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Asymptotic Behavior of Integral Equations Using Monotonicity

DENNIS G. WEIS

Department of Mathematics, Arizona State University, Tempe, Arizona 85281

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In this paper, some Volterra integral equations that arise in heat transfer are studied. In particular, sufficient conditions for asymptotically periodic solutions are given. The results are derived, in part, using the fact that the resolvent form of the equations involved yields a monotone operator.

INTRODUCTION

This paper deals with the asymptotic behavior of some Volterra integral equations. In particular, we are looking at solutions that are asymptotically periodic, i.e., if x is a solution, then there exists periodic p such that $x(t) - p(t)$ goes to zero as t goes to infinity. The proofs are based upon the monotonicity of certain integral operators connected with the equations involved.

The interest in the type of equations studied here started with the work of Mann and Wolf in [3]. They looked at heat radiation from a half space and found an integral equation of the form

$$u(t) = f(t) + \int_0^t a(t-s)\, g(s, u(s))\, ds. \tag{1}$$

In the above equation, $f(t)$ comes from initial conditions, $a(t)$ comes from $(t)^{-1/2}$ times some physical constants, g comes from the normal derivative radiation condition $\partial u/\partial n = g(t, u)$, and $u(t)$ is the temperature at the boundary. Note that a similar equation occurs in half spaces interacting via radiation, see [2].

Now the same type of analysis can be extended to a wall radiating from both sides and mathematically similar problems, see [4, 5, 7]. Suppose the wall is given by $0 \leqslant x \leqslant 1$, and the radiation conditions are $\partial u/\partial n = g_1(t, u)$ at $x = 0$, and $\partial u/\partial n = g_2(t, u)$ at $x = 1$, then we get the equation

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \int_0^t \begin{bmatrix} a_1(t-s) & a_2(t-s) \\ a_2(t-s) & a_1(t-s) \end{bmatrix} \begin{bmatrix} g_1(s, u_1(s)) \\ g_2(s, u_2(s)) \end{bmatrix} ds.$$

Here u_1 and u_2 are the temperatures at the boundaries of the wall and f_1, f_2 come from the initial conditions. When this second equation is written in matrix form, it looks exactly like Eq. (1).

To analyze the problem of the wall or the half space, we will need the resolvent of the kernel $a(t)$, which we define here as the locally L^1 (possibly matrix) solution of

$$r(t) = a(t) - \int_0^t a(t-s)\, r(s)\, ds$$
$$= a(t) - \int_0^t r(t-s)\, a(s)\, ds.$$

The resolvent may be used to rewrite (1) in the "variation of constants" form

$$u(t) = f(t) - \int_0^t r(t-s) f(s)\, ds + \int_0^t r(t-s)\,(u(s) + g(s, u(s)))\, ds. \qquad (2)$$

See [6] for this and other basic results on integral equations.

Before stating the mathematical problem, we will "justify" physically some of our assumptions. For example, consider the problem of radiation from a wall with the boundary conditions $g_1(t, u) = g_2(t, u) = -(u - c)$, where c is a constant, and zero initial conditions. In this case, the matrix form of (2) becomes

$$u(t) = \int_0^t r(t-s) \begin{bmatrix} c \\ c \end{bmatrix} ds.$$

From standard results on the heat equation we know that $\lim_{t\to\infty} u(t) = \left[\begin{smallmatrix} c \\ c \end{smallmatrix}\right]$. Taking the limit of the preceding equation we get

$$\begin{bmatrix} c \\ c \end{bmatrix} = \int_0^\infty r(s)\, ds \begin{bmatrix} c \\ c \end{bmatrix}.$$

If we let $a_N(t) = a(t)/N$, and $r_N(t)$ equal the resolvent of $a_N(t)$, then Eq. (2) becomes

$$u(t) = f(t) - \int_0^t r_N(t-s) f(s)\, ds + \int_0^t r_N(t-s)\,(u(s) + g(s, u(s))/N)\, ds. \qquad (3)$$

When the boundary conditions are $g_1(t, x) = -N(x - T_1(t))$ and $g_2(t, x) = -N(x - T_2(t))$, with $T_1(t)$, $T_2(t)$ nonnegative C^2 functions and $N > 0$, and the initial conditions are zero, the maximum principle for the heat equation can be used to show that the solution is nonnegative. The integral Eq. (3) reduces to

$$u(t) = \int_0^t r_N(t-s) \begin{bmatrix} T_1(s) \\ T_2(s) \end{bmatrix} ds.$$

Since $u(t)$ is nonnegative and $T_1(t)$, $T_2(t)$ are arbitrary C^2, nonnegative functions, the above shows that $r_N(t)$ is nonnegative a.e. for all $N > 0$.

