On the Pringsheim Rearrangement Theorems

Marion Scheepers

Department of Mathematics and Computer Science,
Boise State University, Boise, Idaho 83725
E-mail: marion@cantor.idbsu.edu

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We show that requiring that the set of positions of the positive terms in a conditionally convergent numerical series have asymptotic density provides converses for old rearrangement theorems of Alfred Pringsheim.

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Pringsheim proved a generalization of a rearrangement theorem of Schlömilch, which in turn is a generalization of a classical theorem of Ohm. Pringsheim’s result is stated in concise form on p. 491 of [3]. I found that, subject to a natural regularity hypothesis, some of the conclusions given in Pringsheim’s theorem have converses. One of the purposes of this paper is to give these converses. A careful referee of an earlier version of this paper noted (by giving a counterexample) that one of the cited Pringsheim results (Theorem 2 below) is false. We shall at an appropriate place point out where the error in [3] occurred and also give a very brief description of the referee’s counterexample.

The symbol \( \mathbb{N} \) denotes the set of positive integers, while \( \mathbb{R} \) denotes the set of real numbers. A sequence (of real numbers) is a function from \( \mathbb{N} \) to \( \mathbb{R} \). Let \( f \) be a sequence of nonzero real numbers, fixed for the remainder of the paper. While \( \sum f \) denotes the series associated with \( f \), the limit of this series when it is convergent is denoted by \( \sum_{n<\infty} f(n) \). The symbol \( \sum^* f \) is used exclusively to denote the lim sup of the partial sums of the series, while \( \sum_* f \) denotes the lim inf. It is clear that \( \sum_* f \leq \sum^* f \); the series is convergent only when \( \sum_* f = \sum^* f \).

\[ \sum^* f = \sum^* f \]

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We use $|f|$ to denote the sequence defined from $f$ by letting $|f|(n)$ be $|f(n)|$ for each $n$. The series $\sum f$ is \textit{conditionally convergent} if it is convergent, but $\sum |f|$ is not. If $\sum f$ converges conditionally, then both \{n : f(n) > 0\} and \{n : f(n) < 0\} are infinite sets. Let
\[ a_1, a_2, \ldots, a_n, \ldots \]
list the positive terms of $f$ in their order of occurrence, and let
\[ -b_1, -b_2, \ldots, -b_n, \ldots \]
list the negative terms of $f$ in their order of occurrence. If both the $a_n$’s and the $b_n$’s are monotonic sequences, then we say $f$ is \textit{signwise monotonic}. In this paper we consider only signwise monotonic sequences.

If $g$ is a sequence of nonzero real numbers such that there is a one-to-one and onto function $\phi : \mathbb{N} \to \mathbb{N}$ for which $g = f \circ \phi$, then $g$ is said to be a rearrangement of $f$ and the series $\sum g$ is said to be a rearrangement of $\sum f$. Closer examination of the popular proofs of this (see, e.g., [4, pp. 76–77]) reveals that the well-known rearrangement theorem of Riemann can be stated as follows:

\textbf{Theorem 1} (Riemann). Let $f$ be a sequence of nonzero real numbers such that $\sum f$ is conditionally convergent. Let $\alpha \leq \beta$ be real numbers. Then there is a rearrangement $g$ of $f$ such that:

1. $\alpha = \sum_* g$ and $\beta = \sum^* g$.
2. For each $n$, the $n$th positive term of $g$ is the $n$th positive term of $f$, and the $n$th negative term of $g$ is the $n$th negative term of $f$.

It seems appropriate to refer to a rearrangement $g$ of $f$ having the second property in Theorem 1 as a \textit{Riemann rearrangement}. I do so in this paper.

Let $A$ be an infinite subset of $\mathbb{N}$ such that $\mathbb{N} \setminus A$ is also infinite. Define $f_A$ so that
\[ f_A(n) = \begin{cases} 
    \text{the } j\text{th positive term of } f \text{ if } n \text{ is the } j\text{th element of } A, \\
    \text{the } j\text{th negative term of } f \text{ if } n \text{ is the } j\text{th element of } \mathbb{N} \setminus A.
\end{cases} \]
Then $f_A$ is a Riemann rearrangement of $f$. Every Riemann rearrangement is obtainable in this way.

For a set $X$, $|X|$ denotes the cardinality of $X$. For a subset $A$ of $\mathbb{N}$ we define
\[ \pi_A(n) = |A \cap \{1, 2, \ldots, n\}|, \quad \text{the predensity of } A, \]
\[ d^*(A) = \limsup \frac{\pi_A(n)}{n}, \quad \text{the upper density of } A, \]
\[ d_*(A) = \liminf \frac{\pi_A(n)}{n}, \quad \text{the lower density of } A. \]
It is clear that $0 \leq d_*(A) \leq d^*(A) \leq 1$. When $d_*(A)$ and $d^*(A)$ are equal, this common value is denoted $d(A)$ and is said to be the (asymptotic) density of $A$; in this case we say that $A$ is a set with density.

Toward stating Pringsheim’s results, assume that $A \subseteq \mathbb{N}$ is such that $\sum f_A$ is conditionally convergent. Consider a set $B \subseteq \mathbb{N}$ such that for all but finitely many $n$, $\pi_A(n) < \pi_B(n)$ (the treatment when for all but finitely many $n$, $\pi_B(n) < \pi_A(n)$ is similar). For each $n$, choose $k_n$ to be minimal such that $n - \pi_B(n) = k_n - \pi_A(k_n)$. Then for each $n$,

$$\sum_{j \leq n} f_B(j) - \sum_{i \leq k_n} f_A(i) = a_{\pi_A(k_n)+1} + \cdots + a_{\pi_B(n)}.$$ 

The convergence as well as the value of the limit of $\sum f_B$ is determined by that of $\sum f_A$ and $(a_{\pi_A(k_n)+1} + \cdots + a_{\pi_B(n)} : n \in \mathbb{N})$. Since we are assuming that $f$ is signwise monotonic, for each $n$, the $n$th term of this sequence is bounded below by $(\pi_B(n) - \pi_A(k_n) - 1) \cdot a_{\pi_B(n)}$ and above by $(\pi_B(n) - \pi_A(k_n)) \cdot a_{\pi_A(k_n)}$. The following three theorems state the results summarized on p. 491 of [3].

