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J. Math. Anal. Appl. 306 (2005) 72–82

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Traveling wave solutions of the Degasperis–Procesi equation

Jonatan Lenells

Department of Mathematics, Lund University, PO Box 118, 22100 Lund, Sweden

Received 10 August 2004
Available online 27 January 2005
Submitted by P. Broadbridge

Abstract

We classify all weak traveling wave solutions of the Degasperis–Procesi equation. In addition to smooth and peaked solutions, the equation is shown to admit more exotic traveling waves such as cuspons, stumpons, and composite waves.

Keywords: Traveling waves; Degasperis–Procesi equation

1. Introduction

The Degasperis–Procesi equation

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \quad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

was discovered in [10] as one out of three integrable equations within a certain family of third-order nonlinear dispersive PDE's, the other two being the well-known KdV,

$$u_t - 6uu_x + u_{xxx} = 0, \quad x \in \mathbb{R},\ t > 0, \tag{1.2}$$

and Camassa–Holm [2],

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad x \in \mathbb{R},\ t > 0, \tag{1.3}$$

E-mail address: jonatan@maths.lth.se.

0022-247X/$ – see front matter © 2004 Published by Elsevier Inc.
doi:10.1016/j.jmaa.2004.11.038

models for shallow water waves. Just like (1.2)–(1.3) the Degasperis–Procesi equation has a Lax pair formulation and a bi-Hamiltonian structure leading to an infinite number of conservation laws [11]. While all solutions with initial data $u(0,\cdot) \in H^3(\mathbb{R})$ exist globally for (1.2) cf. [3], both (1.1) and (1.3) have global solutions as well as smooth solutions that blow up in finite time (see [4–6,20,21]). Another interesting feature of (1.1) is the existence of multi-peakon solutions (an explicit formula for the two-peakon was presented in [10]; other multi-peakons can also be obtained [17]) in analogy to the peakon solutions of (1.3) (see [2,9,14,15]). Despite their similarity, Eqs. (1.1) and (1.3) have different properties. While (1.3) is known to be integrable for a large class of initial data via the inverse scattering procedure [1,5,8,13], for (1.1) only the existence of an isospectral problem (very different from that known for the Camassa–Holm equation) has been established [10,11]. Moreover, (1.3) is a re-expression of geodesic flow [4,7,18] whereas no such geometric interpretation is valid for (1.1).

This paper deals with traveling wave solutions, $u(x,t) = \varphi(x - ct)$, $c \in \mathbb{R}$, of Eq. (1.1). Using a natural weak formulation, we will classify all weak traveling wave solutions of (1.1). Just like for the Camassa–Holm equation [16], there exists a multitude of peculiar singular waves: cuspons, composite waves, and stumpons. The composite waves are obtained by combining cuspons and peakons into new traveling waves—see (i) of Fig. 1. An interesting class of waves—called stumpons because of their shape—is obtained by inserting intervals where φ equals a constant at the crests of suitable cusped waves—see (j) of Fig. 1.

Traveling waves of (1.1) were studied in [19]. The main contributions of the present paper are as follows:

- **New waves**—New kinds of traveling waves are obtained: composite waves and stumpons.
- **Classification of all traveling waves**—Using a natural and precise definition of a weak traveling wave solution, we classify all bounded traveling wave solutions of (1.1).[1]
- **Mathematical rigor**—We explain exactly in what sense the cusped, composite, and stumped waves are solutions.

The rigorous derivation is not only of theoretical interest, but is qualitatively important for the construction of composite waves. Indeed, peakons and cuspons cannot be joined arbitrarily into a composite wave, but have to satisfy certain mathematical conditions in order for the composed solution to be a traveling wave. This agrees with numerical experiments: composite weak traveling waves travel unchanged, whereas composite waves that are not mathematically weak solutions disintegrate even if they are formal traveling wave solutions[2] (see [12] for a numerical study in the case of the Camassa–Holm equation).