Also note that if $g(t, x)$ is Lipschitz on bounded sets of x uniformly in t, then given a bounded set of x, we can make the Lipschitz constant of $g(t, x)/N$, the "g" of Eq. (3), less than one.

Due to the type of analysis used here, we will need concepts like positive, nonnegative, greater than or equal to, etc. for vectors in R^n and $n \times n$ matrices. When descriptive terms, such as positive, nonnegative, come up we mean that the description applies to all entries of the vector or matrix. When comparative terms, such as greater, less than or equal, come up they refer to component by component comparison, e.g. col(1, 2, 3) $\geqslant$ col(2, 2, 4) since $1 \geqslant 1$, $2 \geqslant 2$, $3 \geqslant 4$. We also need a specific norm in R^n, namely, $|x| = \max\{|x_i|: i = 1,..., n\}$, and the convention that col(d), for a number d, means col($d, d,..., d$).

Statement of Mathematical Problem and Theorems

The basic equation under consideration will be

$$x(t) = \int_0^t a(t - s)\, g(s, x(s))\, ds, \tag{E}$$

where x is a function from $R^+ (=[0, \infty))$ to R^n, a is a function from R^+ to the $n \times n$ matrices over R, and g is a function from $R^+ \times R^n$ to R^n. In addition, we will use the following assumptions:

A. a is locally L^1 and has an L^1 resolvent r that is nonnegative a.e. and satisfies

$$\int_0^\infty r(s)\, ds \operatorname{col}(c) = \operatorname{col}(c)$$

for any constant c.

B. $g(t, x)$ is continuous in (t, x), Lipschitz continuous in x on bounded sets of x uniformly in t and of the "separated" form $g(t, x) = \operatorname{col}(g_i(t, x_i)$: $i = 1,..., n)$; also assume there are constants a and b such that $a < 0 < b$, $g(t, \operatorname{col}(a)) > 0$ (note that this is the zero in R^n), $g(t, \operatorname{col}(b)) < 0$ and the Lipschitz constant for g on the bounded set $D = \{x: \operatorname{col}(a) \leqslant x \leqslant \operatorname{col}(b)\}$ is less than one.

The variation of contants equation for (E) is

$$x(t) = \int_0^t r(t - s)\, (x(s) + g(s, x(s)))\, ds \tag{VE}$$

and this equation is equivalent to (E) in the sense that they have the same solutions. We will define the operator Q, on continuous functions on R^+, by the equation

$$Q(x)(t) = \int_0^t r(t-s)(x(s) + g(s, x(s)))\, ds.$$

LEMMA 1. *Assume* A *and* B, *then* Q *is monotone in the sense that if* x *and* $y \in C(R^+, D)$ *and* $x(t) \geqslant y(t)$ *for* $t \in R^+$, *then* $Q(x)(t) \geqslant Q(y)(t)$ *for* $t \in R^+$. *Also, we have that* $Q(\mathrm{col}(a))(t) \geqslant \mathrm{col}(a)$, *and* $Q(\mathrm{col}(b))(t) \leqslant \mathrm{col}(b)$ *for* $t \in R^+$.

Now, in a manner similar to fixed point theorems for monotone functions in one dimension, the above information can be used to prove the next theorem.

THEOREM 1. *Given that Eq.* (E) *and assumptions* A *and* B *are true, Eq.* (E) *has a unique, continuous, solution on* R^+ *and this solution,* $x(t)$, *satisfies* $\mathrm{col}(a) \leqslant x(t) \leqslant \mathrm{col}(b)$ *for* $t \in R^+$.

Thus, assumptions A and B ensure that E has a unique solution for all $t \geqslant 0$. To study asymptotic results, we will need an assumption about the asymptotic character of g. That information is contained in assumption C.

C. There exists a function $G(t, x)$ from $R \times R^n$ to R^n that satisfies B with R^+ replaced by R (using the same a and b), which is periodic in t and strictly decreasing in x for fixed t, and that satisfies $\lim_{t \to \infty} g(t, x) - G(t, x) = 0$ uniformly on bounded sets of x.