**Theorem 2 (Pringsheim I).** Assume that $(n \cdot a_n : n \in \mathbb{N})$ diverges to $\infty$. Then $\sum f_B$ converges if, and only if, $(a_{\pi_A(k_n)} \cdot (\pi_B(n) - \pi_A(k_n)) : n \in \mathbb{N})$ does. Moreover, if the limit of $(a_{\pi_A(k_n)} \cdot (\pi_B(n) - \pi_A(k_n)) : n \in \mathbb{N})$ is $a$, then $\sum_{n<\infty} f_B(n) = \sum_{n<\infty} f_A(n) + a$. \(^2\)

**Theorem 3 (Pringsheim II).** Assume that $\lim_{n \to \infty} n \cdot a_n = 0$. If $(a_{\pi_A(k_n)} \cdot (\pi_B(n) - \pi_A(k_n)) : n \in \mathbb{N})$ is bounded, then $\sum_{n<\infty} f_B = \sum_{n<\infty} f_A$.

**Theorem 4 (Pringsheim–Schlömilch).** Assume that $(n \cdot a_n : n \in \mathbb{N})$ converges to the nonzero real number $g$. Then $\sum f_B$ converges if and only if $(a_{\pi_A(k_n)} \cdot (\pi_B(n) - \pi_A(k_n)) : n \in \mathbb{N})$ converges. Moreover, if $\lim_{n \to \infty} a_{\pi_A(k_n)} \cdot (\pi_B(n) - \pi_A(k_n)) = a$, then $\sum_{n<\infty} f_B = \sum_{n<\infty} f_A + g \cdot \ln(1 + 1/g)$.

Theorem 4 extends a theorem of Schlömilch [5] that proved the result for the case where $A$ is a set of the following form: For a rational number $p/(q + p)$ in $(0, 1)$, $A$ is the set of all the first $p$ elements of the disjoint $(p + q)$-element intervals constituting $\mathbb{N}$. Ohm [2] proved Schlömilch’s theorem for the special case where for all $n$, $f(n) = (-1)^{n+1}/n$.

The rest of this paper is divided into four sections. The first of these is devoted to some generalities and each of the remaining three is devoted to one of these three Pringsheim theorems. The additional regularity

\(^2\)A referee of an earlier version of this paper noted that the result cited here is false. See Section 2 in this regard.
hypothesis alluded to earlier is that \( A \) has density and that \( 0 < d(A) < 1 \). Density considerations inspire two more concepts:

\[
\sigma(f) := \{ x \in (0, 1) : (\forall A \subseteq \mathbb{N})(d(A) = x \Rightarrow \text{converges}) \},
\]

\[
\omega(f) := \{ x \in [0, 1] : (\exists A \subseteq \mathbb{N})(d(A) = x \text{ and } \text{converges}) \}.
\]

We shall show that if \( x \in \sigma(f) \), then for all \( A \) and \( B \) contained in \( \mathbb{N} \) and of density \( x \), \( \sum_{n<\infty} f_A(n) = \sum_{n<\infty} f_B(n) \) (Corollary 13). This implies that when \( \sigma(f) \) is nonempty, the following function, denoted \( \phi_f \) and defined on \( \sigma(f) \), is well defined: \( \phi_f(x) := \sum_{n<\infty} f_A(x) \), an \( A \) of density \( x \).

1. GENERALITIES

Throughout our arguments we shall repeatedly refer to some elementary facts, which I collect here for easy reference. For sets \( A \) and \( B \), \( A =^* B \) means their symmetric difference is finite, while \( A \subseteq^* B \) means \( A \setminus B \) is finite.

**Lemma 5.** If \( A \) and \( B \) are sets of natural numbers such that for all but finitely many \( n \), \( \pi_A(n) \leq \pi_B(n) \), then \( \sum_* f_A \leq \sum_* f_B \) and \( \sum^* f_A \leq \sum^* f_B \).

**Lemma 6.** If \( A \) and \( B \) are sets of natural numbers such that \( A =^* B \), then \( d_+(A) = d_+(B) \), \( d^+(A) = d^+(B) \), \( \sum_* f_A = \sum_* f_B \), and \( \sum^* f_A = \sum^* f_B \).

**Lemma 7.** If \( A \) and \( B \) are sets of natural numbers such that \( A \subset^* B \), then \( d^+(A) \leq d^+(B) \), \( \sum_* f_A \leq \sum_* f_B \), and \( \sum^* f_A \leq \sum^* f_B \).

**Lemma 8.** If \( B \) and \( C \) are sets of natural numbers such that \( \sum^* f_B < \sum_* f_C \), then \( \lim_{n \to \infty} \pi_C(n) - \pi_B(n) = \infty \).

**Lemma 9.** If \( 0 < p_1/(p_1 + q_1) < p_2/(p_2 + q_2) < 1 \) are rational numbers, then \( \lim_{n \to \infty}((p_2 \cdot q_1)/(q_2 \cdot p_1))^n = \infty \) and \( \lim_{m \to \infty}((p_1 \cdot q_2)/(q_1 \cdot p_2))^m = 0 \).

**Lemma 10.** For \( 0 < \alpha < \beta < 1 \), the set

\[
\left\{ \frac{1}{1 + \left( \frac{p_1}{q_1} \right)^{m}, \frac{p_2}{q_1}, \frac{p_2}{p_2 + q_2} < \beta} : m \in \mathbb{N}, \alpha < \frac{p_1}{p_1 + q_1}, \frac{p_2}{p_2 + q_2} < \beta \right\}
\]

is dense in \([0, 1]\).

**Lemma 11.** Let \( f \) be signwise monotonic. If there are subsets \( A \) and \( B \) of \( \mathbb{N} \) such that \( 0 < d(A) < d(B) < 1 \), and \( \sum f_A \) and \( \sum f_B \) converge, then \( \lim \sup n \cdot a_n \leq (d(B) \cdot (1 - d(A)))/(d(A) \cdot (1 - d(B))) \cdot \lim \inf n \cdot a_n \).
2. ON PRINGSHEIM I

Theorem 2 as cited is false. The error in Pringsheim’s argument occurs directly after formula (25) on p. 489 of [3]. Here Pringsheim claims that if, for a differentiable function $g$, one has $\lim_{x \to \infty} g(x)/x = 0$, then $\lim_{x \to \infty} g'(x) = 0$. This conclusion dramatically simplifies (25), which in turn leads to the false conclusions in Theorem 2.

