{1}{q^2} C_n^2(q) \qquad \text{if } N = 2n,$$

or

$$\frac{1}{q^2} C_n(q) C_{n+1}(q) \qquad \text{if } N = 2n + 1.$$

For example, the alternating Baxter permutations listed earlier for $N = 4$ give

$$\sum_{\sigma \in S_4,\ \text{alt. Bax.}} q^{\widehat{\mathrm{des}}(\sigma)} = 1 + 2q + q^2 = \frac{1}{q^2}\left(C_2(q)\right)^2.$$

Returning now to polytopes, we have the following connection with products of q-Catalan and q-Schröder numbers.

THEOREM 6.4 (Dulucq and Simion, 1995). *For $N \geq 2$, there exists an $(N-2)$-dimensional polytope, $Q_N^{(B)}$, whose h- and f-vectors are given by*

$$\sum_{i=0}^{N-2} h_i\left(Q_N^{(B)}\right) q^{N-2-i} = \sum_{\sigma \in S_N,\ \text{alt. Bax}} q^{\widehat{\mathrm{des}}(\sigma)} = \frac{1}{q^2} C_{[N/2]}(q) C_{[(N+1)/2]}(q)$$

$$\sum_{i=0}^{N-2} f_{i-1}\left(Q_N^{(B)}\right) q^{N-2-i} = \frac{1}{(1+q)^2} \mathrm{Sch}_{[N/2]}(q) \mathrm{Sch}_{[(N+1)/2]}(q).$$

The polytope $Q_N^{(B)}$ can be described easily; it is a vertex figure of the associahedron Q_N. Given a polytope P and one of its vertices, v, a *vertex figure* with respect to v is a polytope obtained as the intersection of P with a hyperplane which separates v from the other vertices of P. The combinatorial type of a vertex figure is given by the interval $[v, P]$ in the face lattice of P. The polytope $Q_N^{(B)}$ is a vertex figure of Q_N with respect to a vertex representing a diagonal in the $(N+2)$-gon which divides this polygon as evenly as possible (i.e., a longest possible diagonal). In Fig. 13, the vertex marked v is such a vertex. The corresponding vertex figure is easily seen to be a convex quadrilateral.

This can be readily extended to face figures (which can be obtained through successive vertex−figure operations), providing an h- and f-vector interpretation for multiple products of q-Catalan and q-Schröder numbers [DuSi].

7. EXAMPLE 5—THE *cd*-INDEX OF A POLYTOPE

Given a polytope P of dimension d, select a set $S \subseteq [d]$ of proper ranks in the face lattice of P, and consider flags of faces from the selected ranks. That is, sequences of faces of P, $\varnothing \neq F_1 \subset F_2 \subset \cdots \subset F_{|S|} \neq P$, such that $S = \{\mathrm{rk}(F_1), \mathrm{rk}(F_2), \dots, \mathrm{rk}(F_{|S|})\}$. Let f_S be the number of such flags of faces. It counts chains consisting of elements at prescribed ranks, in the face lattice of P. In particular, if S is a singleton set, $S = \{i\}$, then $f_{\{i\}}$ is the number of faces of dimension $i-1$, that is, the entry f_{i-1} in the f-vector of P. Thus, the information provided by the f-vector is enhanced by the 2^d-tuple $(f_S)_{S \subseteq [d]}$, which is called the *flag f-vector* of P. Note that $(f_S)_S$ makes sense for ranked posets, not necessarily face lattices of polytopes.

To each $S \subseteq [d]$ associate a word $w_S = w_1 w_2 \cdots w_d$ in the letters a and b, which encodes the characteristic function of S: $w_i = a$ if $i \notin S$ and $w_i = b$ if $i \in S$. Form the polynomial $\sum_{S \subseteq [d]} f_S w_S \in \mathbf{Z}\langle a, b\rangle$ in *noncommuting* variables a and b. Now substitute $a - b$ for a to obtain the *ab-index* of P. For example, the ab-index of a three dimensional simplex (whose face lattice, a boolean lattice, appears in Fig. 2) is