[1] Note that we only consider single-valued traveling waves. Hence, the multi-valued loop-like solutions presented in [19] fall outside the scope of this paper. The loop-like solutions are composite waves made up of an anticusped wave segment glued to two overlapping cusped solitary-wave segments. It remains to be investigated in what sense multi-valued waves are solutions.

[2] I.e., they satisfy the ODE obtained by formal integration from (1.1).

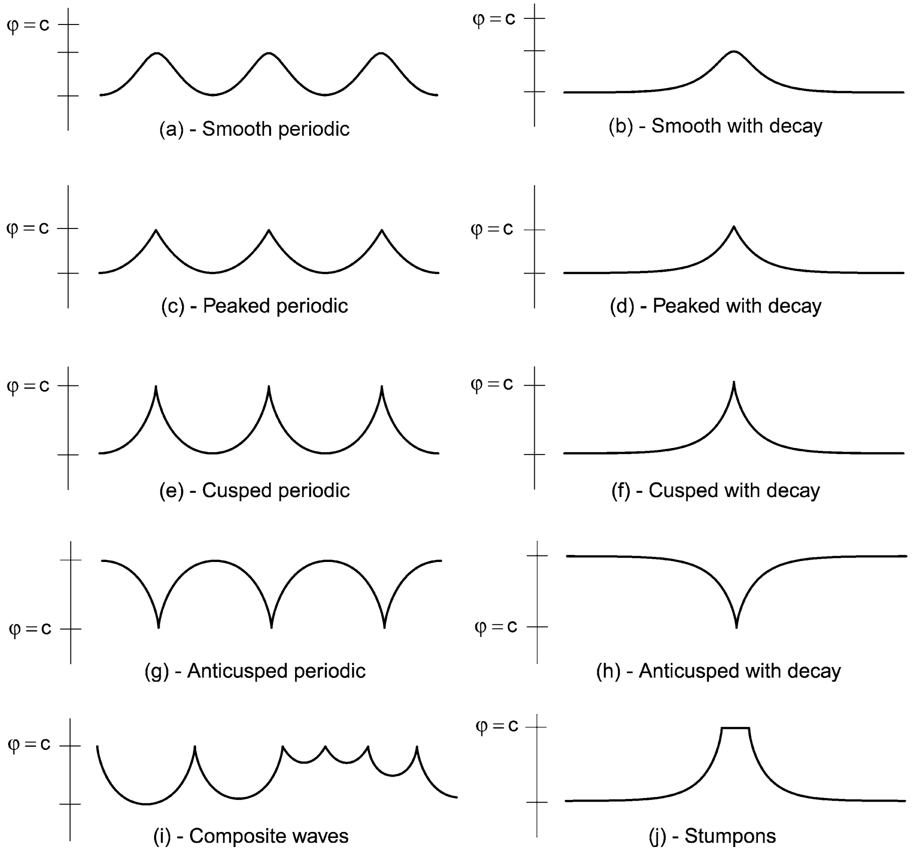

Fig. 1. The different kinds of traveling wave solutions of (1.1).

We refer to [19] for mathematical expressions for the traveling waves in terms of elliptic integrals.

In Section 2 we give the precise definition of a weak traveling wave solution of (1.1). Section 3 contains the main theorem: a classification of the traveling wave solutions of the Degasperis–Procesi equation. The proof is presented in Section 4. In Appendix A we review some notation.

2. Weak formulation

For a traveling wave $u(x,t) = \varphi(x - ct)$, Eq. (1.1) takes the form

$$-c\varphi_x + c\varphi_{xxx} + 4\varphi\varphi_x = 3\varphi_x\varphi_{xx} + \varphi\varphi_{xxx}. \tag{2.1}$$

Integration yields

$$-c\varphi + c\varphi_{xx} + 2\varphi^2 = \varphi\varphi_{xx} + \varphi_x^2 + a, \tag{2.2}$$

for some constant $a \in \mathbb{R}$. We rewrite this as

$$-c\varphi + 2\varphi^2 = \frac{1}{2}\big((\varphi - c)^2\big)_{xx} + a. \tag{2.3}$$

Equation (2.3) makes sense for all $\varphi \in H^1_{\mathrm{loc}}(\mathbb{R})$. The following definition is therefore natural.