The referee provided the following example to illustrate this point: For $i = 0, 1, 2, \ldots$, let $C_i = i \cdot 3^i$ and define $f(n) = (-1)^n/3^i$ for $2 \cdot C_{i-1} < n < 2 \cdot C_i$. Then $f$ is signwise monotonic and $\sum f$ is convergent with sum zero. Moreover, $\lim n \cdot a_n = \infty$. With $A$ denoting the set of even numbers, $f = f_A$. For each $i$ define

$$I_i = [(2 \cdot i - 2) \cdot 3^i, (2 \cdot i + 6) \cdot 3^i]$$

and then put $B = A \cup \{n \in \mathbb{N} : (\exists i)(\exists m)(n \in I_i, \text{and } n = 4 \cdot m + 1)\}$. One can check that for all but finitely many $n$, $\pi_A(n) < \pi_B(n)$, that $\sum f_B$ is convergent with sum 2, but $d_{\pi_A(k_n)} \cdot (\pi_B(n) - \pi_A(k_n))$ does not converge to 2.

If $f$ is signwise monotonic and if $n \cdot a_n$ diverges to $\infty$, then $\sum f$ is not absolutely convergent (for each $n$ there is a partial sum of $\sum |f|$ which exceeds $n \cdot a_n$).

**Theorem 12.** Assume that $f$ is a signwise monotonic sequence that converges to 0 and that $0 < x < 1$ is a real number. The following are equivalent:

1. $\lim_{n \to \infty} n \cdot a_n = \infty$ and $x \in \omega(f)$.
2. For each set $B$ such that $\sum f_B$ converges, $d(B) = x$ (in particular, $\omega(f) = \{x\}$).
3. There are sets $B$ and $C$ such that $d(B) = x = d(C)$, and $\sum_{n<\infty} f_B(n) = \infty$ and $\sum_{n<\infty} f_C(n) = -\infty$.
4. There are sets $B$ and $C$ such that $d(B) = x = d(C)$ and $-\infty < \sum^* f_B < \sum^* f_C < \infty$.

**Proof.** 1 $\Rightarrow$ 2. By 1, choose a subset $A$ of $\mathbb{N}$ such that $d(A) = x$ and $\sum f_A$ converges. Let $B$ be a set of natural numbers such that $\sum f_B$ converges. We show that then $B$ has density and that this density is also $x$.

Suppose, on the contrary that $d_+(B) < x$. Choose a sequence $n_1 < n_2 < \cdots < n_k < \cdots$ such that $d_+(B) = \lim_{k \to \infty} (\pi_B(n_k)/n_k)$. Thus, we have $d(A) = x$ larger than this limit of quotients.
For each \( k \), let \( m_k \) be minimal with \( n_k - \pi_B(n_k) = m_k - \pi_A(m_k) \). Since for all but finitely many \( k \) we have \( \pi_B(n_k) < \pi_A(n_k) \) and since \( n - \pi_A(n) \) is nondecreasing in \( n \), it follows that for all but finitely many \( k \), we have \( n_k < m_k \). Moreover, for all but finitely many \( k \) we have

\[
1 - \frac{\pi_B(n_k)}{n_k} = \frac{m_k}{n_k} \cdot \left( 1 - \frac{\pi_A(m_k)}{m_k} \right).
\]

The left-hand side of this equation converges to \( 1 - d_*(B) \), and so the right-hand side converges. Since \( A \) has density, it follows that

\[
\lim \frac{m_k}{n_k} = \frac{1 - d_*(B)}{1 - d(A)}.
\]

Now for each sufficiently large \( k \) we have by signwise monotonicity that

\[
\sum_{j \leq n_k} f_B(j) - \sum_{j \leq m_k} f_A(j) = a_{\pi_B(n_k)+1} + \cdots + a_{\pi_A(m_k)} \\
\geq a_{\pi_A(m_k)} [\pi_A(m_k) - \pi_B(n_k)] \\
= a_{\pi_A(m_k)} (m_k - n_k) \\
= \pi_A(m_k) a_{\pi_A(m_k)} \cdot \frac{m_k - n_k}{\pi_A(m_k)}.
\]

But the factor \((m_k - n_k)/(\pi_A(m_k))\) converges to the positive number \((d(A) - d_*(B))/(1 - d_*(B)) \cdot 1/(d(A))\), while the factor \(\pi_A(m_k) \cdot a_{\pi_A(m_k)}\) diverges to infinity. This implies that the subsequence of partial sums of the form \(\sum_{j \leq n_k} f_B(j)\) of the series \(\sum f_B\) diverges to infinity, contradicting the fact that \(\sum f_B\) converges.

This contradiction shows that \(d(A) \leq d_*(B)\). An analogous argument shows that we must have \(d^*(B) \leq d(A)\). In particular, \(B\) has density, and its density is equal to that of \(A\).

2 ⇒ 3. To obtain a set \(B\) of density \(x\) for which \(\sum f_B\) diverges to \(-\infty\), proceed as follows. By Riemann’s theorem, choose for each \(k \in \mathbb{N}\) a set \(A_k\) such that \(\sum f_{A_k}\) converges to \(-k\). By 2, each \(A_k\) has density \(x\). Then recursively choose

\(1 < n_1 < n_2 < \cdots < n_m < \cdots\)

and define finite sets

\(C_1, C_2, \ldots, C_m, \ldots\)

such that

1. For all \(n \geq n_1\), we have
   a) \(\|(\pi_{A_1}(n))/n - x\| < \frac{1}{10}\);
   b) \(\sum_{j \leq n} f_{A_1}(j) + 1 < \frac{1}{10}\);
   c) \(\|(\pi_{A_1}(n) - \pi_{A_2}(n))/n\| < \frac{1}{10}\);
   d) \(\sum_{j \leq n} (f_{A_1}(j) - f_{A_2}(j)) \geq 1 - \frac{1}{10}\);

and \(C_1 = A_1 \cap \{1, \ldots, n_1\}\).
2. For each $k$, $n_{k+1} > n_k$ is minimal such that for all $n \geq n_{k+1}$,
   (a) $|\sum_{y \leq n} f(C \cup \{A_i \setminus \{n_k\}\})(y) + (k + 1)| < (\frac{1}{10})^{k+1}$;
   (b) $|\sum_{y \leq n} f(C \cup \{A_i \setminus \{n_k\}\})(y) + (k + 1)| < (\frac{1}{10})^{k+1}$;
   (c) $|\sum_{y \leq n} f(C \cup \{A_i \setminus \{n_k\}\})(y) - \pi(C \cup \{A_i \setminus \{n_k\}\})(y)| < (\frac{1}{10})^{k+1}$
   for all $j \leq k + 2$.