Now consider the following limiting equation

$$y(t) = \int_0^\infty r(s)(y(t-s) + G(t-s, y(t-s)))\, ds, \qquad \text{(LE)}$$

and the limiting operator

$$M(y)(t) = \int_0^\infty r(s)(y(t-s) + G(t-s, y(t-s)))\, ds.$$

This limiting equation is important because it is actually the equation of the asymptotic form of the solution of (E). Next, we want to show that (LE) has a unique solution and that the solution is periodic.

LEMMA 2. *Assume* A *and* C. *Then,* M *is monotone on* $C(R, D)$ *in the same sense that* Q *was on* $C(R^+, D)$, $M(\mathrm{col}(a))(t) \geqslant \mathrm{col}(a)$ *and* $M(\mathrm{col}(b))(t) \leqslant \mathrm{col}(b)$ *for* $t \in R^+$.

THEOREM 2. *Assume A, B, and C, then* (LE) *has a unique solution,* $y(t)$, *on R such that* $\mathrm{col}(a) \leqslant y(t) \leqslant \mathrm{col}(b)$. *The solution is also periodic with the same period as* $G(t, y)$ *has with respect to t.*

We just note in passing that G must be strictly decreasing, not just nonincreasing, for uniqueness in Theorem 2. In the one-dimensional case, for example, define $g = G$ by $g(t, x) = -(x + 1)/2$ for $x \leqslant -1$, $g(t, x) = 0$ for $-1 \leqslant x \leqslant 1$, and $g(t, x) = -(x - 1)/2$ for $x \geqslant 1$. Then the limiting equation becomes

$$y(t) = \int_0^\infty r(s)\,(y(t - s) + g(t - s, y(t - s)))\, ds.$$

Since $\int_0^\infty r(s)\, ds = 1$, $y(t) = c$ is a solution for any c such that $-1 \leqslant c \leqslant 1$. The same example can be constructed in any dimension to show nonuniqueness for a nonincreasing G.

Once that we know that the solution to (LE) is unique, the problem about limiting behavior is essentially solved. From the uniqueness it easily follows that the solution of (E) must be asymptotic to the solution of (LE).

THEOREM 3. *If* $x(t)$ *is the solution of* (E) *and* $y(t)$ *is the approxipriately bounded solution of* (LE), *then* $\lim_{t\to\infty} y(t) - x(t) = 0$, i.e., $x(t)$ *is asymptotically periodic.*

The next section contains the proofs to the above theorems and lemmas. In a way, the proofs are as interesting as the theorems. They are interesting in that they use a constructive approach to solving the main equations that is made possible by the monotonicity of the operators involved. The constructions and the unique solution to (LE) then give us asymptotic behavior.

Now that we have treated the homogeneous equation (E), there is the question of how to handle the nonhomogeneous problem

$$x(t) = f(t) - \int_0^t a(t - s)\,(x(s) - g(s, x(s)))\, ds.$$

It turns out that we can handle it in several ways. We can transform it into (E) by defining $x'(t) = x(t) - f(t)$, and thereby getting a new $g(t, x')$, or we can handle it with an analysis similar to that for the homogeneous problem by making several assumptions on $f(t)$. A similar analysis will work if we assume: $f(t)$ is continuous and

$$\begin{aligned}\mathrm{col}(b) &- \int_0^t r(t - s)\, ds\, \mathrm{col}(b)\\ &\geqslant f(t) - \int_0^t r(t - s) f(s)\, ds \geqslant \mathrm{col}(a) - \int_0^t r(t - s)\, ds\, \mathrm{col}(a).\end{aligned}$$

The last assumption may be taken physically as saying that the initial temperature distribution, in the material, is bounded above by b and is bounded below by a (see [7]). Also note that the last assumption implies

$$\lim_{t\to\infty} f(t) - \int_0^t r(t-s)\, f(s)\, ds = 0$$

since r is as in assumption A, which says $\int_0^\infty r(s)\, ds \operatorname{col}(c) = \operatorname{col}(c)$.