$$a^3 + 3a^2b + 5aba + 3ba^2 + 3ab^2 + 5bab + 3b^2a + b^3,$$

and a three-dimensional cube has ab-index

$$a^3 + 5a^2b + 11aba + 7ab^2 + 7ba^2 + 11bab + 5b^2a + b^3.$$

If it is possible to express the ab-index in terms of the new noncommuting variables $c = a + b$ and $d = ab + ba$, then the resulting polynomial in

c, d is called the *cd-index* of P, denoted $\Phi(P; c, d)$. (We acknowledge the regrettable clash of notation, in this section, between the variable d and the dimension d of a polytope. We hope that the intended meaning of the symbol d will be clear from the context of each of its occurrences.) In the case of a three-dimensional simplex, Δ^3, and three-dimensional cube, C^3, the *ab*-indices above give

$$\begin{aligned} \Phi(\Delta^3; c, d) &= c^3 + 2cd + 2\,dc, \\ \Phi(C^3; c, d) &= c^3 + 4cd + 6\,dc. \end{aligned} \tag{4}$$

The *cd*-index has topological significance with regard to shelling and the intersection homology of the toric variety associated with the polytope. Note, as suggested by the above examples, that it gives a compact encoding of the information about the flag f-vector of the polytope. Clearly, the *ab*-index is homogeneous of degree $\dim(P)$, and the number of distinct monomials in the c and d is a Fibonacci number (since $\deg(c) = 1$, and $\deg(d) = 2$). Bayer and Billera [BayBi] showed that the affine span of all flag f-vectors of Eulerian posets of a given rank has dimension equal to a Fibonacci number.

In the early 1980s, J. Fine introduced the idea of the *cd*-index and conjectured that

(1) every polytope has a *cd*-index, and

(2) the *cd*-index of a polytope has nonnegative (integer) coefficients.

Fine's conjecture is true. In fact, the first part holds for every Eulerian poset. It follows from the fact that the flag f-vector of an Eulerian poset satisfies a generalized Dehn−Sommerville set of relations, which turn out to be equivalent to the existence of the *cd*-index.

THEOREM 5.1 (Bayer-Billera, 1985) (Generalized Dehn-Sommerville equations). *Let P be an Eulerian poset whose maximum element has rank $d + 1$. Then the flag f-vector of P satisfies*

$$\sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_{S \cup \{j\}} = \left(1 + (-1)^{k-i}\right) f_S,$$

for every $0 \le i < j < k \le d + 1$, such that $i, k \in S \cup \{0, d + 1\}$ and $[i, k] \cap S \subseteq \{i, k\}$.

THEOREM 5.2 (Bayer and Klapper, Fine, 1991). *For a poset P, the generalized Dehn−Sommerville equations hold if and only if the cd-index exists for P.*

The nonnegativity of the coefficients of the cd-index for polytopes was later established by Stanley [St7].

THEOREM 5.3 (Stanley, 1994). *The cd-index of a polytope has nonnegative coefficients.*

Under the assumption that a certain type of shelling exists (called "S-shelling"), Stanley expressed the cd-index as the sum of polynomials ("shelling components") arising from the successive stages of the S-shelling and showed that each of these polynomials has nonnegative coefficients. In particular, since the definition of an S-shelling is such that line shellings (which exist for boundaries of polytopes) are S-shellings, the second part of Fine's conjecture follows.

Following the notation in [St7], let $\check{\Phi}_0^d(c,d), \check{\Phi}_1^d(c,d), \dots, \check{\Phi}_{d-1}^d(c,d)$ be the shelling components of the boundary of a d-dimensional simplex. For example, for a three-dimensional simplex it turns out that

$$\begin{aligned} \check{\Phi}_0^3(c,d) &= c^3 + dc \\ \check{\Phi}_1^3(c,d) &= dc + cd \\ \check{\Phi}_2^3(c,d) &= cd. \end{aligned} \tag{5}$$

As expected, the sum of these polynomials gives the cd-index of the three-dimensional simplex computed in an earlier example. A further result by Stanley shows that the cd-index of a d-dimensional *simplicial* polytope is determined by the h-vector and the shelling components of the d-dimensional simplex.