   Finally put $B = \bigcup_{k < \infty} C_k$. Then $B$ has the desired properties. A similar
   argument shows the existence of a set $C$ with density $x$ such that $\sum f_C$
   diverges to $\infty$.

3 $\Rightarrow$ 4. Riemann's theorem and 2 imply 4. We show that 3 implies 2.
   Let $D$ be a set of natural numbers such that $\sum f_D$ converges. Choose $B$ and
   $C$ as in 3. First observe that $x \leq d_*(D)$, for otherwise we find a sequence $n_1 < n_2 < \cdots < n_k < \cdots$
   such that $\lim_{k \to \infty} ((\pi_B(n_k)) / n_k) = d_*(D) < x$. Then we have for all sufficiently large
   $k$ that $\pi_B(n_k) < \pi_B(n_k)$, and so $\sum_{j \leq n_k} f_D(j) \leq \sum_{j \leq n_k} f_B(j)$. This implies that
   $\sum f_D = -\infty$, contradicting the fact that $\sum f_D$ converges. Similarly, but using $C$ instead of $B$, we find
   that $d^*(D) \leq x$.

4 $\Rightarrow$ 1. Let $B$ and $C$ be sets as in 4. Then by the hypotheses of 4 and
   by the definition of $\omega(f)$, we have $x \in \omega(f)$. Moreover, by Lemma 8
   we have $\lim_{n \to \infty} \pi_C(n) - \pi_B(n) = \infty$. Thus, for all sufficiently large $n$ we
   have $\pi_B(n) < \pi_C(n)$. For each $n$ let $j_n$ be minimal such that $n - \pi_B(n) = j_n - \pi_C(j_n)$. Then $j_n \geq n$ for all sufficiently large $n$.

   Also define $b = \sum^* f_B$ and $c = \sum^* f_C$. Then for all but finitely many $n$ we have
   
   $\begin{align*}
   0 &\leq \frac{c - b}{2} < a_{\pi_C(n)} + \cdots + a_{\pi_C(j_n)} < a_{\pi_B(n)} \cdot (\pi_C(j_n) - \pi_B(n)). \tag{1}
   \end{align*}$

   Since we have for each $n$ $j_n/n = (j_n / (j_n - \pi_C(j_n))) \cdot ((n - \pi_B(n))/n)$, and
   since the right-hand side of this equation converges to 1, so does the left-hand side. This implies that the quotient
   $(\pi_C(j_n))/\pi_B(n)$ converges to 1. Writing the right-hand side of (1) as
   
   $\pi_B(n) \cdot a_{\pi_B(n)} \cdot \frac{\pi_C(j_n)}{\pi_B(n)} = 1$,

   we see from the fact that the liminf of this expression is a finite positive
   number and the limit of $(\pi_C(j_n))/\pi_B(n) - 1$ is zero that the limit of
   $\pi_B(n) \cdot a_{\pi_B(n)}$ is infinite.

COROLLARY 13. If $0 \leq x \leq 1$ is such that for each $A \subseteq \mathbb{N}$ with $d(A) = x$,
   $\sum f_A$ is convergent, then $0 < x < 1$ and there is a unique $y$ such that for all
   $A$ with $d(A) = x$, $\sum_{n < \infty} f_A(n) = y$. 

Proof. It is clear that there is an $A$ of zero density for which $\sum f_A$ diverges to $-\infty$ and a $B$ of density 1 for which $\sum f_B$ diverges to $\infty$. Thus, $0 < x < 1$. If there are sets $A$ and $B$ of density $x$ for which $\sum_{n<\infty} f_A(n) < \sum_{n<\infty} f_B(n)$, then by $4 \rightarrow 3$ of Theorem 12 there is a set $C$ of density $x$ for which $\sum f_C$ diverges to $\infty$; this contradicts our assumption about $x$. 

3. ON PRINGSHEIM II

Lemma 14. If there are sets $A$ and $B$ such that $0 < d(A)$, $d(B) < 1$, and $d(A) \neq d(B)$, and both $\sum f_A$ and $\sum f_B$ converge, then for any $C$ such that $d(C) = d(A)$, $\sum f_C$ converges to the same value as $\sum f_A$.

Proof. For definiteness, assume that $d(A) < d(B)$; the argument for the other case is similar. We show that $\sum f_C \leq \sum_{n<\infty} f_A(n) \leq \sum f_C$.

First we show that $\sum f_C \neq -\infty$: Our assumptions about $A$ and $B$ imply that for all sufficiently large $n$,

$$\sum_{x \leq n} f_B(x) - f_A(x) \geq (\pi_B(n) - \pi_A(n) - 1) \cdot (a_{\pi_B(n)} + b_{n-\pi_B(n)}).$$

Since $\sum f_B$, $\sum f_A$ as well as $(1 - (\pi_A(n)))/(\pi_B(n)) - 1/(\pi_B(n)) : n \in \mathbb{N})$ are convergent and the last sequence does not converge to 0, both $(n \cdot a_n : n \in \mathbb{N})$ and $(n \cdot b_n : n \in \mathbb{N})$ are bounded. If we had $\sum f_C = -\infty$, then the sequence $(\sum_{x \leq n} f_A(x) - f_C(x) : n \in \mathbb{N})$ would be unbounded. A typical large positive term from this sequence is of the form

$$(a_{\pi_A(n)+1} + \cdots + a_{\pi_A(n)}) + (b_{n-\pi_A(n)+1} + \cdots + b_{n-\pi_A(n)}).$$

Because of the signwise monotonicity of $f$, such a term is no larger than $(\pi_A(n) - \pi_C(n)) \cdot (a_{\pi_A(n)} + b_{n-\pi_A(n)})$. Since $d(A) = d(C)$, the sequence $((\pi_A(n)/\pi_C(n)) - 1 : n \in \mathbb{N})$ converges to 0, meaning that $(\pi_C(n) \cdot (a_{\pi_A(n)} + b_{n-\pi_A(n)})) : n \in \mathbb{N})$ is an unbounded sequence. This implies that the $n \cdot a_n$’s or the $n \cdot b_n$’s form an unbounded sequence, a contradiction.

Since $d(C) < d(B)$, Lemma 5 implies that $\sum f_C \leq \sum_{n<\infty} f_B(n) < \infty$.