PROOFS OF LEMMAS AND THEOREMS

Proof of Lemma 1. Let x and $y \in C(R^+, D)$ and $x(t) \geqslant y(t)$ for $t \in R^+$. Then

$$Q(x)\,(t) - Q(y)\,(t) = \int_0^t r(t-s)\,(x(s) - y(s) + g(s, x(s)) - g(s, y(s)))\, ds,$$

and $x(s) - y(x) + g(s, x(s)) - g(s, y(s)) = \operatorname{col}(x_i(s) - y_i(s) + g_i(s, x_i(s)) - g_i(s, y_i(s))\colon n = 1,\ldots, n)$. Since the Lipschitz constant of g on D is less than one, the Lipschitz constant for each g_i is less than one and the sign of $x_i(s) - y_i(x) + g_i(s, x_i(s)) - g_i(s, y_i(s))$ is the same as the sign of $x_i(s) - y_i(s)$. Hence, $x(s) - y(s)$ nonnegative implies that $\operatorname{col}(x_i(s) - y_i(s) + g_i(s, x_i(s)) - g_i(s, y_i(s))\colon n = 1,\ldots, n)$ is nonnegative. Therefore, $\int_0^t r(t-s)(x(s) - y(s) + g(s, x(s)) - g(s, y(s)))\, ds$ is the integral of the product of a nonnegative matrix and a nonnegative column, and thus, is nonnegative itself. Now

$$\begin{aligned} Q(\operatorname{col}(a)) &= \int_0^t r(t-s)\,(\operatorname{col}(a) + g(s, \operatorname{col}(a)))\, ds \\ &\geqslant \int_0^t r(t-s) \operatorname{col}(a)\, ds \\ &\geqslant \int_0^\infty r(t-s) \operatorname{col}(a)\, ds = \operatorname{col}(a), \end{aligned}$$

since $\operatorname{col}(a)$ is negative, $r(t)$ is nonnegative a.e., and $g(s, \operatorname{col}(a))$ is nonnegative. A similar argument can be used to show that $Q(\operatorname{col}(b))(t) \leqslant \operatorname{col}(b)$.

Proof of Theorem 1. Define $Q^1(a) = Q(\operatorname{col}(a))$ and $q^{n+1}(a)(t) = Q(Q^n(a))(t)$, for $n = 1, 2,\ldots$. Then $Q^n(a)(t)$ is a sequence of continuous functions and it easily can be shown that $Q^{n+1}(a)(t) \geqslant Q^n(a)(t)$, for all $t \geqslant 0$, and $n = 1, 2,\ldots,$ and that $Q^n(a)(t) \leqslant \operatorname{col}(b)$ for similar t and n. Since $Q^n(a)(t)$ forms a bounded increasing sequence of numbers for each t, the sequence of functions $Q^n(a)(t)$

converges pointwise to some function, call it $h(t)$. Because of its source and because $r(t)$ is L^1, $h(t)$ satisfies

$$h(t) = \int_0^t r(t-s)\,(h(s) + g(s, h(s)))\,ds.$$

The above integral representation for h shows that it must be continuous. By the usual arguments for integral equations, h is the unique solution for (E) since g is Lipschitz. Note that h also may be generated by a sequence $Q^n(b)(t)$, defined in analogy to $Q^n(a)(t)$. Also note that $Q^n(a)(t)$, or $Q^n(b)(t)$, converges to $h(t)$ uniformly on compact sets of t. The last statement on convergence comes from the fact that the sequence is a sequence of continuous functions converging pointwise and monotonically to a continuous function.

Proof of Lemma 2. The proof is an obvious extension of the proof of Lemma 1 and will not be stated.

LEMMA 3. *If $M^n(a)(t)$ and $M^n(b)(t)$ are defined in a way that is analogous to the definition of $Q^n(a)$ in the proof of Theorem* 1, *and if* A, B, *and* C *are true, then $M^n(a)(t)$ and $M^n(b)(t)$ both converge uniformly and monotonically to periodic functions that solve* (LE). *The functions will have values in the set D and have the same period as $G(t, x)$.*

Proof. First look at

$$M^1(a)(t) = \int_0^\infty r(s)\,(\mathrm{col}(a) + G(t-s, \mathrm{col}(a)))ds.$$

The integrand is obviously periodic in t with the same period as $G(t, x)$. Therefore, $M^1(a)(t)$ must be periodic with the same period as $G(t, x)$.

$$M^2(a)(t) = \int_0^\infty r(s)\,(M^1(a)(t-s) + G(t-s, M^1(a)(t-s)))\,ds.$$

In the case of $M^2(a)(t)$, we also have a periodic integrand with the same period as $G(t, x)$. Thus, $M^2(a)(t)$ is periodic with the same period as $G(t, x)$. Similar arguments can be made for all $M^n(a)(t)$. By the same reasoning as in Theorem 1, $M^n(a)(t)$ converges monotonically and uniformly on compact sets of t to a continuous function $L(t)$ that satisfies (LE). Using the monotonicity of M and that fact that $\mathrm{col}(a) \leqslant M(\mathrm{col}(a))(t) \leqslant M(\mathrm{col}(b))(t) \leqslant \mathrm{col}(b)$, we have that

$$\begin{aligned}\mathrm{col}(a) &\leqslant M^n(a)(t) \leqslant M^{n+1}(a)(t) \leqslant M^{n+1}(b)(t)\\ &\leqslant M^n(b)(t) \leqslant \mathrm{col}(b).\end{aligned} \tag{4}$$