THEOREM 5.4 (Stanley, 1994). *If P is a simplicial d-dimensional polytope with h-vector $h(P) = (h_0(P), h_1(P), \dots, h_d(P))$, then*

$$\Phi(P;c,d) = \sum_{i=0}^{d-1} h_i(P)\check{\Phi}_i^d(c,d).$$

As an easy example, let P be a d-dimensional simplex. Then all the entries of the h-vector are equal to 1, and we recover, as expected, $\Phi(\Delta^d;c,d) = \sum_{i=0}^{d-1}\check{\Phi}_i^d(c,d)$.

We now return to enumeration, to point out some purely combinatorial means of computing the cd-index of a simplex, an already nontrivial object. Before Stanley's proof of Theorem 5.3, some of the attempts to complete the proof of Fine's conjecture relied on the natural idea of showing that the coefficients of the cd-index of a polytope count something. In particular, for the simplex, the sum of the coefficients of the cd-index was known to be $\Phi(\Delta^d;1,1) = a_{d+1}$, the number of alternating permutations in S_{d+1}.

(Recall that by an *alternating permutation* we mean a permutation σ such that $\sigma(1) < \sigma(2) > \sigma(3) < \cdots$.) The enumeration of alternating permutations is a classical problem, with a well-known answer in (exponential) generating function form:

$$\sum_{n \geq 0} a_n \frac{x^n}{n!} = \sec x + \tan x = 1 + 1 \cdot \frac{x}{1!} + 1 \cdot \frac{x^2}{2!} + 2 \cdot \frac{x^3}{3!} + 5 \cdot \frac{x^4}{4!} + 16 \cdot \frac{x^5}{5!} + 66 \cdot \frac{x^6}{6!} + \cdots .$$

The fact that $\Phi(\Delta^d; 1, 1) = a_{d+1}$ suggests that the coefficients of the *cd*-index of the simplex enumerate, according to some criterion, the alternating permutations or another class of permutations having the same cardinality. This turns out indeed to be the case. In fact, several classes of permutations are now known from which the *cd*-index of the simplex can be computed directly (e.g., André permutations of the first and second type, simsun permutations, forgotten André permutations).

Purtill [Pu] was successful in showing, among other facts, the nonnegativity of the coefficients of the *cd*-index for the *d*-dimensional simplex, *d*-dimensional octahedron, and *d*-dimensional cube, by combinatorial means. He described how to partition the maximal flags of faces (terms from the *ab*-index) into classes, each of which yields a *cd*-monomial toward the *cd*-index. Each class contains one permutation of a special type, from which the *cd*-monomial corresponding to that class can be computed directly. Such permutations (André permutations) and their reduced variation (defined below) had already been investigated by Foata and Schützenberger [FoSchü], independently of the *cd*-index of a polytope which had yet to be introduced. Purtill used André permutations to establish the nonnegativity of the coefficients in the *cd*-index for the simplex and signed André permutations (hyperoctahedral counterpart of André permutations in the symmetric group) for the octahedron and cube.

Here we describe only two types of permutations and how the *cd*-index of a simplex, indeed, the shelling components of the *cd*-index, can be obtained from one of them.

We call $\sigma \in S_n$ an *André permutation* (of type II) if it satisfies the following condition: whenever there exist $1 \leq i < j - 1 \leq n - 1$ such that $\sigma(i) = \max\{\sigma(i), \sigma(i+1), \sigma(j-1), \sigma(j)\}$ and $\sigma(j) = \min\{\sigma(i), \sigma(i+1), \sigma(j-1), \sigma(j)\}$, then there is some k such that $i + 1 < k < j - 1$ and $\sigma(k) < \sigma(j)$. For instance, the permutation $4\ 2\ 3\ 1 \in S_4$ does not satisfy the André condition; also, in S_3 there are five permutations satisfying the André condition (the exception being $3\ 2\ 1$). One can establish a recurrence relation from which it follows that the number of André permutations in S_n equals the number of alternating permutations in S_{n+1}.