Second, $\sum f_C \leq \sum_{n \leq C} f_A(n)$ for suppose the contrary. Then there are infinitely many $n$ for which $\sum_{x \leq n} (f_C(x) - f_A(x))$ is larger than a fixed $\epsilon > 0$. For these infinitely many $n$ we have

$$\sum_{x \leq n} (f_C(x) - f_A(x)) \leq \left(\frac{\pi_C(n)}{\pi_A(n)} - 1\right) \cdot \pi_A(n) \cdot (a_{\pi_A(n)} + b_{n-\pi_A(n)}).$$

This together with the fact that $d(A) = d(C)$ implies that $(n \cdot a_n : n \in \mathbb{N})$ or $(n \cdot b_n : n \in \mathbb{N})$ is unbounded, a contradiction.

An analogous argument shows that $\sum_{n<\infty} f_A(n) \leq \sum f_C$. 

COROLLARY 15. If $0 < x < y < 1$, and $x$ and $y$ are elements of $\omega(f)$, then they are elements of $\sigma(f)$.

THEOREM 16. For $f$ signwise monotonic and $x$ a real number, the following statements are equivalent:

1. There are $s < t$ in $\omega(f)$ such that $\phi_j(s) = x = \phi_j(t)$.
2. $(n \cdot a_n : n < \infty)$ converges to $0$ and for an $A \subseteq \mathbb{N}$ with $0 < d(A) < 1$, $\sum_{n<\infty} f_A(n) = x$.
3. $\omega(f) \supseteq (0, 1)$ and $\phi_j$ is constant of value $x$ on $(0, 1)$.
4. There are $A_0, A_1 \subseteq \mathbb{N}$ such that $d(A_0) = 0$, $d(A_1) = 1$, and $\sum_{n<\infty} f_{A_1}(n) = x = \sum_{n<\infty} f_{A_0}(n)$.
5. For each $A \subseteq \mathbb{N}$ with $0 < d_*(A) \leq d^*(A) < 1$, $\sum_{n<\infty} f_A(n) = x$.

Proof. 1 $\Rightarrow$ 2. Assume 1. We must show that $\lim_{n \to \infty} n \cdot a_n = 0$. Assume that $A$ and $B$ as in 1 are such that $d(A) < d(B)$. Then we have, because of the signwise monotonicity of $f$, that for all sufficiently large $n$, $\sum_{x \leq n} (f_B(x) - f_A(x)) \geq \frac{\pi_A(n)}{n} - \frac{\pi_B(n)}{n} \cdot n \cdot (a_n + b_n) \geq 0$.

Since $\sum f_A$ and $\sum f_B$ have the same limit, the quantity in the middle converges to $0$, i.e.,

$$0 = (d(B) - d(A)) \cdot \lim_{n \to \infty} (n \cdot (a_n + b_n)).$$

Since $d(A) < d(B)$, this in particular implies that $\lim_{n \to \infty} n \cdot a_n = 0$.

2 $\Rightarrow$ 3. Let $A$ be as in 2 and consider a set $B$ with $d(B) \neq d(A)$ and $0 < d(A) < 1$. We may assume that $d(A) < d(B)$; the argument for the other case is similar. Then for all but finitely many $n$ we have $\pi_A(n) < \pi_B(n)$. Since $A$ and $B$ both have density and $(\pi_B(n) - \pi_A(n)) \cdot a_{\pi_A(n)}$ can be written as $((\pi_B(n)/\pi_A(n)) - 1) \cdot \pi_A(n) \cdot a_{\pi_A(n)}$, we see that $((\pi_B(n) - \pi_A(n)) \cdot a_{\pi_A(n)} : n \in \mathbb{N})$ converges to $0$. By Theorem 3, $\sum f_A$ and $\sum f_B$ converge to the same limit. With this case done, the case where $d(A) = d(B)$ follows from it and Lemma 14.

3 $\Rightarrow$ 4. To find a set $A_0$ of density $0$ for which $\sum f_{A_0}$ converges to $x$, proceed as follows: Put $C_0 = \mathbb{N}$ and, for each $n$, put $C_n = \{k \cdot (n + 1) : k \in C_{n-1}\}$. Then $C_0 \supset C_1 \supset \cdots \supset C_n \supset \cdots$ for each $n$, $d(C_n) = 1/(n + 1)!$, and $\sum f_{C_n}$ converges to $x$.

Recursively choose $1 < n_1 < n_2 < \cdots < n_k < \cdots$ and define finite sets $B_1, B_2, \ldots, B_k, \ldots$ so that:

1. $n_1 > 1$ is minimal such that for all $n \geq n_1$,
   - $(a(n)/n - 1/2] < 1/10$,
   - $|\sum_{y \leq n} f_{C}(y) - x| < 1/10$;
and $B_1$ is $C_1 \cap \{1, \ldots, n_1\}$.

2. $n_{k+1} > n_k$ is minimal such that for each $n \geq n_{k+1}$,
   
   $\left| \left( \prod (B_{1} \cup \ldots \cup B_{k}) \cdot \{ \pi_{n_{k+1}} \} \right)^{n} / \left( 1 / (k+2) \right) \right| < \left( \frac{1}{n} \right)^{k+1}$;

   $\left| \sum_{y<\infty} f_{A_{j}}(y) - f_{A}(y) \right| < \left( \frac{1}{n} \right)^{k+1}$;

   $\left| \sum_{y<\infty} f_{A_{j}}(y) \right| < \left( \frac{1}{n} \right)^{k+1}$;

   $\left| \sum_{y<\infty} f_{A_{j}}(y) \right| < \left( \frac{1}{n} \right)^{k+1}$;

   and $B_{k+1}$ is $C_{k+1} \cap \{ n_{k} + 1, \ldots, n_{k+1} \}$.

Put $A_0 = \bigcup_{k<\infty} B_k$. Then $d(A_0) = 0$ and $\sum f_{A_k}$ converges to $x$.

An analogous argument yields a set $A_1$ of density 1 such that $\sum f_{A_1}$ converges to $x$.

4 $\Rightarrow$ 5. Let $A_0$ and $A_1$ be as in 4. Consider a subset $A$ of $\mathbb{N}$ with $0 < d(A) \leq d^*(A) < 1$. Choose $N$ so large that for all $n \geq N$ we have $\pi_{A_0}(n) \leq \pi_A(n) \leq \pi_{A_1}(n)$.