Definition 1. A function $\varphi \in H^1_{\mathrm{loc}}(\mathbb{R})$ is a *traveling wave* of the Degasperis–Procesi equation if φ satisfies (2.3) in distribution sense for some $a \in \mathbb{R}$.

3. Classification of traveling waves

If $\varphi(x)$ is a traveling wave of (1.1), then also $x \mapsto -\varphi(-x)$ is a traveling wave of (1.1) with c replaced by $-c$. Therefore, we will only consider waves traveling with a positive speed $c \geqslant 0$.

Theorem 1 classifies all bounded traveling waves $\varphi(x - ct)$, $\varphi \in H^1_{\mathrm{loc}}(\mathbb{R})$, of (1.1).

Theorem 1. *Let $c > 0$. All traveling waves $\varphi(x - ct)$ of* (1.1) *are smooth except at points where $\varphi = c$. The waves are parametrized by $a \in \mathbb{R}$ as follows*:

(1) *For $a \leqslant -\frac{c^2}{8}$ there are no bounded solutions of* (1.1).
(2) *For each $a \in \big(-\frac{c^2}{8}, 0\big)$, Eq.* (1.1) *admits a one-parameter group of smooth periodic traveling waves and one smooth traveling wave with decay.*
(3) *For $a = 0$, Eq.* (1.1) *admits a one-parameter group of smooth periodic traveling waves and one peaked solitary wave.*
(4) *For each $a \in (0, c^2)$, Eq.* (1.1) *admits a one-parameter group of smooth periodic traveling waves, one peaked periodic traveling wave, a one-parameter group of cusped periodic traveling waves, and one cusped traveling wave with decay.*
(5) *For $a = c^2$, Eq.* (1.1) *admits a one-parameter group of cusped traveling waves and one cusped traveling wave with decay.*
(6) *For each $a > c^2$, Eq.* (1.1) *admits a one-parameter group of cusped traveling waves, one cusped traveling wave with decay, a one-parameter group of anticusped traveling waves, and one anticusped traveling wave with decay.*
(7) (Composite waves) *Any countable number of cuspons, anticuspons, and peakons from the categories* (1)–(6) *corresponding to the same value of a may be joined at points where $\varphi = c$ to form a composite wave φ–see* (i) *of Fig.* 1. *If the Lebesgue measure $\mu(\varphi^{-1}(c)) = 0$, then φ is a traveling wave of* (1.1).
(8) (Stumpons) *For $a = c^2$ the composite waves are traveling waves of* (1.1) *even if $\mu(\varphi^{-1}(c)) > 0$. Consequently, these waves may contain intervals where $\varphi \equiv c$ (see* (j) *of Fig.* 1).

These are all bounded traveling waves of (1.1).

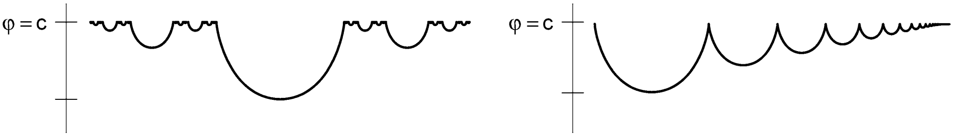

Fig. 2. Two composite traveling waves of (1.1) with a fractal appearance.

Since a countable number of waves are permitted in a composite wave, the wave-profiles can get very intricate (cf. [16]). Figure 2 shows two fractal-like traveling wave solutions of (1.1).