In an unrelated context, a different type of permutations was defined by Sundaram and this author (see [Su]). A permutation $\sigma \in S_n$ is a *simsun permutation* if it contains no consecutive descents and, for $n > 1$, if the permutation $\sigma^- \in S_{n-1}$ obtained by deleting the value n from the sequence $\sigma(1)\sigma(2) \cdots \sigma(n)$ is also simsun. Alternatively, $1 \in S_1$ is a simsun permutation, and the simsun permutations in S_n, $n > 1$, can be obtained from those in S_{n-1} by inserting n in a position where it does not create two consecutive descents. For example, $4\ 2\ 3\ 1 \in S_4$ is simsun (but not André), and $3\ 2\ 4\ 1 \in S_4$ is not simsun (but is André). Clearly, all permutations in S_1 and S_2 are simsun, and five of the permutations in S_3 are simsun (the exception being 3 2 1).

Consider a simsun permutation $\sigma \in S_{d+1}$ such that $\sigma(d+1) = d+1$ (an "augmented" simsun permutation). There are five augmented simsun permutations in S_4:

$$1\,2\,3\,4,\ 2\,1\,3\,4,\ 1\,3\,2\,4,\ 3\,1\,2\,4,\ 2\,3\,1\,4.$$

By definition, a simsun permutation does not have consecutive descents. Hence, the following is well defined and gives a monomial, U_σ, in noncommuting variables c and d: for each descent $\sigma(k) > \sigma(k+1)$, replace the two values by d; replace the remaining values $\sigma(m)$, $1 \le m \le d$, with c. For example, $U_{1324} = cd$ and $U_{315426} = d^2c$. The monomial U_σ is the *reduced variation* of σ. Note that the André condition rules out the presence of consecutive descents (since the André condition fails for i and $j = i + 2$ if $\sigma(i) > \sigma(i+1) > \sigma(i+2)$). Thus the reduced variation is well defined for André permutations.

Stanley [St7] conjectured that the shelling component $\check{\Phi}_i^d(c,d)$ of the simplex can be obtained as the sum of the reduced variations of the permutations of any one of three types (André I, André II, simsun), satisfying the additional conditions $\sigma(d+1) = d+1$ ("augmentation") and having a prescribed value for $\sigma(d)$. A further type of equinumerous permutation appears in [He1], who proved Stanley's conjecture. We state and illustrate this result in the case of simsun permutations.

THEOREM 5.5 (Hetyei, 1994). *Let $X(d+1, i)$ denote the set of simsun permutations σ in S_{d+1} such that $\sigma(d+1) = d+1$ and $\sigma(d) = d - i - 1$. Then the ith shelling component of the d-dimensional simplex is given by*

$$\check{\Phi}_i^d(c,d) = \sum_{\sigma \in X(d+1,i)} U_\sigma .$$

For the three-dimensional simplex, we obtain

$$\check{\Phi}_0^3(c,d) = U_{1234} + U_{2134} = c^3 + dc,$$

$$\check{\Phi}_1^3(c,d) = U_{1324} + U_{3124} = cd + dc,$$

$$\check{\Phi}_2^3(c,d) = U_{2314} = cd,$$

in agreement with the expressions (5).

In view of the combinatorial interpretations for the coefficients of the *cd*-index of the simplex and octahedron, the following question arises naturally: is there a systematic way to find a class of permutations enumerated by the *cd*-index of a polytope?

We close with a conjecture by Ehrenborg and Readdy [EhreRe], which generalizes a (still not settled) conjecture by Stanley. Given a $(d-1)$-dimensional polytope F, let $\mathscr{P}_F$ be the collection of d-dimensional polytopes having F as a facet. The conjecture is that, among the polytopes in $\mathscr{P}_F$, the pyramid over F minimizes the coefficients of the *cd*-index.

8. CONCLUDING REMARKS

Recent extensions of the idea of the associhedron include related polytopes such as the permutoassociahedron, in work by Kapranov (1993), and Coxeter-associahedra, in work by Reiner and Ziegler (1993), as well as far-reaching generalizations stemming from work by Gel'fand, Kapranov, and Zelevinsky (1994). These involve secondary polytopes, in turn generalized to fiber polytopes by Billera and Sturmfels (1992, 1993), and discriminantal arrangements. For more information and references, the interested reader may consult [Zi; BayBr].

For the reader interested in fiber polytopes, we propose the following questions: When is the fiber polytope simple? Simplicial? What is the relation between the *cd*-indices of the polytopes P, Q, and the fiber polytope $\Sigma(P,Q)$?

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