For all such $n$, we then have

$$\sum_{j \leq n} f_{A_{0}}(j) \leq \sum_{j \leq n} f_{A}(j) \leq \sum_{j \leq n} f_{A_{1}}(j).$$

Since both $\sum f_{A_0}$ and $\sum f_{A_1}$ converge to $x$ it follows that $\sum f_A$ converges to $x$.

5 $\Rightarrow$ 1. This implication is trivial. \(\blacksquare\)

Any of the five clauses of Theorem 16 implies that there is a set $A$ with lower density 0 and upper density 1 such that $\sum f_A$ converges to $x$.

4. THE PRINGSHEIM–SCHŁÖMILCH THEOREM

The Pringsheim–Schlömilch theorem implies that if $f$ is an alternating sequence that is absolutely monotonic and converges to 0, and if $\sum f$ converges conditionally, then $\sigma(f) = (0, 1)$ and for each $x$ in $\sigma(f)$, $\phi_f(x) = \phi_f(\frac{1}{2}) + (\lim_{n \rightarrow \infty} n \cdot a_n) \cdot \ln(d(A)/1 - d(A))$. This is a special case of the implication (5) $\Rightarrow$ (4) in the following theorem.

**Theorem 17.** Assume that $f$ is signwise monotonic and $\sum f$ is conditionally convergent. Then the following are equivalent:

1. $\omega(f)$ is dense in some interval.
2. $\sigma(f)$ has nonempty interior.
3. $\sigma(f) = (0, 1)$.
4. $\lim n \cdot a_n$ exists and for each $x, y \in (0, 1)$, $\phi_f(x) = \phi_f(y) + (\lim n \cdot a_n) \ln(x + (1 - y)/y \cdot (1 - x))$.
5. $\omega(f) \cap (0, 1) \neq \emptyset$ and $\lim n \cdot a_n$ exists.
Proof: 1 $\Rightarrow$ 2. Let $0 < \alpha < \beta < 1$ be such that $\omega(f)$ is dense in $(\alpha, \beta)$. According to Lemma 14 it would be enough to show that $\omega(f)$ has nonempty interior. By Lemma 14, $\omega(f) \cap (\alpha, \beta) \subseteq \sigma(f)$, and by Corollary 13, $\phi_f$ is well defined on this set. Define $a = \inf\{\phi_f(x) : x \in \omega(f) \cap (\alpha, \beta)\}$ and define $b = \sup\{\phi_f(x) : x \in \omega(f) \cap (\alpha, \beta)\}$. When $(a, b) = \emptyset$, $a = b$ and Theorem 16 implies that $[0, 1] = \omega(f)$. Thus, assume that $(a, b) \neq \emptyset$. We show that $(a, b) \subseteq \text{range}(\phi_f)$.

Consider a $c \in (a, b)$. By Riemann’s rearrangement theorem, choose a $C \subseteq \mathbb{N}$ such that $\sum f_C$ converges to $c$. We show that $C$ has density and $\alpha < d(C) < \beta$. This gives $\phi_f(d(C)) = c$.

First $\alpha < d_*(C)$: For assume the contrary. Choose $n_1 < n_2 < \cdots < n_k < \cdots$ such that $\lim(n_k/n_k) = d_*(C)$. Since $a < c$, choose an $x \in \omega(f) \cap (\alpha, \beta)$ with $a < \phi_f(x) < c$ and let $X \subseteq \mathbb{N}$ be a set of density $x$. By Corollary 13, $\sum f_X$ converges to $\phi_f(x) (< c)$. Since we have $d_*(C) \leq \alpha < x = d(X)$, we see that for all but finitely many $k$, $\pi_{\sigma_\omega}(n_k) = 0$. Then for all but finitely many $k$, $\sum f_C(y) \leq \sum f_X(y)$. Thus

$$c = \sum_{y < \infty} f_C(y) \leq \sum_{y < \infty} f_X(y) = \phi_f(x) < c,$$

a contradiction.

Next, $\beta > d^*(C)$: For suppose on the contrary that $d^*(C) \geq \beta$. Then arguing as before, we find an $x$ in $\omega(f) \cap (\alpha, \beta)$ with $c < \phi_f(x) < b$ and a set $X \subseteq \mathbb{N}$ of density $x$ for which $\sum f_X$ converges to $\phi_f(x)$. Since $d^*(C) \geq \beta$, we then get the contradiction that $c < \phi_f(x) \leq c$.

Finally, we show that $d_*(C) = d^*(C)$: For assume the contrary, and choose $x$ and $y$ in $\omega(f)$ with

$$\alpha < d_*(C) < y < x < d^*(C) \leq \beta.$$

Since we have $a < b$, Lemma 5 and Theorem 16 imply that $\phi_f(x) < \phi_f(y)$. Let $X$ and $Y$ be subsets of $\mathbb{N}$ that respectively, have densities $x$ and $y$. Choose $1 < m_1 < n_1 < \cdots < m_k < n_k < \cdots$ such that $\lim_{k \to \infty}(\pi_{\sigma_\omega}(m_k)/m_k) = d_*(C) < d^*(C) = \lim_{k \to \infty}(\pi_{\sigma_\omega}(n_k)/n_k)$. Then for all sufficiently large $k$,

$$\sum_{y \leq n_k} f_C(y) - \sum_{y \leq m_k} f_C(y) = \left(\sum_{y \leq n_k} f_C(y) - f_Y(y)\right) + \sum_{y \leq n_k} f_Y(y)
+ \left(\sum_{t \leq m_k} f_X(t) - f_C(t)\right) - \sum_{y \leq m_k} f_X(y)
\geq \sum_{y \leq n_k} f_Y(y) - \sum_{y \leq m_k} f_X(y)
\geq \frac{1}{2} \left(\sum_{y \leq \infty} f_Y(y) - \sum_{y \leq \infty} f_X(y)\right) > 0.$$
This implies that $\sum f_C - \sum f_C > 0$, contradicting the fact that $\sum f_C$ is convergent.