4. Proof of Theorem 1

Lemma 1. *Let $p(v)$ be a polynomial with real coefficients. Assume that $v \in H^1_{\mathrm{loc}}(\mathbb{R})$ satisfies*

$$\left(v^2\right)_{xx} = p(v) \quad \text{in } \mathcal{D}'(\mathbb{R}). \tag{4.1}$$

Then

$$v^k \in C^j(\mathbb{R}) \quad \text{for } k \geqslant 2^j. \tag{4.2}$$

Proof. From (4.1) we see that $(v^2)_{xx} \in L^1_{\mathrm{loc}}(\mathbb{R})$. Hence $(v^2)_x \in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$ so that $(v^2)_x$ is absolutely continuous and

$$v^2 \in C^1(\mathbb{R}).$$

Note that $v \in H^1_{\mathrm{loc}}(\mathbb{R}) \subset C(\mathbb{R})$. Moreover,

$$\begin{aligned}\left(v^k\right)_{xx} &= \left(\left(v^k\right)_x\right)_x = \frac{k}{2}\left(v^{k-2}\left(v^2\right)_x\right)_x = \frac{k}{2}\left(\left(v^{k-2}\right)_x\left(v^2\right)_x + v^{k-2}\left(v^2\right)_{xx}\right)\\ &= k(k-2)v^{k-2}v_x^2 + \frac{k}{2}v^{k-2}\left(v^2\right)_{xx}.\end{aligned}$$

Using (4.1), we infer that

$$\left(v^k\right)_{xx} = k(k-2)v^{k-2}v_x^2 + \frac{k}{2}v^{k-2}p(v). \tag{4.3}$$

For $k = 3$ the right-hand side of (4.3) is in $L^1_{\mathrm{loc}}(\mathbb{R})$. We deduce that

$$v^3 \in C^1(\mathbb{R}).$$

Next we notice that (4.3) implies

$$\left(v^k\right)_{xx} = \frac{k}{4}(k-2)v^{k-4}\left(\left(v^2\right)_x\right)^2 + \frac{k}{2}v^{k-2}p(v), \quad k \geqslant 4. \tag{4.4}$$

Since $v^2 \in C^1(\mathbb{R})$, it follows that

$$v^k \in C^2(\mathbb{R}), \quad k \geqslant 4. \tag{4.5}$$

Now let $k \geqslant 8$. We have $v^{k-2}p(v) \in C^2(\mathbb{R})$ by (4.5). Moreover, since $v^4, v^{k-4} \in C^2(\mathbb{R})$, we get

$$v^{k-2}v_x^2 = \frac{1}{4}\big(v^4\big)_x \frac{1}{k-4}\big(v^{k-4}\big)_x \in C^1(\mathbb{R}).$$

We conclude from (4.3) that

$$v^k \in C^3(\mathbb{R}), \quad k \geqslant 8.$$

Extending these arguments to higher values of k proves (4.2). □

We need the following two lemmas.

Lemma 2 [16]. *Let $f:\mathbb{R} \to \mathbb{R}$ be an absolutely continuous function. Then $f_x = 0$ a.e. on $f^{-1}(c)$ for any $c \in \mathbb{R}$.*

Lemma 3 [16]. *Let $f \in W^{2,1}_{\mathrm{loc}}(\mathbb{R})$. Then $f_{xx} = 0$ a.e. on $f^{-1}(c)$ for any $c \in \mathbb{R}$.*

Recall Definition 1 of a traveling wave solution: a function $\varphi \in H^1_{\mathrm{loc}}(\mathbb{R})$ is a traveling wave of (1.1) with speed c if the equation

$$-c\varphi + 2\varphi^2 = \frac{1}{2}\big((\varphi - c)^2\big)_{xx} + a \quad \text{in } \mathcal{D}'(\mathbb{R}), \tag{4.6}$$

holds for some $a \in \mathbb{R}$.