Hence, $L(t)$ is bounded, i.e. has values in D. Since the functions $M^n(a)(t)$ are periodic with the same period, $L(t)$ is periodic with that period. Periodicity may then be used to show that the uniform convergence on compact sets is in fact uniform. Similarly, it can be shown that $M^n(b)(t)$ converges uniformly and monotonically to a function $U(t)$ that solves (LE) and has the same period as $G(t, x)$. From (4), we know that $U(t) \geqslant L(t)$.

LEMMA 4. *Assume everything as in Lemma 3. Then, the constructed solutions $U(t)$ and $L(t)$ are in fact equal.*

Proof. Suppose that $U(t) \not\equiv L(t)$. Then let

$$m = \max\{| U(t) - L(t)|: t \in R\} > 0$$

and note that $\mathrm{col}(0) \leqslant U(t) - L(t) \leqslant \mathrm{col}(m)$. Inasmuch as $U(t) - L(t)$ is periodic, there is a $t_0 \in R$ such that $U(t_0) - L(t_0) \leqslant \mathrm{col}(m)$, but $U(t_0) - L(t_0) \not< \mathrm{col}(m)$ (i.e., one of the components of $U(t_0) - L(t_0)$ must be equal to m). Consider the equation

$$\begin{aligned} U(t_0) - L(t_0) = \int_0^\infty r(s)\,(U(t_0 - s) - L(t_0 - s) \\ + G(t_0 - s, U(t_0 - s)) - G(t_0 - s, L(t_0 - s)))\,ds. \end{aligned} \tag{5}$$

If $U(t_0 - s) \neq L(t_0 - s)$, then

$$\begin{aligned} 0 \leqslant U(t_0 - s) - L(t_0 - s) + G(t_0 - s, U(t_0 - s)) - G(t_0 - s, L(t_0 - s)) \\ < U(t_0 - s) - L(t_0 - s) \leqslant \mathrm{col}(m), \end{aligned}$$

since $G(t, x)$ is strictly decreasing in x and has Lipschitz constant on D less than 1. If $U(t_0 - s) = L(t_0 - s)$, then

$$\begin{aligned} U(t_0 - s) - L(t_0 - s) + G(t_0 - s, U(t_0 - s)) \\ - G(t_0 - s, L(t_0 - s)) = 0 < \mathrm{col}(m). \end{aligned}$$

Therefore, from (5),

$$U(t_0) - L(t_0) < \int_0^\infty r(s)\,\mathrm{col}(m)\,ds = \mathrm{col}(m).$$

But this last statement is a contradiction to $U(t_0) - L(t_0) \not< \mathrm{col}(m)$. Hence, $U(t) \equiv L(t)$.

Proof of Theorem 2. Let $y(t)$ be any solution of (LE) with values in D. Applying the monotone operator M to the inequality $\mathrm{col}(a) \leqslant y(t) \leqslant \mathrm{col}(b)$ gives $M^1(a)(t) \leqslant M(y)(t) = y(t) \leqslant M^1(b)(t)$. Similarly, $M^n(a)(t) \leqslant y(t) \leqslant M^n(b)(t)$. Therefore, since $M^n(a)(t)$ converges to $L(t)$, $M^n(b)(t)$ converges to $U(t)$ and $L(t) = U(t)$, $y(t) = U(t) = L(t)$. This means that $L(t) = U(t)$ is the unique solution with values in the interval noted. The periodicity follows from Lemmas 3 and 4.

LEMMA 5. *Assume* A, B, *and* C. *Then, given* n *and* $e > 0$, *there exists a* T *such that* $|Q^n(a)(t) - M^n(a)(t)| < e/2$ *and* $|Q^n(b)(t) - M^n(b)(t)| < e/2$, *for* $t \geqslant T$.