Next we show that $(\alpha, \beta) \subseteq \omega(f)$. Consider an $x \in (\alpha, \beta)$. For each $n$ choose $x_n$ and $y_n$ in $(\alpha, \beta) \cap \omega(f)$ such that $m < n$ implies $x_m < x_n < x < y_n < y_m$ and $\lim_{m \to \infty} x_m = x = \lim_{m \to \infty} y_n$. Lemma 5 and Theorem 16 imply that for $m < n$, $a \leq \phi_f(x_m) < \phi_f(x_n) < \phi_f(y_n) < \phi_f(y_m) \leq b$. Let $\xi$ be the supremum of $\{\phi_f(x_n) : n < \infty\}$. Then we have $a < \xi < b$. By the preceding paragraphs, fix a $C \subseteq \mathbb{N}$ with $\sum f_C$ converging to $\xi$. As we have seen, $C$ has density. Applying Lemma 5 and Theorem 16 again, we see that for all $n$, $x_n < d(C) < y_n$. This implies that $x = d(C)$, and so $x \in \omega(f)$.

2 $\Rightarrow$ 3. Let $\alpha < \beta$ be such that $(\alpha, \beta) \subseteq \sigma(f)$. We claim first that if $p_1$, $q_1$, $p_2$, and $q_2$ are positive integers such that

$$
\alpha < \frac{p_1}{p_1 + q_1} < \frac{p_2}{p_2 + q_2} < \beta,
$$

then for each $m$ there are sets $A_m$ and $B_m$ such that

1. $d(A_m) = 1/(1 + (p_2 \cdot q_1/p_1 \cdot q_2)^m \cdot (q_1/p_1))$ and $\sum f_{A_m}$ converges.
2. $d(B_m) = 1/(1 + (p_1 \cdot q_2/p_2 \cdot q_1)^m \cdot (q_2/p_2))$ and $\sum f_{B_m}$ converges.

To begin, by Theorem 16 we may assume that $\sum f_A$ and $\sum f_B$ do not have the same limit. By Corollary 13 we may assume that $A$ is such that in each interval $[(p_1 + q_1) \cdot n + 1, \ldots, (p_1 + q_1) \cdot (n + 1)] \subseteq \mathbb{N}$, $A$ contains exactly the first $p_1$ elements. $B$ may be assumed to have similar structure, where $p_1$ and $q_1$ are now replaced by $p_2$ and $q_2$. Observe that $p_1/(p_1 + q_1) < p_2/(p_2 + q_2)$ implies that $q_2/p_2 < q_1/p_1$, and thus $p_1 \cdot q_2 < q_1 \cdot p_2$.

For each $n$, define

$$
s_n = \sum_{j \leq n} f_A(j),
$$

and

$$
u_n = \sum_{j \leq n} f_B(j).
$$

Then for each $k$ and $n$, we have

$$
s_{(p_1 + q_1) \cdot k \cdot n} = (a_1 + \cdots + a_{p_1 \cdot k \cdot n}) - (b_1 + \cdots + b_{q_1 \cdot k \cdot n})
$$

and

$$
u_{(p_2 + q_2) \cdot k \cdot n} = (a_1 + \cdots + a_{p_2 \cdot k \cdot n}) - (b_1 + \cdots + b_{q_2 \cdot k \cdot n}).
$$

Since for each $k$, $u_{k \cdot (p_1 + q_1) \cdot n} - s_{k \cdot (p_1 + q_1) \cdot n}$ converges to $\sum_{j \leq n} f_B(j) - \sum_{j \leq n} f_A(j)$, we see that for each $k$,

$$
\lim_{n \to \infty} (b_{k \cdot p_1 \cdot q_2 \cdot n + 1} + \cdots + b_{k \cdot p_2 \cdot q_1 \cdot n}) = \sum_{j \leq n} f_B(j) - f_A(j).
$$
By considering $u_{k-q_1(p_2+q_2)} - s_{k-q_2(p_1+q_1)}$, we see that

$$\lim_{n \to \infty} \left( a_{k-p_1,q_2(n+1)} + \cdots + a_{k+p_2,q_1} \right) = \sum_{j<\infty} f_B(j) - f_A(j).$$

(3)

By applying (2) and (3) for appropriately chosen values of $k$, we then show by induction on $m$ that for each $m$,

$$\lim_{n \to \infty} \left( a_1 + \cdots + a_{p_2^m+q_1^m} n \right) = \left( (m+1) \cdot \sum_{j<\infty} f_B(j) - m \cdot \sum_{j<\infty} f_A(j) \right).$$

The terms listed in this sequence form a subsequence of the partial sums of $\sum f_{B_m}$, where $B_m$ is the set that contains in each interval of the form

$$\left[ n \cdot \left( (p_2^m)^{m+1} q_1^m + p_1^m q_2^m \right) + 1, \ldots, (n+1) \right]$$

the first $p_2^m q_1^m$ elements.

As such, $B_m$ has density

$$d(B_m) = \frac{1}{1 + \left( \frac{q_2}{p_2} \right)^m \cdot \left( \frac{q_1}{p_1} \right)^m}.$$

To see that the convergence of the particular sequence of partial sums of $\sum f_{B_m}$ implies the convergence of $\sum f_{B_m}$, consider a large $n$. Then $n = i \mod p_2^{m+1} q_1^m + p_1^m q_2^m$, and so if we let $t_j$ denote the $j$th partial sum of $\sum f_{B_m}$, we see that

$$t_n = t_r(p_2^{m+1} q_1^m + p_1^m q_2^m) + i$$

more terms;

since $i < p_2^m q_1^m + p_1^m q_2^m$, these $i$ terms converge to 0 as $n$ diverges to $\infty$.

Similarly, we find a set $A_m$ of density

$$d(A_m) = \frac{1}{1 + \left( \frac{p_2}{p_1} \right)^m \cdot \left( \frac{q_2}{q_1} \right)^m}$$

for which $\sum f_{A_m}$ is convergent.

By Lemma 10 we have a dense subset of $[0,1]$ contained in $\omega(f)$. By the argument in the proof of $1 \Rightarrow 2$, we see that $(0,1) \subseteq \sigma(f)$. 

3 \implies 4. First we show that \( \lim_{n \to \infty} n \cdot a_n \) exists. For each \( n \) choose 0 < \( i_n < \frac{1}{2} < j_n < 1 \) such that \( |j_n - i_n| < \left( \frac{1}{2} \right)^n \). For each \( n \) choose \( A_n \) and \( B_n \) such that \( d(A_n) = i_n \) and \( d(B_n) = j_n \). Then by 3, \( \sum f_{A_n} \) and \( \sum f_{B_n} \) converge. Applying Lemma 11 we see that for each \( k \),

\[
\limsup_n n \cdot a_n \leq \frac{j_k \cdot (1 - i_k)}{i_k \cdot (1 - j_k)} \liminf_n n \cdot a_n.
\]

Since \( \lim_{k \to \infty} (j_k \cdot (1 - i_k)/i_k \cdot (1 - j_k)) = 1 \), we see that \( \limsup_n n \cdot a_n \leq \liminf_n n \cdot a_n \). This shows that \( n \cdot a_n \) converges. If \( \lim_{n \to \infty} n \cdot a_n = 0 \), then we are done, by Theorem 16. Thus we may assume that \( \lim_{n \to \infty} n \cdot a_n = L \neq 0 \).