Lemma 4. *A function $\varphi \in H^1_{\mathrm{loc}}(\mathbb{R})$ is a traveling wave of* (1.1) *with speed c if and only if the following three statements hold*:

(TW1) *There are disjoint open intervals E_i, $i \geqslant 1$, and a closed set C such that $\mathbb{R}\backslash C = \bigcup_{i=1}^{\infty} E_i$, $\varphi \in C^\infty(E_i)$ for $i \geqslant 1$, $\varphi(x) \neq c$ for $x \in \bigcup_{i=1}^{\infty} E_i$, and $\varphi(x) = c$ for $x \in C$.*

(TW2) *There is an $a \in \mathbb{R}$ such that*

- (i) *For each $i \geqslant 1$, there exists $b_i \in \mathbb{R}$ such that*

$$\varphi_x^2 = F(\varphi) \quad \text{for } x \in E_i, \qquad \varphi \to c \quad \text{at any finite endpoint of } E_i, \tag{4.7}$$

and

$$F(\varphi) = \frac{(\varphi^2 - a)(\varphi - c)^2 + b_i}{(\varphi - c)^2}. \tag{4.8}$$

- (ii) *If C has strictly positive Lebesgue measure, $\mu(C) > 0$, then $a = c^2$.*

(TW3) $(\varphi - c)^2 \in W^{2,1}_{\mathrm{loc}}(\mathbb{R})$.

Proof. Assume $\varphi \in H^1_{\mathrm{loc}}(\mathbb{R})$ is a traveling wave of (1.1) with speed c. Applying Lemma 1 with $v = \varphi - c$ and $p(v) = 4\varphi^2 - 2c\varphi - 2a$, we get

$$(\varphi - c)^k \in C^j(\mathbb{R}) \quad \text{for } k \geqslant 2^j.$$

We conclude that φ is smooth except possibly at points in the boundary of the set $\varphi^{-1}(c)$. Since φ is continuous, $\varphi^{-1}(c)$ is a closed set. We let $C = \varphi^{-1}(c)$. Since every open set is a countable union of disjoint open intervals, there are disjoint open intervals E_i, $i \geqslant 1$, such that $\mathbb{R}\backslash C = \bigcup_{i=1}^{\infty} E_i$. We deduce, by construction, that (TW1) is satisfied.

To prove (TW2) we let E_i be one of these open intervals. Since φ is smooth in E_i, we deduce that (4.6) holds pointwise in E_i. Therefore, we may multiply by $((\varphi - c)^2)_x$ to get

$$\left(-c\varphi + 2\varphi^2\right)\left((\varphi - c)^2\right)_x = \frac{1}{2}\left((\varphi - c)^2\right)_x\left((\varphi - c)^2\right)_{xx} + a\left((\varphi - c)^2\right)_x, \quad x \in E_i.$$

The left-hand side can be rewritten as

$$2\varphi\varphi_x(\varphi - c)^2 + 2\varphi^2(\varphi - c)\varphi_x.$$

Hence integration gives

$$\varphi^2(\varphi - c)^2 - a(\varphi - c)^2 + b_i = \frac{1}{4}\left(\left((\varphi - c)^2\right)_x\right)^2, \quad x \in E_i,$$

for some constant of integration b_i. Expanding the derivative on the right-hand side and dividing by $(\varphi - c)^2$, we obtain $\varphi_x^2 = F(\varphi)$ with F as in (4.8). That $\varphi \to c$ at the finite endpoints of E_i follows from the continuity of φ and (TW1). This proves (i) of (TW2).

The left-hand side of (4.6) is in $L^1_{\mathrm{loc}}(\mathbb{R})$. Hence $((\varphi - c)^2)_{xx} \in L^1_{\mathrm{loc}}(\mathbb{R})$ so that (TW3) follows.