Proof. To show the above, we will show by induction, that

$$\lim_{t \to \infty} Q^n(a)(t) - M^n(a)(t) = 0,$$

and a similar statement for b. Given $n = 1$, look at

$$\begin{aligned} M^1(a)(t) - Q^1(a)(t) &= \int_0^\infty r(s)\,(\mathrm{col}(a) + G(t-s, \mathrm{col}(a)))\,ds \\ &\quad - \int_0^t r(s)\,(\mathrm{col}(a) + g(t-s, \mathrm{col}(a)))\,ds \\ &= \int_t^\infty r(s)\,(\mathrm{col}(a) + G(t-s, \mathrm{col}(a)))\,ds \\ &\quad - \int_0^t r(s)\,(G(t-s, \mathrm{col}(a)) - g(t-s, \mathrm{col}(a)))\,ds. \end{aligned}$$

In the last statement above, the first integral goes to zero as t goes to ∞ since $r(s)$ integrable and the rest of the integrand is bounded. The second goes to zero since it is the convolution of an L^1 function and a function that goes to zero as its argument goes to infinity. Consider $M^{n+1}(a)(t) - Q^{n+1}(a)(t)$, assuming $M^n(a)(t) - Q^n(a)(t)$ goes to zeros at t goes to infinity. First note that, under the above assumptions, $G(t, M^n(a)(t)) - g(t, Q^n(a)(t))$ goes to zero as t goes to infinity. Rewrite the last difference as $G(t, M^n(a)(t)) - G(t, Q^n(a)(t)) + G(t, Q^n(a)(t)) - g(t, Q^n(a)(t))$. The first term goes to zero inasmuch as $M^n(a)(t) - Q^n(a)(t)$ goes to zero and $G(t, x)$ is Lipschitz in x. The second term goes to zero since, from C, $\lim_{t\to\infty} G(t, x) - g(t, x) = 0$ uniformly on bounded sets of x.

$$\begin{aligned} &M^{n+1}(a)(t) - Q^{n+1}(a)(t) \\ &\quad = \int_0^\infty r(s)\,(M^n(a)(t-s) + G(t-s, M^n(a)(t-s)))\,ds \\ &\quad\quad - \int_0^t r(s)\,(Q^n(a)(t-s) + g(t-s, Q^n(t-s)))\,ds \\ &\quad = \int_t^\infty r(s)\,(M^n(a)(t-s) + G(t-s, M^n(a)(t-s)))\,ds \\ &\quad\quad + \int_0^t r(s)\,(M^n(a)(t-s) - Q^n(a)(t-s) \\ &\quad\quad + G(t-s, M^n(a)(t-s)) - g(t-s, Q^n(a)(t-s)))\,ds. \end{aligned}$$

For the same reasons as in the first case, these last integrals go to zero as t goes to infinity. Similar arguments can be made for $M^n(b)(t) - Q^n(b)(t)$.

Proof of Theorem 3. Given $e > 0$, pick n such that $|y(t) - M^n(a)(t)| < e/2$ and $|y(t) - M^n(b)(t)| < e/2$. Now pick T such that $|Q^n(a)(t) - M^n(a)(t)| < e/2$ and $|Q^n(b)(t) - M^n(b)(t)| < e/2$ for $t \geqslant T$. The above inequalities also can be written as $0 \leqslant y(t) - M^n)a?)t? < \mathrm{col}(e/2)$, $0 \leqslant M^n(b)(t) - y(t) < \mathrm{col}(e/2)$, $-\mathrm{col}(e/2) < Q^n(b)(t) - M^n(b)(t) < \mathrm{col}(e/2)$, for $t \geqslant T$; and $-\mathrm{col}(e/2) < Q^n(a)(t) - M^n(a)(t) < \mathrm{col}(e/2)$, for $t \geqslant T$. Next note that

$$\begin{aligned} y(r) - x(t) &\geqslant y(t) - Q^n(b)(t) \\ &\geqslant M^n(b)(t) - \mathrm{col}(e/2) - Q^n(b)(t) \\ &\geqslant -\mathrm{col}(e/2) - \mathrm{col}(e/2) = -\mathrm{col}(e), \end{aligned}$$

for $t \geqslant T$. We can also show that $y(t) - x(t) \leqslant \mathrm{col}(e)$ for $t \geqslant T$. From the definitions, $\mathrm{col}(e) \geqslant y(t) - x(t) \geqslant -\mathrm{col}(e)$ implies that $|y(t) - x(t)| \leqslant e$ for $t \geqslant T$. Thus, we are finished inasmuch as the last phrase is equivalent to $\lim_{t\to\infty} y(t) - x(t) = 0$.

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