To prove the remaining part of 4, consider \( x, y \in (0, 1) \). We may assume that \( x < y \). Choose sets \( X \) and \( Y \) with respective densities \( x \) and \( y \). We know that \( \sum f_X \) and \( \sum f_Y \) both converge. For all sufficiently large \( n \) choose the least \( m_n \geq n \) such that

\[
m_n - \pi_X(m_n) = n - \pi_Y(n).
\]

Then \( \lim_{n \to \infty} (n/m_n) = (1 - d(X)/1 - d(Y)) \). For all sufficiently large \( n \), consider

\[
\sum_{j \leq m_n} f_X(j) - \sum_{i \leq n} f_Y(i) = (a_{\pi_Y(n)+1} + \cdots + a_{\pi_X(m_n)}).
\]

Let \( \epsilon > 0 \) be given. Then for all sufficiently large \( n \),

\[
L - \epsilon < n \cdot a_n < L + \epsilon,
\]

whence from (4) we have

\[
(L - \epsilon) \cdot \left( \frac{1}{\pi_Y(n)+1} + \cdots + \frac{1}{\pi_X(m_n)} \right)
\]

\[
\leq \sum_{j \leq m_n} f_X(j) - \sum_{i \leq n} f_Y(i)
\]

\[
\leq (L + \epsilon) \cdot \left( \frac{1}{\pi_Y(n)+1} + \cdots + \frac{1}{\pi_X(m_n)} \right).
\]

The left-hand side of these inequalities converges to \( (L - \epsilon) \cdot \ln(d(X)) \cdot (1 - d(Y)) / d(Y) \cdot (1 - d(X)) \) because we have

\[
\frac{1}{\pi_Y(n)+1} + \cdots + \frac{1}{\pi_X(m_n)} \leq \int_{\pi_Y(n)}^{\pi_X(m_n)} \frac{1}{x} \, dx
\]

\[
\leq \frac{1}{\pi_Y(n)} + \cdots + \frac{1}{\pi_X(m_n) - 1}.
\]

The quantity in the middle is \( \ln(\pi_X(m_n)/\pi_Y(n)) \), and its upper bound and lower bound converge to each other. Moreover, \( n/m_n \) converges to
\((1 - d(X))/(1 - d(Y))\). Similarly, the right-hand side converges to \((L + \varepsilon) \cdot \ln(d(X)(1 - d(Y))/d(Y)(1 - d(X)))\). Then (5) implies that for each \(\varepsilon > 0\), for sufficiently large \(n\), the quantity in (4) is bounded below by \((L - \varepsilon) \cdot \ln(d(X)(1 - d(Y))/d(Y)(1 - d(X)))\) and above by \((L + \varepsilon) \cdot \ln(d(X)(1 - d(Y))/d(Y)(1 - d(X)))\). In particular, we find that

\[
\sum_{j < \infty} f_X(j) = \sum_{j < \infty} f_Y(j) + L \cdot \ln \left( \frac{d(X)(1 - d(Y))}{d(Y)(1 - d(X))} \right).
\]

4 \(\Rightarrow\) 5. This implication is clear.

4 \(\Rightarrow\) 1. This implication is clear.

5 \(\Rightarrow\) 4. Assume 5. If \(\lim_{n \to \infty} n \cdot a_n = 0\), then by Theorem 16 there is nothing to prove. We may assume that \(\lim_{n \to \infty} n \cdot a_n = L \neq 0\). Also, choose \(x \in \omega(f) \cap (0, 1)\) and let \(A\) be a set with \(d(A) = x\) and \(\sum f_A\) convergent. Consider any set \(B\) with density in \((0, 1)\), say \(d(B) = y\). At first, assume that \(x < y\). Then for sufficiently large \(n\) we have \(\pi_A(n) < \pi_B(n)\). For all sufficiently large \(n\) choose the least \(k_n\) such that \(n - \pi_B(n) = k_n - \pi_A(k_n)\).

Notice that for such \(n\), \(k_n \leq n\). From the definitions it follows that \(\frac{k_n}{n}\) converges to \((1 - d(B))/(1 - d(A))\).

Then for all sufficiently large \(n\) we have:

\[
a_{\pi_A(k_n)} \cdot (\pi_B(n) - \pi_A(k_n)) = \pi_A(k_n) \cdot a_{\pi_A(k_n)} \cdot \left( \frac{\pi_B(n)}{\pi_A(k_n)} - 1 \right).
\]

The factor \(a_{\pi_A(k_n)} \cdot \pi_A(k_n)\) converges to \(L\) and the factor \((\pi_B(n))/(\pi_A(k_n)) - 1\) converges to \((d(B) \cdot (1 - d(A))/d(A) \cdot (1 - d(B))) - 1\). Thus, \(a_{\pi_A(k_n)} \cdot (\pi_B(n) - \pi_A(k_n))\) converges to \(L \cdot (d(B) \cdot (1 - d(A))/d(A) \cdot (1 - d(B))) - 1\). By Theorem 4, \(\sum f_B\) is convergent and

\[
\sum_{n < \infty} f_B(n) = \sum_{n < \infty} f_A(n) + L \cdot \ln \left( 1 + L \cdot \left( \frac{d(B) \cdot (1 - d(A))}{d(A) \cdot (1 - d(B))} - 1 \right) \right)/L.
\]

Now 4 follows. 

For a time I thought that if (for signwise monotonic \(f\)) \(\sigma(f)\) is nonempty, then \(\sigma(f)\) is the open unit interval. Professor David Fremlin pointed out the following simple counterexample for this: One could have a signwise monotonic \(f\) for which \(\sigma(f)\) consists of a single point. Define \(a_i = b_i = 1/(n+1)!\) for \(n! < i \leq (n+1)!\). Then the corresponding series \(\sum f\) with \(m\)th positive term \(a_m\) and \(m\)th negative term \(-b_m\) is conditionally convergent and has \(\omega(f) = \sigma(f) = \{1/2\}\).
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