To show (ii) of (TW2), let us assume $\mu(C) > 0$. Since $\varphi \in H^1_{\mathrm{loc}}(\mathbb{R})$ and $(\varphi - c)^2 \in W^{2,1}_{\mathrm{loc}}(\mathbb{R})$, we deduce from Lemma 3 that

$$\left((\varphi - c)^2\right)_{xx} = 0 \quad \text{a.e. on } C. \tag{4.9}$$

In view of the fact that $(\varphi - c)^2 \in W^{2,1}_{\mathrm{loc}}(\mathbb{R})$, we see that (4.6) holds a.e. on $\mathbb{R}$, i.e.,

$$-c\varphi + 2\varphi^2 = \frac{1}{2}\left((\varphi - c)^2\right)_{xx} + a \quad \text{a.e. on } \mathbb{R}.$$

In particular, this equation holds a.e. on C, so that, by (4.9), we have

$$-c\varphi + 2\varphi^2 = a \quad \text{a.e. on } C. \tag{4.10}$$

Since $\mu(C) > 0$ and $\varphi \equiv c$ on C, we conclude that $a = c^2$. This shows that all traveling waves of the Degasperis–Procesi equation satisfy (TW1)–(TW3).

To prove the converse suppose φ satisfies (TW1)–(TW3). We will show that φ is a traveling wave of (1.1).

Let C and E_i, $i \geqslant 1$, be as in (TW1). Let a be as in (TW2). (4.7) gives us

$$\varphi^2(\varphi - c)^2 - a(\varphi - c)^2 + b_i = \frac{1}{4}\left(\left((\varphi - c)^2\right)_x\right)^2 \quad \text{on } \bigcup_{i=1}^{\infty} E_i.$$

Differentiating and dividing by $((\varphi - c)^2)_x$, we obtain

$$-c\varphi + 2\varphi^2 = \frac{1}{2}\left((\varphi - c)^2\right)_{xx} + a \quad \text{on } \bigcup_{i=1}^{\infty} E_i. \tag{4.11}$$

If $\mu(C) = 0$, then (4.11) says that

$$-c\varphi + 2\varphi^2 = \frac{1}{2}\big((\varphi - c)^2\big)_{xx} + a \quad \text{a.e. on } \mathbb{R}. \tag{4.12}$$

Since $((\varphi - c)^2)_{xx} \in L^1_{\mathrm{loc}}(\mathbb{R})$ by (TW3), (4.12) implies (4.6) so that φ is a traveling wave solution of (1.1).

It remains to show that (4.12) holds also in the case when $\mu(C) > 0$. Suppose $\mu(C) > 0$. Since $\varphi \in H^1_{\mathrm{loc}}(\mathbb{R})$ and $(\varphi - c)^2 \in W^{2,1}_{\mathrm{loc}}(\mathbb{R})$, we deduce from Lemma 3 that (4.9) holds. From (ii) of (TW2) we have $a = c^2$. Therefore, since $\varphi \equiv c$ on C, we get

$$-c\varphi + 2\varphi^2 = \frac{1}{2}\big((\varphi - c)^2\big)_{xx} + a \quad \text{a.e. on } C. \tag{4.13}$$

Together with (4.11) this gives (4.12). Hence φ is a traveling wave solution of (1.1) and the proof is complete. $\square$

We will show that the set of bounded functions satisfying (TW1)–(TW3) consists exactly of the waves stated in the Theorem 1.

Suppose φ satisfies (TW1)–(TW3). Then, by (TW1), φ consists of a countable number of smooth wave segments separated by a closed set C. By (TW2) each such wave segment solves

$$\begin{aligned} &\varphi_x^2 = F(\varphi) \quad \text{for } x \in E, \\ &F(\varphi) = \frac{(\varphi^2 - a)(\varphi - c)^2 + b}{(\varphi - c)^2}, \qquad \varphi \to c \quad \text{at any finite endpoint of } E, \end{aligned} \tag{4.14}$$

for some interval E and constants a, b.

Suppose we could find all solutions φ of (4.14) for different intervals E and different values of a and b. Then we can join solutions defined on intervals whose union is $\mathbb{R}\backslash C$ for some closed set C of measure zero. It is easy to see that the function, defined on $\mathbb{R}$, that we get, will satisfy (TW1) and (TW2) if and only if all wave segments satisfy (4.14) with the same a. Moreover, if we for $a = c^2$ allow $\mu(C) > 0$, this procedure will give us all functions satisfying (TW1) and (TW2). We will show that these functions belong to $H^1_{\mathrm{loc}}(\mathbb{R})$, that they satisfy (TW3), and that they are exactly the waves stated in Theorem 1. This will prove Theorem 1.

The analysis of Eq. (4.14), $\varphi_x^2 = F(\varphi)$, is based on the following observations (see [16] for a similar analysis presented with more details):

(1) Assume $F(\varphi)$ has a simple zero at $\varphi = m$, so that $F'(m) \neq 0$. Then a solution φ of (4.14) satisfies

$$\varphi_x^2 = (\varphi - m)F'(m) + O\big((\varphi - m)^2\big) \quad \text{as } \varphi \downarrow m.$$

Hence

$$\varphi(x) = m + \frac{1}{4}(x - x_0)^2 F'(m) + O\big((x - x_0)^4\big) \quad \text{as } x \to x_0, \tag{4.15}$$

where $\varphi(x_0) = m$.

(2) If $F(\varphi)$ instead has a double zero at m, so that $F'(m) = 0$, $F''(m) \neq 0$, we obtain

$$\varphi_x^2 = (\varphi - m)^2 F''(m) + O\big((\varphi - m)^3\big) \quad \text{as } \varphi \downarrow m.$$

We get

$$\varphi(x) - m \sim \alpha \exp\big(-x\sqrt{|F''(m)|}\big) \quad \text{as } x \to \infty, \tag{4.16}$$

for some constant α. Thus $\varphi \to m$ exponentially as $x \to \infty$.

(3) Suppose φ approaches a double pole $\varphi = c$ of F. Then, if $\varphi(x_0) = c$,

$$\varphi(x) - c = \alpha|x - x_0|^{1/2} + O(x - x_0) \quad \text{as } x \to x_0, \tag{4.17}$$

for some constant α. In particular, whenever F has a double pole, the solution φ has a cusp.

(4) Peakons occur when the evolution of φ according to (4.14) suddenly changes direction: $\varphi_x \mapsto -\varphi_x$.

If we apply these remarks to

$$\varphi_x^2 = F(\varphi) = \frac{(\varphi^2 - a)(\varphi - c)^2 + b}{(\varphi - c)^2}, \tag{4.18}$$

we can classify the solutions of (4.14). Consider the polynomial $P(\varphi) = (\varphi^2 - a)(\varphi - c)^2$ with a double root at $\varphi = c$. The categories of Theorem 1 correspond to different behaviors of this polynomial. Once a is fixed, a change in b will shift the graph vertically up or down. Hence we can easily determine which b's that yield bounded traveling waves.

There are six qualitatively different cases (see Fig. 3):

(1) If $a \leqslant -\frac{c^2}{8}$, then $P(\varphi)$ is decreasing for $\varphi < c$. There are no bounded solutions.

(2) If $-\frac{c^2}{8} < a < 0$, there are smooth solutions for some negative b's.

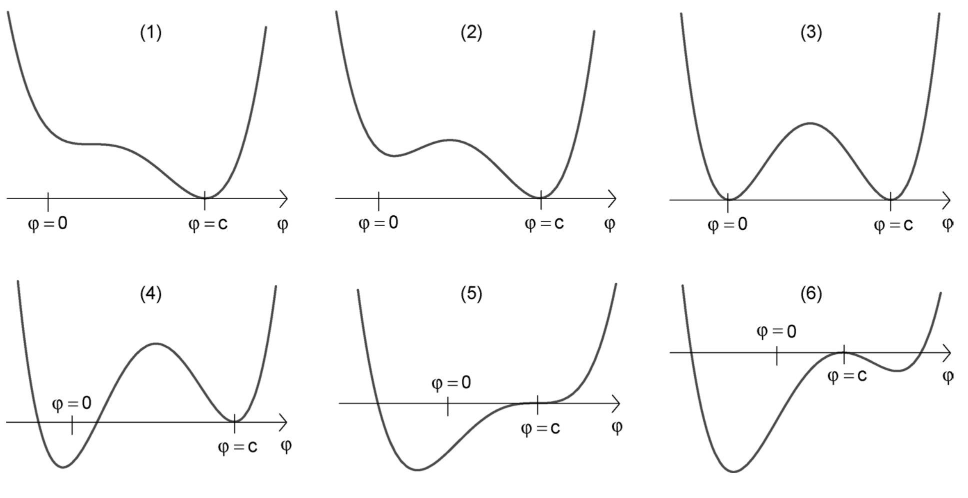

Fig. 3. The graph of the polynomial $P(\varphi) = (\varphi^2 - a)(\varphi - c)^2$ displayed for different values of a. The six cases give rise to the categories (1)–(6) in Theorem 1.

(3) If $a = 0$, there are smooth solutions for some negative b's and a peaked solitary wave for $b = 0$.
(4) If $0 < a < c^2$, there are smooth solutions for some negative b's, a peaked periodic solution for $b = 0$, and cusped solutions for some $b > 0$.
(5) If $a = c^2$, there are cusped solutions for some positive b's and the constant $\varphi \equiv c$ is a solution.
(6) If $a > c^2$, there are cusped and anticusped solutions for some positive b's.

This establishes the categories in Theorem 1.

Once we prove that the waves stated in Theorem 1 belong to $H^1_{\mathrm{loc}}(\mathbb{R})$ and satisfy (TW3), the proof is finished. But this follows from the next lemma, the proof of which proceeds just like in [16].

Lemma 5. *Any bounded function φ satisfying* (TW1) *and* (TW2) *belongs to $H^1_{\mathrm{loc}}(\mathbb{R})$ and satisfies* (TW3).

Acknowledgment

The author thanks the referee for valuable suggestions.

Appendix A. Notation

Let $X \subset \mathbb{R}$ be an open set. We denote, for integers $n, p \geqslant 1$, by $W^{n,p}(X)$ the space of all functions $f \in L^p(X)$ with distributional derivatives $\partial_x^i f \in L^p(X)$ for $i = 1, \dots, n$. If $p = 2$, we write $H^n(X)$ instead of $W^{n,2}(X)$. The Hilbert spaces $H^n(X)$ are endowed with the inner products

$$\langle f, g \rangle_{H^n(X)} = \sum_{i=0}^{n} \int_X \big(\partial_x^i f\big)(x)\big(\partial_x^i g\big)(x)\, dx.$$

We let $C_c^\infty(X)$ be the space of smooth functions with compact support in X. $H^n_{\mathrm{loc}}(X)$ denotes the Hilbert space of all f such that $\phi f \in H^n(X)$ for all test functions $\phi \in C_c^\infty(X)$.

Furthermore, we let $\mathcal{D}'(X)$ be the space of distributions on X, i.e. linear continuous functionals on $C_c^\infty(X)$. $C^n(X)$ denotes the space of n times continuously differentiable functions on X. If $\varphi \in \mathcal{D}'(X)$, we write φ_x for its distributional derivative defined by

$$\langle \varphi_x, \psi \rangle = -\langle \varphi, \psi_x \rangle, \quad \psi \in C_c^\infty(X),$$

where $\langle \cdot, \cdot \rangle$ is the pairing between $\mathcal{D}'(X)$ and $C_c^\infty(X)$.

μ denotes Lebesgue measure on $\mathbb{R}$.

We say that φ is a *wave with decay* if there is a constant $\alpha \in \mathbb{R}$ such that $\varphi - \alpha \in H^1(\mathbb{R})$. A *solitary wave* is a wave with decay to zero at infinity.

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